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Perfect state transfer via quantum probability theory

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Abstract

The transfer of quantum states has played an important role in quantum information processing. In fact, transfer of quantum states from point A to B with unit fidelity is very important for us and we focus on this case. In recent years, in represented works, they designed Hamiltonian in a way that a mirror symmetry creates with with respect to network center. In this paper, we stratify the spin network with respect to an arbitrary vertex of the spin network o then we design coupling coefficient in a way to create a mirror symmetry in Hamiltonian with respect to center. By using this Hamiltonian and represented approach, initial state that have been encoded on the first vertex in suitable

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time and with unit fidelity from it's antipodes vertex can be received. In his work, there is no need to external control.

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1 Introduction

Transfer of a quantum state from one qubit to another is very important for quantum information processing [1]. In the quantum computers high quality communication between different parts of the system is essential, therefore we need to transfer quantum states within the quantum processor. Depending on the technology at hand, this task can be accomplished in a number of ways. One of them is a quantum spin network which can be defined as a collection of interacting quantum two-state systems (or qubits) on a graph. Spin network dynamics is governed by a suitable Hamiltonian such as the Heisenberg or XY Hamiltonian. Spin networks do not require external controls during the transport of information. It is an advantage because any external control can be a source of noise. A Perfect state transfer between two qubits of a spin network is accomplished if a single excitation can travel from one qubit to another with the fidelity being equal to 1. Bose is the first one who suggested to use a 1D chain of N qubits permanently coupled which is described by the nearest-neighbor Heisenberg Hamiltonian [2]

$$H_{ij} = J_{ij}\vec{S}_i \cdot \vec{S}_j \tag{1-1}$$

in

$$H_{Bose} = J\sum_{n=1}^{N-1}(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \sigma_n^z\sigma_{n+1}^z) \tag{1-2}$$

where $\sigma_n^i, \sigma_{n+1}^i$ are the pauli matrices acting on the n-th and $(n+1)$-th qubit with $i = x, y, z$, and J is the coupling strength between qubits. The aim of the suggestion is to transfer a state from end of the chain to the other end after a definite time τ. Bose's idea has motivated many studies focusing on the perfect quantum state transfer in spin networks [3-32]. For a spin chain with two and three qubits, transfer can be achieved perfectly, but for larger N, i.e. $N \geq 4$, it can be shown that perfect transfer is impossible [3]. By modulating the coupling and mirror symmetry of a spin chain, Christandl *et al*, have shown that perfect state transfer can be achieved for spin chain with any N . They have also demonstrated that how spin networks can be used to transfer entangled quantum states and to generate entanglement between different sites in the network. In Ref. [4], Christandl's studies on quantum states have been extended to the high-excitation states. It is also shown that the entangled states in the form of $\alpha|00\cdots0\rangle + \beta|11\cdots1\rangle$ can be transferred perfectly through a spin chain.

Star networks is an example of spin networks which is potentially useful for connecting different parts of a quantum network. Investigation of the quantum state transfer in a star network and generalization of the spin chain engineering problem to the topology of a star network have been considered [5]. For a 3-spin chain the perfect state transfer has been experimentally tested by using liquid nuclear magnetic resonance [6]. It has been suggested that the "dual-rail" encoding which adopts two parallel quantum channels, can achieve arbitrary perfect quantum state transfer [7]. Authors in Ref.[33] have introduced a new method for calculating the probability amplitudes of quantum walk based on spectral distribution. In this method a canonical relation between the Fock space of stratification graph and set of orthogonal polynomials has been established which leads to obtain the probability measure (spectral distribution) of adjacency matrix graph. The method of spectral distribution only requires simple structural data of graph and allows us to avoid a heavy combinational argument often necessary to obtain full description of spectrum of the adjacency matrix. In this paper, we will consider spin networks as graphs in which the vertices are qubits. Then, by using the method which has been introduced in the Ref .[33] we design the spin network and can transfer a quantum state perfectly between antipodes vertices.

The paper is organised as follows: In Sec. 2, we provide a brief review on the concept of the graph, adjacency matrix, transfer of quantum states and fidelity. In Sec. 3, we look at the works in Ref.[33] and its references. We also introduce the concept of the stratification and quantum decomposition by which we can transfer quantum states between various networks antipodes perfectly. To support our method, we provide some interesting examples in Sec. 4. Finally, section 5 is the conclusion.

2 Graph, perfect state transfer between antipodes of spin networks

A graph $G = (V, E)$ is generally defined by two sets. One set $(V(G))$ is called the vertices set, including N integer numbers from 1 to N, $V(G) = \{1, 2, \cdots N\}$. Another set is actually a subset $(E(G))$ of the Cartesian product of the vertices set by itself, $E \subset V \times V$. Two vertices $i, j \in V(G)$ are adjacent if and only if $(i, j) \in E(G)$, and in this case we write $i \sim j$. The degree or valency of a vertex $i \in V$ is defined by

$$\kappa(i) = |\{j \in V | i \sim j\}|, \tag{2-3}$$

where $|A|$ is the cardinality of the set A. Any graph can be characterized by adjacency matrix $A(G)$,

$$A_{ij}(G) = \begin{cases} 1 & \text{if } (i,j) \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

Let us consider a system of N spin-1/2 particles. Each particle is represented by a vertex of G and the Hilbert space associated with G is $(C^2)^{\otimes N}$, where C is the set of the complex numbers. We consider the Hamiltonian of a modified Heisenberg XX model

$$H_G = \sum_{(i,j) \in E(G)} J_{i,j}(\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y), \tag{2-4}$$

Where $J_{i,j}$ is coupling strength between vertices i and j, and σ_i^l and σ_j^l $(l = x, y, z)$ are the Pauli matrices acting on the i-th and j-th vertices, respectively. The standard basis for the 1-qubit Hilbert space can be the set $\{|0\rangle \equiv |\downarrow\rangle\ ,\ |1\rangle \equiv |\uparrow\rangle\}$. According to the quantum state transfer protocol, a quantum state is written as

$$|\psi_{in}\rangle = \alpha|\underline{0}\rangle + \beta|\underline{1}\rangle, \qquad |\alpha|^2 + |\beta|^2 = 1, \tag{2-5}$$

where $|\underline{0}\rangle = |0_1 0_2 \cdots 0_N\rangle$ represent a network state in which all spin are down, and $|\underline{1}\rangle = |1_1 0_2 \cdots 0_N\rangle$ shows a state with the first spin up and the rest down. It is clear that the state

$|\underline{0}\rangle$ is an eigenvector of the Hamiltonian corresponding to zero-value of eigenvalue,

$$H_G|\underline{0}\rangle = E_0|\underline{0}\rangle, \qquad E_0 = 0. \tag{2-6}$$

Thus, α does not change in time. Since the total z -component of the spin by

$$\sigma^z_{tot} = \sum_{i \in V(G)} \sigma_i, \tag{2-7}$$

commutes with H_G, i.e. $[\sigma^z_{tot}, H_G] = 0$. From this relation can be understood that initial state must evolve into single-excitation space, in states that one spin is $|\uparrow\rangle$ state and all other spins are $|\downarrow\rangle$ state. Define $U(t) = e^{-iH_Gt}$ as quantum mechanical time evolution operator, thus network state in time t

$$|\psi(t)\rangle = \alpha|\underline{0}\rangle + \beta \sum_{n=1}^{N} f_{n,1}(t)|n\rangle, \tag{2-8}$$

where, $f_{n,1}(t) = \langle n|U_G(t)|1\rangle$ is the fidelity at time t between n and 1. We say that there is perfect state transfer between vertices N and 1 at time t if and only if

$$|f_{N,1}(t)| = 1. \tag{2-9}$$

To obtain this condition we use the method of the quantum probability theory which we discuss it in the next section.

3 Quantum decomposition and spectral distribution of adjacency matrix

We define a walk of length k (or k steps) for a finite sequence $i_0; i_1; \cdots; i_k \in V(G)$ if i_{n-1} adjacent to i_n ($i_{n-1} \sim i_n$). Also in the Hilbert space, adjacent matrix acts as following

$$A|i\rangle = \sum_{i \sim j} |j\rangle, \tag{3-10}$$

and $i \in V(G)$. Let us define $\partial(i, j)$ be the length of shortest walk connecting i and j vertices. We note that $\partial(i, i) = 0$ for all vertices. We consider one vertex such $o \in V(G)$ as reference vertex and stratify the spin network (graph) with respect to o vertex. Then, the graph can be stratified into a disjoint union of strata

$$V = \bigcup_{k=0}^{\infty} V_k, \qquad V_k = \{i \in V; \partial(o, i) = k\}. \tag{3-11}$$

With each associate class V_k we associate a unit vector in Hilbert space defined by

$$|\phi_k\rangle = \frac{1}{\sqrt{|V_k|}} \sum_{i \in V_k} |i\rangle, \tag{3-12}$$

where $|i\rangle$ denote the eigenket of the i-th vertex at the stratum k. We define $\Gamma(G)$ as a closed subspace of Hilbert space spanned by $\{|\phi_k\rangle\}$, thus we can write

$$\Gamma(G) = \sum_k \oplus\ \mathbf{C}|\phi_k\rangle. \tag{3-13}$$

Now we want to review the quantum decomposition of matrix adjacent associated with the stratification(3-12). We consider A as a adjacency matrix of a graph $G = (V, E)$, then using stratification that was explained above, we can define three matrices A^-, A^+ and A^0 as a follows and for $i \in V_k$

$$A^-|i\rangle = \sum_{j \in V_{k-1}} |j\rangle, \quad A^0|i\rangle = \sum_{j \in V_k} |j\rangle, \quad A^+|i\rangle = \sum_{j \in V_{k+1}} |j\rangle, \tag{3-14}$$

where $j \sim i$. Since $i \in V_k$ and i adjacent to j then $j \in V_{k-1} \bigcup V_{k+1} \bigcup V_k$. Thus we can write

$$A = A^- + A^0 + A^+, \tag{3-15}$$

(for more details see Ref.[34]), we fix a vertex $o \in V$ as an origin of the graph, called reference vertex and consider $|\phi_o\rangle$ as a vector state $|o\rangle$, i.e. $|o\rangle = |\phi_o\rangle$.

According to Ref.[34], $\langle A^m \rangle$ the expectation value of power adjacency matrix A with respect to a state $|\phi_0\rangle$, coincides with the number of m-step walk starting and terminating at o, then by using [34], we define two Szegö-Jacobi sequences $\{\omega_k\}_{k=1}^{\infty}$ and $\{\alpha_k\}_{k=1}^{\infty}$, such that

$$A^-|\phi_k\rangle = \sqrt{\omega_k}|\phi_{k-1}\rangle, \quad A^-|\phi_0\rangle = 0, \quad k \geq 1, \tag{3-16}$$

$$A^0|\phi_k\rangle = \alpha_{k+1}|\phi_k\rangle, \qquad k \geq 0, \tag{3-17}$$

$$A^+|\phi\rangle_k = \sqrt{\omega_{k+1}}|\phi_{k+1}\rangle, \qquad k \geq 0, \tag{3-18}$$

where $\sqrt{\omega_{k+1}} = \frac{|V_{k+1}|^{1/2}}{|V_k|^{1/2}}\kappa_{-(j)}, \kappa_{-(j)} = |\{i \in V_k; i \sim j\}|$ for $j \in V_{k+1}$ and also $\alpha_{k+1} = \kappa_{0(j)}$,such that $\kappa_{0(j)} = |\{i \in V_k; i \sim j\}|$ for $j \in V_k$. Or, equivalently

$$A|\phi_k\rangle = \sqrt{\omega_k}|\phi_{k-1}\rangle + \alpha_{k+1}|\phi_k\rangle + \sqrt{\omega_{k+1}}|\phi_{k+1}\rangle, \tag{3-19}$$

we note that the $|\phi_{-1}\rangle = 0$.

Now, by using the method of quantum decomposition we define the spectral distribution of adjacency matrix [33, 34]. The spectral analysis of operators is an important issue in quantum mechanics, operator theory and mathematical physics. It is well-known, for any pair $(A, |\phi_0\rangle)$ of a matrix A and a vector $|\phi_0\rangle$, one can be assigned a measure μ as

$$\mu(x) = \langle\phi_0|E(x)|\phi_0\rangle, \tag{3-20}$$

where $E(x) = \sum_l |u_l\rangle\langle u_l|$ is the operator of projection onto the eigenspace of A corresponding to eigenvalue x, i.e,

$$A = \int xE(x)dx. \tag{3-21}$$

Also, it is easy to see that for any polynomial $P(A)$ we have

$$P(A) = \int P(x)E(x)dx, \tag{3-22}$$

where for discrete spectrum the above integrals are replaced with summation.

Here we are interested to the spectral distribution of the adjacency matrix of graphs, since the spectrum of a given spin network can be determined by spectral distribution of its adjacency matrix A.

Therefore, by using relations (3-20) and (3-21), the expectation value of the power of adjacency matrix A can be written

$$\langle A^m\rangle = \int_R x^m\mu(dx), \qquad m = 0, 1, 2, ... \tag{3-23}$$

where $\langle . \rangle$ is the expectation value with respect to a reference vector $|\phi_0\rangle$. The existence of the spectral distribution which satisfy Eq. (3-23) is a consequence of Hamburgers theorem [35](see theorem 1.2). Therefore, the spectral distribution μ under question will be characterized by the property of orthogonalizing polynomials $\{P_n\}$ defined recurrently by

$$P_0(x) = 1, \qquad P_1(x) = x - \alpha_1,$$

$$xP_n(x) = P_{n+1}(x) + \alpha_{n+1}P_n(x) + \omega_n P_{n-1}(x), \qquad n \geq 1. \tag{3-24}$$

If such a spectral distribution is unique, the spectral distribution μ is determined by the identity

$$G_\mu(x) = \int_R \frac{\mu(dy)}{x-y} = \frac{1}{x-\alpha_1-\frac{\omega_1}{x-\alpha_2-\frac{\omega_2}{x-\alpha_3-\frac{\omega_3}{x-\alpha_4-\cdots}}}} = \frac{P_{n-1}^{(1)}(x)}{P_n(x)} = \sum_{l=1}^{n} \frac{A_l}{x-x_l}, \tag{3-25}$$

where, x_l are the roots of polynomial P_n. $G_\mu(x)$ is called the Stieltjes transform, A_l is the coefficient in the Gauss quadrature formula and where the polynomials $\{P_n^1\}$ are defined recurrently as

$$P_0^{(1)}(x) = 1, \qquad P_1^{(1)}(x) = x - \alpha_2,$$

$$xP_n^{(1)}(x) = P_{n+1}^{(1)}(x) + \alpha_{n+2}P_n^{(1)}(x) + \omega_{n+1}P_{n-1}^{(1)}(x), \qquad n \geq 1. \tag{3-26}$$

Then, the spectral distribution μ can be recovered from $G_\mu(x)$ by means of the Stieltjes inversion formula:

$$\mu(y) - \mu(x) = -\frac{1}{\pi} \lim_{v \longrightarrow 0^+} \int_x^y Im\{G_\mu(u+iv)\}du. \tag{3-27}$$

Substituting the right hand side of Eq.(3-25) in the Eq.(3-27), the spectral distribution μ can be determined in terms of $x_l, l = 1, 2, \cdots$, and Guass quadrature constant $A_l, l = 1, 2, \cdots$ as

$$\mu = \sum_l A_l \delta(x - x_l). \tag{3-28}$$

Also, by using the recursion relation (3-24) and quantum decomposition of adjacency matrix A the other matrix element $\langle \phi_k | A^m | \phi_0 \rangle$ can be obtained as [33]

$$\langle \phi_k | A^m \mid \phi_0 \rangle = \frac{1}{\sqrt{\omega_1 \omega_2 \cdots \omega_k}} \int_R x^m P_k(x) \mu(dx), \qquad m = 0, 1, 2, ... \tag{3-29}$$

Finally, by using the above approach, we want to transfer quantum states between antipodes of various networks perfectly. After stratificating of each graph in respect to reference vertex o and replacing A whit $e^{-iH_G t}$, we have

$$\langle \phi_k | e^{-iH_G t} \mid \phi_0 \rangle = \frac{1}{\sqrt{\omega_1 \omega_2 \cdots \omega_k}} \int_R e^{-ixt} P_k(x) \mu(dx), \qquad m = 0, 1, 2, ... \tag{3-30}$$

where this relation is $f_{k,1}(t)$ and we use this point that adjacent matrix and Hamiltonian are equivalent. Therefore, if we consider k the antipode strata we can obtain the condition of perfect state transfer on spin networks.

4 Examples

In this section, we want to investigate perfect state transfer on the variety of graphs by using the approaches given.

4.1 Graph G$_n$

As the first example, the graphs taken from column method are investigated. In column method, all vertices can be placed in N column G_n of size

$$|G_n| = \begin{pmatrix} N-1 \\ n-1 \end{pmatrix}, \tag{4-31}$$

that satisfy the following two conditions for n=1,2,...,N:

(i) each vertex in column n is connected to $N-n$ vertices in column n+1

(ii)each vertex in column n+1 is connected to n vertices in column n.

That Hamiltonian XY is defined as following ($J_{n,n+1} = J = 1$)[3],

$$H_G = \sum_{(i,j)\in E(G)} J_{i,j}(\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) = \frac{1}{2} \sum_{(i,j)\in E(G)} J_{i,j}(\sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+), \tag{4-32}$$

If $N = 2$, i.e, only two vertices, for this formalism we have

$$|\phi_0\rangle = |1\rangle, \qquad |\phi_1\rangle = |2\rangle,$$
$$H = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \tag{4-33}$$
$$\omega_1 = \frac{1}{4}, \qquad \alpha_1 = \alpha_2 = \cdots \alpha_n = 0,$$
$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = x^2 - \frac{1}{4}, \tag{4-34}$$

And also

$$G_\mu(x) = \frac{x}{x^2 - \frac{1}{4}}, \qquad \mu(x) = \frac{1}{2}\delta(x - \frac{1}{2}) + \frac{1}{2}\delta(x + \frac{1}{2}), \tag{4-35}$$

Based on equation (3-30),

$$f_{2,1}(t) = \langle\phi_1|e^{-i\lambda H_G t}|\phi_0\rangle = \frac{1}{\sqrt{\omega_1}}\int e^{-i\lambda x t}P_1(x)\mu(x)dx = -i\sin(\frac{\lambda t}{2}). \tag{4-36}$$

If $N = 4$ then ,we will have tree columns that one vertex in first and third columns and two vertex in second column, that degree of each vertex is two. In result basic vectors can be defined as following:

$$|\phi_0\rangle = |1\rangle, \qquad |\phi_1\rangle = \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle), \qquad |\phi_2\rangle = |4\rangle,$$
$$H = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \tag{4-37}$$
$$\omega_1 = \frac{1}{2}, \qquad \omega_2 = \frac{1}{2}, \qquad \alpha_1 = \alpha_2 = \cdots \alpha_n = 0,$$
$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = x^2 - \frac{1}{2}, \qquad P_3(x) = x^3 - x, \tag{4-38}$$

And also

$$G_\mu(x) = \frac{x^2 - \frac{1}{2}}{x^3 - x}, \qquad \mu(x) = \frac{1}{2}\delta(x) + \frac{1}{4}\delta(x-1) + \frac{1}{4}\delta(x+1), \tag{4-39}$$

Using equation (3-30),

$$f_{3,1}(t) = \langle\phi_2|e^{-i\lambda H_G t}|\phi_0\rangle = \frac{1}{\sqrt{\omega_1\omega_2}}\int e^{-i\lambda xt}P_2(x)\mu(x)dx,$$

$$= \sin^2(\frac{\lambda t}{2}) = -(-i\sin(\frac{\lambda t}{2}))^2. \tag{4-40}$$

If we consider $N = 8$ then, we will have four column that one vertex in first and forth column and three vertices in second and third columns, that degree of each vertex is three. In result basic vectors can be defined as following

$$|\phi_0\rangle = |1>, \quad |\phi_1>= \frac{1}{\sqrt{3}}(|2\rangle + |3\rangle + |4\rangle), \quad |\phi_2\rangle = \frac{1}{\sqrt{3}}(|5\rangle + |6\rangle + |7\rangle), \quad |\phi_3\rangle = |8\rangle,$$

$$H = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}, \tag{4-41}$$

$$\omega_1 = \frac{3}{4}, \quad \omega_2 = 1, \quad \omega_3 = \frac{3}{4}, \quad \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0,$$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = x^2 - \frac{3}{4}, \quad P_3(x) = x^3 - \frac{7}{4}x, \quad P_4(x) = x^4 - \frac{5}{2}x^2 + \frac{9}{16}, \tag{4-42}$$

so

$$G_\mu(x) = \frac{x^3 - \frac{7}{4}x}{x^4 - \frac{5}{2}x^2 + \frac{9}{16}}, \quad \mu(x) = \frac{3}{8}\delta(x - \frac{1}{2}) + \frac{1}{8}\delta(x - \frac{3}{2}) + \frac{1}{8}\delta(x + \frac{3}{2}) + \frac{3}{8}\delta(x + \frac{1}{2}), \tag{4-43}$$

Using equation (3-30),

$$f_{4,1}(t) = \langle\phi_3|e^{-i\lambda H_G t}|\phi_0\rangle = \frac{1}{\sqrt{\omega_1\omega_2\omega_3}}\int e^{-i\lambda xt}P_3(x)\mu(x)dx = i\sin^3(\frac{\lambda t}{2}) = (-i\sin(\frac{\lambda t}{2}))^3. \tag{4-44}$$

If $N = 16$ then, we will have five columns that one vertex in first and fifth columns, four vertex in second and fourth columns and sex vertex in third columns. Degree of each vertex is four, in result basic vectors can be defined as following:

$$|\phi_0\rangle = |1\rangle,$$

$$|\phi_1\rangle = \frac{1}{\sqrt{4}}(|2\rangle + |3\rangle + |4\rangle + |5\rangle),$$

$$|\phi_2\rangle = \frac{1}{\sqrt{6}}(|6\rangle + |7\rangle + |8\rangle + |9\rangle + |10\rangle + |11\rangle),$$

$$|\phi_3\rangle = \frac{1}{\sqrt{4}}(|12\rangle + |13\rangle + |14\rangle + |15\rangle),$$

$$|\phi_4\rangle = |16\rangle,$$

$$H = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{4-45}$$

$$\omega_1 = 1, \quad \omega_2 = \frac{3}{2}, \quad \omega_3 = \frac{3}{2} \quad , \omega_4 = 1, \quad \alpha_1 = \alpha_2 = \cdots \alpha_n = 0,$$

$$P_0(x) = 1 \quad P_1(x) = x, \quad P_2(x) = x^2 - 1, \quad P_3(x) = x^3 - \frac{5}{2}x,$$

$$P_4(x) = x^4 - 4x^2 + \frac{3}{2}, \quad P_5(x) = x^5 - 5x^3 + 4x, \tag{4-46}$$

And also,

$$G_\mu(x) = \frac{x^2 - 4x + \frac{3}{2}}{x^5 - 5x + 4x}, \quad \mu(x) = \frac{1}{16}\delta(x-2) + \frac{1}{4}\delta(x-1) + \frac{3}{8}\delta(x) + \frac{1}{4}\delta(x+1) + \frac{1}{16}\delta(x+2), \tag{4-47}$$

Based on equation (3-30),

$$f_{5,1}(t) = \langle\phi_4|e^{-i\lambda H_G t}|\phi_0\rangle = \frac{1}{\sqrt{\omega_1\omega_2\omega_3\omega_4}} \int e^{-i\lambda x t} P_4(x)\mu(x) = \sin^4(\frac{\lambda t}{2}) = (-i\sin(\frac{\lambda t}{2}))^4. \tag{4-48}$$

If $t = \pi/\lambda$, then in all cases above, we have $|f_{2,1}(t)| = |f_{3,1}(t)| = |f_{4,1}(t)| = |f_{5,1}(t)| = 1$ that the prefect quantum state transfer is obtained.

4.2 W network

Now, we want to investigate PST in case of W network. W network is a eight-vertex and $E(W) = \{\{1, i\}, \{j, 8\}2 \leq i, j \leq 7; \}$. We can consider this graph as a three-columns graph that there is one vertex in first and third column and are six vertex in second column. Vertex degree in second column is two and vertex degree in first and third is six. In this network, we have

$$|\phi_0\rangle = |1\rangle, \quad |\phi_1\rangle = \frac{1}{\sqrt{6}}(|2\rangle + |3\rangle + |4\rangle + |5\rangle + |6\rangle + |7\rangle), \quad |\phi_2\rangle = |8\rangle,$$

$$H = \begin{pmatrix} 0 & \sqrt{\frac{3}{2}} & 0 \\ \sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{3}{2}} \\ 0 & \sqrt{\frac{3}{2}} & 0 \end{pmatrix}, \tag{4-49}$$

$$\omega_1 = \frac{3}{2}, \quad \omega_1 = \frac{3}{2}, \quad \alpha_1 = \alpha_2 = \cdots \alpha_n = 0,$$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = x^2 - \frac{3}{2}, \quad P_3(x) = x^3 - 3x, \tag{4-50}$$

so

$$G_\mu(x) = \frac{x^2 - \frac{3}{2}}{x^3 - 3}, \quad \mu(x) = \frac{1}{2}\delta(x) + \frac{1}{4}\delta(x - \sqrt{3}) + \frac{1}{4}\delta(x + \sqrt{3}), \tag{4-51}$$

Based on equation (3-30),

$$f_{3,1}(t) = \langle\phi_2|e^{-i\lambda H(G)t}|\phi_0\rangle = \frac{1}{\sqrt{\omega_1\omega_2}} \int e^{-i\lambda xt} P_2(x)\mu(x) = -\sin^2(\frac{\sqrt{3}\lambda t}{2}). \tag{4-52}$$

If we consider $t = \frac{\pi}{\sqrt{3}\lambda}$, we have prefect quantum state transfer $|f_{3,1}(t = \frac{\pi}{\sqrt{3}\lambda})| = 1$.

4.3 Binary tree network

Here, we want to investigate quantum state transfer in Binary tree network with $N = 7$ vertices. In this network, we will have tree column that one vertex is in first column with two degrees and two vertices are in second column with tree degrees and four vertices in third column with one degree. Then we have

$$|\phi_0\rangle = |1\rangle, \quad |\phi_1\rangle = \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle), \quad |\phi_2\rangle = \frac{1}{\sqrt{4}}(|4\rangle + |5\rangle + |6\rangle + |7\rangle),$$

$$H = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \tag{4-53}$$

$$\omega_1 = \frac{1}{2}, \quad \omega_2 = \frac{1}{2}, \quad \alpha_1 = \alpha_2 = \cdots \alpha_n = 0,$$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = x^2 - \frac{1}{2}, \quad P_3(x) = x^3 - x, \tag{4-54}$$

also

$$G_\mu(x) = \frac{x^2 - \frac{1}{2}}{x^3 - x}, \quad \mu(x) = \frac{1}{2}\delta(x) + \frac{1}{4}\delta(x-1) + \frac{1}{4}\delta(x+1). \tag{4-55}$$

Using equation (3-30),

$$f_{3,1}(t) = \langle\phi_2|e^{-i\lambda H_G t}|\phi_0\rangle = -\sin^2(\frac{\lambda t}{2}). \tag{4-56}$$

To obtain prefect quantum state transfer we consider $t = \pi/\lambda$, i.e, $|f_{3,1}(t = \pi/\lambda)| = 1$.

4.4 Linear spin chain

In this example, we want to follow the above discussion(PST) through a spin chain:

$$H_G = \sum_{n=1}^{N-1} J_{n,n+1}(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y) = \frac{1}{2}\sum_{n=1}^{N-1} J_{n,n+1}(\sigma_n^+ \sigma_{n+1}^- \sigma_{n+1}^+),$$

where $J_{n,n+1} = \sqrt{n(N-n)}$.

If $N = 2$, i.e. total number of vertices is two, then

$$|\phi_0\rangle = |1\rangle, \quad |\phi_1\rangle = |2\rangle,$$

$$H = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \tag{4-57}$$

$$\omega_1 = \frac{1}{4}, \qquad \alpha_1 = \alpha_2 = \cdots \alpha_n = 0,$$

also

$$G_\mu(x) = \frac{x}{x^2 - \frac{1}{4}}, \qquad \mu(x) = \frac{1}{2}\delta(x - \frac{1}{2}) + \frac{1}{2}\delta(x + \frac{1}{2}), \tag{4-58}$$

Based on equation (3-30),

$$f_{2,1}(t) = \langle \phi_1 | e^{-i\lambda H_G t} | \phi_0 \rangle = \frac{1}{\sqrt{\omega_1}} \int e^{-i\lambda x t} P_1(x)\mu(x)dx = -i\sin(\frac{\lambda t}{2}). \tag{4-59}$$

If $N = 3$ then

$$|\phi_0\rangle = |1\rangle, \qquad |\phi_1\rangle = |2\rangle, \qquad |\phi_2\rangle = |3\rangle,$$

$$H = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \tag{4-60}$$

$$\omega_1 = \frac{1}{2}, \qquad \omega_2 = \frac{1}{2}, \qquad \alpha_1 = \alpha_2 = \cdots \alpha_n = 0,$$

$$G_\mu(x) = \frac{x^2 - \frac{1}{2}}{x^3 - x}, \qquad \mu(x) = \frac{1}{2}\delta(x) + \frac{1}{4}\delta(x-1) + \frac{1}{4}\delta(x+1), \tag{4-61}$$

$$f_{3,1}(t) = \langle \phi_2 | e^{-i\lambda H_G t} | \phi_0 \rangle = \frac{1}{\sqrt{\omega_1 \omega_2}} \int e^{-i\lambda x t} P_2(x)\mu(x)dx = \sin^2(\frac{\lambda t}{2}) = -(-i\sin(\frac{\lambda t}{2}))^2. \tag{4-62}$$

if $N = 4$ then

$$|\phi_0\rangle = |1\rangle, \qquad |\phi_1\rangle = |2\rangle, \qquad |\phi_2\rangle = |3\rangle, \qquad |\phi_3\rangle = |4\rangle,$$

$$H = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}, \tag{4-63}$$

$$\omega_1 = \frac{3}{4}, \quad \omega_2 = 1, \quad \omega_3 = \frac{3}{4}, \quad \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0,$$

$$G_\mu(x) = \frac{x^3 - \frac{7}{4}x}{x^4 - \frac{5}{2}x^2 + \frac{9}{16}}, \quad \mu(x) = \frac{3}{8}\delta(x - \frac{1}{2}) + \frac{1}{8}\delta(x - \frac{3}{2}) + \frac{1}{8}\delta(x + \frac{3}{2}) + \frac{3}{8}\delta(x + \frac{1}{2}). \quad (4\text{-}64)$$

Using equation (3-30)

$$f_{4,1}(t) = \langle\phi_3|e^{-i\lambda H_G t}|\phi_0\rangle = \frac{1}{\sqrt{\omega_1\omega_2\omega_3}}\int e^{-i\lambda x t}P_3(x)\mu(x)dx = i\sin^3(\frac{\lambda t}{2}) = (-i\sin(\frac{\lambda t}{2}))^3. \quad (4\text{-}65)$$

If $N = 5$,

$$|\phi_0\rangle = |1\rangle, \quad |\phi_1\rangle = |2\rangle, \quad |\phi_2\rangle = |3\rangle, \quad |\phi_3\rangle = |4\rangle, \quad |\phi_4\rangle = |5\rangle,$$

$$H = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (4\text{-}66)$$

$$\omega_1 = 1, \quad \omega_2 = \frac{3}{2}, \quad \omega_3 = \frac{3}{2} \quad ,\omega_4 = 1, \quad \alpha_1 = \alpha_2 = \cdots \alpha_n = 0,$$

also

$$G_\mu(x) = \frac{x^2 - 4x + \frac{3}{2}}{x^5 - 5x + 4x}, \quad \mu(x) = \frac{1}{16}\delta(x-2) + \frac{1}{4}\delta(x-1) + \frac{3}{8}\delta(x) + \frac{1}{4}\delta(x+1) + \frac{1}{16}\delta(x+2), \quad (4\text{-}67)$$

Based on equation (3-30)

$$f_{5,1}(t) = \langle\phi_4|e^{-i\lambda H_G t}|\phi_0\rangle = \frac{1}{\sqrt{\omega_1\omega_2\omega_3\omega_4}}\int e^{-i\lambda x t}P_4(x)\mu(x) = \sin^4(\frac{\lambda t}{2}) = (-i\sin(\frac{\lambda t}{2}))^4, \quad (4\text{-}68)$$

and if we continue this process, we will have [4],

$$f_{N,1}(t) = \langle\phi_{N-1}|e^{-i\lambda H_G t}|\phi_0\rangle = \left[-i\sin(\frac{\lambda t}{2})\right]^{N-1}. \quad (4\text{-}69)$$

In result, in $t = \pi/\lambda$ we have perfect state transfer between antipodal, i.e,

$$|f_{N,1}(t = \pi/\lambda)| = 1.$$

In following cases, with change $J_{n,n+1}$ we can transfer quantum states between antipodal perfectly.

4.5 Star network

We consider star network with $N = 5$. In this network, one vertex with one degree is in first column and one vertex with five degree is in second column and three vertices with one degree are in third column. Also in this case we have

$$J_{1,2} = \sqrt{3}, \quad J_{2,3} = J_{2,4} = J_{2,5} = 1, \tag{4-70}$$

$$|\phi_0\rangle = |1\rangle, \quad |\phi_1\rangle = |2\rangle, \quad |\phi_2\rangle = \frac{1}{\sqrt{3}}(|3\rangle + |4\rangle + |5\rangle),$$

$$H = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}, \tag{4-71}$$

$$\omega_1 = \omega_2 = \frac{3}{2}, \quad \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

$$P_0 = 1, \quad P_1 = x, \quad P_2 = x^2 - \frac{3}{4}, \quad P_3 = x^3 - \frac{3}{2}x, \tag{4-72}$$

and also

$$G_\mu(x) = \frac{x^2 - \frac{3}{4}}{x^3 - \frac{3}{2}x}, \qquad \mu(x) = \frac{1}{2}\delta(x) + \frac{1}{4}\delta(x - \sqrt{\frac{3}{2}}) + \frac{1}{4}\delta(x + \sqrt{\frac{3}{2}}), \tag{4-73}$$

$$f_{3,1}(t) = \langle\phi_2|e^{-i\lambda H_G t}|\phi_0\rangle = \frac{1}{\sqrt{\omega_1\omega_2}}\int e^{-i\lambda x t}P_2(x)\mu(x)dx = -\sin^2(\sqrt{\frac{3}{2}}\lambda t). \tag{4-74}$$

In result, for $t = \sqrt{\frac{2}{3}}\pi/\lambda$ we have perfect state transfer between antipodal.

4.6 Circulant network

Circulant network with $N = 6$ is define as following: one vertex is in first and fourth column and two vertices are in second and third column that degree of each vertex in this network is two. For this network, we have

$$J_{1,2} = J_{1,3} = \sqrt{\frac{3}{2}}, \quad J_{2,4} = J_{3,5} = 2, \quad J_{4,6} = J_{5,6} = \sqrt{\frac{3}{2}}, \tag{4-75}$$

$$|\phi_0\rangle = |1\rangle, \quad |\phi_1\rangle = \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle), \quad |\phi_2\rangle = \frac{1}{\sqrt{2}}(|4\rangle + |5\rangle) \quad |\phi_3\rangle = |6\rangle,$$

$$H = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}, \tag{4-76}$$

$$\omega_1 = \frac{3}{4}, \quad \omega_2 = 1, \quad \omega_3 = \frac{3}{4}, \quad \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0,$$
$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = x^2 - \frac{3}{4}, \quad P_3(x) = x^3 - \frac{7}{4}x, \quad P_4(x) = x^4 - \frac{5}{2}x^2 + \frac{9}{16}, \tag{4-77}$$

so

$$G_\mu(x) = \frac{x^3 - \frac{7}{4}x}{x^4 - \frac{5}{2}x^2 + \frac{9}{16}}, \quad \mu(x) = \frac{3}{8}\delta(x - \frac{1}{2}) + \frac{1}{8}\delta(x - \frac{3}{2}) + \frac{1}{8}\delta(x + \frac{3}{2}) + \frac{3}{8}\delta(x + \frac{1}{2}), \tag{4-78}$$

Using equation (3-30)

$$f4,1(t) = \langle\phi_3|e^{-i\lambda H_G t}|\phi_0\rangle = \frac{1}{\sqrt{\omega_1\omega_2\omega_3}}\int e^{-i\lambda x t}P_3(x)\mu(x)dx = i\sin^3(\frac{\lambda t}{2}). \tag{4-79}$$

In result, for $t = \pi/\lambda$ we have perfect state transfer.

4.7 Binary tree network with modulating coupling

In this example, we consider binary tree spin networks with $N = 16$ vertices with

$$J_{1,2} = 2, \quad J_{2,3} = J_{2,4} = \sqrt{3}, \quad J_{3,5} = J_{3,6} = J_{4,7} = J_{4,8} = \sqrt{3}$$

$$J_{5,9} = J_{5,10} = J_{6,11} = J_{6,12} = J_{7,13} = J_{7,14} = J_{8,15} = J_{8,16} = \sqrt{2}. \tag{4-80}$$

For this graph. we have

$$|\phi_0\rangle = |1\rangle, \qquad |\phi_1\rangle = |2\rangle, \qquad |\phi_2\rangle = \frac{1}{\sqrt{2}}(|3\rangle + |4\rangle),$$

$$|\phi_3\rangle = \frac{1}{\sqrt{4}}(|5\rangle + |6\rangle + |7\rangle + |8\rangle)+$$

$$\frac{1}{\sqrt{8}}(|9\rangle + |10\rangle + |11\rangle + |12\rangle + |13\rangle + |14\rangle + |15\rangle + |16\rangle),$$

where the form of Hamiltonian in the above basis is

$$H = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{4-81}$$

$$\omega_1 = 1, \qquad \omega_2 = \frac{3}{2}, \qquad \omega_3 = \frac{3}{2} \quad , \omega_4 = 1, \qquad \alpha_1 = \alpha_2 = \cdots \alpha_n = 0,$$

$$P_0(x) = 1 \quad P_1(x) = x, \qquad P_2(x) = x^2 - 1, \qquad P_3(x) = x^3 - \frac{5}{2}x,$$

$$P_4(x) = x^4 - 4x^2 + \frac{3}{2}, \qquad P_5(x) = x^5 - 5x^3 + 4x, \tag{4-82}$$

also

$$G_\mu(x) = \frac{x^2 - 4x + \frac{3}{2}}{x^5 - 5x + 4x}, \qquad \mu(x) = \frac{1}{16}\delta(x-2) + \frac{1}{4}\delta(x-1) + \frac{3}{8}\delta(x) + \frac{1}{4}\delta(x+1) + \frac{1}{16}\delta(x+2). \tag{4-83}$$

Based on equation (3-30)

$$f_{5,1}(t) = \langle\phi_4|e^{-i\lambda H_G t}|\phi_0\rangle = \frac{1}{\sqrt{\omega_1\omega_2\omega_3\omega_4}}\int e^{-i\lambda x t} P_4(x)\mu(x) = \sin^4(\frac{\lambda t}{2}). \tag{4-84}$$

In result, for $t = \pi/\lambda$ we have perfect state transfer between antipodal.

5 Conclusion

In this paper, we have focused on perfect quantum states transfer between antipodes of networks i.e, fidelity equivalent one. Thus, we have presented a total approach for perfect quantum states transfer and by using this, we were able to transfer quantum states in column networks, W network, spin chain, star network,$\cdots$ perfectly. By using this approach, we can transfer quantum states in various network between antipodes with correct choice $J_{m,n}$ perfectly. Advantage of this approach have is that there is no need to have complicated eigenvalue and eigenvector computations. For example in column network with 16 vertices, we have a 5×5 Hamiltonian instead of a 16×16 Hamiltonian.

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