Shared quantum control via sharing operation on remote single qutrit

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Abstract Two qubit operation sharing schemes [*J. Phys.* B **44** (2011) 165508] are generalized to qutrit ones. Operations to be shared are classified into three different classes in terms of different probabilities (i.e, 1/3, 2/3 and 1). For the latter two classes, ten and three restricted sets of operations are found out, respectively. Moreover, the two generalized schemes are amply compared from four aspects, namely, quantum and classical resource consumption, necessary-operation complexity, success probability and efficiency. It is found that the second scheme is overall more optimal than the first one as far as three restricted sets of operations are concerned.

Keywords: quantum operation sharing, generalized Bell state, restricted sets of qutrit operations, success probability, efficiency.

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1 Introduction

Entanglement is admitted as a kind of important quantum resource nowadays. In the last two decades it has been extensively exploited and utilized in the many fields of quantum information science to fulfill various quantum tasks involving classical information processing in a quantum manner and quantum information (i.e., quantum state) processing[1-25], such as quantum key distribution, quantum state teleportation, remote state preparation, operation teleportation, state sharing, direct secure communication, quantum computing, and so on. Enlightened by the generalization of quantum state teleportation to quantum state sharing, in 2011 Zhang and Cheung[26] definitely presented quantum operation sharing with shared entanglements. They introduced the sharing idea into quantum operation teleportation and proposed two specific schemes with different groups of shared entanglements. The first one is a universal but nontrivial scheme for sharing any single-qubit operation with one group of shared entanglements, while the latter treats the sharing of two restricted sets of operations with less resource consumptions. Incidentally, these schemes are uniformly referred to as the ZC schemes later. As same as quantum operation teleportation, QOS can be viewed as a remote control (encryption, decryption or destruction) on quantum information in the future quantum network. If the target state as an important quantum information is initially encrypted with a given unitary operation, then the inverse operation on it is obviously the decryption on it. Moreover, it is natural to regard a random operation as the destruction on it. As a consequence, quantum operation sharing can be taken as a key to activate some important actions in the future lives by some sharer entities, such as missile emissions, quantum collective seal or unseam, remote joint destruction of quantum money, and so on. Therefore, recently this topic has already attracted some attentions[27-29]. In spite of this, it is worthy emphasizing that all these works treat only the sharing of single-qubit operations with shared qubit entanglements.

In the intending quantum networks, various quantum states involving high-dimensional qudit state might be employed and distributed among different nodes due to some special demands, such as peculiar quantum tasks in some concrete quantum scenarios, some definite security requirements, and so on. Because of this, some researchers have been attracted by the issues related to high-dimensional qudit cases and started to explore them[30-39]. Surely, their works includes the employment of qutrit entanglements in addressing various relative quantum problems[38]. In this condition, it is intriguing to ask what will happen in the field of QOS if the accessible quantum channels are composed of shared qutrit entanglements? Specifically, what kind of qutrit states can be used to fulfill QOS? If quantum channels are determined, what is the maximal success probability for sharing a given operation? Can the operation be shared more economically with less operation complexity? Can operations be classified with different classes corresponding to different success sharing probabilities? How to characterize those classes and find out them? and so on and so forth. To our best knowledge these mentioned issues have not been touched by far. Hence, it is of interest to consider the generalization of the ZC schemes[26] from the aspect of particle degree. Since there exist many open questions, in this paper we will consider a comparatively simple generalization, i.e., generalize the simplest QOS schemes (i.e., the ZC schemes) to the qutrit ones by employing the qutrit Bell and GHZ states as quantum channels. Such extensions is actually a little intricate but will lead to quite abundant results. More importantly, they indicate that QOS as remote control can be achieved with shared qutrit entanglement, too. We will show them later. Additionally, at present it is broadly recognized that resource consumption and operation complexity as hot topics in many fields are always concentrated. How to consume less amount of resources and how to degrade the difficulty and intensity of necessary operations are continuously attracting much attention and pursued by many researchers. Because of this, in this paper we will focus our attention on this issue during the generalizations, too.

The rest of this paper is organized as follows. In section 2, two three-party schemes for sharing single-qutrit operations on a remote qutrit in any state are preciously proposed with different quantum channels. In section 3, two schemes are amply compared and discussed from the four aspects of quantum and classical resource consumption, necessary-operation complexity, success probability and efficiency. Finally, a concise summary is made in section 4.

2 Remote sharing of single-qutrit operation

Now let us start to present our schemes. In either scheme there are three legitimate users, say, Alice, Bob and Charlie. Alice is the initial performer of a single-qutrit operation $\mathcal{U}$. Incidentally, whether she knows it or not is uncertain and will be discussed and treated separately later. Bob and Charlie are Alice's two remote agents. Alice needs to perform the operation $\mathcal{U}$ on a qutrit in state $|\chi\rangle$ at one remote agent's site. She wants to fulfill the task with her agents' assistance and by making full use of the quantum and classical channels linking the three legitimate users. However, she trusts neither agent but their entity. Specifically, she should assure that the operation can not be successfully executed on the qutirt by either agent solely but conclusively achieved via the mutual collaboration of her two agents. Moreover, the state $|\chi\rangle$ in the qutrit can be arbitrary. Suppose the state $|\chi\rangle$ is initially in Bob's qutrit b'' and can be written as

$$|\chi\rangle_{b''} = \alpha|0\rangle_{b''} + \beta|1\rangle_{b''} + \gamma|2\rangle_{b''}, \tag{1}$$

where α, β and γ are complex and satisfy $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$.

2.1 General scheme for arbitrary single-qutrit operation

Now let us put forward our general three-party QOS scheme, which is universally applicable for sharing an arbitrary single-qutrit operation. The schematic demonstration is illustrated in figure 1. The scheme can be concisely depicted as follows.

(i) *Initial stage.* In this scheme, the quantum channels linking the three legitimate users are a shared generalized Bell state $|\mathcal{B}_{00}\rangle$ and a shared generalized GHZ state $|\mathcal{G}\rangle$, i.e.,

$$|\mathcal{B}_{00}\rangle_{a'b'} = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)_{a'b'}, \tag{2}$$

$$|\mathcal{G}\rangle_{abc} = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)_{abc}, \tag{3}$$

where the qutrit trio (b, b', b'') belongs to Bob, the qutrit pair (a, a') to Alice and the qutrit c to Charlie.

(ii) *QT process.* In this stage, the state $|\chi\rangle$ in Bob's qutrit b'' is teleported to Alice's qutirt a' via the standard QT process with the generalized Bell state $|\mathcal{B}_{00}\rangle$ as the quantum channel between Alice and Bob. Alternatively, after the process the state of qutrit b'' has been swapped to the qutrit a' and hence the state of qutrit a' is transformed to $|\chi\rangle$.

(iii) *Operation performance.* Alice carries out the operation $\mathcal{U}$ on her qutrit a' in state $|\chi\rangle$, i.e.,

$$\mathcal{U} \to |\chi\rangle_{a'} \Longrightarrow (\mathcal{U}|\chi\rangle)_{a'}. \tag{4}$$

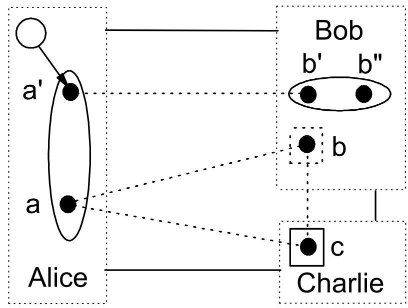

FIG. 1: Illustration of our general three-party QOS scheme with generalized GHZ and generalized Bell states. Dotted rectangles are participants' locations. Solid lines among rectangles stand for classical communication channels. Solid dots denote qutrits. Dotted lines linking qutrits means entanglements. The solid circle labels the unitary operation $\mathcal{U}$ to be shared. Solid ellipses represent generalized Bell-state measurements. The solid and dotted squares illustrate the single-qutrit measurement and unitary operation, respectively. See text for more details.

(iv) *QSTS process.* In this stage, the state $\mathcal{U}|\chi\rangle$ in Alice's qutrit a' is shared by Bob and Charlie via a standard QSTS process with the generalized GHZ state $|\mathcal{G}\rangle_{abc}$ as the quantum channel. In other words, if Bob and Charlie collaborate with each other, they can finally reconstruct the state $\mathcal{U}|\chi\rangle$ on either the qutrit b or the qutrit c. This result is actually equivalent to Alice's aim, that is, conclusively performing her single-qutrit operation $\mathcal{U}$ on a remote qutrit in state χ at an agent's position.

2.2 Specific scheme for restricted sets of operations

Now let us present another tripartite QOS scheme with quantum channels different from those in the first scheme. The illustration of this scheme is shown in figure 2. The details of the scheme are described as follows.

(I) *Initial stage.* The present quantum channels linking the three legitimate users are

$$|\mathcal{B}_{00}\rangle_{a'b'} = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)_{a'b'}, \tag{5}$$

$$|\mathcal{B}_{00}\rangle_{ac} = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)_{ac}, \tag{6}$$

where the qutrit pair (a, a') is at Alice's hand, and the qutrits b' and c are in Bob's and Charlie's sites, respectively. Note that, different from the quantum channels employed in the general scheme, here a generalized Bell state $|\mathcal{B}_{00}\rangle_{ac}$ is used instead of the generalized GHZ state $|\mathcal{G}\rangle_{abc}$ there.

(II) *Bob's performances.* First, Bob performs a unitary operation $\mathcal{V}$ on his qutrit pair (b', b''), where

$$\begin{aligned} \mathcal{V} &= |00\rangle\langle 00| + |01\rangle\langle 01| + |02\rangle\langle 02| + |10\rangle\langle 11| + |11\rangle\langle 12| \\ &+ |12\rangle\langle 10| + |20\rangle\langle 22| + |21\rangle\langle 20| + |22\rangle\langle 21|. \end{aligned} \tag{7}$$

After the operation, the state of the qutrit trio (a', b', b'') is converted into

$$\begin{aligned} \mathcal{V}_{b'b''}|\Psi_{00}\rangle_{a'b'}|\chi\rangle_{b''} &= \{[\alpha|00\rangle + \beta|11\rangle + \gamma|22\rangle]_{a'b'}|0\rangle_{b''} + [\alpha|22\rangle + \beta|00\rangle + \gamma|11\rangle]_{a'b'}|1\rangle_{b''} \\ &+ [\alpha|11\rangle + \beta|22\rangle + \gamma|00\rangle]_{a'b'}|2\rangle_{b''}\}/\sqrt{3}. \end{aligned} \tag{8}$$

Then Bob measures his qutrit b'' in the bases $\{|0\rangle, |1\rangle, |2\rangle\}$. If Bob measures $|0\rangle_{b''}$, then he does the identity operation (i.e., nothing) on his qutrit b'. Otherwise, if he gets $|1\rangle_{b''}$ ($|2\rangle_{b''}$), he executes the

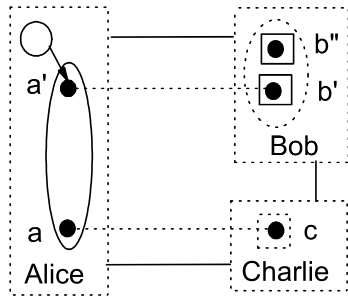

FIG. 2: Illustration of our specific three-party QOS scheme with two generalized Bell states. The same as fig.1 with the dotted ellipse representing the unitary operation $\mathcal{V}$. See text for more details.

unitary operation $S = |0\rangle\langle 2| + |2\rangle\langle 1| + |1\rangle\langle 0|$ ($T = |0\rangle\langle 1| + |2\rangle\langle 0| + |1\rangle\langle 2|$) on his qutrit b'. In terms of their prior agreement that the single-qutrit state $|i\rangle$ corresponds to the classical single trit i and vice versa (same hereafter), Bob notifies Charlie of the measurement result via the classical channel linking them.

(III) *Alice's performances.* Upon receiving Bob's message, Alice first does the same single-qutrit operation as Bob's on her qutrit a'. Bob's and Alice's performances lead to one of the following transformations:

$$[\alpha|00\rangle + \beta|11\rangle + \gamma|22\rangle]_{a'b'}|0\rangle_{b''} \xrightarrow{I_{a'}I_{b'}} [\alpha|00\rangle + \beta|11\rangle + \gamma|22\rangle]_{a'b'}|0\rangle_{b''}, \quad (9)$$

$$[\alpha|22\rangle + \beta|00\rangle + \gamma|11\rangle]_{a'b'}|1\rangle_{b''} \xrightarrow{S_{a'}S_{b'}} [\alpha|00\rangle + \beta|11\rangle + \gamma|22\rangle]_{a'b'}|1\rangle_{b''}, \quad (10)$$

$$[\alpha|11\rangle + \beta|22\rangle + \gamma|00\rangle]_{a'b'}|2\rangle_{b''} \xrightarrow{T_{a'}T_{b'}} [\alpha|00\rangle + \beta|11\rangle + \gamma|22\rangle]_{a'b'}|2\rangle_{b''}. \quad (11)$$

Note that in the formulae above the operations on qutrits a' and b' are same. Obviously, one can see that the state of the qutrit pair (a', b') is always $[\alpha|00\rangle + \beta|11\rangle + \gamma|22\rangle]_{a'b'}$ after their operations. Afterwards, she performs the unitary operation $\mathcal{U}$ on her qutrit a'. In this case the state of Alice's qutrit pair becomes

$$|J\rangle_{a'b'} = \mathcal{U}_{a'}[\alpha|00\rangle + \beta|11\rangle + \gamma|22\rangle]_{a'b'}. \quad (12)$$

At this moment the state of the qutrit quadruple (a', a, b', c) is

$$|Q\rangle_{a'ab'c} = |J\rangle_{a'b'} \otimes |\mathcal{B}_{00}\rangle_{ac}. \quad (13)$$

Subsequently, Alice measures her qutrit pair with the generalized Bell states as the measuring bases

$$\{|\Psi_{n,m}\rangle = \sum_{j=0}^{2} e^{\frac{2nj\pi}{3}i}|j\rangle \otimes |(j+m) \bmod 3\rangle/\sqrt{3};\quad n \in (0,1,2),\ m \in (0,1,2)\}. \quad (14)$$

After Alice's measurements, the state of the qutrit quadruple has collapsed to one of the following nine states:

$$|\Psi_{n,m}\rangle_{a'a}\langle\Psi_{n,m}|Q\rangle = \frac{1}{3}|\Psi_{n,m}\rangle_{a'a}\sigma_c^{(n,m)}\mathcal{U}_c(\alpha|00\rangle + \beta|11\rangle + \gamma|22\rangle)_{cb'}, \quad n, m = 0, 1, 2, \quad (15)$$

where

$$\sigma^{(n,m)} = |m\rangle\langle 0| + e^{\frac{4n\pi}{3}i}|(m+1) \bmod 3\rangle\langle 1| + e^{\frac{2n\pi}{3}i}|(m+2) \bmod 3\rangle\langle 2|. \quad (16)$$

Then at last, she publishes via the classical channels the outcome according to the prior agreement, i.e., the generalized Bell state $|\Psi_{n,m}\rangle$ corresponds to the classical trit pair (n, m) and vice versa (same hereafter).

(IV) *Reconstruction via agents' collaboration.*

If Bob and Charlie collaborate with each other, they can deterministically or probabilistically reconstruct the operation $\mathcal{U}$ on the state $|\chi\rangle$ in either qutrit c or qutrit b'. Without loss of generality, suppose they decide to restore it in qutrit c. In this condition, Charlie starts to perform a unitary operation $\sigma^{(n,m)\dagger}$ on his qutrit c after Alice publishes the outcome. His operation makes the qutrit pair (b', c) in the state

$$|J\rangle_{cb'} = \mathcal{U}_c[\alpha|00\rangle + \beta|11\rangle + \gamma|22\rangle]_{cb'}. \tag{17}$$

Subsequently, Bob measures his qutrit b' with the orthonormal measuring bases defined as

$$|\xi_0\rangle = |\vec{V}(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})\rangle, \quad |\xi_1\rangle = e^{i\tau_1}|\vec{V}_1(x_1, y_1, z_1)\rangle, \quad |\xi_2\rangle = e^{i\tau_2}|\vec{V}_2(x_2, y_2, x_2)\rangle, \tag{18}$$

where x's, y's and z's are complex, τ's are arbitrarily real, and $|\vec{V}_i(x, y, z)\rangle \equiv x|0\rangle + y|1\rangle + z|2\rangle$. The orthogonality and normality of the measuring bases require that

$$\begin{aligned}
&z_1 = -x_1 - y_1, |x_1|^2 + |y_1|^2 + |x_1 + y_1|^2 = 1,\\
&x_2 = x_2'/N = (-1 + 2x_1x_1^* + x_1y_1^*)/N,\\
&y_2 = y_2'/N = (2y_1x_1^* + y_1y_1^*)/N,\\
&z_2 = z_2'/N = (1 - 2x_1x_1^* - 2y_1x_1^* - x_1y_1^* - y_1y_1^*)/N,\\
&N = \sqrt{|x_2'|^2 + |y_2'|^2 + |z_2'|^2}.
\end{aligned}$$

It is obvious that the first basis is unchanged as a fixed vector in the three-dimensional Hilbert space, while the latter two are variant as the function of three dependent parameters. Nonetheless, here it is necessary to stress that parameters x_1 and y_1 are not completely independent but correlated to each other by a constraint. By virtue of the measuring bases, the state of the qutrit pair (c, b') can be rewritten as

$$\begin{aligned}
|J\rangle_{cb'} &= \frac{1}{\sqrt{3}}\{[\mathcal{U}|\chi\rangle]_c|\xi_0\rangle_{b'} + [\mathcal{U}W_1(x_1, y_1, z_1)|\chi\rangle]_c|\xi_1\rangle_{b'}\\
&+ [\mathcal{U}W_2(x_2, y_2, x_2)|\chi\rangle]_c|\xi_2\rangle_{b'}\},
\end{aligned} \tag{19}$$

where

$$W_k(x, y, z) = e^{-i\tau_k}(x^*|0\rangle\langle 0| + y^*|1\rangle\langle 1| + z^*|2\rangle\langle 2|), \qquad k = 1, 2. \tag{20}$$

Incidentally, it is obvious that $W_k(x, y, z)$ is unitary. Consequently, if Bob measures $|\xi_0\rangle_{b'}$, Alice's operation $\mathcal{U}$ has been conclusively performed on Charlie's qutrit c. However, $|\xi_0\rangle_{b'}$ occurs only with probability 1/3. It is quite possible that Bob measures $|\xi_1\rangle_{b'}$ or $|\xi_2\rangle_{b'}$. In these two cases, intuitively, one is readily to see that the operation $\mathcal{U}$ has not been successfully executed on the qutrit c. Whether Alice's goal can be achieved in the end is still uncertain and completely determined by the relations among $\mathcal{U}$ and $W_1(x_1, y_1, z_1)$ as well as $W_2(x_2, y_2, x_2)$. With probability 1/3 Bob may measure $|\xi_1\rangle_{b'}$. In this case, if $\mathcal{U}W_1(x_1, y_1, z_1) = \pm\mathcal{U}W_1(x_1, y_1, z_1)$ holds, then based on Bob's message about the measurement result, Charlie can achieve Alice's goal by executing the unitary operation $W_1^\dagger(x_1, y_1, z_1)$ on his qutrit c. Otherwise, Alice's goal can not be achieved. Similarly, Bob may measure $|\xi_2\rangle_{b'}$ with probability 1/3, too. In this case, if $\mathcal{U}W_2(x_2, y_2, x_2) = \pm\mathcal{U}W_2(x_2, y_2, x_2)$ holds, then Charlie can fulfill Alice's operation on his qutrit c by performing the reversal unitary operation of $W_2(x_2, y_2, x_2)$ in terms of Bob's message on the measurement result. Otherwise, Alice's operation on a remote qutrit in agents' site fails. Surely, if both $\mathcal{U}W_1(x_1, y_1, z_1) = \pm\mathcal{U}W_1(x_1, y_1, z_1)$ and $\mathcal{U}W_2(x_2, y_2, x_2) = \pm\mathcal{U}W_2(x_2, y_2, x_2)$ hold simultaneously, then Alice's goal can be deterministically reached via Bob and Charlie's collaboration. Hence, as mentioned before, the success probability is fully determined by the properties of $\mathcal{U}$ and $W_1(x_1, y_1, z_1)$ as well as

$W_2(x_2, y_2, x_2)$, particularly their mutual relations. After our intensive investigations, we have found that in the following four cases the success probability can be doubled or increased to unit, provided that some information on the operation $\mathcal{U}$ is partially known in priori.

(1) $x_1 = 0$ and $y_1 = -\frac{1}{\sqrt{2}} e^{i\phi_1}$ (ϕ_1 is arbitrarily real).

In this case, one can easily get the two variant measuring bases and the two W operations, i.e.,

$$
\begin{aligned}
|\xi_{\mathbf{11}}\rangle &= e^{i(\tau_1+\phi_1)}|\vec{V}(0, -1/\sqrt{2}, 1/\sqrt{2})\rangle, &(21)\\
|\xi_{\mathbf{12}}\rangle &= e^{i\tau_2}|\vec{V}(-2/\sqrt{6}, 1/\sqrt{6}, 1/\sqrt{6})\rangle, &(22)\\
W_{\mathbf{11}} &= e^{-i(\tau_1+\phi_1)}(-|1\rangle\langle 1| + |2\rangle\langle 2|)/\sqrt{2}, &(23)\\
W_{\mathbf{12}} &= e^{-i\tau_2}(-2|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|)/\sqrt{6}. &(24)
\end{aligned}
$$

If $\mathcal{U}W_{\mathbf{11}} = W_{\mathbf{11}}\mathcal{U}$ or $\mathcal{U}W_{\mathbf{11}} = -W_{\mathbf{11}}\mathcal{U}$, then $\mathcal{U}$ should take one of the following two forms

$$
\mathcal{U}^{(1)} = \begin{pmatrix} e^{i\mu_{11}} & 0 & 0 \\ 0 & e^{i\mu_{12}} & 0 \\ 0 & 0 & e^{i\mu_{13}} \end{pmatrix}, \quad \mathcal{U}^{(2)} = \begin{pmatrix} e^{i\mu_{21}} & 0 & 0 \\ 0 & 0 & e^{i\mu_{22}} \\ 0 & e^{i\mu_{23}} & 0 \end{pmatrix}, \tag{25}
$$

where μ's are arbitrarily real (same hereafter). This means that if Bob measures $|\xi_{\mathbf{11}}\rangle_{b'}$ and $\mathcal{U}$ belongs to either of the two restricted sets above, Charlie can finally achieve Alice's goal.

Similarly, if $\mathcal{U}W_{\mathbf{12}} = W_{\mathbf{12}}\mathcal{U}$, then $\mathcal{U}$ should be any matrix of the following couple of sets

$$
\mathcal{U}^{(3)} = \begin{pmatrix} e^{i\mu_{31}} & 0 & 0 \\ 0 & \cos\mu_{35} e^{i(\mu_{32}+\mu_{34})} & \sin\mu_{35} e^{i(\mu_{33}+\mu_{34})} \\ 0 & -\sin\mu_{35} e^{-i(\mu_{33}-\mu_{34})} & \cos\mu_{35} e^{-i(\mu_{32}-\mu_{34})} \end{pmatrix}, \tag{26}
$$

$$
\mathcal{U}^{(4)} = \begin{pmatrix} e^{i\mu_{41}} & 0 & 0 \\ 0 & \cos\mu_{45} e^{i(\mu_{42}+\mu_{44})} & \sin\mu_{45} e^{i(\mu_{43}+\mu_{44})} \\ 0 & \sin\mu_{45} e^{-i(\mu_{43}-\mu_{44})} & -\cos\mu_{45} e^{-i(\mu_{42}-\mu_{44})} \end{pmatrix}. \tag{27}
$$

This implies that if Bob gets $|\xi_{\mathbf{12}}\rangle_{b'}$ via measurement and $\mathcal{U}$ belongs to either $\mathcal{U}^{(3)}$ or $\mathcal{U}^{(4)}$, Charlie can fulfill Alice's operation $\mathcal{U}$ on his qutrit c in the end. Incidentally, there does not exist any $\mathcal{U}$ which satisfies $\mathcal{U}W_{\mathbf{12}} = -W_{\mathbf{12}}\mathcal{U}$.

Let $\mathcal{U}^{(12)} = \mathcal{U}^{(1)} \cup \mathcal{U}^{(2)}$ and $\mathcal{U}^{(34)} = \mathcal{U}^{(3)} \cup \mathcal{U}^{(4)}$. Surely, if $\mathcal{U} \in [\mathcal{U}^{(12)} \cap \mathcal{U}^{(34)}]$, then the success probability of the scheme is unit when taking account of the success probability 1/3 with the measured $|\xi_0\rangle_{b'}$. In this case, one is readily to see that $\mathcal{U}^{(12)} \cap \mathcal{U}^{(34)} = \mathcal{U}^{(12)}$. In contrast, in the case that $\mathcal{U} \in [\mathcal{U}^{(34\backslash 12)} = \mathcal{U}^{(34)} \setminus \mathcal{U}^{(12)}]$, then the total success probability is 2/3.

(2) $x_1 = -\frac{1}{\sqrt{2}} e^{i\phi_2}$ and $y_1 = 0$.

Two variant measuring bases and two W operations corresponding to this case are

$$
\begin{aligned}
&|\xi_{\mathbf{21}}\rangle = e^{i(\tau_1+\phi_2)}|\vec{V}(-1/\sqrt{2}, 0, 1/\sqrt{2})\rangle, &(28)\\
&|\xi_{\mathbf{22}}\rangle = e^{i\tau_2}|\vec{V}(1/\sqrt{6}, -2/\sqrt{6}, 1/\sqrt{6})\rangle, &(29)\\
&W_{\mathbf{21}} = e^{-i(\tau_1+\phi_2)}(-|0\rangle\langle 0| + |2\rangle\langle 2|)/\sqrt{2}, &(30)\\
&W_{\mathbf{22}} = e^{-i\tau_2}(|0\rangle\langle 0| - 2|1\rangle\langle 1| + 1|2\rangle\langle 2|)/\sqrt{6}. &(31)
\end{aligned}
$$

If $\mathcal{U} \in \mathcal{U}^{(15)}$, where $\mathcal{U}^{(15)} = \mathcal{U}^{(1)} \cup \mathcal{U}^{(5)}$ and

$$
\mathcal{U}^{(5)} = \begin{pmatrix} 0 & 0 & e^{i\mu_{51}} \\ 0 & e^{i\mu_{52}} & 0 \\ e^{i\mu_{53}} & 0 & 0 \end{pmatrix}, \tag{32}
$$

then $\mathcal{U}W_{\mathbf{21}} = W_{\mathbf{21}}\mathcal{U}$ or $\mathcal{U}W_{\mathbf{21}} = -W_{\mathbf{21}}\mathcal{U}$. It means that if Bob's measurement result is $|\xi_{\mathbf{21}}\rangle_{b'}$ and $\mathcal{U}$ belongs the restricted set $\mathcal{U}^{(15)}$, Charlie can finally reconstruct the state $\mathcal{U}|P\rangle$ on his qutrit c, as implies that with the two agents' help with the shared entanglement and LOCC Alice's operation $\mathcal{U}$ has been successfully executed on the remote qutrit c in Charlie's site. Of course, such circumstance only appears with probability 1/3.

Similarly, if $\mathcal{U}\in\mathcal{U}^{(67)}$, where $\mathcal{U}^{(67)}=\mathcal{U}^{(6)}\cup\mathcal{U}^{(7)}$ and

$$\mathcal{U}^{(6)}=\begin{pmatrix}\cos\mu_{65}e^{i(\mu_{62}+\mu_{64})} & 0 & \sin\mu_{65}e^{i(\mu_{63}+\mu_{64})}\\ 0 & e^{i\mu_{61}} & 0\\ -\sin\mu_{65}e^{-i(\mu_{63}-\mu_{64})} & 0 & \cos\mu_{65}e^{-i(\mu_{62}-\mu_{64})}\end{pmatrix}, \tag{33}$$

$$\mathcal{U}^{(7)}=\begin{pmatrix}\cos\mu_{75}e^{i(\mu_{72}+\mu_{74})} & 0 & \sin\mu_{75}e^{i(\mu_{73}+\mu_{74})}\\ 0 & e^{i\mu_{71}} & 0\\ \sin\mu_{75}e^{-i(\mu_{73}-\mu_{74})} & 0 & -\cos\mu_{75}e^{-i(\mu_{72}-\mu_{74})}\end{pmatrix}, \tag{34}$$

then $\mathcal{U}W_{\mathbf{22}}=\pm W_{\mathbf{22}}\mathcal{U}$. Hence in the case, $\mathcal{U}$ can be finally performed on the qutrit c in a deterministic manner provided that Bob measures $|\xi_{\mathbf{22}}\rangle_{b'}$ and notifies him of the result. Surely, if $\mathcal{U}\in[\mathcal{U}^{(15)}\cap\mathcal{U}^{(67)}=\mathcal{U}^{(15)}]$, then the success probability of the scheme is unit. Apparently, in the case that $\mathcal{U}\in[\mathcal{U}^{(67\backslash 15)}=\mathcal{U}^{(67)}\setminus\mathcal{U}^{(15)}]$, then the total success probability is still 2/3. At last, it is worthy mentioning that **(1)** is completely same as **(2)** if $|0\rangle$ exchanges with $|1\rangle$.

(3) $x_1=-y_1=\frac{1}{\sqrt{2}}e^{i\phi_3}$.

The two variant measuring bases and two W operations corresponding to the conditions above are

$$|\xi_{\mathbf{31}}\rangle=e^{i(\tau_1+\phi_3)}|\vec{V}(1/\sqrt{2},-1/\sqrt{2},0)\rangle, \tag{35}$$

$$|\xi_{\mathbf{32}}\rangle=e^{i\tau_2}|\vec{V}(1/\sqrt{6},1/\sqrt{6},-2/\sqrt{6})\rangle, \tag{36}$$

$$W_{\mathbf{31}}=e^{-i(\tau_1+\phi_3)}(-|0\rangle\langle 0|+|1\rangle\langle 1|)/\sqrt{2}, \tag{37}$$

$$W_{\mathbf{32}}=e^{-i\tau_2}(|0\rangle\langle 0|+|1\rangle\langle 1|-2|2\rangle\langle 2|)/\sqrt{6}. \tag{38}$$

If $\mathcal{U}W_{\mathbf{31}}=\pm W_{\mathbf{31}}\mathcal{U}$, then $\mathcal{U}\in\mathcal{U}^{(18)}$, where $\mathcal{U}^{(18)}=\mathcal{U}^{(1)}\cup\mathcal{U}^{(8)}$ and

$$\mathcal{U}^{(8)}=\begin{pmatrix}0 & e^{i\mu_{81}} & 0\\ e^{i\mu_{82}} & 0 & 0\\ 0 & 0 & e^{i\mu_{83}}\end{pmatrix}. \tag{39}$$

If $\mathcal{U}W_{\mathbf{32}}=W_{\mathbf{32}}\mathcal{U}$, then $\mathcal{U}\in\mathcal{U}^{(910)}$, where $\mathcal{U}^{(910)}=\mathcal{U}^{(9)}\cup\mathcal{U}^{(10)}$ and

$$\mathcal{U}^{(9)}=\begin{pmatrix}\cos\mu_{95}e^{i(\mu_{92}+\mu_{94})} & \sin\mu_{95}e^{i(\mu_{93}+\mu_{94})} & 0\\ -\sin\mu_{95}e^{-i(\mu_{93}-\mu_{94})} & \cos\mu_{95}e^{-i(\mu_{92}-\mu_{94})} & 0\\ 0 & 0 & e^{i\mu_{91}}\end{pmatrix}, \tag{40}$$

$$\mathcal{U}^{(10)}=\begin{pmatrix}\cos\mu_{105}e^{i(\mu_{102}+\mu_{104})} & \sin\mu_{105}e^{i(\mu_{103}+\mu_{104})} & 0\\ \sin\mu_{105}e^{-i(\mu_{103}-\mu_{104})} & -\cos\mu_{105}e^{-i(\mu_{102}-\mu_{104})} & 0\\ 0 & 0 & e^{i\mu_{101}}\end{pmatrix}. \tag{41}$$

In addition, after extensive investigations it is found that $\mathcal{U}W_{\mathbf{32}}=-W_{\mathbf{32}}\mathcal{U}$ does not hold for any $\mathcal{U}$. Consequently, when $\mathcal{U}\in[\mathcal{U}^{(18)}\cap\mathcal{U}^{(910)}=\mathcal{U}^{(18)}]$, the scheme success probability can reach 1. On the contrary, if $\mathcal{U}\in[\mathcal{U}^{(910\backslash 18)}=\mathcal{U}^{(910)}\setminus\mathcal{U}^{(18)}]$, the success probability of the scheme is 2/3. Obviously, one can see that the item **(3)** is as same as the item **(2)** [or the item **(1)**] provided that $|2\rangle$ is exchanged with $|1\rangle$ [or $|0\rangle$].

(4) $x_1\neq 0$, $y_1\neq 0$ and $x_1\neq -y_1$.

With these conditions, one gets the two variant measuring bases and the two W operations

$$|\xi_{\mathbf{41}}\rangle=e^{i\tau_1}|\vec{V}(x_1,y_1,-x_1-y_1)\rangle, \tag{42}$$

$$|\xi_{\mathbf{42}}\rangle=e^{i\tau_2}|\vec{V}(x_2,y_2,z_2)\rangle, \tag{43}$$

$$W_{\mathbf{41}}=e^{-i\tau_1}(x_1^*|0\rangle\langle 0|+y_1^*|1\rangle\langle 1|+(-x_1^*-y_1^*)|2\rangle\langle 2|), \tag{44}$$

$$W_{\mathbf{42}}=e^{-i\tau_2}(x_2^*|0\rangle\langle 0|+y_2^*|1\rangle\langle 1|+z_2^*|2\rangle\langle 2|). \tag{45}$$

After extensive studies, we have found that no $\mathcal{U}$ exists to satisfy $\mathcal{U}W_{41} = -W_{41}\mathcal{U}$ or $\mathcal{U}W_{42} = -W_{42}\mathcal{U}$. Instead, if $\mathcal{U}W_{41} = W_{41}\mathcal{U}$ or $\mathcal{U}W_{42} = W_{42}\mathcal{U}$ should hold, then one can find that $\mathcal{U}$ must be in $\mathcal{U}^{(1)}$. Alternatively, if $\mathcal{U} \in \mathcal{U}^{(1)}$, both W_{41} and W_{42} commute with it. Nevertheless, as stressed before, x_1 and y_1 are not independent but constrained by a relation. Because of this, not all measuring bases can be used to finally fulfill Alice's task. In other words, only some specific bases are applicable, including those occurring in items **(1-3)**. In reality, one can conveniently take different measuring bases in terms of the feasible state discrimination ability. Here we pick out another two discrete examples to demonstrate that other measuring bases are also applicable:

$$|\xi'_{41}\rangle = e^{i\tau_1}|\vec{V}(1/\sqrt{3}, e^{\frac{2\pi}{3}i}/\sqrt{3}, e^{\frac{4\pi}{3}i}/\sqrt{3})\rangle, \tag{46}$$
$$|\xi'_{42}\rangle = e^{i\tau_2}|\vec{V}(1/\sqrt{3}, e^{\frac{4\pi}{3}i}/\sqrt{3}, e^{\frac{2\pi}{3}i}/\sqrt{3})\rangle, \tag{47}$$
$$W'_{41} = e^{-i\tau_1}(|0\rangle\langle 0| + e^{\frac{4\pi}{3}i}|1\rangle\langle 1| + e^{\frac{2\pi}{3}i}|2\rangle\langle 2|)/\sqrt{3}, \tag{48}$$
$$W'_{42} = e^{-i\tau_2}(|0\rangle\langle 0| + e^{\frac{2\pi}{3}i}|1\rangle\langle 1| + e^{\frac{4\pi}{3}i}|2\rangle\langle 2|)/\sqrt{3}. \tag{49}$$

$$|\xi''_{41}\rangle = e^{i\tau_1}|\vec{V}[(i/2, 1/2, -(1+i)/2)]\rangle, \tag{50}$$
$$|\xi''_{42}\rangle = e^{i\tau_2}|\vec{V}[(-2\sqrt{3}+\sqrt{3}i)/6, (-2\sqrt{3}i+\sqrt{3})/6, (\sqrt{3}+\sqrt{3}i)/6]\rangle, \tag{51}$$
$$W''_{41} = e^{-i\tau_1}(-i|0\rangle\langle 0| + |1\rangle\langle 1| + (-1+i)|2\rangle\langle 2|)/2, \tag{52}$$
$$W''_{42} = e^{-i\tau_2}[(-2\sqrt{3}-\sqrt{3}i)|0\rangle\langle 0| + (2\sqrt{3}i+\sqrt{3})|1\rangle\langle 1| + (\sqrt{3}-\sqrt{3}i)|2\rangle\langle 2|]/6. \tag{53}$$

4 Comparisons and discussions

Now let us move to compare our schemes from the following four aspects: the resource consumption consisting of its classical and quantum parts, the necessary-operation complexity including its difficulty and intensity, the scheme success probability and the intrinsic efficiency of the scheme. We have already summarized the two schemes in Table 1 with respect to the four aspects. The intrinsic efficiency of any single-qutrit operation sharing scheme reads

$$\eta = \frac{P}{Q_t + C_t}, \tag{54}$$

where Q_t is the number of the qutrits which are used as quantum channels (except for those chosen for security checking), C_t is the classical trits transmitted and P is the final success probability.

Table 1. Comparisons between our two schemes S1 and S2. QRC: quantum resource consumption; NO: necessary operations; CRC: classical resource consumption; GB: generalized Bell state; GG: generalized GHZ state; GM: generalized Bell-state measurement; SM: single-qutrit measurement; SO: single-qutrit unitary operation. Note that, in the text the two single-qutrit operations on c are used for the sake of convenient expressions. Because they can be commuted with other operations on other qutrits, they can be incorporated as one.

S	$\mathcal{U}$	QRC	NO	CRC	P	η
S1	arbitrary	GB, GG	2 GMs, SM, 2 SOs	5 ctrits	1	1/10
S2	arbitrary	2 GBs	$\mathcal{V}$, GM, 2 SMs, 3 SOs	4 ctrits	1/3	1/24
S2	$\mathcal{U}^{(34\backslash 12)}$	2 GBs	$\mathcal{V}$, GM, 2 SMs, 3 SOs	4 ctrits	2/3	1/12
S2	$\mathcal{U}^{(67\backslash 15)}$	2 GBs	$\mathcal{V}$, GM, 2 SMs, 3 SOs	4 ctrits	2/3	1/12
S2	$\mathcal{U}^{(910\backslash 18)}$	2 GBs	$\mathcal{V}$, GM, 2 SMs, 3 SOs	4 ctrits	2/3	1/12
S2	$\mathcal{U}^{(12)}$	2 GBs	$\mathcal{V}$, GM, 2 SMs, 3 SOs	4 ctrits	1	1/8
S2	$\mathcal{U}^{(15)}$	2 GBs	$\mathcal{V}$, GM, 2 SMs, 3 SOs	4 ctrits	1	1/8
S2	$\mathcal{U}^{(18)}$	2 GBs	$\mathcal{V}$, GM, 2 SMs, 3 SOs	4 ctrits	1	1/8

Obviously, one can see that the general scheme (i.e., S1) is a deterministic one, that is, Alice's task can be achieved with unit probability. As can be seen from the first line of the table. As a matter of fact, the first scheme is essentially an ordering incorporation of three processes, i.e., QST and Alice's operation performance as well as QSTS. Although it looks like a trivial one, it actually offers an upper limit of resource consumptions and shows the complexity of necessary operations in accomplishing the quantum task. Specifically, in the scheme two units of entanglements and five trits of the classical communication cost are indispensable, the necessary operations are two generalized Bell-state measurements and two single-qutrit unitary operations as well as a single-qutrit measurement, the intrinsic efficiency of the scheme is 1/10. The distinct feature of this scheme is its universality for sharing any single-qutrit operation in a deterministic manner. All these can be taken as a useful frame of reference for some other optimal schemes which might be proposed later aiming at some special considerations.

In our second scheme, the quantum channels are changed by instituting the generalized GHZ state with the generalized Bell state. Very intuitively, the quantum resource consumption is decreased. Moreover, the classical resource consumption is also reduced, too. All these can be seen from the table by inspecting the third and fifth columns of the the schemes S1 and S2. Nonetheless, the second scheme can not be simply decomposed as the three processes mentioned in the last paragraph anymore. This essential change will bring some variance, as will be see later.

As far as the sharing of any single-qutrit operation is concerned, besides the common advantages mentioned in the last paragraph, the quantum operation complexity in the second scheme is also apparently simplified. Note that, generally speaking, the complexity of a generalized Bell-state measurement can be decomposed into a series of ordering two-qutrit control operation and a single single-qutrit operation as well as two single-qutrit measurements. Hence its implementation difficulty is approximately equal to that of the operation $\mathcal{V}$ and a single-qutrit operation as well as two single-qutrit measurements. All these advantages can be seen from the second line contrasting to the first one in the table. However, the cost of these resultant advantages is also very clear. It is obvious that in the second scheme both the scheme success probability and its intrinsic efficiency are much smaller than those in the first one. Evidentally, these two indicators mentioned just are violently decreased. The scheme S2 becomes a probabilistic one.

As for sharing the restricted sets of operations, two cases in the second scheme are grouped. One considers the restricted sets listed in the second column of the third, fourth and fifth lines in the table, i.e., $\mathcal{U}^{(34\backslash 12)}$, $\mathcal{U}^{(67\backslash 15)}$ and $\mathcal{U}^{(910\backslash 18)}$. The other treats the restricted sets shown in the second column of the last three in the table, namely, $\mathcal{U}^{(12)}$, $\mathcal{U}^{(15)}$ and $\mathcal{U}^{(18)}$. With a priori knowledge on these sets (only on the sets themselves, not the detailed information of the elements in the sets), contrasting to the sharing an arbitrary operation, one can easily find that in the former case both the scheme success probability and the intrinsic efficiency are doubled while in the latter the two indicators are tripled. We have mentioned before that the scheme S2 has become a probabilistic one after the quantum channel change. In the present former case on restricted sets, The scheme remains probabilistic. However, in the latter case, an essential variance happens, i.e., the scheme has already changed to be a deterministic one. Alternatively, the scheme success probability has been increased to 1. In this situation, it is intriguing to compare the specific scheme S2 in this case with the general scheme S1. They both have the unit success probability. Nonetheless, with the priori knowledge on the restricted sets, S2 completely overwhelms S1 in all the four aspects of quantum resource consumption, classical resource consumption, difficulty and intensity of necessary operations, and the intrinsic efficiency. In this sense, one can think that S2 is more optimal with the precondition of the priori knowledge.

At last, we want to briefly point out that the present schemes are actually the generalization of the ZC schemes from the aspect of particle degrees. The former is a qutrit one while the latter is a qubit one. In the ZC scheme on restricted sets, there only exist two kinds of success probabilities (1/2 or 1), and the success probability 1/2 corresponds to the universal applicability while the unit probability to the successful application to two restricted sets of unitary operations. After the degree extension, the results become more abundant. Easily one can see that more restricted sets occur and more possibilities appear. Our present qutrit schemes actually contains the ZC schemes. Alternatively, the present schemes can be reduced to the ZC schemes, as can be easily seen from the restricted sets. In the case of the unit success probability, if one degree is frozen, the restricted sets $\mathcal{U}^{(12)}$, $\mathcal{U}^{(15)}$ and $\mathcal{U}^{(18)}$ can be easily reduced to the diagonal or anti-diagonal sets in the ZC scheme.

5 Summary

To summarize, in this paper we have actually considered the shared remote control in the form of quantum operation sharing on a single-qutrit state. By integrating the ideas of quantum operation teleportation and quantum secret sharing, we have presented two possible schemes. The first scheme is universally applicable for any arbitrary single-qutrit unitary operation. The second scheme is generally a probabilistic one. However, after intensive investigations we have found that, if the operation $\mathcal{U}$ in question is known to belong to some restricted sets, both the scheme success probability and its efficiency can be doubled or even tripled. We have concretely compared the schemes in different cases from the four aspects of quantum and classical resource consumption, necessary-operation complexity, success probability and efficiency. In the tripled case the latter scheme becomes a deterministic one and is more optimal than the general scheme.

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