Upper Bound on Singlet Fraction of Two Qubit Mixed Entangled States

Satyabrata Adhikari [*], Atul Kumar [†]
Centre for System Science
Indian Institute of Technology Jodhpur, Jodhpur-342011, Rajasthan, India

July 14, 2018

Abstract

We demonstrate the possibility of achieving the maximum possible singlet fraction using a entangled mixed two-qubit state as a resource. For this, we establish a tight upper bound on singlet fraction and show that the maximal singlet fraction obtained in [6] does not attain the obtained upper bound on the singlet fraction. Interestingly, we found that the required upper bound can in fact be achieved using local filtering operations.

1   Introduction

Quantum entanglement [1] has been used as an efficient resource for several quantum communication protocols such as teleportation, dense coding, and cryptography [2, 3, 4]. The existence of long range quantum correlations between entangled qubits allows the use of such systems for information transfer in communication protocols. In general, if a state is maximally entangled then the optimal success of a communication protocol is a certainty. However, in real experimental set-ups it is difficult to prepare a pure maximally entangled state due to the difficulty in handling the multi-particle quantum systems in terms of interaction with environment, control, and preserving the necessary quantum coherence [5]. This leads to a situation where one may have to deal with

[*]satya@iitj.ac.in

[†]atulk@iitj.ac.in

mixed entangled resources for quantum information processing. This raises the question of usefulness of such mixed entangled systems for efficient quantum information processing. Verstraete and Verschelde [6] have shown that any entangled two-qubit mixed state can be used as a resource for quantum teleportation using certain trace preserving local operations and classical communications. The measure of usefulness of any two-qubit mixed entangled resource (ρ) for quantum teleportation is given by teleportation fidelity, i.e. $f_T = \frac{2F+1}{3}$, where F is the singlet fraction of the entangled resource defined as

$$F(\rho) = \max_U \langle\psi| U^\dagger \rho U |\psi\rangle \text{ and } |\psi\rangle = \frac{1}{\sqrt{2}}\sum_{i=0}^{1}|ii\rangle \text{ [7]}.$$

For $F > \frac{1}{2}$, quantum teleportation is always successful and if the singlet fraction of the underlying state is unity then in principle perfect teleportation can be achieved. Hence for a state to be useful as a resource, the singlet fraction or the teleportation fidelity must be maximized. In this Letter, we address the following question: given an entangled mixed state of two qubits as a resource for quantum teleportation, what is the upper bound on singlet fraction that can be achieved using all possible filtering operations? Our results show an interesting observation that the maximal singlet fraction obtained in [6] can still be increased to coincide with the required upper bound on singlet fraction.

We analyze the upper bound on the singlet fraction (and thus the upper bound on teleportation fidelity) that can be achieved using a two qubit entangled

mixed state. We call this bound as Dembo's bound on singlet fraction for any given two-qubit mixed state. Although Verstraete *et al* have found that any two-qubit mixed entangled state can be used to achieve $F > \frac{1}{2}$ using optimal local operations and classical communication, our analysis in this Letter demonstrates that the singlet fraction or teleportation fidelity obtained in [6] does not achieve the Dembo's upper bound on singlet fraction. We further demonstrate that one can in fact obtain the optimal singlet fraction in coherence with the Dembo's bound by applying additional filtering operations. In general, our results show that for any state with $F > 1/2$, the Dembo's bound on singlet fraction can be successfully achieved. The increase in singlet fraction using local operations and hence the increase in teleportation fidelity will have a significant impact on quantum information processing. In principle, this will allow one to use even a mixed state to achieve successful teleportation with optimal fidelity [8]. Furthermore, our results may also release the constraints on the experimental set-ups to prepare a pure maximally entangled state for efficient and optimal quantum teleportation.

Verstraete *et al* have addressed the problem of maximal achievable singlet fraction of entangled two-qubit mixed states optimized over all trace preserving local operations and classical communications. Our study is motivated by the fact that the optimal singlet fraction (F^*) of the filtered state obtained in [6] does not exceed the value of $2/3$. At this juncture, we would like to raise the question of analyzing the upper bound on the singlet fraction of the filtered state. Precisely, we are interested in analyzing whether the value F^* can be increased to coincide with the upper bound on singlet fraction. In order to discuss the importance of our results, we now proceed to establish a relation between the upper bound on the singlet fraction and eigen values of any Hermitian-positive semidefinite density operator.

In quantum information processing, eigenvalues play an important role not only in shedding light on many physical aspects of quantum systems but also for analyzing many essential features such as entanglement, discord, communication, and security. For higher dimensional systems, evaluating the ex-

act maximum or minimum eigenvalue of a Hermitian positive-definite matrix is not easy, but its upper or lower bound suffices. To proceed further with our analysis we state the following theorem [9]: For any real $n \otimes n$ matrix C and a positive semidefinite operator B, the following inequality holds

$$\lambda_1(\bar{C})tr(B) \leq Tr(CB) \leq \lambda_n(\bar{C})tr(B) \qquad (1)$$

where $\bar{C} = \frac{C+C^T}{2}$, C^T is the transpose of matrix C, $\lambda_n(\bar{C})$ is the nth eigen value of the matrix $\bar{C}$, and $\lambda_1 \leq \lambda_2 \leq \lambda_3 ..... \leq \lambda_n$.
For $C = \frac{I_4}{2} - X^\Gamma$ and $B = \rho$, Eq. (1) can be re-expressed as

$$\lambda_1 \left(\frac{I_4}{2} - X^\Gamma \right) \leq F^* = tr\left[\left(\frac{I_4}{2} - X^\Gamma \right) \rho \right]$$
$$\leq \lambda_4 \left(\frac{I_4}{2} - X^\Gamma \right) \qquad (2)$$

where $X = (A \otimes I_2)\left|\psi\right\rangle^- \left\langle\psi\right|^- \left(A^\dagger \otimes I_2\right)$, X^Γ is partial transpose of X, $\left|\psi\right\rangle^- = \frac{1}{\sqrt{2}}\left\lceil\left|01\right\rangle - \left|10\right\rangle\right\rceil$, and A represents the filter. Although the functional form of the upper bound of F^* is optimal for the above inequality, it depends on the state parameter and hence, must have a maximum achievable value for every particular value of the state parameter. This value would be provided by Dembo's bound [10] stated below as:
For any $n \otimes n$ Hermitian positive semi-definite operator R_n with eigen values $\lambda_1 \leq \lambda_2 \leq \lambda_3 ..... \leq \lambda_n$, Dembo's bound can be given by

$$\frac{c + \eta_1}{2} + \sqrt{\frac{(c - \eta_1)^2}{4} + b^* b} \leq \lambda_n(R_n)$$
$$\leq \frac{c + \eta_{n-1}}{2} + \sqrt{\frac{(c - \eta_{n-1})^2}{2} + b^* b} \qquad (3)$$

where $R_n = \begin{pmatrix} R_{n-1} & b \\ (b^*)^T & c \end{pmatrix}$, η_1 is the lower bound on the minimal eigenvalue of R_{n-1}, η_{n-1} is the upper bound on the maximal eigen value of R_{n-1}, and b is an eigenvector of dimension $(n-1)$. Using Dembo's

bound Eq. (2) can be re-expressed as

$$\lambda_1 \left(\frac{I_4}{2} - X^\Gamma \right) \leq F^* \leq \lambda_4 \left(\frac{I_4}{2} - X^\Gamma \right)$$

$$\leq \quad \frac{c + \eta_3}{2} + \sqrt{\frac{(c - \eta_3)^2}{2} + b^* b} \qquad (4)$$

Therefore, the upper bound on optimal singlet fraction is

$$F_D^* = \frac{c + \eta_3}{2} + \sqrt{\frac{(c - \eta_3)^2}{2} + b^* b} \qquad (5)$$

For the family of states given by

$$\rho(F) = F \left| \psi \right\rangle \left\langle \psi \right| + (1 - F) \left| 01 \right\rangle \left\langle 01 \right|; F \geq \frac{1}{3}, \quad (6)$$

the upper bound on singlet fraction F_D^* is

$$\begin{matrix} F_D^*[\rho(F)] = \frac{2-F}{4(1-F)}; \frac{1}{3} \leq F \leq \frac{2}{3} \\ F_D^*[\rho(F)] = F; F \geq \frac{2}{3}, \end{matrix} \quad (7)$$

The value of F_D^* obtained in Eq. (7) is the optimal value of singlet fraction that can be achieved for the family of states represented by Eq. (6). Verstraete and Verschelde have shown that using trace preserving optimal local operations, the maximal achievable singlet fraction F^* for the family of states given in Eq. (6) is

$$\begin{matrix} F^*[\rho(F)] = \frac{1}{2}\left[1 + \frac{F^2}{4(1-F)}\right]; \frac{1}{3} \leq F \leq \frac{2}{3} \\ F^*[\rho(F)] = F; F \geq \frac{2}{3}, \end{matrix} \quad (8)$$

Eq. (7) and (8) describe an interesting result that F^* does not achieve the upper bound on singlet fraction given by F_D^*. A comparison of maximal singlet fraction F^* obtained after performing the filtering operations and F_D^* obtained using Dembo's bound is given in Fig. (1) and clearly demonstrates that the optimal singlet fraction F^* is always less than the optimal singlet fraction F_D^* for $\frac{1}{3} \leq F \leq \frac{2}{3}$. In general, we will always find that $F^* \leq F_D^*$. Dembo's upper bound on singlet fraction is obtained using mathematical rigour. However, we still need to find a way to obtained this bound experimentally i.e. would it be

Figure 1: Comparison of maximal singlet fraction F^* and upper bound on singlet fraction F_D^* obtained using Dembo's bound

possible to increase the value of optimal singlet fraction performing local operations and classical communication on the filtered state i.e. can we achieve the upper bound of singlet fraction given by Dembo's bound? Surprisingly, our results show that the bound is indeed achievable.

In order to enhance the value of optimal singlet fraction F^*, we perform another filtering operation on the filtered state such that the singlet fraction of the output state can be given as

$$F_{opt}^* = pF^*(\rho_f) + (1 - p)F(\rho_f) \qquad (9)$$

where p is the success probability multiplied with the optimal singlet fraction of the state coming out of the second filter. If we define $1 - p = p_{AB}$ where p_{AB} denotes the success probability of first filter (for details, please refer to [6]), then for $F(\rho_f) = \langle \psi | \rho_f | \psi \rangle$ where $\rho_f = \frac{(A \otimes I)\rho(A \otimes I)^\dagger}{p_{AB}}$, F_{opt}^* can be re-expressed as

$$F_{opt}^* = (1 - p_{AB})F^*(\rho_f) + tr\left[(A \otimes I)\rho(A^\dagger \otimes I)|\psi\rangle\langle\psi|\right] \qquad (10)$$

The definition of p_{AB} suggests that for success probability p to be high, p_{AB} must be minimized. Moreover, the minimum value of p_{AB} should be chosen in such a way that the value of singlet fraction for the second filter must not exceed Dembo's bound. Hence,

the minimum value of p_{AB} would be

$$p_{AB}^{min} = 1 - \frac{F_D^* - tr\left[(A \otimes I)\rho(A^T \otimes I)\left|\psi\right\rangle\left\langle\psi\right|\right]}{F^*(\rho_f)} \quad (11)$$

Using Eq. (10) and (11), we have

$$F_{opt}^* = F_D^* \quad (12)$$

Eq. (12) clearly shows that one can achieve the maximum singlet fraction equals to Dembo's bound by using the filtering operations twice.

As an illustration, we use the family of states given by Eq. (6) to calculate the F_{opt}^*. In this case, Eq. (11) will become

$$p_{AB}^{min} = \frac{F^2}{2(1-F)(2-F)}, \quad (13)$$

and hence

$$F_{opt}^* = \frac{2-F}{4(1-F)}, \quad (14)$$

which is equal to F_D^*.

Hence, applying the filter twice will always result in achieving the upper bound on singlet fraction for any two-qubit mixed density operator. However, achieving this upper bound on singlet fraction will lead to the decrease in success probability of the first filter. Nevertheless, the second filter always compensates for the decrease in success probability of the first filter by attaining the Dembo's bound on singlet fraction.

In summary, we have established a relation between Dembo's upper bound and singlet fraction (and hence with teleportation fidelity) of a mixed two-qubit entangled state. This relation is used to demonstrate that any two-qubit mixed entangled state can be used as a resource to achieve maximum possible teleportation fidelity. It was found earlier that the the trace preserving local operations always yield a teleportation fidelity larger than 2/3 if the original state is entangled. However, we found that the maximal fidelity obtained earlier can be increased with additional local operations with certain non-zero probability. It would be interesting to find an example where the optimal fidelity after the operation of a single filter can coincide with Dembo's bound.

References

[1] A. Einstien, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935); E. Schrodinger, Naturwissenschaften **23**, 807 (1935); R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).

[2] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).

[3] C. H. Bennett and S. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).

[4] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Rev. Mod. Phys. **74**, 145 (2009).

[5] P. Zanardi, and S. Lloyd, Phys. Rev. Lett. **90**, 067902 (2003); L. M. Duan, and G. C. Guo, Phys. Rev. Lett. **79**, 1953 (1997).

[6] F. Verstraete, and H. Verschelde, Phys. Rev. Lett. **90**, 097901 (2003).

[7] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A **60**, 1888 (1999).

[8] B. G. Taketani, F. de Melo, and R. L. de Matos Filho1, Phys. Rev. A **85**, 020301(R) (2012).

[9] Y. Fang, K. A. Loparo, and X. Feng, IEEE Transactions on Automatic Control **39**, 2489 (1994).

[10] A. Dembo, IEEE Trans. Inform. Theory **34**, 352 (1988); D. Park, and B. G. Lee, IEEE Signal Processing Letters **10**, 267 (2003).