

# A General Method for Selecting Quantum Channel for Bidirectional Controlled State Teleportation and Other Schemes of Controlled Quantum Communication

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## Abstract

Recently, a large number of protocols for bidirectional controlled state teleportation (BCST) have been proposed using  $n$ -qubit entangled states ( $n \in \{5, 6, 7\}$ ) as quantum channel. Here, we propose a general method of selecting multi-qubit ( $n > 4$ ) quantum channels suitable for BCST and show that all the channels used in the existing protocols of BCST can be obtained using the proposed method. Further, it is shown that the quantum channels used in the existing protocols of BCST forms only a negligibly small subset of the set of all the quantum channels that can be constructed using the proposed method to implement BCST. It is also noted that all these quantum channels are also suitable for controlled bidirectional remote state preparation (CBRSP). Following the same logic, methods for selecting quantum channels for other controlled quantum communication tasks, such as controlled bidirectional joint remote state preparation (CJBRSP) and controlled quantum dialogue, are also provided.

**Keywords:** Bidirectional controlled state teleportation, controlled quantum communication, multi-qubit quantum channel.

## 1 Introduction

The idea of quantum teleportation was introduced by Bennett *et al.* [1] in 1993. Since this pioneering work, a large number of modified teleportation schemes (such as, schemes for quantum information splitting (QIS) or controlled teleportation (CT) [2, 3], quantum secret sharing (QSS) [4], hierarchical quantum information splitting (HQIS) [5, 6], remote state preparation [7], etc.) have been proposed (see [8] for a review). Teleportation and its modified versions drew considerable attention of the quantum communication community because of two main reasons: firstly, teleportation is a purely quantum phenomenon having no classical analogue and secondly, teleportation and modified teleportation schemes have potential applications in secure quantum communication and remote quantum operations [9].

Bennett *et al.*'s original teleportation scheme [1] enables the sender (Alice) to transmit an unknown single qubit quantum state to the receiver (Bob) by using two bits of classical communication and a pre-shared Bell state. This unidirectional scheme was subsequently generalized by Huelga *et al.* [9, 10] and others by introducing protocols for bidirectional quantum state teleportation (BST) which permit both Alice and Bob to simultaneously transmit unknown quantum states to each other. Interestingly, Huelga *et al.* had also shown that nonlocal quantum gates can be implemented using BST. Actually, the existence of a BST scheme ensures the existence of a nonlocal quantum gate or a quantum remote control (for a clearer discussion see our earlier works [11, 12]). This specific feature of BST attributed much importance to the study of BST in context of both quantum computation and quantum communication. In the recent past, the idea of BST has been further extended, and a few schemes for bidirectional controlled state teleportation (BCST) have been proposed [12, 13, 14, 15, 16, 17, 18, 19, 20]. A standard BCST

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SMO	Initial state shared by Alice and Bob			
	$ \psi^+\rangle$	$ \psi^-\rangle$	$ \phi^+\rangle$	$ \phi^-\rangle$
	Receiver's operation	Receiver's operation	Receiver's operation	Receiver's operation
00	$I$	$Z$	$X$	$iY$
01	$X$	$iY$	$I$	$Z$
10	$Z$	$I$	$iY$	$X$
11	$iY$	$X$	$Z$	$I$

Table 1: The relation between the entangled states shared by the receiver and sender, the measurement outcome of the sender, and the unitary operations to be applied by the receiver to realize perfect teleportation. Here SMO stands for the sender's measurement outcome.

scheme is a three party scheme, where implementation of BST is possible iff the controller (Charlie) allows the other two users (Alice and Bob) to execute a protocol of BST [12]. A careful review of all the recently proposed BCST schemes [13, 14, 15, 16, 17, 18, 19, 20] that do not use permutation of particles (PoP) [21, 22, 23] technique reveals that different  $n$ -qubit (with  $n \geq 5$ ) entangled states are used in these protocols. In contrast, the same task can also be achieved using Bell states and PoP as shown by some of the present authors in Ref. [24]. Clearly, PoP-based schemes for BCST require lesser quantum resource as that require only Bell states, whereas all the existing non-PoP based schemes for BCST require at least 5 qubit entangled states. However, no prescription for experimental realization of PoP exists until now. Keeping this in mind, in the present paper, we restrict ourselves to non-PoP based schemes for BCST. In an earlier work [12], some of the present authors explored the intrinsic symmetry of the 5-qubit quantum states that were used to propose the protocols of BCST until then. In Ref. [12], following general structure of the quantum states that can be used for BCST was provided:

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\psi_1\rangle_{A_1 B_1} |\psi_2\rangle_{A_2 B_2} |a\rangle_{C_1} \pm |\psi_3\rangle_{A_1 B_1} |\psi_4\rangle_{A_2 B_2} |b\rangle_{C_1}), \quad (1)$$

where single qubit states  $|a\rangle$  and  $|b\rangle$  satisfy  $\langle a|b\rangle = \delta_{a,b}$ ,  $|\psi_i\rangle \in \{|\psi^+\rangle, |\psi^-\rangle, |\phi^+\rangle, |\phi^-\rangle : |\psi_1\rangle \neq |\psi_3\rangle, |\psi_2\rangle \neq |\psi_4\rangle\}$ ,  $|\psi^\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$ ,  $|\phi^\pm\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$ , and the subscripts  $A$ ,  $B$  and  $C$  indicate the qubits of Alice, Bob and Charlie, respectively. To illustrate that  $|\psi\rangle$  can be used to implement a BCST scheme, we may consider that Charlie prepares the state  $|\psi\rangle$  and sends 1st and 3rd (2nd and 4th) qubits to Alice (Bob) and keeps the 5th qubit with himself. The condition

$$|\psi_1\rangle \neq |\psi_3\rangle, |\psi_2\rangle \neq |\psi_4\rangle \quad (2)$$

ensures that Charlie's qubit is properly entangled with the remaining 4 qubits, and this in turn ensures that Alice and Bob are not aware of the entangled (Bell) states they share until Charlie measures his qubit using  $\{|a\rangle, |b\rangle\}$  basis and announces the outcome of his measurement. After Charlie's disclosure of the measurement outcome, Alice and Bob know with certainty which two Bell states they share, and that knowledge empowers them to use the conventional teleportation scheme to teleport unknown quantum states to each other by using classical communication and the unitary operations described in Table 1.

In the above scheme for BCST, we require at least 5-qubit entanglement and in that case we have to choose  $|a\rangle$  and  $|b\rangle$  as single qubit states. However, we are allowed to take  $|a\rangle$  and  $|b\rangle$  as multi-qubit states and that leads to 6 or more qubit quantum states capable of performing BCST. For example, in Refs. [17, 18, 19], BCST is reported using 6-qubit entangled states, and in Ref. [20], BCST is reported using a 7-qubit entangled state. These recent papers [17, 18, 19, 20] on  $n$ -qubit ( $n > 5$ ) implementation of BCST scheme, and the fact that not all of them can be expressed in the form (1) motivated us to extend our earlier work and to look for a general structure of the  $n$ -qubit ( $n \geq 5$ ) entangled states that can be used to implement BCST. Keeping this in mind, here we construct a general method for selecting quantum states for implementation of multiqubit BCST schemes and subsequently extend the method to construct suitable quantum states for other quantum communications tasks, such as controlled bidirectional remote state preparation (CBRSP), controlled joint bidirectional remote state preparation (CJBRSP), controlled quantum dialogue, etc. In brief, we establish that there is a general structure of the quantum states that are suitable for controlled quantum communication tasks and there is not much merit in investigating specific quantum channels in isolation.

Remaining part of the present paper is organized as follows. A general method for selecting a quantum channel for multiqubit BCST is introduced in Section 2. In Section 3, all the existing states used so far for BCST schemes are shown as the special cases of the general structure introduced in the previous section. In Section 4, methods for selecting quantum channels for different controlled quantum communication tasks (such as RSP, CBRSP, CJBRSP, controlled quantum dialogue, etc.) are discussed. Finally, the paper is concluded in Section 5.

## 2 A general method for selecting a quantum channel for BCST

The general structure of 5-qubit states used for BCST can further be extended to the case where  $|a\rangle$  and  $|b\rangle$  in Eq. (1) are multiqubit states, and we can write the general structure as

$$|\psi\rangle = \sum_{m=1}^n \frac{1}{\sqrt{n}} ( (|\psi_i\rangle|\psi_j\rangle)_m |a_m\rangle ), \quad (3)$$

where  $|a_m\rangle$  are  $n$  mutually orthogonal  $l$ -qubit states with  $2^l \geq n \geq 2$ , and  $|\psi_i\rangle$  and  $|\psi_j\rangle$  are the elements of a multiqubit basis set whose elements are maximally (nonmaximally) entangled states capable of performing perfect (probabilistic) teleportation (such as the set of Bell states or GHZ states) and  $(|\psi_i\rangle|\psi_j\rangle)_m = (|\psi_i\rangle|\psi_j\rangle)_{m'}$  iff  $m = m'$ . Further, if  $|\psi_i\rangle$  and  $|\psi_j\rangle$  are  $p$ -qubit entangled states, then we can easily observe that  $n \leq (2^p)^2$ . Thus, for the Bell states  $n \leq 16$ , which implies that  $l > 4$  or more than 8-qubit quantum states  $|\psi\rangle$  used for the implementation of BCST scheme involving Bell states for transmission of unknown qubits in both the direction will not yield any new information (will not represent a new quantum state in the perspective of information theory).

Now, the states  $|\psi_i\rangle$  and  $|\psi_j\rangle$  are chosen in such a way that unless Charlie measures his qubits in  $\{|a_m\rangle\}$  basis and discloses the measurement outcome, Alice and Bob (in general the receivers and the senders) do not know which entangled states they share. This control of Charlie should exist in both the directions of communication (i.e., prior to Charlie's disclosure the receiver and the sender neither know the entangled state to be used for Alice to Bob teleportation, nor they know the entangled state to be used for Bob to Alice communication). A systematic method for obtaining quantum states suitable for BCST may be described using the following steps:

**Step 1:** As  $|\psi_i\rangle$  and  $|\psi_j\rangle$  are the elements of a basis set whose elements are  $p$ -qubit entangled states we describe the basis set as  $\{|\psi_i\rangle : i \in \{1, 2, \dots, 2^p\}\}$  and use that to construct a  $2^p \times 2^p$  matrix  $S$  such that  $s_{ij} = |\psi_i\rangle|\psi_j\rangle$  is the  $i^{th}$  row  $j^{th}$  column element of the matrix

$$S \equiv \begin{bmatrix} |\psi_1\rangle|\psi_1\rangle & |\psi_1\rangle|\psi_2\rangle & \cdots & |\psi_1\rangle|\psi_{2^p}\rangle \\ |\psi_2\rangle|\psi_1\rangle & |\psi_2\rangle|\psi_2\rangle & \cdots & |\psi_2\rangle|\psi_{2^p}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ |\psi_{2^p}\rangle|\psi_1\rangle & |\psi_{2^p}\rangle|\psi_2\rangle & \cdots & |\psi_{2^p}\rangle|\psi_{2^p}\rangle \end{bmatrix}. \quad (4)$$

**Step 2:** To construct a quantum state of the form (3) that can perform BCST we choose  $n \geq 2$  elements of  $S$  as  $(|\psi_i\rangle|\psi_j\rangle)_m$  with the following restrictions.

**Rule 1:** We cannot pick all the  $n$  elements from the same row or the same column of the matrix<sup>1</sup> (4).

**Rule 2:** We cannot pick one element more than once<sup>2</sup> as  $(|\psi_i\rangle|\psi_j\rangle)_m = (|\psi_i\rangle|\psi_j\rangle)_{m'}$  iff  $m = m'$ .

Let us now elaborate the method described above using a simple example. Consider that  $\{|\psi_i\rangle\}$  is a set of Bell states and  $|\psi^+\rangle = |\psi_1\rangle$ ,  $|\psi^-\rangle = |\psi_2\rangle$ ,  $|\phi^+\rangle = |\psi_3\rangle$ , and  $|\phi^-\rangle = |\psi_4\rangle$ . Thus, the matrix (4) reduces to

$$S_{\text{Bell}} \equiv \begin{bmatrix} |\psi^+\rangle|\psi^+\rangle & |\psi^+\rangle|\psi^-\rangle & |\psi^+\rangle|\phi^+\rangle & |\psi^+\rangle|\phi^-\rangle \\ |\psi^-\rangle|\psi^+\rangle & |\psi^-\rangle|\psi^-\rangle & |\psi^-\rangle|\phi^+\rangle & |\psi^-\rangle|\phi^-\rangle \\ |\phi^+\rangle|\psi^+\rangle & |\phi^+\rangle|\psi^-\rangle & |\phi^+\rangle|\phi^+\rangle & |\phi^+\rangle|\phi^-\rangle \\ |\phi^-\rangle|\psi^+\rangle & |\phi^-\rangle|\psi^-\rangle & |\phi^-\rangle|\phi^+\rangle & |\phi^-\rangle|\phi^-\rangle \end{bmatrix}. \quad (5)$$

<sup>1</sup>If we choose all the elements from the  $i^{th}$  row ( $j^{th}$  column) of  $S$ , then the desired quantum channel of the form (3) will become separable, and Charlie will lose control in one direction of the BST. Thus, the scheme will not remain BCST.

<sup>2</sup>This condition ensures the required bijective mapping between Charlie's measurement outcome and the entangled states shared by Alice and Bob. In the absence of this unique mapping, the receivers will not be able to decide which unitary operation is to be applied to achieve teleportation.

Now, consider that Charlie keeps a single qubit, and he measures his qubit in  $\{|+\rangle, |-\rangle\}$  basis. Thus, to construct a quantum state of the form (3) we may consider  $|a_1\rangle = |+\rangle$  and  $|a_2\rangle = |-\rangle$ . Further, following the rules listed above, we may choose  $(|\psi_i\rangle|\psi_j\rangle)_1 = s_{\text{Bell}_{11}} = |\psi^+\rangle|\psi^+\rangle$  and  $(|\psi_i\rangle|\psi_j\rangle)_2 = s_{\text{Bell}_{22}} = |\psi^-\rangle|\psi^-\rangle$  as  $s_{\text{Bell}_{11}} \neq s_{\text{Bell}_{22}}$ , and they are not elements of the same row or the same column. This choice would reduce the quantum state described by (3) to  $|\psi\rangle = \frac{1}{\sqrt{2}}(|\psi^+\rangle_{A_1 B_1}|\psi^+\rangle_{A_2 B_2}|+\rangle_{C_1} + |\psi^-\rangle_{A_1 B_1}|\psi^-\rangle_{A_2 B_2}|-\rangle_{C_1})$ . This is the quantum channel used by Zha et al. in their proposal of BCST [13].

Let us consider another example in which  $n = 4$  and  $|a_1\rangle = |0+\rangle$ ,  $a_2 = |0-\rangle$ ,  $a_3 = |1+\rangle$ ,  $a_4 = -|1-\rangle$ . In order to construct a quantum state of the form (3), we have to select 4 elements of  $S_{\text{Bell}}$  in such a way that Rules 1 and 2 are not violated. Keeping Rules 1 and 2 in mind, let us select the elements of  $S_{\text{Bell}}$  shown in rectangular boxes below as  $(|\psi_i\rangle|\psi_j\rangle)_m$

$$S_{\text{Bell}} \equiv \begin{bmatrix} \boxed{|\psi^+\rangle|\psi^+\rangle} & \boxed{|\psi^+\rangle|\psi^-\rangle} & |\psi^+\rangle|\phi^+\rangle & |\psi^+\rangle|\phi^-\rangle \\ |\psi^-\rangle|\psi^+\rangle & |\psi^-\rangle|\psi^-\rangle & |\psi^-\rangle|\phi^+\rangle & |\psi^-\rangle|\phi^-\rangle \\ |\phi^+\rangle|\psi^+\rangle & |\phi^+\rangle|\psi^-\rangle & \boxed{|\phi^+\rangle|\phi^+\rangle} & \boxed{|\phi^+\rangle|\phi^-\rangle} \\ |\phi^-\rangle|\psi^+\rangle & |\phi^-\rangle|\psi^-\rangle & |\phi^-\rangle|\phi^+\rangle & |\phi^-\rangle|\phi^-\rangle \end{bmatrix}. \quad (6)$$

Note that neither all the elements shown in rectangular boxes belong to the same row nor they belong to the same column. Thus, they satisfy the rules. Further, if we arrange the selected elements as  $(|\psi_i\rangle|\psi_j\rangle)_1 = |\psi^+\rangle|\psi^+\rangle$ ,  $(|\psi_i\rangle|\psi_j\rangle)_2 = |\psi^+\rangle|\psi^-\rangle$ ,  $(|\psi_i\rangle|\psi_j\rangle)_3 = |\phi^+\rangle|\phi^+\rangle$ ,  $(|\psi_i\rangle|\psi_j\rangle)_4 = |\phi^+\rangle|\phi^-\rangle$ , then we obtain

$$|\psi\rangle = \frac{1}{2}(|\psi^+\rangle_{A_1 B_1}|\psi^+\rangle_{A_2 B_2}|0+\rangle_{C_1 C_2} + |\psi^+\rangle_{A_1 B_1}|\psi^-\rangle_{A_2 B_2}|0-\rangle_{C_1 C_2} + |\phi^+\rangle_{A_1 B_1}|\phi^+\rangle_{A_2 B_2}|1+\rangle_{C_1 C_2} - |\phi^+\rangle_{A_1 B_1}|\phi^-\rangle_{A_2 B_2}|1-\rangle_{C_1 C_2}) \quad (7)$$

which is the quantum state used in Ref. [18] to implement BCST (cf. Eq. (1) of [18]). Now, we may choose the same  $|a_1\rangle, |a_2\rangle, |a_3\rangle$ , and  $|a_4\rangle$  as used in this example and select 4 other elements of  $S_{\text{Bell}}$  that are not from the same row/column to obtain a new 6-qubit quantum channel (one possible quantum channel for each selection) that can be used to realize BCST. Thus, we observe that if we follow the method prescribed here, we can easily generate several new quantum states that can be used to implement BCST. In what follows, we will show that the number of possible quantum channels is extremely high and only a few possibilities have been studied in the existing works on BCST.

In the first example above, Charlie keeps only one qubit with himself, and consequently we were required to choose two elements of  $S_{\text{Bell}}$  without violating rules 1 and 2 stated above. A specific example is shown above. However, the rules allow us to choose any of the 16 elements of  $S_{\text{Bell}}$  as  $(|\psi_i\rangle|\psi_j\rangle)_1$ . The moment we make a specific choice, Rule 1 allows us to choose  $(|\psi_i\rangle|\psi_j\rangle)_2$  only from 9 elements of  $S_{\text{Bell}}$  (i.e., for the second choice, the row and column of the first choice are exempted). Thus, the total possible choices of 5-qubit quantum states that can implement BCST is  $16 \times 9 = 144$  for a specific choice of Charlie's measurement basis  $\{|a_m\rangle\}$ . This coincides exactly with the results reported in Ref. [12]. Extending this logic to more general cases, we may note that a simple algebraic analysis reveals that for a specific choice of a subset of order  $n$  of a basis set  $\{|a_m\rangle\}$ , the total number of possible states ( $N_s$ ) of the form (3) that can be constructed using  $S$  without violating the rules mentioned above is

$$N_s = \begin{cases} \frac{2^{2p_1}}{(2^{2p-n})!} & \text{for } n > 2^p \\ 2^{pn} (2^{pn} - 2^{p+1} + 1) & \text{for } n < 2^p \end{cases}. \quad (8)$$

Using (8), we may quickly obtain the possible number of quantum channels for Bell-state based BCST for a specific choice of  $\{|a_m\rangle\}$  using  $p = 2$ . Specifically, in this case, if  $n > 4$ , then  $N_s = \frac{16!}{(16-n)!}$  and consequently, for  $n = 2, 3$  and  $4$ , we can construct quantum states of the form (3) in 144, 3648 and 63744 ways, respectively. These numbers clearly show that until now BCST is investigated using a very small subset of all possible states that can be used to realize BCST. Here, we have obtained a quantitative measure of  $N_s$  for a specific choice of an  $n$ th order subset of  $\{|a_m\rangle\}$  without considering the following: (i) the relative phases of the superposition in (3), (ii) possible permutations of  $\{|a_m\rangle\}$  and (iii) the number of ways in which subset of order  $n$  can be constructed. Inclusion of these factors will further enhance the number of ways in which BCST can be done. Specifically, if Charlie keeps  $l$  qubits with himself, then the inclusion of the last two factors listed above will further increase the total number of possible states by  $\frac{2^l!}{(2^l-n)!}$  times for specific value of  $n$ . Our intention is not to obtain the total number. We are interested to establish that there exists a systematic way to obtain quantum states that can implement BCST, and the set of all the states that are shown to be useful in implementing BCST only forms a small subset of the set of all possible states that can be used to implement BCST. Above discussion firmly establishes the fact we intended to establish.

### 3 Existing states as the special cases of the general structure

In Section 1, we have already mentioned that schemes for BCST have been proposed in the recent past using various quantum states. Here, we show that all the states used till date can be expressed in the general form (3). This fact is explicitly shown in Table 2, where following notation is used to express GHZ states:

$$\text{GHZ}^{x\pm} = \frac{(|ij\rangle \pm |\bar{i}\bar{j}\rangle)}{\sqrt{2}}, \quad (9)$$

where  $x$  is the decimal value of binary number  $ijj$  with  $i, j \in \{0, 1\}$ , and  $\pm$  denotes the relative phase between the two components of the superposition. For example, if we consider  $i = j = 0$ , we obtain  $\text{GHZ}^{0\pm} = \frac{(|000\rangle \pm |111\rangle)}{\sqrt{2}}$ .

Quantum states used in existing works	How to express the quantum states used in the existing work in the generalized form described in the present paper?	Remarks
Eq. (1) in Ref. [13]	$\frac{1}{\sqrt{2}} ( \psi^+\rangle_{A_1 B_1}  \psi^+\rangle_{A_2 B_2}  +\rangle_{C_1} +  \psi^-\rangle_{A_1 B_1}  \psi^-\rangle_{A_2 B_2}  -\rangle_{C_1})$	
Eq. (8) in Ref. [14]	$\frac{1}{\sqrt{2}} ( \psi^+\rangle_{A_1 B_1}  \psi^+\rangle_{A_2 B_2}  0\rangle_{C_1} -  \psi^-\rangle_{A_1 B_1}  \psi^-\rangle_{A_2 B_2}  1\rangle_{C_1})$	
Eq. (3) in Ref. [15]	$\frac{1}{\sqrt{2}} ( \psi^+\rangle_{A_1 B_1}  +\rangle_{C_1} +  \psi^-\rangle_{A_1 B_1}  -\rangle_{C_1})  \psi^+\rangle_{A_2 B_2}$	Charlie's control is limited to one side of teleportation.
Eq. (3) in Ref. [16]	$\frac{1}{\sqrt{2}} ( \psi^+\rangle_{A_2 B_2}  0\rangle_{C_1} +  \psi^-\rangle_{A_2 B_2}  1\rangle_{C_1})  \phi^+\rangle_{A_1 B_1}$	
Eq. (12) in Ref. [17]	$\frac{1}{\sqrt{2}} ( \psi^+\rangle_{A_1 B_1}  \phi^+\rangle_{A_2 B_2}  00\rangle_{C_1 C_2} +  \phi^+\rangle_{A_1 B_1}  \psi^+\rangle_{A_2 B_2}  11\rangle_{C_1 C_2})$	
Eq. (1) in Ref. [18]	$\frac{1}{2} ( \psi^+\rangle_{A_1 B_1}  \psi^+\rangle_{A_2 B_2}  0+\rangle_{C_1 C_2} +  \psi^+\rangle_{A_1 B_1}  \psi^-\rangle_{A_2 B_2}  0-\rangle_{C_1 C_2} +  \phi^+\rangle_{A_1 B_1}  \phi^+\rangle_{A_2 B_2}  1+\rangle_{C_1 C_2} -  \phi^+\rangle_{A_1 B_1}  \phi^-\rangle_{A_2 B_2}  1-\rangle_{C_1 C_2})$	
Eq. (3) in Ref. [19]	$\frac{1}{2} ( \psi^+\rangle_{A_1 B_1}  \psi^+\rangle_{A_2 B_2}  ++\rangle_{C_1 C_2} +  \psi^+\rangle_{A_1 B_1}  \phi^-\rangle_{A_2 B_2}  +-\rangle_{C_1 C_2} +  \phi^-\rangle_{A_1 B_1}  \psi^+\rangle_{A_2 B_2}  -+\rangle_{C_1 C_2} +  \phi^-\rangle_{A_1 B_1}  \phi^-\rangle_{A_2 B_2}  --\rangle_{C_1 C_2})$ or $\frac{1}{2} ( \psi^+\rangle_{A_1 B_1}  \phi^+\rangle_{A_2 B_2}  0+\rangle_{C_1 C_2} +  \psi^+\rangle_{A_1 B_1}  \phi^+\rangle_{A_2 B_2}  0-\rangle_{C_1 C_2} +  \phi^+\rangle_{A_1 B_1}  \psi^+\rangle_{A_2 B_2}  1+\rangle_{C_1 C_2} -  \phi^+\rangle_{A_1 B_1}  \psi^+\rangle_{A_2 B_2}  1-\rangle_{C_1 C_2})$	These are shown in Ref. [17] as Eqs. (11) and (13).
Eq. (4) in Ref. [20]	$\frac{1}{2} ( \psi^+\rangle_{A_1 B_1}  \psi^+\rangle_{A_2 B_2}  \text{GHZ}^{0+}\rangle_{C_1 C_2 C_3} -  \psi^-\rangle_{A_1 B_1}  \phi^-\rangle_{A_2 B_2}  \text{GHZ}^{2+}\rangle_{C_1 C_2 C_3} -  \phi^+\rangle_{A_1 B_1}  \phi^+\rangle_{A_2 B_2}  \text{GHZ}^{3+}\rangle_{C_1 C_2 C_3} -  \phi^-\rangle_{A_1 B_1}  \psi^-\rangle_{A_2 B_2}  \text{GHZ}^{1+}\rangle_{C_1 C_2 C_3})$	

Table 2: Quantum channels used in different proposals for BCST as special cases of the generalized structure shown here.

### 4 The condition for selecting a quantum channel for other controlled quantum communication tasks

Several schemes of controlled quantum communication have been discussed in the recent past ([11, 12, 24, 25, 26, 27, 28, 29] and references therein). To be precise, schemes for controlled bidirectional remote state preparation [11, 25], controlled joint bidirectional remote state preparation [11], controlled quantum dialogue [26, 27], etc., have been proposed using various quantum states. Extending the argument above, in what follows, we provide a general method for selecting quantum states for these tasks. Here, we limit ourselves to the explicit discussion of the general structure for the quantum states required for (i) controlled bidirectional remote state preparation, (ii) controlled joint bidirectional remote state preparation and (iii) controlled quantum dialogue, but the logic can be extended easily to other controlled quantum communication tasks.

#### 4.1 How to select a quantum channel for Controlled Bidirectional Remote State Preparation?

In Ref. [11], some of the present authors have shown that the quantum states suitable for BCST are also suitable for CBRSP. This is reasonable for the obvious reason that the capability of transportation of an unknown state automatically implies the capability of transporting a known quantum state. Further, it is well known that a shared Bell state and one bit of classical communication is sufficient for probabilistic RSP [7], whereas a shared Bell state and two bits of classical communication is sufficient for deterministic RSP. This fact and our discussion above in the context of the choice of quantum states for controlled BCST imply that the quantum states of the form (3) are sufficient for CBRSP if  $|\psi_i\rangle, |\psi_j\rangle$  are chosen using the rules described above.

#### 4.2 How to select a quantum channel for Controlled Joint Bidirectional Remote State Preparation?

For CJBRSP, the structure of quantum state to be used would remain same (i.e., the states described by Eq. (3) and element selection rules described after that) with the only difference that  $|\psi_i\rangle$  and  $|\psi_j\rangle$  must be the elements of a basis set whose elements are at least tripartite entangled and capable of performing joint remote state preparation. Specifically,  $|\psi_i\rangle$  and  $|\psi_j\rangle$  can be GHZ or GHZ-like states.

#### 4.3 How to select a quantum channel for Controlled Quantum Dialogue?

In case of quantum dialogue protocols of Ba-An type [30, 31], the quantum communication happens in both directions using the same quantum state, hence we do not require product of two entangled states after the measurement of controller Charlie. Here, it is sufficient to choose a quantum state such that unless the controller measures his/her qubit and announces the outcome, other two users (Alice and Bob) will be unaware of the quantum state they share. Thus, any quantum state of the form

$$|\psi\rangle = \sum_{m=1}^n \frac{1}{\sqrt{n}} (|\psi_i\rangle|a_m\rangle), \quad (10)$$

where  $|a_m\rangle$  are  $n$  mutually orthogonal  $l$ -qubit states with  $2^l \geq n \geq 2$ , and  $|\psi_i\rangle$  is an element of a set of entangled quantum states that are capable of performing quantum dialogue using the same set of unitary operators and that are unitarily connected with each other. Such that after the encoding operation of Alice (say  $U_j$ ) and that of Bob (say  $U_i$ ) the final states must also be a member of the set of mutually orthogonal states to ensure the deterministic discrimination of the state and thus to decode the encoded message, where  $U_j$  and  $U_i$  are the unitary operators which forms a group under multiplication. To be precise,  $|\psi\rangle_{final} = U_j U_i |\phi_0\rangle = U_j |\phi_i\rangle \in \{|\phi_0\rangle, |\phi_1\rangle, \dots, |\phi_i\rangle, \dots, |\phi_{2^n-1}\rangle \forall i, j \in \{0, 1, \dots, 2^n - 1\} \Rightarrow U_B U_A \in \{U_0, U_1, U_2, \dots, U_{2^n-1}\}$ . A list of quantum states that can be used for quantum dialogue protocol with corresponding unitary operators are given in Table 4 in Ref. [31]. Superpositions of such states with mutually orthogonal states in the control part  $|a_m\rangle$  can be used for the generalized protocol of controlled quantum dialogue.

## 5 Conclusions

The general structures of the quantum states suitable for BCST and other controlled quantum communication protocols are provided, and a method for obtaining all such states is proposed. Further, it is shown that all the quantum channels used in the existing protocols of BCST can be easily obtained using the general method proposed here. In fact, all the quantum states that are used in the existing protocols are explicitly expressed in the general form proposed here (cf. Table 2). The states described in Table 2, i.e., the states used in the existing literature, only forms a negligibly small subset of all possible states that can perform BCST. It is easy to visualize that there are infinitely many possible states that can be used to perform BCST. To elaborate this point we may note that there is no constraint on the choice of the set of controllers multiqubit orthogonal states. The infinitely many possible choices for the set of controllers multiqubit orthogonal states imply availability of infinitely many possible quantum channels for BCST. Even if we restrict Charlie to prepare and measure his qubits in a specific basis (say,

computational basis) and Alice and Bob to use Bell states for quantum communication, there exist a large number of ways in which quantum states of the form (3) can be constructed. This point is firmly established in Section 2.

We have already seen that the general structure provided here for BCST scheme gives an infinitely many possibilities for the choice of the suitable quantum channels to experimentalists. The generation of this kind of quantum states suitable for BCST scheme requires easily available resources, such as CNOT gate, Hadamard gate, etc. A multiqubit quantum states of the form given here (i.e., of the form (3)) have already been experimentally realized in the recent past [32], and further discussion on the possibilities of experimental preparation of the quantum states of the structure similar to the structure given here can be found in our earlier work [11]. Further, if we consider that  $|\psi_i\rangle$  and  $|\psi_j\rangle$  are Bell states, then after the measurement of the controller's (Charlie's) qubits in suitable basis, the remaining qubits reduce to a product state  $|\psi_i\rangle \otimes |\psi_j\rangle$ , which is the product of two entangled states (product of two Bell states in the case of Bell state based BCST scheme) shared between Alice and Bob, and which can be used for simultaneous teleportation of two unknown quantum states, one from Alice to Bob and the other from Bob to Alice. The resources required in BCST after the Charlie's measurement are just two copies of the resources required for teleportation of the unknown quantum state using the entangled state shared by Alice and Bob.

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