Partial Standard Quantum Process Tomography

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With the general theory of standard quantum process tomography, we shall develop a scheme to decide an arbitrary matrix element of χ , which is in the Choi matrix representation, in a scalable way.

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The characterization of the evolution of a quantum system is one of the main tasks to accomplish to achieve quantum information processing. A general class of methods, which have been developed in quantum information theory to accomplish this task is known as quantum process tomography (QPT)- for a review of quantum tomography, see Refs. [1-3]. Under general assumptions, the evolution of quantum system can be represented by a linear, completely-positive map which can be written as

$$\varepsilon(\rho) = \sum_{m} E^{m} \rho(E^{m})^{\dagger} \tag{1}$$

with $\sum_{m} (E^{m})^{\dagger} E^{m} = \mathbf{I}$. Using a fixed set of operators $\{\tilde{E}_{i}\}$, which form a basis for the set of operators on the state space, so that $E^{m} = \sum_{i} e_{i}^{m} \tilde{E}_{i}$, one may introduce the χ matrix representation with $\chi_{ij} = \sum_{m} e_{i}^{m} (e_{j}^{m})^{*}$ for its matrix elements, and rewrite (1) in the way like

$$\varepsilon(\rho) = \sum_{ij} \chi_{ij} \tilde{E}_i(\rho) \tilde{E}_j^{\dagger}.$$
 (2)

The task of QPT can be viewed as to decide χ via different protocols. Standard quantum process tomography (SQPT) was the first method proposed [1,4,5]. It involves preparing a set of input $\{\rho_j\}$ and measuring its output $\varepsilon(\rho_j)$ via the quantum state tomography. With the denotations that $\varepsilon(\rho_j) = \sum_k \lambda_{jk}\rho_k$ and $\tilde{E}_m \rho_j \tilde{E}_m^{\dagger} = \sum_j \beta_{ij}^{mn} \rho_j$, where β_{ij}^{mn} are complex numbers which can be determined by standard algorithms from linear algebra given the $\{\tilde{E}_m\}$ operators and $\{\rho_j\}$ operators [1], one may get the relation

$$\sum_{mn} \beta_{ij}^{mn} \chi_{mn} = \lambda_{ij}, \qquad (3)$$

where χ_{mn} can be determined by the given parameters β_{ij}^{mn} and the data of λ_{ij} known from quantum state tomography. Besides SQPT, other methods like the ancillaassisted quantum process tomography have also been proposed [6].

Usually, the complete characterization of χ matrix is a non-scalable task: For the N d-levels system, there are about d^{4N} elements to be decided. Recently, a series of works have demonstrated that it is possible to extract partial but nevertheless relevant information about the quantum process in an efficient and scalable way [7-13]. These approaches share an essential feature that: They are based on the idea that the tomography of a quantum map can be efficiently performed by studying certain properties of a twirling of such a map. Another method, the so-called direct characterization of quantum dynamics (DCQD), was also constructed for use in partial characterization of quantum dynamics [14-16].

Here, we shall focus on the topic whether the general theorem of SQPT developed in [1] can be applied for partial QPT. The case, where the χ is defined by the Choi operators, should be discussed in present work. With the general theory of SQPT in [1], we shall develop a scheme to decide an arbitrary χ matrix element, which certainly carries some partial information about the quantum process [13], in a scalable way. We shall show that: For the χ in the Choi matrix representation, to decide its diagonal element one need just a single measurement while to decide its off-diagonal we require about sixteen measurements. This result is independent of the actual dimension of the system. Our scheme also represents a new kind of method of partial QPT without of ancilla. In following argument, we shall to show how to construct our scheme in detail.

The general theorem.- With $\{|a\rangle\}_{a=0,1,\dots,D-1}$ the basis of the D-dimensional system, we introduce the Choi matrix

$$\tilde{E}_{ab} = |a\rangle\langle b|. \tag{4}$$

This set of operators, $\{\tilde{E}_{ab}\}_{a,b=0,1,\dots,D-1}$, forms an orthogonal basis with the algebra that $\operatorname{Tr}(\tilde{E}_{ab}^{\dagger}\tilde{E}_{cd}) = \delta_{ac}\delta_{bd}$. Any $D \times D$ matrix, say, M, can be expanded with it, $M = \sum_{ab=0}^{D-1} M_{ab}\tilde{E}_{ab}$, where the expanding coefficients $M_{ab} = \operatorname{Tr}(\tilde{E}_{ab}^{\dagger}M)$ are just the matrix elements of M. Let each Kraus-operator in $\{E^m\}$ to be expanded with the conjugated Choi matrices in (6), $E^m = \sum_{e,f=0}^{D-1} e_{ef}^m \tilde{E}_{ef}$ with $e_{ef}^m = \operatorname{Tr}[\tilde{E}_{ef}^{\dagger}E^m]$ and $(E^m)^{\dagger} = \sum_{g,h=0}^{D-1} (e_{gh}^m)^{*}(\tilde{E}_{gh})^{\dagger}$, we can define the χ in the Choi matrix representation with its elements to be

$$\chi_{ef;gh} = \sum_{m} e_{ef}^{m} (e_{gh}^{m})^{*}.$$
 (5)

We call $\chi_{ef;gh}$ the diagonal matrix element if e = g and f = h, else, we call it the off-diagonal matrix element.

The relation in (3), which is one of the main results of the general theorem of SQPT [1], puts no constraints on our choices to define λ and χ matrices. In present work, we use $\{\tilde{E}_{ab}\}$ to describe the input and $\varepsilon(\tilde{E}_{ab})$ for its output. Let $\varepsilon(\tilde{E}_{ab})$ be expanded with the set of Choi operators in (4), $\varepsilon(\tilde{E}_{ab}) = \sum_{c,d=0}^{D-1} \lambda_{ab;cd} \tilde{E}_{cd}$ with its elements to be

$$\lambda_{ab;cd} = \operatorname{Tr}[\tilde{E}_{cd}^{\dagger} \varepsilon(\tilde{E}_{ab})].$$
(6)

With the above definitions in hands, it can be shown that there exists a one-to-one mapping between the λ matrix element and the χ matrix element,

$$\lambda_{ab;cd} = \chi_{ca;db}.\tag{7}$$

This result can be proved by the general theory of SQPT. From the expressions of $\lambda_{ab;cd}$ and $\chi_{ef;gh}$ in above, there should be

$$\lambda_{ab;cd} = \sum_{e,f,g,h=0}^{D-1} \beta_{ab;cd}^{ef;gh} \chi_{ef;gh}$$
(8)

where the transformation matrix β has its elements to be

$$\beta_{ab;cd}^{ef;gh} = \operatorname{Tr}(\tilde{E}_{cd}^{\dagger}\tilde{E}_{ef}\tilde{E}_{ab}\tilde{E}_{gh}^{\dagger}).$$
(9)

Each matrix element of β can be directly calculated, $\beta_{ab;cd}^{ef;gh} = \operatorname{Tr}[|d\rangle\langle c|) \cdot |e\rangle\langle f| \cdot |a\rangle\langle b| \cdot (|h\rangle\langle g|)] = \delta_{ec}\delta_{fa}\delta_{gd}\delta_{hb}.$ With columns indexed by ef;gh and rows by ab;cd, β can be expressed as a $D^4 \times D^4$ matrix. In each row and each column of it, there is just one non-zero matrix elements (with the value of 1). Furthermore, one may verify that the determinant of β equals 1 while its inverse is the transpose of itself,

$$\det(\beta) = 1, \beta^{-1} = \beta^{\mathsf{T}}.$$
(10)

According to the famous Cramer's rule [17]: If $A\mathbf{x} = \mathbf{b}$ is a system of n linear equations in n unknowns such that det(A) $\neq 0$, then the system has a unique solutions. Equation (8) can be viewed as such a set of linear equations with: $\mathbf{A} \to \beta$, $\mathbf{x} \to \chi$, and $\mathbf{b} \to \lambda$. From (10), we conclude that χ is unambiguously determined by λ , $\chi = \beta^{\mathrm{T}} \lambda$, since det(β) = 1. Finally, using the known result of $\beta_{ab;cd}^{ef;gh}$ for (8), we shall get the one-to-one mapping described by (7).

As a direct application of the one-to-one mapping in (8), we find that the diagonal matrix element $\chi_{ab;ab}(\lambda_{bb;aa})$ is physical meaningful: it represents the probability of the transition from the initial a-th level to the final b-th level,

$$\chi_{ab;ab} = \langle b|\varepsilon(|a\rangle\langle a|)|b\rangle. \tag{11}$$

In other words, the classical measurement of the transition probability of the D-levels atom can be also viewed as to decide the diagonal χ matrix element in the scheme of SQPT. With a simple reasoning, we have

$$\operatorname{Tr}\chi = \sum_{a,b=0}^{D-1} \chi_{ab;ab} = \sum_{a=0}^{D-1} \operatorname{Tr}[\varepsilon(|a\rangle\langle a|)].$$
(12)

For the trace-preserving cases, $Tr\chi = D$.

Choi matrix SQPT.-As we have shown, the χ matrix in (5) should be given if the λ matrix in (6) has been decided. However, the λ matrix elements, $\lambda_{ab;cd} = \operatorname{Tr}[\tilde{E}_{cd}^{\dagger}\varepsilon(\tilde{E}_{ab})]$, can not be directly measured if one of the operators, \tilde{E}_{cd}^{\dagger} and \tilde{E}_{ab} , is non-Hermitian. This problem can be solved with the following protocol: At first, one may introduce a set of linearly independent states, $\{|\Psi_m\rangle\}_{m=1,\dots,D^2}$, and expand each \tilde{E}_{ab} as

$$\tilde{E}_{ab} = \sum_{m=1}^{D^2} r_m^{ab} |\Psi_m\rangle \langle \Psi_m|$$
(13)

where the coefficients r_m^{ab} are known. Now, $\varepsilon(\tilde{E}_{ab}) = \sum_{m=1}^{D^2} r_m^{ab} \varepsilon(|\Psi_m\rangle \langle \Psi_m|)$ since that the operation of ε is linear. Then, giving a set of linear independent Hermitian operators $\{\hat{O}_n\}_{n=1,...,D^2}$ $(\hat{O}_n = \hat{O}_n^{\dagger})$, we rewrite \tilde{E}_{cd}^{\dagger} as

$$\tilde{E}_{cd}^{\dagger} = \sum_{n=1}^{D^2} s_n^{cd} \hat{O}_n \tag{14}$$

with known parameters s_n^{cd} . Because that performing trace is also a linear operation, there should be $\operatorname{Tr}[\tilde{E}_{cd}(\ldots)] = \sum_{n=1}^{D^2} s_n^{cd} \operatorname{Tr}[\hat{O}_n(\ldots)]$. Finally, the way of measuring $\lambda_{ab;cd}(\chi_{ca;db})$ is clear,

$$\lambda_{ab;cd} = \sum_{m,n=1}^{D^2} r_m^{ab} s_n^{cd} \operatorname{Tr}[\hat{O}_n \varepsilon(|\Psi_m\rangle \langle \Psi_m|)].$$
(15)

The configuration space of the measurements is known,

$$\mathcal{M}: \{ \operatorname{Tr}[\hat{O}_n \varepsilon(|\psi_m \rangle \langle \psi_m |)] \}_{m,n=1,2,\dots,D^2}.$$

To perform the complete SQPT, we shall carry out all $D^2 \times D^2$ measurements in \mathcal{M} and decide each $\lambda(\chi)$ matrix element one by one according to their known coefficients, r_i^{ab} and s_j^{cd} , here. However, if only a selected $\lambda_{ab;cd}(\chi_{ca;db})$ is to be measured, one may just perform the measurements satisfying the constraint that $r_n^{ab} \cdot s_m^{cd} \neq 0$.

In fact, it has been shown that each \tilde{E}_{ab} can always be expanded with four pure states [1]. Introducing

$$|ab,+\rangle = \frac{\sqrt{2}}{2}(|a\rangle + |b\rangle), |ab,-\rangle = \frac{\sqrt{2}}{2}(|a\rangle + i|b\rangle), \quad (16)$$

for a < b, each non-Hermitian \tilde{E}_{ab} can be expanded with

$$\tilde{E}_{ab} = |ab, +\rangle \langle ab, +| + i|ab, -\rangle \langle ab, -| -\frac{1+i}{2} (|a\rangle \langle a| + |b\rangle \langle b|).$$
(17)

For the case a > b, \tilde{E}_{ab} can be directly derived with $\tilde{E}_{ab} = \tilde{E}_{ba}^{\dagger}$. Formally, for $a \neq b$, we have $\tilde{E}_{ab} = \sum_{i=1}^{4} r_i^{ab} |\psi_i^{ab}\rangle \langle \psi_j^{ab}|$ with $\forall |\psi_i^{ab}\rangle \in \{|a\rangle, |b\rangle, |ab, \pm\rangle\}$. In the similar way, we expand \tilde{E}_{cd}^{\dagger}

with $\tilde{E}_{cd}^{\dagger} = \sum_{j=1}^{4} s_j^{cd} |\psi_j^{cd}\rangle \langle \psi_j^{cd}|$ by requiring $\forall |\psi_j^{cd}\rangle \in \{|c\rangle, |d\rangle, |cd, \pm\rangle\}$. Now, equation (15) is simplified into

$$\lambda_{ab;cd} = \sum_{i,j=1}^{4} r_i^{ab} s_j^{cd} \langle \psi_j^{cd} | \varepsilon(|\psi_i^{ab}\rangle \langle \psi_i^{ab}|) | \psi_j^{cd} \rangle.$$
(18)

To decide the off-diagonal matrix element of χ , we need sixteen measurements by taking $|\psi_i^{ab}\rangle$ for the input state and measuring its output state $\varepsilon(|\psi_i^{ab}\rangle\langle\psi_i^{ab}|)$ with the projective operator $|\psi_j^{cd}\rangle\langle\psi_j^{cd}|$.

It can be seen that the number of all the possible input states, $|\psi_j^{ab}\rangle$ with $0 \le a < b \le D - 1$ and j=1, 2, 3, 4, is limited to be D^2 : The basis vector $|a\rangle$ has D terms, while $\frac{\sqrt{2}}{2}(|a\rangle + |b\rangle)$ and $\frac{\sqrt{2}}{2}(|a\rangle + i|b\rangle)$ has the same number of $\frac{1}{2}D(D-1)$, respectively. For convenience, we use S to denote all these states,

$$\mathcal{S}: \{|a\rangle, \frac{\sqrt{2}}{2}(|a\rangle + |b\rangle), \frac{\sqrt{2}}{2}(|a\rangle + i|b\rangle)\}_{0 \le a < b \le D-1}.$$

For the trace-preserving cases, only $D^2(D^2-1)$ measurements are independent. An interpretation for it is like this: the following D measurements, $\langle a|\varepsilon(|\psi_i^{ab}\rangle\langle\psi_i^{ab}|)|a\rangle$ with a=0, 1,..., D-1, should be performed for a given input state $|\psi_i^{ab}\rangle$. With $\sum_{a=0}^{D-1}|a\rangle\langle a|=\mathtt{I}_D$, we can always leave $\langle D-1|\varepsilon(|\psi_i^{ab}\rangle\langle\psi_i^{ab}|)|D-1\rangle$ unmeasured and decide its value according to

$$\langle D-1|\varepsilon(|\psi_i^{ab}\rangle\langle\psi_i^{ab}|)|D-1\rangle = 1 - \sum_{a=0}^{D-2} \langle a|\varepsilon(|\psi_i^{ab}\rangle\langle\psi_i^{ab}|)|a\rangle$$

This fact is consentient with our general analysis about the χ matrix: For the D-dimensional system, it has $D^2(D^2 - 1)$ independent real parameters and D^2 additional constraints for trace.

The N d-levels system.- The way of performing partial QPT is important for the the N d-levels system specially when d is a large number. For such cases, to performing the complete QPT becomes a non-scalable task since the number of the required experiments is exponentially increased with N. Our general protocol also holds for the N d-levels system by taking it as a special case of $D = d^N$. With

$$|a\rangle = |a_1\rangle \otimes \dots \otimes |a_j\rangle \otimes \dots \otimes |a_N\rangle \tag{19}$$

where $\{|a_j\rangle\}_{a_j=0,1,\ldots,d-1}$ is the basis of the j-th ddimensional subsystem, we could define a relation between the single index with its corresponding string of local indices, say, $a \to a_1 a_2 \cdots a_N$, $b \to b_1 b_2 \cdots b_N$, etc.. If the set of product basis $\{\prod_{j=1}^N \otimes |a_j\rangle\}$ is to be used instead of $\{|a\rangle\}$ used above, all the equations in the product basis can be easily given by substituting each single index with its corresponding string of local indices. For example, the one-to-one mapping in (7) with the local indices should be,

$$\lambda_{a_1 \cdots a_N b_1 \cdots b_N; c_1 \cdots c_N d_1 \cdots d_N} = \chi_{c_1 \cdots c_N a_1 \cdots a_N; d_1 \cdots d_N b_1 \cdots b_N}$$

For the two qubits case, with $|+\rangle = \frac{\sqrt{2}}{2}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{\sqrt{2}}{2}(|0\rangle + i|1\rangle)$, S contains following product states, $|0\rangle|0\rangle, |0\rangle|1\rangle, |1\rangle|0\rangle, |1\rangle|1\rangle, |0\rangle|\pm\rangle, |1\rangle|\pm\rangle, |\pm\rangle|0\rangle, |\pm\rangle|1\rangle$, and the four Bell-type states, $\frac{\sqrt{2}}{2}(|00\rangle + |11\rangle)$, $\frac{\sqrt{2}}{2}(|00\rangle + i|11\rangle), \frac{\sqrt{2}}{2}(|01\rangle + |10\rangle)$, and $\frac{\sqrt{2}}{2}(|01\rangle + i|10\rangle)$. For the more general N d-levels case with the product basis, S contains a series of product states and M-parties maximally entangled states with M arranged from 2 to N. Suppose we have the freedoms of choosing one of the followings two mappings, $|a_i\rangle \rightarrow |\uparrow_i\rangle, |b_i\rangle \rightarrow |\downarrow_i\rangle$ or $|a_i\rangle \rightarrow |\uparrow_i\rangle, |b_i\rangle \rightarrow |\downarrow_i\rangle$, for each site, by renumbering the sequence of all the N sites, we can write $|ab,\pm\rangle$ in (16) with

$$|ab,+\rangle = |G^M\rangle \prod_{j=M+1}^N \otimes |\uparrow_j\rangle, |ab,-\rangle = |\tilde{G}^M\rangle \prod_{j=M+1}^N \otimes |\uparrow_j\rangle$$

where the two Greenberger-Horne-Zeilinger type states, $|G^M\rangle = \frac{\sqrt{2}}{2}(|\uparrow\rangle^{\otimes M} + |\downarrow\rangle^{\otimes M})$ and $|\tilde{G}^M\rangle = \frac{\sqrt{2}}{2}(|\uparrow\rangle^{\otimes M} + i|\downarrow\rangle^{\otimes M})$, are the maximally entangled states among M-parties.

Discussion-With the general theory of SQPT in [1], we have developed a scheme of performing partial SQPT for χ in the Choi matrix representation. Besides the fact that the set of Choi operators is a convenient basis for an arbitrary D-dimensional system, χ matrix in this representation is shown to be physical meaningful in the sense that its diagonal matrix element is just the transition probability of the system. For the N qubits case, it can be shown that χ^C (in the Choi matrix representation), are equivalent with each other. An interpretation is like this: Let's at first consider the single qubit case, it can be seen that the set of Pauli operators is related with the Choi operators by a unitary transformation U,

$$\begin{pmatrix} \mathbf{I} \\ \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & -i\frac{\sqrt{2}}{2} & i\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} \bar{E}_{00} \\ \bar{E}_{01} \\ \bar{E}_{10} \\ \bar{E}_{11} \end{pmatrix},$$

where we define $\bar{E}_{ab} = \sqrt{2}\tilde{E}_{ab}$. One may easily verified that $UU^{\dagger} = \mathbb{I}_4$ and get the relation $\chi^P = U\chi^C U^{\dagger}$. For the N qubits case, if χ^P is expanded with the set of Pauli operators $\{\prod_{k=1}^N \otimes \sigma^k\}$ while χ^C is defined with $\{\prod_{k=1}^N \otimes \bar{E}^k\}$, the relation $\chi^P = \tilde{U}\chi^C \tilde{U}^{\dagger}$ still holds with $\tilde{U} = \prod_{k=1}^N U^k$. From above discussion, it's hard for us to claim that one representation has privilege over the other representation for defining χ matrix.

From equation (15), it should be noted that the ways of performing complete SQPT are not limited, any set of linearly independent states $\{|\Psi_m\rangle\}$ with another set of linearly independent Hermitian operators $\{\hat{O}_n\}$ can be applied for this task. Among all the possible ways, our protocol in (18) works for deciding an arbitrary χ matrix element with a fixed number of measurements. If the complete SQPT should be performed, one may use the following scheme: Let $\{|\phi_j^k\}_{j=1,...,d^2}$ the set of linearly independent states for the the k - th subsystem, the input states $|\Psi_m\rangle$ in (13) may be defined as a pure product states, say, $|\Psi_m\rangle = \prod_{k=1}^N \otimes |\phi^k\rangle$; Considering that $\hat{\Gamma}_0^k = \mathbf{I}_d$ and $\{\hat{\Gamma}_j^k\}_{j=1,...,d^2-1}$, where $\hat{\Gamma}_j^k$ is the generator of SU(d), forms a orthogonal basis of the k-th subspace, the operators \hat{O}_n in (14) can be chosen to be $\hat{O}_n = \prod_{k=1}^N \otimes \hat{\Gamma}^k$, the complete SQPT performed in this way has the unique property that it does not require any two-bodies or multi-bodies interaction.

There are two related problems still unsolved with the present work. At first, if all the experiment data are used to reconstruct the whole χ matrix, how to keep the positivity of χ is not given since that the partial SQPT is concerned here. Second, our protocol of partial SQPT is

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limited to the case where χ is in the Choi matrix representation, it is still an open problem whether the general theory of SQPT in [1] can be also applied for the partial QPT when χ is in other physical representations.

Finally, let's make a short summary for our work. With the general theorem of SQPT, we developed a method to estimate an arbitrary χ matrix element in the Choi matrix representation. Our observation is that: To decide the diagonal matrix element, one just needs a single measurement; To decide an arbitrary off-diagonal matrix element, we should carry out sixteen measurements. This result is independent of the actual dimension of the system. Compared with the known methods of partial QPT, our scheme does not require any additional resources. It can be applied for the case where a clean ancilla system is not available.

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