

# Geometry of quantum state space and quantum correlations

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Quantum state space is endowed with a metric structure and Riemannian monotone metric is an important geometric entity defined on such a metric space. Riemannian monotone metrics are very useful for information-theoretic and statistical considerations on the quantum state space. In this article, considering the quantum state space being spanned by  $2 \times 2$  density matrices, we determine a particular Riemannian metric for a state  $\rho$  and show that if  $\rho$  gets entangled with another quantum state, the negativity of the generated entangled state is, upto a constant factor, equals to square root of that particular Riemannian metric. Our result clearly relates a geometric quantity to a measure of entanglement. Moreover, the result establishes the possibility of understanding quantum correlations through geometric approach.

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## I. INTRODUCTION

Geometric tools are often used to treat physical problems. Undoubtedly, these tools have provided advantage to find out less trivial and robust physical constraints on physical systems. *Differential geometry* is one such mathematical tool which finds a lot of applications in various disciplines. Information theory is among those disciplines where the techniques of differential geometry have been applied. As a result of which a new discipline, called *Information Geometry* emerged and it got maturity through the works of Amari, Nagaoka and other mathematicians in the 1980s [1]. In their works they applied the methods of differential geometry to the field of probability theory, which alongwith statistics is the mathematical base of information theory. Initially, the goal of Information geometry was to understand the interplay between the information-theoretic quantities and the geometry of probability space by constructing a Riemannian space corresponding to probability space. Quite obviously it was of fundamental importance to study the Riemannian metrics defined on the space of probability distributions. Later, motivated by information geometry, Morozova and Čencov [2] initiated the study of Riemannian monotone metrics on the state space (Hilbert space) of quantum systems which has been gradually progressed through the works of Petz and other authors [3–8]. The monotone Riemannian metric corresponding to Wigner-Yanase-Dyson skew-information [9] was found out in [6] which vividly expresses the relation between geometry of space and an information-theoretic quantity of great importance. And not only limited to information-theoretic entities, geometric distances (metrics) are also closely related to the quantum state discrimination problem [10, 11]. In [12] the authors have demonstrated that a lower bound for quantum coherence measure can be found out using Riemannian monotone metric.

On the other hand, quantum correlation is the physical quantity responsible for the non-classical phenomenon exhibited by composite quantum systems. Though there are different aspects of quantum correlations, entanglement and discord are the two aspects which have been extensively studied due to their immense importance in quantum information processing tasks. However, till date, quantum correlation is not fully understood. So, the study of quantum correlations demands importance in quantum information theory. Here we consider entanglement because of the fact that all the measures of entanglement are monotonic in nature.

Quantum entanglement [13–15] is one of the bizarre phenomena exhibited by composite quantum systems and a resource for quantum information processing tasks, such as teleportation [16], dense coding [17], quantum cryptography [18], state merging [19], quantum computation and many more. A composite quantum system  $\rho_{AB}$  consisting of subsystems  $A$  and  $B$  is said to be entangled if it can not be written as  $\rho_{AB} = \sum_i p_i \rho_A^i \otimes \rho_B^i$ , where  $p_i$  are probabilities,  $\rho_A$  and  $\rho_B$  are respectively the density matrices of subsystem  $A$  and  $B$ . If the subsystems are two-level quantum states then these are termed as *qubits* [21] in analogy with classical bits and qubits are the fundamental units in quantum information theory.

In this article we ask the question; *Is there any connection between the geometry of quantum state space and entanglement?* To find the answer of the question we consider the negativity ( $\mathcal{N}$ ) [22, 23] measure which is also connected to some other measures [24]. Without loss of generality, we consider a generic two-level quantum state  $\rho_S$  (qubit). Quite obviously the dimension of the Hilbert space associated to such a state will be 2. It is known that entanglement can be generated between the qubit and another ancillary qubit by applying a global unitary interaction. We consider a particular global unitary and find the generated entangled state. The amount of entanglement present in the composite state is calculated through negativity measure. We also determine a particular Riemannian metric for the state  $\rho_S$  using a theorem proposed by Morozova and Čencov. Interestingly,

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we find that there is an explicit relation between the negativity of the generated entangled state and the particular monotone Riemannian metric on the state space of qubits. More precisely to say, the negativity of the two qubit entangled state is, upto a constant factor, equals to the square root of the monotone metric. The result clearly demonstrates the fact that monotone Riemannian metrics on quantum state space are not only connected to information-theoretic quantities but also to entanglement. Alternatively, we can say that the geometry of quantum state space finds connection with a non-classical property of the quantum states

The rest of the article is arranged as follows. In Section(II) we provide an overview on Riemannian metric and Riemannian metrics on matrix space(quantum state space). Section(III) contains a brief description about entanglement generation in qubit scenario. Section(IV) is dedicated to show our results and finally in Section(V) we conclude our work with discussions.

## II. RIEMANNIAN METRIC AND MONOTONE RIEMANNIAN METRICS ON MATRIX SPACE

Riemannian space( $M, g$ ) is a differentiable manifold(topological space)  $M$  embedded with an inner product  $g_x$  on the tangent space  $T_x M$  at each point  $x$  and  $g_x$  varies smoothly from point to point. More precisely if  $X$  and  $Y$  are two vectors on the tangent space passing through  $x$  the  $x \mapsto g_x\{X(x), Y(x)\}$  is a smooth function. Riemannian metric on a manifold  $M$  is the family of  $g_x$ .

Morozova and Čencov initiated the study of monotone Riemannian metrics on the space of matrices with the motivation to extend the geometric approach to quantum setting. They proposed the problem to find monotone Riemannian metrics on the quantum state space which is endowed with a metric structure. The quantum state space is a complex matrix space  $\mathcal{H}_n$  of dimension  $n$ , usually termed as *Hilbert space*. The operators in such a space is designated by  $n \times n$  complex, Hermitian and self-adjoint matrices. Whereas, the quantum states are defined by positive definite  $n \times n$  matrices with trace 1, also termed as density matrices. Hilbert space is an inner product space and the simplest inner product is certainly the Hilbert-Schmidt one, defined as

$$\langle X, Y \rangle = \text{Tr}(X^* Y) \quad (1)$$

where,  $\text{Tr}$  is the usual matrix trace and  $X, Y \in \mathcal{H}_n$ . The inner-product in Hilbert space is indeed unitarily invariant, i.e.  $\langle X, Y \rangle = \langle U X U^\dagger, U Y U^\dagger \rangle$  and this property is so strong that it determines the Hilbert-Schmidt inner-product upto a constant multiple.

Now, by making the inner-products depending on quantum states( $\rho$ ), Riemannian metrics can be determined on the quantum state space in the following way. Let for every  $A, B \in \mathcal{H}_n$ , for every  $\rho \in \mathcal{M}_n$ , and for every  $n \in N$ , a complex quantity  $K_\rho(A, B)$  is given, where  $\mathcal{M}_n$  is the set of all positive definite matrices with trace

1. The complex quantity  $K_\rho(A, B)$  will be a metric if the following conditions hold [3]:

- (a)  $(A, B) \mapsto K_\rho(A, B)$  is sesquilinear.
- (b)  $K_\rho(A, A) \geq 0$ , and the equality holds iff  $A = 0$
- (c)  $\rho \mapsto K_\rho(A, A)$  is continuous on  $\mathcal{M}_n$  for every  $A$

The family of the metrics  $K_\rho(A, B)$  with the above mentioned properties constitute a Riemannian metric on the differentiable manifold formed by the density matrices. The Riemannian metric will be monotone if

- (d) Under completely positive trace preserving(CPTP) map  $K_\rho(A, A)$  is contractive, i.e.  $K_{\Lambda(\rho)}(A, A) \leq K_\rho(A, A)$  for every  $A$ ,  $\rho$  and  $\Lambda$ ;  $\Lambda(\cdot)$  being the CPTP map.

For clear illustration of the metric  $K_\rho(A, B)$ , it is important to focus on the geometry of the quantum state space.  $K_\rho(A, B)$  is basically the inner product on the tangent space  $T_\rho$  and  $A, B$  are the two tangent vectors. Considering the Hilbert space  $\mathcal{H}_n$  to be finite dimensional, let us denote the set of all Hermitian operators on  $\mathcal{H}_n$  by

$$\mathcal{A} = \{A | A = A^*\} \quad (2)$$

and according to definition of  $\mathcal{M}_n$ ;

$$\mathcal{M}_n = \{\rho | \rho = \rho^* \geq 0 \text{ and } \text{Tr} \rho = 1\} \quad (3)$$

The tangent space  $T_\rho(\mathcal{M}_n)$  of each point  $\rho$  may then be identified with

$$\mathcal{A}_0 = \{A | A \in \mathcal{A} \text{ and } \text{Tr} A = 0\} \quad (4)$$

It can be shown that if  $\mathcal{K}$  is an operator and  $\mathcal{K} \in \mathcal{A}$ , then  $i[\rho, \mathcal{K}]$  will be an ordinary element of the tangent space, that is,  $i[\rho, \mathcal{K}] \in \mathcal{A}_0$ [1]. So, by identifying tangent vectors Riemannian metric can be defined on the differential manifold formed by the density matrices and upon satisfying the condition (d) the metric will be called monotone Riemannian metric.

Though Morozova and Čencov were unable to find any monotone Riemannian metric, they provided a useful theorem. Later, Petz and other authors were able to find monotone metrics by introducing operator monotone functions and their works showed that there is an abundance of monotone metrics on the space of self-adjoint matrices [4, 5]. For our purpose we will make use of the theorem provided by Morozova and Čencov, which can be stated as,

**THEOREM**[2, 3]: Assume that for every  $D \in \mathcal{M}_n$  a real bilinear form  $K_D'$  is given on the  $n$ -by- $n$  self-adjoint matrices such that the conditions (b),(c) and (d) are satisfied for self-adjoint  $A$ . Then there exists a positive continuous function  $c(\lambda, \mu)$  and a constant  $C$  with the following property: If  $D$  is diagonal with respect to the matrix units  $E_{ij}$ , i.e.  $D = \sum_i \lambda_i E_{ii}$ , then

$$K'(A, A) = C \sum_{i=1}^n \lambda_i^{-1} A_{ii}^2 + 2 \sum_{i < j} |A_{ij}|^2 c(\lambda_i, \lambda_j). \quad (5)$$

for every self-adjoint  $A = (A_{ij})$ . Moreover if  $c$  is symmetric in its two variables,  $c(\lambda, \lambda) = C\lambda^{-1}$  and  $c(t\lambda, t\mu) = t^{-1}c(\lambda, \mu)$

The function  $c(\lambda, \mu)$  is generally termed as *Morozova-Čencov* function. It can be concluded from the above stated theorem that when  $\mathcal{M}_n$  is considered as a differentiable manifold, the Riemannian metric must be a linear bilinear form and the tangent vectors may be identified with self-adjoint matrices. Also  $K'(A, A)$  for all  $D$  and all self-adjoint  $A$  can be derived from the theorem.

### III. ENTANGLEMENT GENERATION IN QUBIT SCENARIO

Let us consider a generic qubit  $\rho_S$ , generally expressed as:

$$\rho_S = \frac{1}{2}(\mathbf{1}_2 + \vec{n} \cdot \vec{\sigma}) \quad (6)$$

where,  $\vec{n} \equiv (n_x, n_y, n_z)$  is a vector in  $\mathbb{R}^3$  with  $|\vec{n}|^2 \leq 1$  and  $\vec{\sigma} := (\sigma_x, \sigma_y, \sigma_z)$  with  $\sigma_i |i\rangle = x, y, z$  being the Pauli matrices. Taking another ancillary qubit, say  $|0\rangle_A$ , which is an eigen state of  $\sigma_z$ , a unitary interaction is switched on over the product state  $\rho_S \otimes |0\rangle_A$ . The global unitary  $U_{SA}$  acts on the initial product state as:

$$\begin{aligned} U_{SA}(|0\rangle_S \otimes |0\rangle_A) &= |0\rangle_S \otimes |0\rangle_A \\ U_{SA}(|1\rangle_S \otimes |0\rangle_A) &= |1\rangle_S \otimes |1\rangle_A. \end{aligned} \quad (7)$$

So, after the application of the unitary the resulting state becomes

$$\begin{aligned} \rho_{SA} &= U_{SA} \left( \frac{1}{2}(\mathbf{1} + \vec{n} \cdot \vec{\sigma})_S \otimes |0\rangle_M \langle 0| \right) U_{SA}^\dagger \\ &= \frac{1+n_z}{2} |0\rangle_S \langle 0| \otimes |0\rangle_M \langle 0| \\ &\quad + \frac{n_x + in_y}{2} |1\rangle_S \langle 0| \otimes |1\rangle_M \langle 0| \\ &\quad + \frac{n_x - in_y}{2} |0\rangle_S \langle 1| \otimes |0\rangle_M \langle 1| \\ &\quad + \frac{1-n_z}{2} |1\rangle_S \langle 1| \otimes |1\rangle_M \langle 1|. \end{aligned} \quad (8)$$

The unitary operator corresponding to the evolution of the product state  $\rho_S \otimes |0\rangle_A$  can be expressed in terms of the total Hamiltonian ( $H_{tot}$ ) as  $U(t) := \exp(-i\frac{H_{tot}t}{\hbar})$  and  $H_{tot} = H_S \otimes \mathbf{1}_M + \mathbf{1}_S \otimes H_M + H_{int}$ . So it is clear that the initial composite product state will retain its product form if  $H_{int} = 0$ .

We are interested to find out the amount of entanglement in the state  $\rho_{SA}$  and among different measures of entanglement we consider the negativity measure for our purpose. Negativity is given by [22, 23]:

$$\mathcal{N}(\rho_{AB}) = \frac{\|\rho_{AB}^{T_A}\|_1 - 1}{2}. \quad (9)$$

where  $T_A$  denotes partial transpose with respect to the subsystem  $A$ ,  $\lambda_i$ 's denote the eigenvalues of  $\rho_{AB}^{T_A}$  and  $\|X\|_1 = \text{Tr}|X| = \text{Tr}\sqrt{X^\dagger X}$  be the trace-norm of an operator. Using Eqn(9) the negativity of the state in Eqn(8) is found to be

$$\mathcal{N}(\rho_{SA}) = \left(1 - \sqrt{2\mathcal{M}(\rho_S)}\right)^{-\frac{1}{2}} \frac{(1 - \sqrt{1 - |\vec{n}|^2})^{\frac{1}{2}} (n_x^2 + n_y^2)^{\frac{1}{2}}}{2} \quad (10)$$

for  $|\vec{n}| \neq 0$  and

$$\mathcal{N}(\rho_{SA}) = 0, \quad \text{if } |n| = 0. \quad (11)$$

where,  $\mathcal{M}(\rho_S)$  is the mixedness of the state  $\rho_S$  and is given by:

$$\begin{aligned} \mathcal{M}(\rho_S^{in}) &= \text{Tr}(\rho_S^{in}) - \text{Tr}(\rho_S^{in})^2 \\ &= \frac{1}{2}(1 - |\vec{n}|^2). \end{aligned} \quad (12)$$

From Eqn(11) it is clear that negativity of the state will be zero if  $\rho_S$  is maximally mixed, i.e.  $\rho = \frac{\mathbf{1}_2}{2}$ .

### IV. RESULTS

Now we show that the negativity  $\mathcal{N}$  of the state  $\rho_{SA}$  is, upto a constant, equal to a Riemannian metric on the quantum state space of dimension 2. Let us consider the self-adjoint, Hermitian operator  $\sigma_z$ . Then the operator  $i[\rho_S, \sigma_z]$  is also a Hermitian, self-adjoint operator and it is an element of the tangent space of density matrices differentiable manifold [8]. Our aim is to find the metric  $K_{\rho_S}(A, B)$ , where  $A = B = i[\rho_S, \sigma_z]$ .

The state  $\rho_S$  can be represented in the diagonal form as:

$$\begin{aligned} \rho_S &= \frac{1}{2}(\mathbf{1} + |\vec{n}| \hat{n} \cdot \vec{\sigma}) \\ &= \frac{1+|\vec{n}|}{2} \frac{1}{2}(\mathbf{1} + \hat{n} \cdot \vec{\sigma}) + \frac{1-|\vec{n}|}{2} \frac{1}{2}(\mathbf{1} - \hat{n} \cdot \vec{\sigma}). \end{aligned} \quad (13)$$

where,  $\hat{n} = \frac{\vec{n}}{|\vec{n}|}$ . Moreover, the matrix representation of self-adjoint operator  $i[\rho_S, \sigma_z]$  will be:

$$i[\rho_S, \sigma_z] = |\vec{n}| \begin{pmatrix} 0 & (-n_y - in_x) \\ (in_x - n_y) & 0 \end{pmatrix}. \quad (14)$$

It is easy to verify that for the operator represented in Eqn(14) the first summation term in Eqn(5) will be zero, i.e.,

$$C \sum_{i=1}^2 \lambda_i^{-1} A_{ii}^2 = 0 \quad (15)$$

where,  $A_{ii}$  represent the diagonal elements of the matrix corresponding to the operator  $i[\rho_S, \sigma_z]$ . Therefor,

$$K_{\rho_S}(i[\rho_S, \sigma_z], i[\rho_S, \sigma_z]) = 2 \sum_{i < j} |A_{ij}|^2 c(\lambda_i, \lambda_j) \quad (16)$$

$A_{ij}$  being the off-diagonal elements of the matrix given in Eqn(14) and  $c(\lambda_i, \lambda_j)$  is the Morozova-Čencov function [2, 3]. For our purpose we consider one of the functions proposed originally by Morozova and Čencov [2]; we take

$$c(\lambda_i, \lambda_j) = \left( \frac{2}{\sqrt{\lambda_i} + \sqrt{\lambda_j}} \right)^2. \quad (17)$$

Therefore, using Eqns[13,14] and Eqn(17), we get

$$\begin{aligned} K_{\rho_S}(i[\rho_S, \sigma_z], i[\rho_S, \sigma_z]) &= 2 \cdot |A_{12}|^2 \left( \frac{2}{\sqrt{\frac{1+|\vec{n}|}{2}} + \sqrt{\frac{1-|\vec{n}|}{2}}} \right)^2 \\ &= 32 |A_{12}|^2 \frac{(1 - \sqrt{1 - |\vec{n}|^2})}{|\vec{n}|^2} \\ &= 32(n_x^2 + n_y^2)(1 - \sqrt{1 - |\vec{n}|^2}) \end{aligned} \quad (18)$$

It is to be noted that we have considered the general density matrix  $\rho_S$  and determined the Riemannian metric for  $\text{diag}[\lambda_1, \lambda_2]$ ; where  $\lambda_1$  and  $\lambda_2$  are the eigenvalues of  $\rho_S$ . Nevertheless, the evaluated metric is the one which we want and the unitary covariance  $K_{\rho_S}(A, A) = K_{U^* \rho_S U}(U^* A U, U^* A U)$  confirms the fact [3].

Finally, using Eqn(10) and Eqn(18) we get;

$$\mathcal{N}(\rho_{SA}) = A \sqrt{K_{\rho_S}(i[\rho_S, \sigma_z], i[\rho_S, \sigma_z])}. \quad (19)$$

where,  $A = 2\sqrt{2} \left( 1 - \sqrt{2\mathcal{M}(\rho_S)} \right)^{-\frac{1}{2}}$

The above equation is the main result of this article. It is worth noticing that the above equation vividly depicts a relation between a geometric entity of the differential manifold formed by density matrices and a measure of entanglement. Now, as the above equation is a manifestation of equality relation between two different entities, it is necessary to check the validity of the equality and this can be verified with the following arguments;

- Negativity( $\mathcal{N}$ ) is unitarily invariant, i.e.  $\mathcal{N}(\rho_{SA}) = \mathcal{N}(U_{SA} \rho_{SA} U_{SA}^\dagger)$  and it is also a monotonic function;  $\mathcal{N}(\rho_{SA}) \geq \mathcal{N}(\Lambda(\rho_{SA}))$ , where  $\Lambda$  is a CPTP map.
- Riemannian metric  $K_{\rho_S}(i[\rho_S, \sigma_z], i[\rho_S, \sigma_z])$  is a monotone metric due to fulfilment of conditions (d) provided in Sec[III] and monotonicity includes the unitary covariance of the metric.

From the two arguments it can be concluded undoubtedly that our result provides an elegant relation between the geometry of the quantum state space and a non-classical phenomena exhibited by composite quantum system.

It is important to highlight that Eqn(19) encompasses some important facts directly. The Riemannian metric  $K_\rho(A, A) = 0$  iff  $A = 0$ . So,  $i[\rho_S, \sigma_z] = 0$  implies  $K_{\rho_S}(i[\rho_S, \sigma_z], i[\rho_S, \sigma_z]) = 0$ , which in turn gives  $\mathcal{N}_{SA} = 0$ . All these are in agreement with the physical fact that if the state  $\rho_S$  is a mixture of the eigen states of the observable  $\sigma_z$  then no entanglement can be generated between the state and the ancilla by applying  $\sigma_z$  interaction. The metric that we have calculated was identified as Wigner-Yanase(WY) skew information and WY skew-information is a good measure of coherence[25, 26]. Hence, our result identifies a relation between entanglement and coherence, thereby reconfirming the fact that in order to generate entanglement between two qubits, at least one qubit must be in coherent state[27].

Though we have considered a special global unitary  $U_{SA}$  in the entanglement generation scenario, as well as two specific tangent vectors(self-adjoint operators), i.e.  $A = B = i[\rho_S, \sigma_z]$  to determine the monotone Riemannian metric on the quantum state space, nevertheless, the relation given in Eqn(19) is not restricted to be a special case. Consider  $A = B = i[\rho_S, \sigma_x]$  as the tangent elements and the ancillary qubit to be  $\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)_A$ . Further assume that the global unitary acts as :

$$\begin{aligned} U_{SA}(|+\rangle_S \otimes |+\rangle_A) &= |+\rangle_S \otimes |+\rangle_A \\ U_{SA}(|+\rangle_S \otimes |-\rangle_A) &= |-\rangle_S \otimes |-\rangle_A. \end{aligned} \quad (20)$$

Now, if the Riemannian monotone metric is determined and negativity of the generated entangled state is calculated then we can have a similar relation between these two quantities just like in Eqn(19), i.e.

$$\mathcal{N}(\rho_{SA}) = A \sqrt{K_{\rho_S}(i[\rho_S, \sigma_x], i[\rho_S, \sigma_x])} \quad (21)$$

## V. CONCLUSIONS

Quantum state space is endowed with a metric structure and monotone Riemannian metrics are important candidates for quantum-information-theoretic considerations on such a space. In this article we consider 2-dimensional Hilbert space and show that a particular Riemannian metric is, upto a constant, equals to a measure of entanglement. To express more precisely, if a general qubit is considered and entanglement is generated between the qubit and an ancilla by applying interaction, then the negativity of entanglement is, upto a constant, equals to a Riemannian metric. The metric that we have considered was shown to be the Wigner-Yanase skew-information. Moreover, Wigner-Yanase skew-information is a good measure of coherence. So, our result also establishes a connection between coherence and entanglement. It is important to emphasize

that so far Riemannian metrics on quantum state space found relation with information-theoretic quantities only, whereas, our result vividly shows that a monotone Riemannian metric in a manifold of dimension 2 can also be related to negativity of entanglement, which is also a monotone function. The result established in this article

positively illustrates the fact that quantum correlations can be studied using geometric approach.

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- [1] S. Amari and H. Nagaoka, *Methods of information Geometry*,  
 [2] E. A. Morozova and N. N. Čencov, *Itogi Nauki Tekh.* **36**, 69(1990).  
 [3] D. Petz, *Linear Algebra Appl.* **244**, 81 (1996).  
 [4] D. Petz and H. Hasegawa, *Lett. Math. Phys.* **38**, 221 (1996); G. Tóth and D. Petz, *Phys. Rev. A* **87**, 032324 (2013).  
 [5] D. Petz, *J. Phys. A* **35**, 929 (2002);  
 [6] D. Petz and C. Sudár *J. Math. Phys.* **37**, 2662 (1996).  
 [7] P. Gibilisco and T. Isola, *J. Math. Phys.* **44**, 3752 (2003).  
 [8] P. Gibilisco and T. Isola, *Infinite Dimens. Anal. Quantum Probab. Relat. Top.* **4**, 553 (2001).  
 [9] E. P. Wigner and M. M. Yanase, *Proc. Natl. Acad. Sci. USA* **49** 910 (1963); *Can. J. Math.* **16**, 397 (1964).  
 [10] W. K. Wootters, *Phys. Rev. D* **23**, 357 (1981).  
 [11] S. L. Braunstein and C. M. Caves, *Phys. Rev. Lett.* **72**, 3439(1994).  
 [12] D. P. Pires, Lucas C. Céleri and Diogo O. Soares-Pinto *Phys. Rev. A* **91**, 042330 (2015).  
 [13] E. Schrodinger, *Naturwissenschaften* **23**, 807 (1935).  
 [14] R. F. Werner, *Phys. Rev. A* **40**, 4277 (1989).  
 [15] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009).  
 [16] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres and W. K. Wootters, *Phys. Rev. Lett.* **70**, 1895 (1993).  
 [17] C. H. Bennett and S. J. Wiesner, *Phys. Rev. Lett.* **69**, 2881 (1992).  
 [18] A. K. Ekert, *Phys. Rev. Lett.* **67**, 661 (1991). C. H. Bennett, G. Brassard, and N. D. Mermin, *Phys. Rev. Lett.* **68**, 557 (1992).  
 [19] M. Horodecki, J. Oppenheim, A. Winter, *Nature* **436**, 673(2005)  
 [20] L. Henderson and V. Vedral, *J. Phys. A: Math. Gen* **34**, 06899 (2001).  
 [21] Michael. A Nielsen and Issac. L Chuang, *Quantum Computation and Quantum Information*,  
 [22] K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, *Phys. Rev. A* **58**, 883 (1998).  
 [23] G. Vidal and R. F. Werner, *Phys. Rev. A* **65**, 032314 (2002).  
 [24] C. Eltschka, G. Tóth and J. Siewert, *Phys. Rev. A* **91**, 032327 (2015).  
 [25] T. Baumgratz, M. Cramer, and M. B. Plenio, *Phys. Rev. Lett.* **113**, 140401 (2014).  
 [26] D. Girolami, *Phys. Rev. Lett.* **113**, 170401 (2014).  
 [27] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, G. Adesso, [arXiv:1502.05876v3](https://arxiv.org/abs/1502.05876v3).