# Classical-Quantum Arbitrarily Varying Wiretap Channel: Ahlswede Dichotomy, Positivity, Resources, Super Activation

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#### Abstract

We establish the Ahlswede dichotomy for arbitrarily varying classicalquantum wiretap channels, i.e., either the deterministic secrecy capacity of the channel is zero, or it equals its randomness-assisted secrecy capacity. We analyze the secrecy capacity of these channels when the sender and the receiver use various resources. It turns out that randomness, common randomness, and correlation as resources are very helpful for achieving a positive secrecy capacity. We prove the phenomenon "super-activation" for arbitrarily varying classical-quantum wiretap channels, i.e., two channels, both with zero deterministic secrecy capacity, if used together allow perfect secure transmission.

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# 1 Introduction

The developments in modern communication systems are rapid. Especially quantum communication systems allow us to exploit new possibilities while at the same time imposing fundamental limitations. Quantum information processing systems provide huge theoretical advantages over their classical counterparts, one of the two most prominent ones being *perfect secrecy* (cf. [10] and [9] for two well-known examples of quantum key distributions). The impact of quantum information processing systems on our daily live is nonetheless still zero, the main reason for that being the difficulty to store and manipulate quantum states in a predictable and reliable manner.

In this work, we bring these two aspects together, namely we investigate the transmission of messages from a sending to a receiving party. The messages ought to be kept secret from an eavesdropper. Communication takes place over a quantum channel which is, in addition to noise from the environment, subjected to the action of a jammer which actively manipulates the states.

Preceding work in quantum information theory has mostly focused on either of the two attacks. Our goal is to deliver a more general theory considering both channel robustness and security in quantum information theory. By doing so, we build on the preceding works [15] and [18]. Furthermore, we are interested in the delivery of large volumes of messages over many channel uses, so that we study the asymptotic behavior of the system. Our work fits into a broader range of recent work in both classical and quantum information theory [37, 20, 17, 15, 12, 19, 23, 27, 29, 30] that studies the secret information processing tasks with the aim of delivering *embedded security*: Unlike what is nowadays the standard approach in secret communication, namely to first ensure the successful transmission of messages and then implement a cryptographic protocol on top whose security relies on assumptions concerning the difficulties in breaking the protocol, this new paradigm focuses on delivering a guaranteed security right from the start. The security features of the protocol become embedded already at the physical layer of the communication system. The concept does not only cover secure message transmission but also secure key generation.

Also, communication models including a jammer that tries to prevent the legal parties from communicating properly have received a great lot of attention in recent years, some of which we have already mentioned above include wiretapping aspects, while some do not [6]. These publications concentrated on the model of an *arbitrarily varying channel* where the jammer may change his input in every channel use and is not restricted to use a repetitive probabilistic strategy. Quite on the contrary, it is understood that the sender and the receiver have to select their coding scheme first. After that the jammer makes his choice of the channel state. The model of an arbitrarily varying channel was first introduced by Blackwell, Breiman, and Thomasian in [14]. The nature of the model is quite flexible: It allows specifying the impact that the actions of the jammer may have on the communication link under use: In the most restrictive case where the jammer is left with only one choice, we recover the discrete memoryless channel. On the other extreme, it has been shown by Ahlswede in [1] that the capacity (under maximal error criterion) of certain arbitrarily varying channels can be equated to the zero-error capacity of related discrete memoryless channels. The arbitrarily varying channel does at the same time demonstrate the importance of shared randomness for communication in a very clear form: Ahlswede showed in [3] (cf. also [4] and [5]) the surprising result that either the deterministic capacity of an arbitrarily varying channel is zero, or it equals its shared randomness-assisted capacity (this effect is now known as the Ahlswede dichotomy). After that discovery, it remained an open question exactly when the deterministic capacity is nonzero. In [24] Ericson gave a sufficient condition for that, and in [22] Csiszár and Narayan proved that this is condition is also necessary.

In this work, we will therefore put a focus on the analysis of different forms of shared randomness and their impact on the robustness and security. The model of a wiretap channel adds a third party to the communication problem as well, but here the focus is on secure communication, meaning communication without that third party getting to know the messages. This model was first introduced by Wyner in [40] (in this paper we will use a stronger security criterion than the one that was used in [40], cf. Remark 2.9). The relation of the different security criteria is discussed, e.g. in [16] with some generality and in [37] with respect to arbitrarily varying channels.

In the model of an arbitrarily varying wiretap channel, we consider transmission with both a jammer and an eavesdropper. Its secrecy capacity has been analyzed in [12]. A lower bound of the randomness-assisted secrecy capacity has been given. It is worth noting that the channel under consideration in this work is effectively given by an *interference channel* where the legal sender and the jammer are allowed to make inputs to the system and the legal receiver as well as the eavesdropper receive the corresponding outputs. We do leave open the possibility of the jammer communicating his choice of input (equivalently: his channel state sequence) to the eavesdropper, but limit the receiving parties such that they cannot send any messages back to the jammer or the legal receiver. During proofs and when defining the model, we will, however deviate from this point of view and use a notation which respects the historic development of results on arbitrarily varying channels. The physical model we consider is that of a classical-quantum channel, i.e., the legal sender's and the jammer's inputs are classical data and the legal receiver's as well as the eavesdroppers outputs are quantum systems. The capacity of classical-quantum channels without secrecy constraints or active jamming has been determined in [28] and [34].

A classical-quantum channel with a jammer is called an arbitrarily varying classical-quantum channel. In [7] the capacity of arbitrarily varying classicalquantum channels is analyzed. A lower bound of the capacity has been given. An alternative proof and a proof of the strong converse are given in [11]. In [6] the Ahlswede dichotomy for the arbitrarily varying classical-quantum channels is established, and a sufficient and necessary condition for the zero deterministic capacity is given. In [18] a simplification of this condition for the arbitrarily varying classical-quantum channels is given. A classical-quantum channel with an eavesdropper is called a classical-quantum wiretap channel, its secrecy capacity has been determined in [23] and [20].

A classical-quantum channel with both a jammer and an eavesdropper is called an arbitrarily varying classical-quantum wiretap channel. It is defined as a family of pairs of indexed channels  $\{(W_t, V_t) : t = 1, ..., T\}$  with a common input alphabet and possible different output systems, connecting a sender with two receivers, a legal one and a wiretapper, where t is called a channel state of the channel pair. The legitimate receiver accesses the output of the first part of the pair, i.e., the first channel  $W_t$  in the pair, and the wiretapper observes the output of the second part, i.e., the second channel  $V_t$ , respectively. A channel state t, which varies from symbol to symbol in an arbitrary manner, governs both the legal receiver's channel and the wiretap channel. A code for the channel conveys information to the legal receiver such that the wiretapper knows nothing about the transmitted information in the sense of the stronger security criterion (cf. Remark 2.9). This is a generalization of compound classical-quantum wiretap channels in [17], when the channel states are not stationary, but can change over time.

The secrecy capacity of the arbitrarily varying classical-quantum wiretap channels has been analyzed in [15]. A lower bound of the randomness-assisted capacity has been given, and it has been shown that this bound is either a lower bound for the deterministic capacity, or else the deterministic capacity is equal to zero. As mentioned already, we will be interested in the role that different forms of shared randomness play for the arbitrarily varying classical-quantum wiretap channel. To this end, we will distinguish between three kinds of shared randomness: *randomness, common randomness,* and *correlation.* Randomness and common randomness have been used as a method of proof, e.g., in [3] and much of the follow-up work for the determination of the random capacity. If looked at as a resource for communication which is to be deployed in order to make a communication link work reliably, they are, however, a rather strong form of a resource: It is required that both sender and receiver have access to a perfect copy of the outcome of a random experiment. Moreover, the outcomes of said experiment have to be distributed uniformly. The impact of deviations from these strong requirements has not yet received much attention. What has been investigated (starting with [8] and continued in [18]) is a variant where the common randomness gets replaced by a resource that is in some sense the complete opposite: correlation.

Assume that a bipartite source, modeled by an i.i.d. random variable (X, Y) with values in a finite product set  $\mathbf{X} \times \mathbf{Y}$ , is observed by the sender and (legal) receiver. The sender has access to the random variable X and the receiver to Y. We call (X, Y) correlated shared randomness whenever the mutual information between X and Y satisfies I(X; Y) > 0.

It has been shown in [8] that correlated shared randomness is a helpful resource for information transmission through an arbitrarily varying classical channel: The use of mere correlation does already allow one to transmit messages at any rate that would be achievable using any form of shared randomness. The capacity of an arbitrarily varying quantum channel assisted by correlated shared randomness as resource has been discussed in [18], where equivalent results were found. In this work, we extend the concept of correlation-assisted coding to the arbitrarily varying classical-quantum wiretap channel.

In [18] a classification of various resources is given. A distinction is made between two extremal cases: randomness and correlation. Randomness is the strongest resource, it requires a perfect copy of the outcome of a random experiment, and thus we should assume an additional perfect channel. On the other hand, correlation is the weakest resource. The work [18] also put emphasis on the quantification of the differences between correlation and common randomness and used the arbitrarily varying classical-quantum channel as a method of proof. It can be shown that common randomness is a stronger resource than correlation in the following sense: An example is given when not even a finite amount of common randomness can be extracted from a given correlation. On the contrary, a sufficiently large amount of common randomness allows the sender and receiver to asymptotically simulate the statistics of any correlation.

We concentrate our analysis on the case without feedback, i.e., we neither allow the receiver to send messages back to the sender (or the jammer), nor do we allow the eavesdropper to send messages toward the jammer (or the sender). Such an approach may be deemed unsatisfactory from a practical perspective. However, a brief look into the history of the arbitrarily varying channel reveals that only the reduction to the case of deterministic codes (without feedback) leads one to encounter those cases where the capacity of the system is zero, while a dramatic increase to full capacity is possible as soon as shared randomness (or feedback) is available.

In the situation investigated here, the reduction to forward communication allows us to demonstrate the effect of super-activation of the secrecy capacity of the arbitrarily varying classical-quantum channel. We take the space to write a few lines concerning more elaborate models. The case where a (secure) channel from the receiver to the sender is available is likely to be equivalent to the case where shared randomness can be used when the average error criterion is used. The latter will be treated in forthcoming work. In the model treated here, the presence of an eavesdropper makes us take the freedom to allow randomness at the encoder, which makes the average error criterion equivalent to the maximal error criterion [3] (and [18] for the quantum case). The case of deterministic codes in the presence of feedback but with the code performance being evaluated with respect to the maximal error criterion has been evaluated in [2].

It can easily be seen now that the complexity of the channel model under investigation here necessitates a strict reduction in the abilities of the participating parties, at least if the aim is the establishment of definite results.

Our secrecy criterion is chosen such that the messages sent by the sender are to be kept strongly secret. More precisely, the use of shared randomness creates ensembles  $(R_{\text{uni}}, Z_{t^n}) := (J^{-1}, V_{t^n}(E^{\gamma}(\cdot|j))_{j=1}^J)$  where  $j = 1, \ldots, J$  are messages and  $E^{\gamma}(\cdot|1), \ldots, E^{\gamma}(\cdot|J)$  are probability distributions of the codewords associated with the messages. The index  $\gamma$  refers to a particular choice of encoding scheme. This index may be shared with the receiver (common randomness) or may just be correlated with another index  $\gamma'$  at the receiver (correlated codes, in that case both  $\gamma$  and  $\gamma'$  are actually elements of product alphabets  $\mathbf{X}^n$  and  $\mathbf{Y}^n$ ).

Our strong secrecy criterion requires that the Holevo information  $\chi(R_{\text{uni}}, Z_{t^n,\gamma})$  of the ensembles consisting of the messages and the output at the eavesdropper's system is to be kept small in a yet to be defined sense. More precisely, we require that the Holevo information is to be kept small on average over the random choice of codewords and for all possible choices of the jammer, i.e.,  $\max_{t^n} \chi(R_{\text{uni}}, \mathcal{Z}_{t^n} | \Gamma)$  is to vanish asymptotically.

Using this secrecy criterion is a key to prove super-activation of the deterministic secrecy capacity of the arbitrarily varying classical-quantum channel: We take two arbitrarily varying classical-quantum wiretap channels. One of them is assumed to have zero capacity for message transmission because it is symmetrizable in the sense of [7], but its common-randomness-assisted capacity is positive. The other is assumed to be non-symmetrizable but insecure.

In [32] a new code concept for secrecy capacity and a complete characterization of super-activation for classical arbitrarily varying wiretap channels with no sharing resources has been given. In view of this work on classical arbitrarily varying wiretap channels our further task will be to analyze this characterization on arbitrarily varying classical-quantum wiretap channels.

Through parallel transmission of common randomness on the insecure channel and secure data on the other one, the combined system can be proven to have positive capacity. Roughly speaking, the proof uses the fact that the choices of common randomness and messages are independent from each other and only the codewords depend on both of them, together with the data processing inequality applied to the Holevo quantity. Details are to be found in the respective section.

The operational interpretation of the secrecy criterion that we employ here comes through application of the (quantum) Pinsker's inequality. Note that, in an average sense, the eavesdropper "knows" the index  $\gamma$  of the random code. It is clear that, under such circumstances, backwards communication toward the jammer would render the shared randomness completely useless.

A more in-depth discussion of secrecy criteria in the quantum case, including fully quantum channels but not the arbitrarily varying case, can be found in the recent preprint [39]. Different secrecy criteria for arbitrarily varying quantum or classical-quantum channels will be evaluated in future work.

This paper is organized as follows.

The main definitions are given in Section 2.

In Section 3 we generalize the result of [15] by establishing the Ahlswede dichotomy for the arbitrarily varying classical-quantum wiretap channels (without feedback), i.e., either the deterministic secrecy capacity of an arbitrarily varying classical-quantum wiretap channel is zero, or it equals its randomness-assisted secrecy capacity.

In Section 4 we analyze the secrecy capacity of an arbitrarily varying classicalquantum wiretap channel assisted by correlation as resource. We show that correlation is a helpful resource for secure information transmission through an arbitrarily varying classical-quantum wiretap channel.

In Section 5 we give an example in which both cases of the Ahlswede dichotomy for the arbitrarily varying classical-quantum wiretap channels actually occur. We present a new discovery for the arbitrarily varying classical-quantum wiretap channels which is a consequence of the Ahlswede dichotomy for the arbitrarily varying classical-quantum wiretap channels. This phenomenon is called "super-activation", i.e., two arbitrarily varying classical-quantum wiretap channels, both with zero deterministic secrecy capacity, if used together allow perfect secure transmission.

Finally, we will conclude in Section 6 with a discussion of our results.

### 2 Communication Scenarios and Resources

#### 2.1 Basic Definitions and Communication Scenarios

For a finite set  $\mathbf{A}$ , we denote the set of probability distributions on  $\mathbf{A}$  by  $P(\mathbf{A})$ . Let H be a finite-dimensional complex Hilbert space. We denote the (convex) space of density operators on H by  $\mathcal{S}(H)$ . A classical-quantum channel is a linear map  $W : P(\mathbf{A}) \to \mathcal{S}(H), P(\mathbf{A}) \ni P \to W(P) \in \mathcal{S}(H)$ . Let  $a \in \mathbf{A}$ . For a  $P_a \in P(\mathbf{A})$ , defined by  $P_a(a') = \begin{cases} 1 & \text{if } a' = a \\ 0 & \text{if } a' \neq a \end{cases}$ , we write W(a) instead of  $W(P_a)$ .

**Remark 2.1.** In many literature, a classical-quantum channel is defined as a map  $\mathbf{A} \to \mathcal{S}(H)$ ,  $\mathbf{A} \ni a \to W(a) \in \mathcal{S}(H)$ . This is a special case when the input is limited on the set  $\{P_a : a \in \mathbf{A}\}$ .

For any finite set  $\mathbf{A}$ , any finite-dimensional complex Hilbert space H, and  $n \in \mathbb{N}$ , we define  $\mathbf{A}^n := \left\{ (a_1, \ldots, a_n) : a_i \in \mathbf{A} \ \forall i \in \{1, \ldots, n\} \right\}$ , and  $H^{\otimes n} := span \left\{ v_1 \otimes \ldots \otimes v_n : v_i \in H \ \forall i \in \{1, \ldots, n\} \right\}$ . We also write  $a^n$  for the elements of  $\mathbf{A}^n$ .

Associated with W is the channel map on the n-block  $W^{\otimes n}$ :  $P(\mathbf{A}^n) \to \mathcal{S}(H^{\otimes n})$ , such that  $W^{\otimes n}(P^n) = W(P_1) \otimes \ldots \otimes W(P_n)$  if  $P^n \in P(\mathbf{A}^n)$  can be written as  $(P_1, \ldots, P_n)$ . Let  $\theta := \{1, \ldots, T\}$  be a finite set. Let  $\{W_t : t \in \theta\}$  be a set of classical-quantum channels. For  $t^n = (t_1, \ldots, t_n), t_i \in \theta$  we define the n-block  $W_{t^n}$  such that for  $W_{t^n}(P^n) = W_{t_1}(P_1) \otimes \ldots \otimes W_{t_n}(P_n)$  if  $P^n \in P(\mathbf{A}^n)$  can be written as  $(P_1, \ldots, P_n)$ .

Let  $\mathfrak{P}$  and  $\mathfrak{Q}$  be quantum systems, denote the Hilbert space of  $\mathfrak{P}$  and  $\mathfrak{Q}$  by  $H^{\mathfrak{P}}$  and  $H^{\mathfrak{Q}}$ , respectively. We denote the space of density operators on  $H^{\mathfrak{P}}$  and  $H^{\mathfrak{Q}}$  by  $\mathcal{S}(H^{\mathfrak{P}})$  and  $\mathcal{S}(H^{\mathfrak{Q}})$ , respectively. A quantum channel N:

 $\mathcal{S}(H^{\mathfrak{P}}) \to \mathcal{S}(H^{\mathfrak{Q}}), \mathcal{S}(H^{\mathfrak{P}}) \ni \rho \to N(\rho) \in \mathcal{S}(H^{\mathfrak{Q}})$  is represented by a completely positive trace preserving map, which accepts input quantum states in  $\mathcal{S}(H^{\mathfrak{P}})$  and produces output quantum states in  $\mathcal{S}(H^{\mathfrak{Q}})$ .

Associated with N is the channel maps on the n-block  $N^{\otimes n}$ :  $\mathcal{S}(H^{\mathfrak{P}^{\otimes n}}) \to \mathcal{S}(H^{\mathfrak{P}^{\otimes n}})$  such that for  $\rho^n = \rho_1 \otimes \ldots \otimes \rho_n \in \mathcal{S}(H^{\mathfrak{P}^{\otimes n}})$   $N^{\otimes n}(\rho^n) = N(\rho_1) \otimes \ldots \otimes N(\rho_n)$ . For  $t^n = (t_1, \ldots, t_n)$ ,  $t_i \in \theta$ , we define the n-block  $N_{t^n}$  such that for  $\rho^n = \rho_1 \otimes \ldots \otimes \rho_n \in \mathcal{S}(H^{\mathfrak{P}^{\otimes n}})$  we have  $N_{t^n}(\rho^n) = N_{t_1}(\rho_1) \otimes \ldots \otimes N_{t_n}(\rho_n)$ . We denote the identity operator on a space H by  $\mathrm{id}_H$ .

For a discrete random variable X on a finite set **A** and a discrete random variable Y on a finite set **B**, we denote the Shannon entropy of X by  $H(X) = -\sum_{x \in \mathbf{A}} p(x) \log p(x)$  and the mutual information between X and Y by  $I(X;Y) = \sum_{x \in \mathbf{A}} \sum_{y \in \mathbf{B}} p(x,y) \log \left(\frac{p(x,y)}{p(x)p(y)}\right)$ . Here p(x,y) is the joint probability distribution function of X and Y, and p(x) and p(y) are the marginal probability distribution functions of X and Y, respectively, and "log" means logarithm to base 2.

For a quantum state  $\rho \in \mathcal{S}(H)$ , we denote the von Neumann entropy of  $\rho$  by

$$S(\rho) = -\mathrm{tr}(\rho \log \rho) \; ,$$

where "log" means logarithm to base 2. Let  $\Phi := \{\rho_x : x \in \mathbf{A}\}$  be a set of quantum states labeled by elements of  $\mathbf{A}$ . For a probability distribution Q on  $\mathbf{A}$ , the Holevo  $\chi$  quantity is defined as

$$\chi(Q;\Phi) := S\left(\sum_{x \in \mathbf{A}} Q(x)\rho_x\right) - \sum_{x \in \mathbf{A}} Q(x)S\left(\rho_x\right) \ .$$

Note that we can always associate a state  $\rho^{XY} = \sum_x Q(x) |x\rangle \langle x| \otimes \rho_x$  to  $(Q; \Phi)$  such that  $\chi(Q; \Phi) = I(X; Y)$  holds for the quantum mutual information.

**Definition 2.2.** Let  $\mathbf{A}$  be a finite set. Let H be a finite-dimensional complex Hilbert space, and  $\theta := \{1, \ldots, T\}$  be a finite set. For every  $t \in \theta$ , let  $W_t$  be a classical-quantum channel  $P(\mathbf{A}) \to \mathcal{S}(H)$ . The set of the quantum channels  $\{W_t : t \in \theta\}$  defines an arbitrarily varying classical-quantum channel.

Strictly speaking, the set  $\{W_t : t \in \theta\}$  generates the arbitrarily varying classical-quantum channel  $\{W_{t^n} : t^n \in \theta^n\}$ . When the sender inputs a  $P^n \in P(\mathbf{A}^n)$  into the channel, the receiver receives the output  $W_{t^n}(P^n) \in \mathcal{S}(H^{\otimes n})$ , where  $t^n = (t_1, t_2, \ldots, t_n) \in \theta^n$  is the channel state of  $W_{t^n}$ .

**Definition 2.3.** We say that the arbitrarily varying classical-quantum channel  $\{W_t : t \in \theta\}$  is symmetrizable if there exists a parametrized set of distributions  $\{\tau(\cdot \mid a) : a \in \mathbf{A}\}$  on  $\theta$  such that for all  $a, a' \in \mathbf{A}$ ,

$$\sum_{t \in \theta} \tau(t \mid a) W_t(a') = \sum_{t \in \theta} \tau(t \mid a') W_t(a) .$$

**Definition 2.4.** Let  $\mathfrak{P}$  and  $\mathfrak{Q}$  be quantum systems, denote the Hilbert Space of  $\mathfrak{P}$  and  $\mathfrak{Q}$  by  $H^{\mathfrak{P}}$  and  $H^{\mathfrak{Q}}$ , respectively, and let  $\theta := \{1, \ldots, T\}$  be a finite set. For every  $t \in \theta$ , let  $W'_t$  be a quantum channel  $\mathcal{S}(H^{\mathfrak{P}}) \to \mathcal{S}(H^{\mathfrak{Q}})$ . We call the set of the quantum channels  $\{W'_t : t \in \theta\}$  an arbitrarily varying quantum channel when the state t varies from symbol to symbol in an arbitrary manner. We denote the set of arbitrarily varying quantum channels  $\mathcal{S}(H^{\mathfrak{P}}) \to \mathcal{S}(H^{\mathfrak{Q}})$  by  $C(H^{\mathfrak{P}}, H^{\mathfrak{Q}})$ .

**Definition 2.5.** Let  $\mathbf{A}$  be a finite set. Let H and H' be finite-dimensional complex Hilbert spaces. Let  $\theta := \{1, \ldots, T\}$  be a finite set. For every  $t \in \theta$  let  $W_t$  be a classical-quantum channel  $P(\mathbf{A}) \to \mathcal{S}(H)$  and  $V_t$  be a classical-quantum channel  $P(\mathbf{A}) \to \mathcal{S}(H')$ . We call the set of the classical-quantum channel pairs  $\{(W_t, V_t) : t \in \theta\}$  an **arbitrarily varying classical-quantum wiretap channel**, the legitimate receiver accesses the output of the first channel, i.e.,  $W_t$  in the pair  $(W_t, V_t)$ , and the wiretapper observes the output of the second channel, i.e.,  $V_t$  in the pair  $(W_t, V_t)$ , respectively, when the state t varies from symbol to symbol in an arbitrary manner.

When the sender inputs a sequence  $a^n \in \mathbf{A}^n$  into the channel, the receiver receives the output  $W_{t^n}(a^n) \in \mathcal{S}(H^{\otimes n})$ , where  $t^n = (t_1, t_2, \ldots, t_n) \in \theta^n$  is the channel state, while the wiretapper receives an output quantum state  $V_{t^n}(a^n) \in \mathcal{S}(H'^{\otimes n})$ .

#### 2.2 Code Concepts and Resources

Our goal is to see what the effects on the secrecy capacities of an arbitrarily varying classical-quantum wiretap channel are if the sender and the legal receiver have the possibility to use various kinds of resources. We also want to investigate what amount of randomness is necessary for the robust and secure message transmission through an arbitrarily varying classical-quantum wiretap channel. Hence, we consider various kinds of resources, each of them requiring a different amount of randomness, and we consider different codes, each of them requiring a different kind of resource.

**Definition 2.6.** An  $(n, J_n)$  (deterministic) code C for the arbitrarily varying classical-quantum wiretap channel  $\{(W_t, V_t) : t \in \theta\}$  consists of a stochastic encoder  $E : \{1, \ldots, J_n\} \to P(\mathbf{A}^n), j \to E(\cdot|j)$ , specified by a matrix of conditional probabilities  $E(\cdot|\cdot)$ , and a collection of positive semi-definite operators  $\{D_j : j \in \{1, \ldots, J_n\}\}$  on  $H^{\otimes n}$ , which is a partition of the identity, i.e.,  $\sum_{j=1}^{J_n} D_j = \operatorname{id}_{H^{\otimes n}}$ . We call these operators the decoder operators.

A code is created by the sender and the legal receiver before the message transmission starts. The sender uses the encoder to encode the message that he wants to send, while the legal receiver uses the decoder operators on the channel output to decode the message.

**Remark 2.7.** An  $(n, J_n)$  deterministic code C with deterministic encoder consists of a family of n-length strings of symbols  $(c_j)_{j \in \{1,...,J_n\}} \in (\mathbf{A}^n)^{J_n}$  and a collection of positive semi-definite operators  $\{D_j : j \in \{1,...,J_n\}\}$  on  $H^{\otimes n}$  which is a partition of the identity.

The deterministic encoder is a special case of the stochastic encoder when we require that for every  $j \in \{1, ..., J_n\}$ , there is a sequence  $a^n \in \mathbf{A}^n$  chosen with probability 1. The standard technique for message transmission over a channel

and robust message transmission over an arbitrarily varying channel is to use the deterministic encoder (cf. [6] and [18]). However, we use the stochastic encoder, since it is a tool for secure message transmission over wiretap channels (cf. [15] and [7]).

**Definition 2.8.** A nonnegative number R is an achievable (deterministic) secrecy rate for the arbitrarily varying classical-quantum wiretap channel  $\{(W_t, V_t) : t \in \theta\}$  if for every  $\epsilon > 0$ ,  $\delta > 0$ ,  $\zeta > 0$  and sufficiently large n there exist an  $(n, J_n)$  code  $C = (E, \{D_j^n : j = 1, \ldots, J_n\})$  such that  $\frac{\log J_n}{n} > R - \delta$ , and

$$\max_{t^n \in \theta^n} P_e(\mathcal{C}, t^n) < \epsilon \quad , \tag{1}$$

$$\max_{t^n \in \theta^n} \chi\left(R_{\text{uni}}; Z_{t^n}\right) < \zeta \quad , \tag{2}$$

where  $R_{uni}$  is the uniform distribution on  $\{1, \ldots, J_n\}$ . Here  $P_e(\mathcal{C}, t^n)$  (the average probability of the decoding error of a deterministic code  $\mathcal{C}$ , when the channel state of the arbitrarily varying classical-quantum wiretap channel  $\{(W_t, V_t) : t \in \theta\}$  is  $t^n = (t_1, t_2, \ldots, t_n)$ ), is defined as

$$P_e(\mathcal{C}, t^n) := 1 - \frac{1}{J_n} \sum_{j=1}^{J_n} \operatorname{tr}(W_{t^n}(E(|j))D_j) ,$$

 $Z_{t^n} = \left\{ V_{t^n}(E(|i)) : i \in \{1, \dots, J_n\} \right\} \text{ is the set of the resulting quantum state at the output of the wiretap channel when the channel state of } \{(W_t, V_t) : t \in \theta\} \text{ is } t^n.$ 

**Remark 2.9.** A weaker and widely used security criterion is obtained if we replace (2) with  $\max_{t \in \theta} \frac{1}{n} \chi(R_{\text{uni}}; Z_{t^n}) < \zeta$ . In this paper we will follow [12] and use (2).

**Remark 2.10.** When we defined  $W_t$  as  $\mathbf{A} \to \mathcal{S}(H)$ , then  $P_e(\mathcal{C}, t^n)$  is defined as  $1 - \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{a^n \in \mathbf{A}^n} E(a^n | j) \operatorname{tr}(W_{t^n}(a^n) D_j)$ .

When deterministic encoder is used, then  $P_e(\mathcal{C}, t^n)$  is defined as  $1 - \frac{1}{J_n} \sum_{j=1}^{J_n} \operatorname{tr}(W_{t^n}(c_j)D_j)$ .

Now we will define some further coding schemes, where the sender and the receiver use correlation as a resource. We will later show that these coding schemes are very helpful for the robust and secure message transmission over an arbitrarily varying wiretap channel.

**Definition 2.11.** Let  $\mathbf{X}$  and  $\mathbf{Y}$  be finite sets. Let (X, Y) be a random variable distributed according to a probability distribution  $p \in P(\mathbf{X} \times \mathbf{Y})$ .

An (X, Y)-correlation-assisted  $(n, J_n)$  code  $\mathcal{C}(X, Y)$  for the arbitrarily varying classical-quantum wiretap channel  $\{(W_t, V_t) : t \in \theta\}$  consists of a set of stochastic encoders  $\{E_{\mathbf{x}^n} : \{1, \ldots, J_n\} \to P(\mathbf{A}^n) : \mathbf{x}^n \in \mathbf{X}^n\}$ , and a set of collections of positive semi-definite operators  $\{\{D_j^{(\mathbf{y}^n)} : j = 1, \ldots, J_n\} : \mathbf{y}^n \in \mathbf{Y}^n\}$ on  $H^{\otimes n}$  which fulfills  $\sum_{j=1}^{J_n} D_j^{(\mathbf{y}^n)} = \mathrm{id}_{H^{\otimes n}}$  for every  $\mathbf{y}^n \in \mathbf{Y}^n$ . **Definition 2.12.** Let **X** and **Y** be finite sets, and let (X, Y) be a random variable distributed according to a joint probability distribution  $p \in P(\mathbf{X} \times \mathbf{Y})$ .

A nonnegative number R is an achievable m - a - (X, Y) secrecy rate (message transmission under the average error criterion using (X, Y)-correlationassisted  $(n, J_n)$  codes) for the arbitrarily varying classical-quantum wiretap channel  $\{(W_t, V_t) : t \in \theta\}$  if for every  $\epsilon > 0, \delta > 0, \zeta > 0$  and sufficiently large n there exists an (X, Y)-correlation-assisted  $(n, J_n)$  code  $\mathcal{C}(X, Y) = \left\{ \left( E_{\mathbf{x}^n}, \{D_j^{(\mathbf{y}^n)} : j \in \{1, \ldots, J_n\}\} \right) : \mathbf{x}^n \in \mathbf{X}^n, \ \mathbf{y}^n \in \mathbf{Y}^n \right\}$  such that  $\frac{\log J_n}{n} > R - \delta$ , and  $\max_{\substack{t^n \in \theta^n \\ \mathbf{x}^n \in \mathbf{X}^n}} \sum_{\mathbf{y}^n \in \mathbf{Y}^n} p(\mathbf{x}^n, \mathbf{y}^n) P_e(\mathcal{C}(\mathbf{x}^n, \mathbf{y}^n), t^n) < \epsilon$ ,  $\max_{\substack{t^n \in \theta^n \\ t^n \in \theta^n}} \chi\left(R_{\mathrm{uni}}; Z_{t^n, \mathbf{x}^n} \mid X\right) < \zeta$ ,

where  $P_e(\mathcal{C}(\mathbf{x}^n, \mathbf{y}^n), t^n)$  is defined as

$$P_e(\mathcal{C}(\mathbf{x}^n, \mathbf{y}^n), t^n) := 1 - \frac{1}{J_n} \sum_{j=1}^{J_n} \operatorname{tr}(W_{t^n}(E_{\mathbf{x}^n}(|j|))D_j^{(\mathbf{y}^n)}) ,$$
  
$$\chi(R_{\operatorname{uni}}; Z_{t^n, \mathbf{x}^n} \mid X) := \sum_{j=1}^{J_n} p(\mathbf{x}^n, \mathbf{y}^n)\chi(R_{\operatorname{uni}}; Z_{t^n, \mathbf{x}^n}) ,$$

 $\chi(\mathbf{R}_{\mathrm{uni}}; \mathbf{Z}_{t^{n}, \mathbf{x}^{n}} \mid \mathbf{A}) := \sum_{\mathbf{y}^{n} \in \mathbf{Y}^{n}} p(\mathbf{x}^{n}, \mathbf{y}^{n}) \chi(\mathbf{R}_{\mathrm{uni}}; \mathbf{Z}_{t^{n}, \mathbf{x}^{n}})^{n},$ and  $Z_{t^{n}, \mathbf{x}^{n}} = \left\{ V_{t^{n}}(E_{\mathbf{x}^{n}}(\mid i)) : i \in \{1, \dots, J_{n}\} \right\}, \ p(\mathbf{x}^{n}, \mathbf{y}^{n}) = \prod_{i=1}^{n} p(\mathbf{x}_{i}, \mathbf{y}_{i}).$ Here we allowed  $Z_{t^{n}, \mathbf{x}^{n}}$ , the resulting quantum state of the wiretapper, to be

Here we allowed  $Z_{t^n,\mathbf{x}^n}$ , the resulting quantum state of the wiretapper, to be dependent on  $\mathbf{x}^n$ , this means that we do not require (X,Y) to be secure against eavesdropping.

**Remark 2.13.** Her we follow [18] and use the definition "m-a-(X, Y) secrecy rate" because it is important to point out that here the average error criterion is used. Please see [18] for more discussions on the value of message transmission under the average error criterion and message transmission under the maximum error criterion.

**Definition 2.14.** Let  $\left\{C^{\gamma} = \{(E^{\gamma}, D_{j}^{\gamma}) : j = 1, \dots, J_{n}\} : \gamma \in \Lambda\right\}$  be the the set of  $(n, J_{n})$  deterministic codes, labeled by a set  $\Lambda$ .

An  $(n, J_n)$  randomness-assisted quantum code for the arbitrarily varying classical-quantum wiretap channel  $\{(W_t, V_t) : t \in \theta\}$  is a distribution G on  $(\Lambda, \sigma)$ , where  $\sigma$  is a sigma-algebra so chosen such that the functions  $\gamma \to P_e(C^{\gamma}, t^n)$  and  $\gamma \to \chi(R_{\text{uni}}; Z_{C^{\gamma}, t^n})$  are both G-measurable with respect to  $\sigma$  for every  $t^n \in \theta^n$ , where for  $t^n \in \theta^n$  and  $C^{\gamma} = \{(w(j)^{n,\gamma}, D_j^{\gamma}) : j = 1, \ldots, J_n\}$ ,

$$Z_{\mathcal{C}^{\gamma},t^{n}} := \{ V_{t^{n}}(w(1)^{n,\gamma}), V_{t^{n}}(w(2)^{n,\gamma}), \dots, V_{t^{n}}(w(n)^{n,\gamma}) \}$$

**Remark 2.15.** The randomness-assisted code technique is not to be confused with the random encoding technique. For the random encoding technique, only the sender, but not the receiver, randomly chooses a code word in  $\mathbf{A}^n$  to encode a message j according to a probability distribution. The receiver should be able to decode j even when he only knows the probability distribution, but not which code word is actually chosen by the sender. For the randomness-assisted code technique, the sender randomly chooses a stochastic encoder  $E^{\gamma}$  and the receiver chooses a set of the decoder operators  $\{D_j^{\gamma'}: j = 1, \ldots, J_n\}$ . The receiver can decode the message if and only if  $\gamma = \gamma'$ , i.e., when he knows the sender's randomization.

**Definition 2.16.** Let  $\Lambda$  and  $C^{\gamma}$ ,  $\gamma \in \Lambda$ , be defined as in Definition 2.14. An  $(n, J_n)$  common randomness-assisted quantum code for the arbitrarily varying classical-quantum wiretap channel  $\{(W_t, V_t) : t \in \theta\}$  is a finite subset  $\{C^{\gamma} = C^{\gamma} = C^{\gamma}\}$ 

 $\{(E^{\gamma}, D_j^{\gamma}) : j = 1, \dots, J_n\} : \gamma \in \Gamma \}$  of the set of  $(n, J_n)$  deterministic codes, labeled by a finite set  $\Gamma$ .

**Definition 2.17.** A nonnegative number R is an achievable secrecy rate for the arbitrarily varying classical-quantum wiretap channel  $\{(W_t, V_t) : t \in \theta\}$ under randomness-assisted coding if for every  $\delta > 0$ ,  $\zeta > 0$ , and  $\epsilon > 0$ , if n is sufficiently large, there is an  $(n, J_n)$  randomness-assisted quantum code  $(\{C^{\gamma} : \gamma \in \Lambda\}, G)$  such that  $\frac{\log J_n}{n} > R - \delta$ , and

$$\max_{t^n \in \theta^n} \int_{\Lambda} P_e(\mathcal{C}^{\gamma}, t^n) dG(\gamma) < \epsilon ,$$
$$\max_{n \in \theta^n} \int_{\Lambda} \chi\left(R_{\mathrm{uni}}, Z_{\mathcal{C}^{\gamma}, t^n}\right) dG(\gamma) < \zeta$$

Here we allow  $Z_{\mathcal{C}^{\gamma},t^n}$ , the wiretapper's resulting quantum state, to be dependent on  $\mathcal{C}^{\gamma}$ . This means that we do not require randomness to be secure against eavesdropping.

**Definition 2.18.** A non-negative number R is an achievable secrecy rate for the arbitrarily varying classical-quantum wiretap channel  $\{(W_t, V_t) : t \in \theta\}$  under common randomness-assisted quantum coding if for every  $\delta > 0$ ,  $\zeta > 0$ , and  $\epsilon > 0$ , if n is sufficiently large, there is an  $(n, J_n)$  common randomness-assisted quantum code  $(\{C^{\gamma} : \gamma \in \Gamma\})$  such that  $\frac{\log J_n}{n} > R - \delta$ , and

$$\max_{t^{n} \in \theta^{n}} \frac{1}{|\Gamma|} \sum_{\gamma=1}^{|\Gamma|} P_{e}(\mathcal{C}^{\gamma}, t^{n}) < \epsilon ,$$
$$\max_{t^{n} \in \theta^{n}} \chi\left(R_{\mathrm{uni}}, Z_{\mathcal{C}^{\gamma}, t^{n}} \mid \Gamma\right) < \zeta ,$$

where

$$\chi\left(R_{\mathrm{uni}}, Z_{\mathcal{C}^{\gamma}, t^{n}} \mid \Gamma\right) := \frac{1}{|\Gamma|} \sum_{\gamma=1}^{|\Gamma|} \chi\left(R_{\mathrm{uni}}, Z_{\mathcal{C}^{\gamma}, t^{n}}\right) \;.$$

This means that we do not require the common randomness to be secure against eavesdropping.

We may consider the deterministic code, the (X, Y)-correlation-assisted code, the ((X, Y), r)-correlation-assisted code, the (X, Y)-correlation-assisted  $(n, J_n)$ code, and the common randomness-assisted quantum code as special cases of the randomness-assisted quantum code. This means that randomness is a stronger resource than both common randomness and the (X, Y)-correlation, in the sense that it requires more randomness than common randomness and the (X, Y)correlation. Randomness is therefore a more "costly" resource.

**Definition 2.19.** Let  $\{(W_t, V_t) : t \in \theta\}$  be an arbitrarily varying classicalquantum wiretap channel.

The supremum of all achievable (deterministic) secrecy rates of  $\{(W_t, V_t) : t \in \theta\}$  is called the (deterministic) secrecy capacity of  $\{(W_t, V_t) : t \in \theta\}$ , denoted by  $C_s(\{(W_t, V_t) : t \in \theta\})$ .

The supremum of all achievable m-a-(X,Y) secrecy rates of  $\{(W_t, V_t) : t \in \theta\}$ is called the m-a-(X,Y) secrecy capacity, denoted by  $C_s(\{(W_t, V_t) : t \in \theta\}; corr(X,Y))$ .

The supremum of all achievable secrecy rates under random-assisted quantum coding of  $\{(W_t, V_t) : t \in \theta\}$  is called the random-assisted secrecy capacity of  $\{(W_t, V_t) : t \in \theta\}$ , denoted by  $C_s(\{(W_t, V_t) : t \in \theta\}; r)$ .

The supremum of all achievable secrecy rates under common randomness-assisted quantum coding of  $\{(W_t, V_t) : t \in \theta\}$  is called the common randomness-assisted secrecy capacity of  $\{(W_t, V_t) : t \in \theta\}$ , denoted by  $C_s(\{(W_t, V_t) : t \in \theta\}; cr)$ .

For an arbitrarily varying classical-quantum wiretap channel  $\{(W_t, V_t) : t \in \theta\}$  and random variable (X, Y) distributed on finite sets **X** and **Y**, the following facts are obvious and follow from the definitions.

$$C_{s}(\{(W_{t}, V_{t}) : t \in \theta\})$$

$$\leq C_{s}((W_{t}, V_{t})_{t \in \theta}; corr(X, Y))$$

$$\leq C_{s}(\{(W_{t}, V_{t}) : t \in \theta\}; r) , \qquad (3)$$

 $C_s(\{(W_t, V_t) : t \in \theta\}) \le C_s((W_t, V_t)_{t \in \theta}; cr) \le C_s(\{(W_t, V_t) : t \in \theta\}; r) .$ (4)

# 3 Ahlswede dichotomy for Arbitrarily Varying Classical-Quantum Wiretap Channels

In this section, we analyze the secrecy capacities of various coding schemes with resource assistance. Our goal is to see what the effects are on the secrecy capacities of an arbitrarily varying classical-quantum wiretap channel if we use deterministic code, randomness-assisted code, or common randomness-assisted code.

**Theorem 3.1 (Ahlswede dichotomy).** Let  $\{(W_t, V_t) : t \in \theta\}$  be an arbitrarily varying classical-quantum wiretap channel.

1. (a) If the arbitrarily varying classical-quantum channel  $\{W_t : t \in \theta\}$  is not symmetrizable, then

$$C_s(\{(W_t, V_t) : t \in \theta\}) = C_s(\{(W_t, V_t) : t \in \theta\}; r) .$$
(5)

(b) If  $\{W_t : t \in \theta\}$  is symmetrizable,

$$C_s(\{(W_t, V_t) : t \in \theta\}) = 0 .$$
(6)

2.

$$C_s(\{(W_t, V_t) : t \in \theta\}; cr) = C_s(\{(W_t, V_t) : t \in \theta\}; r) .$$
(7)

*Proof.* Our proof is similar to the proof of Ahlswede dichotomy for arbitrarily varying classical-quantum channels in [7]. The different between our proof and the proofs in [7] is that we have to additionally consider the security.

#### 3.1 Proof of Theorem 3.1. 2

At first we use random encoding technique to show the existence of a common randomness-assisted code.

Choose arbitrary positive  $\epsilon$  and  $\zeta$ . Assume we have an  $(n, J_n)$  randomnessassisted code  $(\{\mathcal{C}^{\gamma} : \gamma \in \Lambda\}, G)$  for  $\{(W_t, V_t) : t \in \theta\}$  such that

$$\begin{split} & \max_{t^n \in \theta^n} \int_{\Lambda} P_e(\mathcal{C}^{\gamma}, t^n) dG(\gamma) < \epsilon \ , \\ & \max_{t^n \in \theta^n} \int_{\Lambda} \chi\left( R_{\mathrm{uni}}, Z_{\mathcal{C}^{\gamma}, t^n} \right) dG(\gamma) < \zeta \ . \end{split}$$

Consider now  $n^3$  independent and identically distributed random variables  $\overline{\mathcal{C}}_1, \overline{\mathcal{C}}_2, \ldots, \overline{\mathcal{C}}_{n^3}$  with values in  $\{\mathcal{C}^{\gamma} : \gamma \in \Lambda\}$  such that  $Pr(\overline{\mathcal{C}}_i = \mathcal{C}) = G(\mathcal{C})$  for all  $\mathcal{C} \in \{\mathcal{C}^{\gamma} : \gamma \in \Lambda\}$  and for all  $i \in \{1, \ldots, n^3\}$ . For a fixed  $t^n \in \theta^n$  we have

$$P\left(\sum_{i=1}^{n^{3}} \chi\left(R_{\mathrm{uni}}, Z_{\overline{\mathcal{C}}_{i}, t^{n}}\right) > n^{3}\lambda\right)$$
  
=  $P\left(\exp\left(\sum_{i=1}^{n^{3}} \frac{1}{n} 2\chi\left(R_{\mathrm{uni}}, Z_{\overline{\mathcal{C}}_{i}, t^{n}}\right)\right) > \exp\left(\frac{1}{n} 2n^{3}\lambda\right)\right)$   
 $\leq \exp\left(-2n^{2}\lambda\right) \prod_{i=1}^{n^{3}} \mathbb{E}_{G} \exp\left(\frac{1}{n} 2\chi\left(R_{\mathrm{uni}}, Z_{\overline{\mathcal{C}}_{i}, t^{n}}\right)\right)$   
=  $\exp\left(-2n^{2}\lambda\right) \mathbb{E}_{G} \exp\left(\sum_{i=1}^{n^{3}} \frac{1}{n} 2\chi\left(R_{\mathrm{uni}}, Z_{\overline{\mathcal{C}}_{i}, t^{n}}\right)\right)$ 

$$\leq \exp\left(-2n^{2}\lambda\right)\prod_{i=1}^{n^{3}}\mathbb{E}_{G}\left[1+\sum_{k=1}^{\infty}\frac{2^{k}\frac{1}{n}\chi\left(R_{\mathrm{uni}},Z_{\overline{\mathcal{C}}_{i},t^{n}}\right)}{k!}\right]$$
$$=\exp\left(-2n^{2}\lambda\right)\left[1+\sum_{k=1}^{\infty}\frac{2^{k}\frac{1}{n}\mathbb{E}_{G}\chi\left(R_{\mathrm{uni}},Z_{\overline{\mathcal{C}}_{i},t^{n}}\right)}{k!}\right]^{n^{3}}$$
$$\leq \exp\left(-2n^{2}\lambda\right)\left[1+\sum_{k=1}^{\infty}\frac{2^{k}\epsilon}{nk!}\right]^{n^{3}}$$
$$=\exp\left(-2n^{2}\lambda\right)\left[1+\frac{1}{n}\epsilon\exp2\right]^{n^{3}},\qquad(8)$$

the second inequality holds because the right side is part of the Taylor series.

We fix  $n \in \mathbb{N}$  and define

$$h_n(x) := n \log(1 + \frac{1}{n}e^2 x) - x$$
.

We have  $h_n(0) = 0$  and

$$h'_n(x) = n \frac{1}{1 + \frac{1}{n}e^2 x} \frac{1}{n}e^2 - 1$$
$$= \frac{ne^2}{e^2 x + n} - 1.$$

 $\frac{ne^2}{e^2x+n} - 1 \text{ is positive if } x < \frac{e^2-1}{e}n, \text{ thus if } \hat{c} < \frac{e^2-1}{e}n, h_n(x) \text{ is strictly} \\ \text{monotonically increasing in the interval } ]0, \hat{c}]. \text{ Thus } h_n(x) \text{ is positive for } 0 < x \leq \hat{c}. \text{ For every positive } \hat{c}, \hat{c} < \frac{e^2-1}{e}n \text{ holds if } n > \frac{e}{e^2-1}\hat{c}. \text{ Thus for any positive} \\ \epsilon, \epsilon \leq n\log(1+\frac{1}{n}\epsilon\exp 2) \text{ if } n \text{ is large enough. Choose } \lambda \leq \epsilon \text{ and let } n \text{ be} \\ \text{sufficiently large, we have } \lambda \leq n\log(1+\frac{1}{n}\epsilon\exp 2), \text{ therefore} \end{cases}$ 

$$\exp\left(-2\lambda n^{2}\right)\left[1+\frac{1}{n}\epsilon\exp 2\right]^{n^{2}}$$
$$=\exp\left(-\lambda n^{2}\right)\exp\left(n^{2}(-\lambda+n\log(1+\frac{1}{n}\epsilon\exp 2))\right)$$
$$\leq\exp\left(-\lambda n^{2}\right).$$
(9)

By (8) and (9)

$$P\left(\sum_{i=1}^{n^{3}} \chi\left(R_{\mathrm{uni}}, Z_{\overline{\mathcal{C}}_{i}, t^{n}}\right) > \lambda n^{3} \ \forall t^{n} \in \theta^{n}\right)$$
  
$$< |\theta|^{n} \exp(-\lambda n^{2})$$
  
$$= \exp(n \log |\theta| - \lambda n^{2})$$
  
$$= \exp(-n\lambda) . \tag{10}$$

When  $P_{er} < \zeta$  holds, in a similar way as (8), choose  $\lambda \leq \zeta$ , we can show that

$$P\left(\sum_{i=1}^{n^3} P_e(\overline{\mathcal{C}}_i, t^n) > \lambda n^3 \; \forall t^n \in \theta^n\right) < e^{-\lambda n} \;. \tag{11}$$

Let  $\lambda := \min\{\epsilon, \zeta\}$ , we have

$$P\left(\sum_{i=1}^{n^3} P_e(\overline{\mathcal{C}}_i, t^n) > \lambda n^3 \text{ or } \sum_{i=1}^{n^3} \chi\left(R_{\text{uni}}, Z_{\overline{\mathcal{C}}_i, t^n}\right) > \lambda n^3 \ \forall t^n \in \theta^n\right) \le 2e^{-\lambda n^3} .$$

We denote the event

$$\mathbf{E}_{n} := \left\{ \mathcal{C}_{1}, \mathcal{C}_{2}, \dots, \mathcal{C}_{n^{3}} \in \mathcal{C}_{\nu}' : \frac{1}{n^{3}} \sum_{i=1}^{n^{3}} P_{e}(\mathcal{C}_{i}, t^{n}) \leq \lambda \right\}$$
  
and  $\frac{1}{n^{3}} \sum_{i=1}^{n^{3}} \chi\left(R_{\mathrm{uni}}, Z_{\mathcal{C}_{i}, t^{n}}\right) \leq \lambda \right\}$ .

If n is large enough, then  $P(\mathbf{E}_n)$  is positive. This means  $\mathbf{E}_n$  is not the empty set, since  $P(\emptyset) = 0$  by definition. Thus there exist codes  $C_i = (E_i^n, \{D_{j,i}^n : j = 1, ..., J_n\}) \in C'_{\nu}$  for  $i \in \{1, ..., n^3\}$  with a positive probability such that

$$\frac{1}{n^3} \sum_{i=1}^{n^3} P_e(\mathcal{C}_i, t^n) < \lambda \text{ and } \frac{1}{n^3} \sum_{i=1}^{n^3} \chi\left(R_{\mathrm{uni}}, Z_{\mathcal{C}_i, t^n}\right) \le \lambda .$$
(12)

By (12), for any  $n \in \mathbb{N}$  and positive  $\lambda$ , if there is an  $(n, J_n)$  randomnessassisted code  $(\{\mathcal{C}^{\gamma} : \gamma \in \Lambda\}, G)$  for  $\{(W_t, V_t) : t \in \theta\}$  such that

$$\begin{split} \max_{t^n \in \theta^n} \int_{\Lambda} P_e(\mathcal{C}^{\gamma}, t^n) dG(\gamma) < \lambda \ , \\ \max_{t^n \in \theta^n} \int_{\Lambda} \chi \left( R_{\mathrm{uni}}, Z_{\mathcal{C}^{\gamma}, t^n} \right) dG(\gamma) < \lambda \ , \end{split}$$

there is also an  $(n, J_n)$  common randomness-assisted code  $\{C_1, C_2, \ldots, C_{n^3}\}$  such that

$$\max_{t^n \in \theta^n} \frac{1}{n^3} \sum_{i=1}^{n^\circ} P_e(\mathcal{C}_i, t^n) < \lambda ,$$
$$\max_{t^n \in \theta^n} \frac{1}{n^3} \sum_{i=1}^{n^3} \chi(R_{\text{uni}}, Z_{\mathcal{C}_i, t^n}) < \lambda .$$

Therefore we have

$$C_s(\{(W_t, V_t) : t \in \theta\}; cr) \ge C_s(\{(W_t, V_t) : t \in \theta\}; r) .$$

This and the fact that

$$C_s(\{(W_t, V_t) : t \in \theta\}; cr) \le C_s(\{(W_t, V_t) : t \in \theta\}; r) \ ,$$

prove Theorem 3.1. 2.

#### 3.2 Proof of Theorem 3.1. 1a

Now we are going to use Theorem 3.1. 2 to prove Theorem 3.1. 1a.

To show the lower bound in Theorem 3.1. 1a, we build a two-part code word, which consists of a non-secure code word and a common randomness-assisted secure code word. The non-secure one is used to create the common randomness for the sender and the legal receiver. The common randomness-assisted secure code word is used to transmit the message to the legal receiver.

Choose arbitrary positive  $\epsilon$  and  $\zeta$ . Assume we have an  $(n, J_n)$  randomnessassisted code  $(\{\mathcal{C}^{\gamma} : \gamma \in \Lambda\}, G)$  for  $\{(W_t, V_t) : t \in \theta\}$  such that

$$\begin{split} \max_{t^n \in \theta^n} \int_{\Lambda} P_e(\mathcal{C}^{\gamma}, t^n) dG(\gamma) < \epsilon \ , \\ \max_{\epsilon \in \theta^n} \int_{\Lambda} \chi\left( R_{\mathrm{uni}}, Z_{\mathcal{C}^{\gamma}, t^n} \right) dG(\gamma) < \zeta \end{split}$$

by Theorem 3.1. 2, there is also an  $(n, J_n)$  common randomness-assisted code  $\{C_1, C_2, \ldots, C_n^3\}$  such that

$$\max_{t^n \in \theta^n} \frac{1}{n^3} \sum_{i=1}^{n^3} P_e(\mathcal{C}_i, t^n) < \lambda , \qquad (13)$$

$$\max_{t^n \in \theta^n} \frac{1}{n^3} \sum_{i=1}^{n^3} \chi\left(R_{\mathrm{uni}}, Z_{\mathcal{C}_i, t^n}\right) < \lambda , \qquad (14)$$

where  $\lambda := \min\{\epsilon, \zeta\}.$ 

If the arbitrarily varying classical-quantum channel  $\{W_t : x \in \mathcal{X}\}$  is not symmetrizable, then by [7], the capacity for message transmission of  $\{W_t : x \in \mathcal{X}\}$  is positive. By Remark 2.7 we may assume that the capacity for message transmission of  $\{W_t : x \in \mathcal{X}\}$  using deterministic encoder is positive. This means for any positive  $\vartheta$ , if n is sufficiently large, there is a code  $\left(\left(c_i^{\mu(n)}\right)_{i \in \{1,...,n^3\}}, \{D_i^{\mu(n)} : i \in \{1,...,n^3\}\}\right)$  with deterministic encoder of length  $\mu(n)$ , where  $2^{\mu(n)} = o(n)$ 

such that  $1 \sum_{n=1}^{n^3} (u(n)) = u(n)$ 

$$1 - \frac{1}{n^3} \sum_{i=1}^{n} \operatorname{tr}(W_{t^n}(c_i^{\mu(n)}) D_i^{\mu(n)}) \le \vartheta .$$
(15)

We now can construct a code  $C^{\text{det}} = \left(E^{\mu(n)+n}, \left\{D_j^{\mu(n)+n} : j = 1, \dots, J_n\right\}\right)$ , where for  $a^{\mu(n)+n} = (a^{\mu(n)}, a^n) \in \mathbf{A}^{\mu(n)+n}$ 

$$E^{\mu(n)+n}(a^{\mu(n)+n}|j) = \begin{cases} \frac{1}{n^3} E_i^n(a^n|j) \text{ if } a^{\mu(n)} = c_i^{\mu(n)} \\ 0 \text{ else} \end{cases}$$

and

$$D_j^{\mu(n)+n} := \sum_{i=1}^{n^3} D_i^{\mu(n)} \otimes D_{i,j}^n$$
.

It is a composition of the code  $(c_i^{\mu(n)})_{i=1,\dots,n^3}, \{D_i^{\mu(n)}: i=1,\dots,n^3\})$  and the code  $\mathcal{C}_i = (E_i^n, \{D_{i,j}^n: j=1,\dots,J_n\}$ . This is a code of length  $\mu(n) + n$ .

#### 3.2.1 This code is secure against eavesdropping

We are going to show that the two-part code word is secure when the common randomness-assisted part is secure. Since the two-part code can be seen as a function of its common randomness-assisted part the idea is similar to applying the quantum data processing inequality (cf. [38]) when we consider quantum mutual information as security criterion.

For any  $i \in \{1, \ldots, n^3\}$  let

$$\mathfrak{Z}_{i,t^{\mu(n)+n}} := \left\{ V_{t^{\mu(n)}}(c_i^{\mu(n)}) \otimes V_{t^n}(E_i^n(\mid 1)), \dots, V_{t^{\mu(n)}}(c_i^{\mu(n)}) \otimes V_{t^n}(E_i^n(\mid J_n)) \right\} \,.$$

For any  $t^{\mu(n)+n} = (t^{\mu(n)}, t^n)$  we have

$$\begin{split} \chi\left(R_{\mathrm{uni}}, \mathfrak{Z}_{i,t^{\mu(n)+n}}\right) \\ &= S\left(\frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{a^n \in \mathbf{A}^n} E_i^n(a^n \mid j) V_{t^{\mu(n)}}(c_i^{\mu(n)}) \otimes V_{t^n}(a^n)\right) \\ &- \frac{1}{J_n} \sum_{j=1}^{J_n} S\left(\sum_{a^n \in \mathbf{A}^n} E_i^n(a^n \mid j) V_{t^{\mu(n)}}(c_i^{\mu(n)}) \otimes V_{t^n}(a^n)\right) \\ &= S\left(V_{t^{\mu(n)}}(c_i^{\mu(n)})\right) + S\left(\frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{a^n \in \mathbf{A}^n} E_i^n(a^n \mid j) V_{t^n}(a^n)\right) \\ &- \frac{1}{J_n} \sum_{j=1}^{J_n} S\left(\sum_{a^n \in \mathbf{A}^n} E_i^n(a^n \mid j) V_{t^n}(a^n)\right) \\ &= S\left(\frac{1}{J_n} \sum_{j=1}^{J_n} V_{t^n}(E_i^n(\mid j))\right) - \frac{1}{J_n} \sum_{j=1}^{J_n} S\left(V_{t^n}(E_i^n(\mid j))\right) \\ &= \chi\left(R_{\mathrm{uni}}, Z_{\mathcal{C}_i, t^n}\right) \,. \end{split}$$
(16)

By definition, we have

$$\begin{split} &Z_{\mathcal{C}^{\det},t^{\mu(n)+n}} \\ &= \left\{ V_{t^{\mu(n)+n}} (E^{\mu(n)+n}(||1)), \dots, V_{t^{\mu(n)+n}} (E^{\mu(n)+n}(||J_n)) \right\} \\ &= \left\{ \frac{1}{n^3} \sum_{i=1}^{n^3} \sum_{a^n \in \mathbf{A}^n} E_i^n (a^n ||1) V_{t^{\mu(n)+n}} \left( \left( c_i^{\mu(n)}, a^n \right) \right), \dots, \right. \\ &\frac{1}{n^3} \sum_{i=1}^{n^3} \sum_{a^n \in \mathbf{A}^n} E_i^n (a^n ||J_n) V_{t^{\mu(n)+n}} \left( \left( c_i^{\mu(n)}, a^n \right) \right) \right\} \\ &= \left\{ \frac{1}{n^3} \sum_{i=1}^{n^3} \sum_{a^n \in \mathbf{A}^n} E_i^n (a^n ||I) V_{t^{\mu(n)}} (c_i^{\mu(n)}) \otimes V_{t^n} (a^n), \dots, \right. \\ &\frac{1}{n^3} \sum_{i=1}^{n^3} \sum_{a^n \in \mathbf{A}^n} E_i^n (a^n ||J_n) V_{t^{\mu(n)}} (c_i^{\mu(n)}) \otimes V_{t^n} (a^n) \right\} \end{split}$$

$$= \left\{ \frac{1}{n^3} \sum_{i=1}^{n^3} V_{t^{\mu(n)}}(c_i^{\mu(n)}) \otimes V_{t^n}(E_i^n(|1)), \dots, \frac{1}{n^3} \sum_{i=1}^{n^3} V_{t^{\mu(n)}}(c_i^{\mu(n)}) \otimes V_{t^n}(E_i^n(|J_n)) \right\}.$$

By (14) and (16) for any  $t^{\mu(n)+n} = (t^{\mu(n)}t^n)$  we have

$$\chi \left( R_{\text{uni}}, Z_{\mathcal{C}^{\text{det}}, t^{\mu(n)+n}} \right) \leq \chi \left( R_{\text{uni}}, Z_{\mathcal{C}^{\text{det}}, t^{\mu(n)+n}} \right) - \frac{1}{n^3} \sum_{i=1}^{n^3} \chi \left( R_{\text{uni}}, Z_{\mathcal{C}_i, t^n} \right) + \lambda = \chi \left( R_{\text{uni}}, Z_{\mathcal{C}^{\text{det}}, t^{\mu(n)+n}} \right) - \frac{1}{n^3} \sum_{i=1}^{n^3} \chi \left( R_{\text{uni}}, \mathfrak{Z}_{i, t^{\mu(n)+n}} \right) + \lambda = S \left( \frac{1}{J_n} \frac{1}{n^3} \sum_{j=1}^{J_n} \sum_{i=1}^{n^3} V_{t^{\mu(n)}} (c_i^{\mu(n)}) \otimes V_{t^n} (E_i^n (\mid j)) \right) - \frac{1}{J_n} \sum_{j=1}^{J_n} S \left( \frac{1}{n^3} \sum_{i=1}^{n^3} V_{t^{\mu(n)}} (c_i^{\mu(n)}) \otimes V_{t^n} (E_i^n (\mid j)) \right) - \frac{1}{n^3} \sum_{i=1}^{n^3} S \left( \frac{1}{J_n} \sum_{j=1}^{J_n} V_{t^{\mu(n)}} (c_i^{\mu(n)}) \otimes V_{t^n} (E_i^n (\mid j)) \right) + \frac{1}{J_n} \frac{1}{n^3} \sum_{j=1}^{J_n} \sum_{i=1}^{n^3} S \left( V_{t^{\mu(n)}} (c_i^{\mu(n)}) \otimes V_{t^n} (E_i^n (\mid j)) \right) + \lambda . \tag{17}$$

Let  $H^{\mathfrak{H}}$  be a  $n^3$ -dimensional Hilbert space, spanned by an orthonormal basis  $\{|i\rangle : i = 1, \ldots, n^3\}$ . Let  $H^{\mathfrak{I}}$  be a  $J_n$  dimensional Hilbert space, spanned by an orthonormal basis  $\{|j\rangle : j = 1, \ldots, J_n\}$ . We define

$$\varphi^{\mathfrak{H}^{\mu(n)+n}} := \frac{1}{J_n} \frac{1}{n^3} \sum_{j=1}^{J_n} \sum_{i=1}^{n^3} |j\rangle \langle j| \otimes |i\rangle \langle i| \otimes V_{t^{\mu(n)}}(c_i^{\mu(n)}) \otimes V_{t^n}(E_i^n(|j)) .$$

We have

$$\varphi^{\mathfrak{J}H^{\mu(n)+n}} = \operatorname{tr}_{\mathfrak{H}}\left(\varphi^{\mathfrak{J}\mathfrak{H}^{\mu(n)+n}}\right) = \frac{1}{J_n} \frac{1}{n^3} \sum_{j=1}^{J_n} \sum_{i=1}^{n^3} |j\rangle \langle j| \otimes V_{t^{\mu(n)}}(c_i^{\mu(n)}) \otimes V_{t^n}(E_i^n(\mid j)) ,$$

$$\varphi^{\mathfrak{H}^{\mu(n)+n}} = \operatorname{tr}_{\mathfrak{J}}\left(\varphi^{\mathfrak{I}\mathfrak{H}^{\mu(n)+n}}\right) = \frac{1}{J_n} \frac{1}{n^3} \sum_{j=1}^{J_n} \sum_{i=1}^{n^3} |i\rangle \langle i| \otimes V_{t^{\mu(n)}}(c_i^{\mu(n)}) \otimes V_{t^n}(E_i^n(\mid j)) ,$$

$$\varphi^{H^{\mu(n)+n}} = \operatorname{tr}_{\mathfrak{J}\mathfrak{H}}\left(\varphi^{\mathfrak{J}\mathfrak{H}^{\mu(n)+n}}\right) = \frac{1}{J_n} \frac{1}{n^3} \sum_{j=1}^{J_n} \sum_{i=1}^{n^3} V_{t^{\mu(n)}}(c_i^{\mu(n)}) \otimes V_{t^n}(E_i^n(\mid j)) \ .$$

Furthermore

$$S(\varphi^{\Im H^{\mu(n)+n}}) = S\left(\frac{1}{J_n}\sum_{j=1}^{J_n}\sum_{i=1}^{n^3}|j\rangle\langle j|\otimes V_{t^{\mu(n)}}(c_i^{\mu(n)})\otimes V_{t^n}(E_i^n(|j))\right)$$
$$= H(R_{\text{uni}}) + \frac{1}{J_n}\sum_{j=1}^{J_n}S\left(\frac{1}{n^3}\sum_{i=1}^{n^3}V_{t^{\mu(n)}}(c_i^{\mu(n)})\otimes V_{t^n}(E_i^n(|j))\right),$$

$$S(\varphi^{\mathfrak{H}^{\mu(n)+n}}) = S\left(\frac{1}{J_n}\frac{1}{n^3}\sum_{j=1}^{J_n}\sum_{i=1}^{n^3}|i\rangle\langle i|\otimes V_{t^{\mu(n)}}(c_i^{\mu(n)})\otimes V_{t^n}(E_i^n(||j))\right) = H(Y_{\text{uni}}) + \frac{1}{n^3}\sum_{i=1}^{n^3}S\left(\frac{1}{J_n}\sum_{j=1}^{J_n}V_{t^{\mu(n)}}(c_i^{\mu(n)})\otimes V_{t^n}(E_i^n(||j))\right),$$

$$\begin{split} S(\varphi^{\mathfrak{IH}^{\mu(n)+n}}) \\ &= S\left(\frac{1}{J_n}\frac{1}{n^3}\sum_{j=1}^{J_n}\sum_{i=1}^{n^3}|j\rangle\langle j|\otimes|i\rangle\langle i|\otimes V_{t^{\mu(n)}}(c_i^{\mu(n)})\otimes V_{t^n}(E_i^n(\mid j))\right) \\ &= H(R_{\text{uni}}) + H(Y_{\text{uni}}) + \frac{1}{J_n}\frac{1}{n^3}\sum_{j=1}^{J_n}\sum_{i=1}^{n^3}S\left(V_{t^{\mu(n)}}(c_i^{\mu(n)})\otimes V_{t^n}(E_i^n(\mid j))\right) \ , \end{split}$$

By strong subadditivity of von Neumann entropy it holds  $S(\varphi^{\mathfrak{J}H^{\mu(n)+n}}) + S(\varphi^{\mathfrak{H}^{\mu(n)+n}}) \geq S(\varphi^{H^{\mu(n)+n}}) + S(\varphi^{\mathfrak{I}\mathfrak{H}^{\mu(n)+n}})$ . Thus by (17) we have

$$\chi\left(R_{\mathrm{uni}}, Z_{\mathcal{C}^{\mathrm{det}}, t^{\mu(n)+n}}\right) \le \lambda \ . \tag{18}$$

#### 3.2.2 The legal receiver is able to decode the message

We now use Theorem 3.1. 2 to show that the legal receiver's average error goes to zero.

For any  $t^{\mu(n)+n} \in \theta^{\mu(n)+n}$ , by (13) and (14),

$$P_{e}(\mathcal{C}^{\det}, t^{\mu(n)+n}) = 1 - \frac{1}{J_{n}} \sum_{j=1}^{J_{n}} \operatorname{tr}\left(\left[\frac{1}{n^{3}} \sum_{i=1}^{n^{3}} U_{t^{\mu(n)}}(c_{i}^{\mu(n)}) \otimes U_{t^{n}}(E_{i}^{n}(||j|))\right] \cdot \left[\sum_{k=1}^{n^{3}} D_{k}^{\mu(n)} \otimes D_{k,j}^{n}\right]\right) \\ \leq 1 - \frac{1}{J_{n}} \sum_{j=1}^{J_{n}} \operatorname{tr}\left(\frac{1}{n^{3}} \sum_{i=1}^{n^{3}} \left[U_{t^{\mu(n)}}(c_{i}^{\mu(n)}) \otimes U_{t^{n}}(E_{i}^{n}(||j|))\right] \cdot \left[D_{k}^{\mu(n)} \otimes D_{k,j}^{n}\right]\right)$$

$$= 1 - \frac{1}{J_n} \sum_{j=1}^{J_n} \operatorname{tr} \left( \frac{1}{n^3} \sum_{i=1}^{n^3} \left[ U_{t^{\mu(n)}}(c_i^{\mu(n)}) D_k^{\mu(n)} \right] \otimes \left[ U_{t^n}(E_i^n(\mid j)) D_{k,j}^n \right] \right)$$
  
$$= 1 - \frac{1}{n^3} \sum_{i=1}^{n^3} \left( \operatorname{tr} \left[ U_{t^{\mu(n)}}(c_i^{\mu(n)}) D_k^{\mu(n)} \right] \cdot \frac{1}{J_n} \sum_{j=1}^{J_n} \operatorname{tr} \left[ U_{t^n}(E_i^n(\mid j)) D_{k,j}^n \right] \right)$$
  
$$= 1 - \frac{1}{n^3} \sum_{i=1}^{n^3} \left( \operatorname{tr} \left[ U_{t^{\mu(n)}}(c_i^{\mu(n)}) D_k^{\mu(n)} \right] \cdot (1 - P_e(C_i, t^n)) \right)$$
  
$$\leq 1 - (1 - \vartheta - \lambda)$$
  
$$= \lambda + \vartheta , \qquad (19)$$

the second inequality holds because for non-negative numbers  $\{\alpha_i, \beta_i : i = 1, \ldots, M\}$  such that  $\frac{1}{M} \sum_{i=1}^{M} \alpha_i \leq \vartheta$  and  $\frac{1}{M} \sum_{i=1}^{M} \beta_i \leq \lambda$  we have  $\frac{1}{M} \sum_{i=1}^{M} (1 - \alpha_i)(1 - \beta_i) \geq 1 - \vartheta - \lambda$ .

For any  $n \in \mathbb{N}$  and positive  $\lambda$ , if there is an  $(n, J_n)$  randomness-assisted code  $(\{\mathcal{C}^{\gamma} : \gamma \in \Lambda\}, G)$  for  $\{(W_t, V_t) : t \in \theta\}$  such that

$$\max_{t^n \in \theta^n} \int_{\Lambda} P_e(\mathcal{C}^{\gamma}, t^n) dG(\gamma) < \epsilon ,$$
$$\max_{n \in \theta^n} \int_{\Lambda} \chi\left(R_{\mathrm{uni}}, Z_{\mathcal{C}^{\gamma}, t^n}\right) dG(\gamma) < \zeta ,$$

choose  $\delta = \min\{\epsilon, \zeta\} + \vartheta$ , by (19) and (18), we can find a  $(\mu(n) + n, J_n)$  deterministic code  $\mathcal{C}^{det} = \left(E^{\mu(n)+n}, \{D_j^{\mu(n)+n} : j = 1, \dots, J_n\}\right)$  such that such that

$$\max_{\substack{t^{\mu(n)+n}\in\theta^{\mu(n)+n}\\ \vdots^{\mu(n)+n}\in\theta^{\mu(n)+n}}} P_e(\mathcal{C}^{\det}, t^{\mu(n)+n}) < \lambda ,$$
$$\max_{\substack{\mu(n)+n\in\theta^{\mu(n)+n}\\ \vdots^{\mu(n)+n}}} \chi\left(R_{\mathrm{uni}}, Z_{\mathcal{C}^{\det}, t^{\mu(n)+n}}\right) < \lambda .$$

We know that  $2^{\mu(n)} = o(n)$ . For any positive  $\varepsilon$ , if n is large enough we have  $\frac{1}{n} \log J_n - \frac{1}{\log n+n} \log J_n \leq \varepsilon$ . Therefore, if the arbitrarily varying classical-quantum channel  $\{W_t : x \in \mathcal{X}\}$  is not symmetrizable, we have

$$C_s(\{(W_t, V_t) : t \in \theta\}; cr) \ge C_s(\{(W_t, V_t) : t \in \theta\}; r) - \varepsilon .$$

$$(20)$$

This and the fact that

$$C_s(\{(W_t, V_t) : t \in \theta\}; cr) \le C_s(\{(W_t, V_t) : t \in \theta\}; r)$$

prove Theorem 3.1. 1a (c.f. [7] for Ahlswede dichotomy for arbitrarily varying classical-quantum channel Channels).

#### 3.3 The proof of Theorem 3.1. 1b

If  $\{W_t : t \in \theta\}$  is symmetrizable, the deterministic capacity of  $\{W_t : t \in \theta\}$  using a deterministic encoder is equal to zero by [7]. Now we have to check whether  $C_s(\{(W_t, V_t) : t \in \theta\})$  using stochastic encoder remains equal to zero. The proof is rather standard. Readers with experiences in information theory may pass over this subsection.

For any  $n \in \mathbb{N}$  and  $J_n \in \mathbb{N} \setminus \{1\}$ , let  $\mathcal{C} = \left(E^n, \{D_j^n : j \in \{1, \ldots, J_n\}\}\right)$ be an  $(n, J_n)$  deterministic code with a random encoder. We denote the set of all deterministic encoders by  $\mathbf{F}_n := \left\{f_n : \{1, \ldots, J_n\} \to \mathbf{A}^n\right\}$ . Since the deterministic capacity of  $\{W_t : t \in \theta\}$  using deterministic encoder is zero, there is a positive c such that for any  $n \in \mathbb{N}$  we have

$$\max_{t^n \in \theta^n} \frac{1}{J_n} \sum_{j=1}^{J_n} \operatorname{tr} \left( W_{t^n}(f_n(j)) D_j^n \right) < 1 - c .$$
(21)

For any  $t^n \in \theta^n$ , we have

$$1 - c$$

$$= (1 - c) \sum_{f_n \in \mathbf{F}_n} \prod_{k=1}^{J_n} E^n (f_n(k) \mid k)$$

$$> \sum_{f_n \in \mathbf{F}_n} \prod_{k=1}^{J_n} E^n (f_n(k) \mid k) \frac{1}{J_n} \sum_{j=1}^{J_n} \operatorname{tr} \left( W_{t^n}(f_n(j)) D_j^n \right)$$

$$= \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{a^n \in \mathbf{A}^n} E^n (a^n \mid j) \operatorname{tr} \left( W_{t^n}(a^n) D_j^n \right)$$

$$= \frac{1}{J_n} \sum_{j=1}^{J_n} \operatorname{tr} \left( W_{t^n}(E^n(\mid j)) D_j^n \right), \qquad (22)$$

the first equation holds because

$$\begin{split} &\sum_{f_n \in \mathbf{F}_n} \prod_{j=1}^{J_n} E^n(f_n(j) \mid j) \\ &= \sum_{a^n} \sum_{f_n(1)=a^n} \left( \sum_{a^n} \sum_{f_n(2)=a^n} \left( \dots \left( \sum_{a^n} \sum_{f_n(J_n-1)=a^n} \left( \sum_{a^n} \sum_{f_n(1)=a^n} \prod_{j=1}^{J_n} E^n(f_n(j) \mid j) \right) \right) \dots \right) \right) \\ &= \sum_{a^n} \sum_{f_n(1)=a^n} \left( \sum_{a^n} \sum_{f_n(2)=a^n} \left( \dots \left( \sum_{a^n} \sum_{f_n(J_n-1)=a^n} \left( \sum_{a^n} \sum_{f_n(J_n-1)=a^n} \left( \sum_{a^n} \sum_{f_n(2)=a^n} \left( \dots \left( \sum_{a^n} \sum_{f_n(J_n-1)=a^n} \prod_{j=1}^{J_n-1} E^n(f_n(j) \mid j) \right) \right) \dots \right) \right) \\ &= \sum_{a^n} \sum_{f_n(1)=a^n} \left( \sum_{a^n} \sum_{f_n(2)=a^n} \left( \dots \left( \sum_{a^n} \sum_{f_n(J_n-1)=a^n} \prod_{j=1}^{J_n-1} E^n(f_n(j) \mid j) \right) \dots \right) \right) \\ &= \sum_{a^n} \sum_{f_n(1)=a^n} \left( \sum_{a^n} \sum_{f_n(2)=a^n} \left( \dots \left( \sum_{a^n} E^n(a^n \mid J_n) \prod_{j=1}^{J_n-1} E^n(f_n(j) \mid j) \right) \dots \right) \right) \end{split}$$

= ...  
= 
$$\sum_{a^n} \sum_{f_n(1)=a^n} E^n(f_n(1) \mid 1)$$
  
=  $\sum_{a^n} E^n(a^n \mid 1)$   
= 1,

the second equation holds because for any  $j \in \{1, \ldots, J_n\}$ , we have

$$\sum_{f_n \in \mathbf{F}_n} \prod_{k=1}^{J_n} E^n(f_n(k) \mid k) \operatorname{tr} \left( W_{t^n}(f_n(j)) D_j^n \right)$$
  
= 
$$\sum_{a^n} \sum_{f_n(j)=a^n} E^n(a^n \mid j) \left( \prod_{k \neq j} E^n(f_n(k) \mid k) \right) \operatorname{tr} \left( W_{t^n}(f_n(j)) D_j^n \right)$$
  
= 
$$\sum_{a^n} \sum_{f_n(j)=a^n} E^n(a^n \mid j) \operatorname{tr} \left( W_{t^n}(f_n(j)) D_j^n \right)$$
  
= 
$$\sum_{a^n} E^n(a^n \mid j) \operatorname{tr} \left( W_{t^n}(a^n) D_j^n \right).$$

By (22), for any  $n \in \mathbb{N}$ ,  $J_n \in \mathbb{N} \setminus \{1\}$ , let  $\mathcal{C}$  be any  $(n, J_n)$  deterministic code with a random encoder, if  $\{W_t : t \in \theta\}$  is symmetrizable, we have

$$\max_{t\in\theta} P_e(\mathcal{C}, t^n) > c \; .$$

Thus the only achievable deterministic secrecy capacity of  $\{(W_t, V_t) : t \in \theta\}$  is  $\log 1 = 0$ . Therefore  $C_s(\{(W_t, V_t) : t \in \theta\}) = 0$ . (Actually, (22) shows that if  $\{W_t : t \in \theta\}$  is symmetrizable, even the deterministic capacity for message transmission of  $\{(W_t, V_t) : t \in \theta\}$  with random encoding technique is equal to zero. Since the deterministic secrecy capacity  $C_s(\{(W_t, V_t) : t \in \theta\})$  cannot exceed the deterministic capacity for message transmission, we have  $C_s(\{(W_t, V_t) : t \in \theta\}) = 0$ .) This completes the proof of Theorem 3.1. 1b.

As we learn from Example 5.1, there are indeed arbitrarily varying classicalquantum wiretap channels which have zero deterministic secrecy capacity and positive random secrecy capacity. Therefore, as Theorem 3.1. 1 shows, randomness is indeed a very helpful resource for the secure message transmission through an arbitrarily varying classical-quantum wiretap channel. But the problem is: how should the sender and the receiver know which code is used in the particular transmission?

Theorem 3.1. 2 shows that common randomness capacity is always equal to the random secrecy capacity, even for the arbitrarily varying classical-quantum wiretap channels of Example 5.1. Therefore, common randomness is an equally helpful resource for the secure message transmission through an arbitrarily varying classical-quantum wiretap channel. However, as [18] showed, common randomness is a very "costly" resource. As Theorem 3.1 shows, for the transmission of common randomness we have to require that the deterministic capacity for message transmission of the sender's and legal receiver's channel is positive. In the following Section 4, we will see that the much "cheaper" resource, the m - a - (X, Y) correlation, is also an equally helpful resource for the message transmission through an arbitrarily varying classical-quantum channel. The advantage here is that we do not have to require that the deterministic capacity for message transmission of the sender's and legal receiver's channel is positive.

# 4 Arbitrarily Varying Classical-Quantum Wiretap Channel with Correlation Assistance

In this section we consider the m - a - (X, Y) correlation-assisted secrecy capacity of an arbitrarily varying classical-quantum wiretap channel.

Theorem 3.1. 2 shows that common randomness is a helpful resource for the secure message transmission through an arbitrarily varying classical-quantum wiretap channel. The m - a - (X, Y) correlation is a weaker resource than common randomness (cf. [18]). We can simulate any m - a - (X, Y) correlation by common randomness asymptotically, but there exists a class of sequences of bipartite distributions which cannot model common randomness (cf. Lemma 1 of [18]). However, the results of [18] show that the "cheaper" m - a - (X, Y) correlation is nevertheless a helpful resource for message transmission through an arbitrarily varying classical-quantum channel. Our following Theorem 4.1 shows that also in case of secure message transmission through an arbitrarily varying classical-quantum wiretap channel, the m - a - (X, Y) correlation assistance is an equally helpful resource as common randomness.

**Theorem 4.1.** Let  $\{(W_t, V_t) : t \in \theta\}$  be an arbitrarily varying classical-quantum wiretap channel. Let  $\mathbf{X}$  and  $\mathbf{Y}$  be finite sets. If I(X, Y) > 0 holds for a random variable (X, Y) which is distributed according to a joint probability distribution  $p \in P(\mathbf{X} \times \mathbf{Y})$ , then the randomness-assisted secrecy capacity is equal to the m - a - (X, Y) correlation-assisted secrecy capacity.

*Proof.* Our proof is similar to the capacity results of arbitrarily varying channels with correlation assistance in [8] and [18].

### 4.1 When the randomness-assisted code has positive secrecy capacity

If the randomness-assisted secrecy capacity of  $(W_t, V_t)_{t \in \theta}$  is positive, we can build a new arbitrarily varying classical-quantum channel  $\{\tilde{U}_t : t \in \theta\}$  to create common randomness for the sender and the legal receiver. We show that this channel does not have to be secure to be useful for a secure code for the original arbitrarily varying classical-quantum wiretap channel. Then, similar to our proof of Theorem 3.1. 1, the sender and the legal receiver can build two-part code word, which consists of a non-secure code word for  $\{\tilde{U}_t : t \in \theta\}$  to pass the index and a common randomness-assisted secure code to transmit the message .

At first we assume that the m - a - (X, Y) secrecy capacity of  $\{(W_t, V_t) : t \in \theta\}$  is positive, then the m - a - (X, Y) capacity of the arbitrarily varying classical-quantum channel  $\{W_t : t \in \theta\}$  is positive. For the definition of the capacity of an arbitrarily varying classical-quantum channel please see [18].

By Theorem 3.1. 2, the randomness-assisted secrecy capacity is equal to the common randomness-assisted secrecy capacity. Let  $\delta > 0$ ,  $\zeta > 0$ , and  $\epsilon > 0$ , and  $\left\{ C^{\gamma} = \left( E_{\gamma}^{n}, \{D_{\gamma,j}^{n} : j \in \{1, \ldots, J_{n}\}\} \right) : \gamma \in \Gamma \right\}$  be an  $(n, J_{n})$  common randomness-assisted quantum code such that  $\frac{\log J_{n}}{n} > C_{s}((W_{t}, V_{t})_{t \in \theta}, r) - \delta$ , and

$$\max_{t^n \in \theta^n} \frac{1}{|\Gamma|} \sum_{\gamma=1}^{|\Gamma|} P_e(\mathcal{C}^{\gamma}, t^n) < \epsilon ,$$
$$\max_{t^n \in \theta^n} \frac{1}{|\Gamma|} \sum_{\gamma=1}^{|\Gamma|} \chi(R_{\mathrm{uni}}, Z_{\mathcal{C}^{\gamma}, t^n}) < \zeta .$$

We denote  $\mathfrak{F} := \{f : f \text{ is a function } \mathbf{X} \to \mathbf{A}\}$ . Let  $H_{\mathbf{Y}}$  be a Hilbert space of dimension  $|\mathbf{Y}|$  and  $\{\check{\kappa}_y : y \in \mathbf{Y}\}$  be a set of pairwise orthogonal and pure states on  $H_{\mathbf{Y}}$ . For every  $t \in \theta$ ,

$$\tilde{U}_t(f) := \sum_{\mathbf{x}} \sum_{\mathbf{y}} p(\mathbf{x}, \mathbf{y}) \breve{\kappa}_y \otimes W_t(f(\mathbf{x}))$$
(23)

defines a classical-quantum channel

$$U_t: \mathfrak{F} \to \mathcal{S}(H \otimes H_\mathbf{Y})$$
.

 $\{\tilde{U}_t : t \in \theta\}$  defines an arbitrarily varying classical-quantum channel  $\mathfrak{F} \to \mathcal{S}(H) \otimes H_{\mathbf{Y}}$ .

In [18] (see also [8] for a classical version), it was shown that if I(X, Y) is positive, the deterministic capacity of  $(\tilde{U}_t)_{t\in\theta}$  is equal to the m - a - (X, Y)capacity of  $\{W_t : t \in \theta\}$ . By Remark 2.7, we may assume that the deterministic capacity of  $(\tilde{U}_t)_{t\in\theta}$  using deterministic encoder is positive. This means that the sender and the receiver can build a code  $\left(\left(f_{\gamma}^{\nu(n)}\right)_{\gamma=1,\ldots,|\Gamma|}, \{D_{\gamma}^{\nu(n)} : \gamma =$  $1,\ldots,|\Gamma|\}\right)$  with deterministic encoder for  $(\tilde{U}_t)_{t\in\theta}$  of length  $\nu(n)$ , where  $2^{\nu(n)}$ is in polynomial order of n and  $f_{\gamma}^{\nu(n)}(\mathbf{x}^{\nu(n)}) = \left(f_{\gamma,1}(\mathbf{x}_1),\ldots,f_{\gamma,\nu(n)}(\mathbf{x}_{\nu(n)})\right)$ for  $\mathbf{x}^{\nu(n)} = (\mathbf{x}_1,\ldots,\mathbf{x}_{\nu(n)})$ , such that the following statement is valid. For any positive  $\vartheta$ , if n is large enough, we have

$$1 - \vartheta$$

$$\leq \min_{t^{\nu(n)} \in \theta^{\nu(n)}} \frac{1}{|\Gamma|} \sum_{\gamma=1}^{|\Gamma|} \operatorname{tr} \left( \tilde{U}_{t^{\nu(n)}} \left( f_{\gamma}^{\nu(n)} \right) D_{\gamma}^{\nu(n)} \right)$$

$$= \min_{t^{\nu(n)} \in \theta^{\nu(n)}} \frac{1}{|\Gamma|} \operatorname{tr} \left( \sum_{\gamma=1}^{|\Gamma|} \sum_{\mathbf{x}^{\nu(n)} \in \mathbf{X}^{\nu(n)}} \sum_{\mathbf{y}^{\nu(n)} \in \mathbf{Y}^{\nu(n)}} p \left( \mathbf{x}^{\nu(n)}, \mathbf{y}^{\nu(n)} \right) \right)$$

$$\cdot \left[ \breve{\kappa}_{y^{n}} \otimes W_{t^{\nu(n)}} \left( f_{\gamma}^{\nu(n)} (\mathbf{x}^{\nu(n)}) \right) \right] D_{\gamma}^{\nu(n)} \right)$$

$$= \min_{t^{\nu(n)} \in \theta^{\nu(n)}} \frac{1}{|\Gamma|} \sum_{\gamma=1}^{|\Gamma|} \sum_{\mathbf{x}^{\nu(n)} \in \mathbf{X}^{\nu(n)}} \sum_{\mathbf{y}^{\nu(n)} \in \mathbf{Y}^{\nu(n)}} p \left( \mathbf{x}^{\nu(n)}, \mathbf{y}^{\nu(n)} \right)$$

$$\cdot \operatorname{tr}\left(W_{t^{\nu(n)}}(c_{\mathbf{x}^{\nu(n)},\gamma}^{\nu(n)})D_{(\mathbf{y}^{\nu(n)}),\gamma}^{\nu(n)}\right),\qquad(24)$$

where for every  $\gamma \in \Gamma$ ,  $\mathbf{x}^{\nu(n)} = (\mathbf{x}_1, \dots, \mathbf{x}_{\nu(n)}) \in \mathbf{X}^{\nu(n)}$ , and  $\mathbf{y}^{\nu(n)} = (\mathbf{y}_1, \dots, \mathbf{y}_{\nu(n)}) \in \mathbf{Y}^{\nu(n)}$ , we set  $p(\mathbf{x}^{\nu(n)}, \mathbf{y}^{\nu(n)}) = \prod_{i,j} p(\mathbf{x}_i, \mathbf{y}_j)$ ,

$$c^{\nu(n)}_{{\bf x}^{\nu(n)},\gamma}:=f^{\nu(n)}_{\gamma}({\bf x}^{\nu(n)})\in {\bf A}^{\nu(n)}\ ,$$

and

$$D_{(\mathbf{y}^{\nu(n)}),\gamma}^{\nu(n)} := \operatorname{tr}_{H_{\mathbf{Y}^{\nu(n)}}} \left( (\breve{\kappa}_{y^{\nu(n)}} \otimes \operatorname{id}_{H^{\otimes \nu(n)}}) D_{\gamma}^{\nu(n)} \right)$$

The last equation of (24) holds because

$$\begin{split} &\operatorname{tr}\left(W_{t^{\nu(n)}}(c_{\mathbf{x}^{\nu(n)},\gamma}^{\nu(n)})\operatorname{tr}_{H_{\mathbf{Y}^{\nu(n)}}}\left((\breve{\kappa}_{y^{\nu(n)}}\otimes\operatorname{id}_{H^{\otimes\nu(n)}})D_{\gamma}^{\nu(n)}\right)\right) \\ &=\operatorname{tr}\left(\left[\operatorname{id}_{H_{\mathbf{Y}^{\nu(n)}}}\otimes W_{t^{\nu(n)}}(c_{\mathbf{x}^{\nu(n)},\gamma}^{\nu(n)})\right]\left[\breve{\kappa}_{y^{\nu(n)}}\otimes\operatorname{id}_{H^{\otimes\nu(n)}}\right]D_{\gamma}^{\nu(n)}\right) \\ &=\operatorname{tr}\left(\left[\breve{\kappa}_{y^{\nu(n)}}\otimes W_{t^{\nu(n)}}(c_{\mathbf{x}^{\nu(n)},\gamma}^{\nu(n)})\right]D_{\gamma}^{\nu(n)}\right). \end{split}$$

$$\begin{split} & \text{Since } \sum_{\gamma=1}^{|\Gamma|} D_{(\mathbf{y}^{\nu(n)}),\gamma}^{\nu(n)} = \sum_{\gamma=1}^{|\Gamma|} \operatorname{tr}_{H_{\mathbf{Y}^{\nu}(n)}} \left( (\breve{\kappa}_{y^{\nu(n)}} \otimes \operatorname{id}_{H^{\otimes \nu(n)}}) D_{\gamma}^{\nu(n)} \right) = \operatorname{tr}_{H_{\mathbf{Y}^{\nu}(n)}} \left( (\breve{\kappa}_{y^{\nu(n)}} \otimes \operatorname{id}_{H^{\otimes \nu(n)}}) \sum_{\gamma=1}^{|\Gamma|} D_{\gamma}^{\nu(n)} \right) = \operatorname{id}_{H^{\otimes \nu(n)}}, \text{ we can define an } (X,Y) \text{-correlation-assisted} \\ & (\nu(n), |\Gamma|) \text{ code (this is a code with deterministic encoder) by } \left( \left( c_{\mathbf{x}^{\nu(n)},\gamma}^{\nu(n)} \right)_{\gamma \in \{1,\ldots,|\Gamma|\}}, \\ & \left\{ D_{(\mathbf{y}^{\nu(n)}),\gamma}^{\nu(n)} : \gamma \in \{1,\ldots,|\Gamma|\} \right\} \right). \end{split}$$

Now we can construct an (X, Y)-correlation-assisted  $(\nu(n) + n, J_n)$  code  $\mathcal{C}(X, Y) = \left\{ \left( E_{\mathbf{x}^{\nu(n)+n}}, \{D_j^{\mathbf{y}^{\nu(n)+n}} : j \in \{1, \dots, J_n\}\} \right) : \mathbf{x}^{\nu(n)+n} \in \mathbf{X}^{\nu(n)+n}, \mathbf{y}^{\nu(n)+n} \in \mathbf{Y}^{\nu(n)+n} \right\}, \text{ where for } \mathbf{x}^{\nu(n)+n} = (\mathbf{x}^{\nu(n)}, \mathbf{x}^n), \mathbf{y}^{\nu(n)+n} = (\mathbf{y}^{\nu(n)}, \mathbf{y}^n) \text{ and } a^{\nu(n)+n} = (a^{\nu(n)}, a^n) \in \mathbf{A}^{\nu(n)+n}$ 

$$E_{\mathbf{x}^{\nu(n)+n}}(a^{\nu(n)+n}|j) = \begin{cases} \frac{1}{|\Gamma|} E_{\gamma}(a^{n}|j) \text{ if } a^{\nu(n)} = c_{\mathbf{x}^{\nu(n)},\gamma}^{\nu(n)} \\ 0 \text{ else} \end{cases}$$

and

$$D_j^{\mathbf{y}^{\nu(n)+n}} := \sum_{\gamma=1}^{|\Gamma|} D_{(\mathbf{y}^{\nu(n)}),\gamma}^{\nu(n)} \otimes D_{\gamma,j}^n .$$

For any  $\gamma \in \{1, \ldots, |\Gamma|\}$  let

$$\begin{aligned} \boldsymbol{\mathfrak{Z}}_{\gamma,t^{\nu(n)+n},\mathbf{x}^{\nu(n)+n}} &:= \left\{ V_{t^{\nu(n)}}\left(c_{\mathbf{x}^{\nu(n)},\gamma}^{\nu(n)}\right) \otimes V_{t^n}\left(E_{\gamma}(a^n|1)\right),\ldots,\right.\\ &\left.V_{t^{\nu(n)}}\left(c_{\mathbf{x}^{\nu(n)},\gamma}^{\nu(n)}\right) \otimes V_{t^n}\left(E_{\gamma}(a^n|J_n)\right)\right\} \,. \end{aligned}$$

Similar to (16), for any  $\mathbf{x}^{\nu(n)+n} \in \mathbf{X}^{\nu(n)+n}$ ,  $\gamma \in \Gamma$ , and  $t^{\nu(n)+n} = (t^{\nu(n)}t^n)$  we have

$$\chi\left(R_{\mathrm{uni}},\mathfrak{Z}_{\gamma,t^{\nu(n)+n},\mathbf{x}^{\nu(n)+n}}\right) = \chi\left(R_{\mathrm{uni}},Z_{\mathcal{C}^{\gamma},t^{n}}\right) \ . \tag{25}$$

By definition we have

$$Z_{t^{\nu(n)+n},\mathbf{x}^{\nu(n)+n}} := \left\{ \frac{1}{|\Gamma|} \sum_{\gamma=1}^{|\Gamma|} V_{t^{\nu(n)}}(c_{\mathbf{x}^{\nu(n)},\gamma}^{\nu(n)}) \otimes V_{t^{n}}(E_{\gamma}(|1)), \dots, \\ \frac{1}{|\Gamma|} \sum_{i=1}^{|\Gamma|} V_{t^{\nu(n)}}(c_{\mathbf{x}^{\nu(n)},\gamma}^{\nu(n)}) \otimes V_{t^{n}}(E_{\gamma}(|J_{n})) \right\}.$$

Similar to (17) let  $\lambda := \min\{\epsilon, \zeta\}$ , for any  $t^{\nu(n)+n} = (t^{\nu(n)}, t^n)$ ,  $\mathbf{x}^{\nu(n)+n} = (\mathbf{x}^{\nu(n)}, \mathbf{x}^n)$  and  $\mathbf{y}^{\nu(n)+n} = (\mathbf{y}^{\nu(n)}, \mathbf{y}^n)$  we have

$$\begin{split} &\sum_{\mathbf{x}^{\nu(n)+n} \in \mathbf{X}^{\nu(n)+n}} \sum_{\mathbf{y}^{\nu(n)+n} \mathbf{y}^{\nu(n)+n} \in \mathbf{Y}^{\nu(n)+n}} p\left(\mathbf{x}^{\nu(n)+n}, \mathbf{y}^{\nu(n)+n}\right) \chi\left(R_{\mathrm{uni}}, Z_{t^{\nu(n)+n}, \mathbf{x}^{\nu(n)+n}}\right) \\ &\leq \sum_{\mathbf{x}^{\nu(n)+n}} \sum_{\mathbf{y}^{\nu(n)+n}} p\left(\mathbf{x}^{\nu(n)+n}, \mathbf{y}^{\nu(n)+n}\right) \chi\left(R_{\mathrm{uni}}, Z_{t^{\nu(n)+n}, \mathbf{x}^{\nu(n)+n}}\right) \\ &= \sum_{\mathbf{x}^{\nu(n)+n}} \sum_{\mathbf{y}^{\nu(n)+n}} p\left(\mathbf{x}^{\nu(n)+n}, \mathbf{y}^{\nu(n)+n}\right) \chi\left(R_{\mathrm{uni}}, Z_{t^{\nu(n)+n}, \mathbf{x}^{\nu(n)+n}}\right) \\ &= \sum_{\mathbf{x}^{\nu(n)+n}} \sum_{\mathbf{y}^{\nu(n)+n}} p\left(\mathbf{x}^{\nu(n)+n}, \mathbf{y}^{\nu(n)+n}\right) \chi\left(R_{\mathrm{uni}}, Z_{t^{\nu(n)+n}, \mathbf{x}^{\nu(n)+n}}\right) \\ &= \sum_{\mathbf{x}^{\nu(n)+n}} \sum_{\mathbf{y}^{\nu(n)+n}} p\left(\mathbf{x}^{\nu(n)+n}, \mathbf{y}^{\nu(n)+n}\right) \frac{1}{|\Gamma|} \sum_{\gamma=1}^{|\Gamma|} \chi\left(R_{\mathrm{uni}}, \mathbf{3}_{\mathcal{C}^{\gamma}, t^{\nu(n)+n}, \mathbf{x}^{\nu(n)+n}\right) + \lambda \\ &= \sum_{\mathbf{x}^{\nu(n)+n}} \sum_{\mathbf{y}^{\nu(n)+n}} p\left(\mathbf{x}^{\nu(n)+n}, \mathbf{y}^{\nu(n)+n}\right) \left[S\left(\frac{1}{J_n} \frac{1}{|\Gamma|} \sum_{j=1}^{J_n} \sum_{i=1}^{|\Gamma|} V_{t^{\nu(n)}}\left(c_{\mathbf{x}^{\nu(n)}, \gamma}^{\nu(n)}\right) \otimes V_{t^n}\left(E_{\gamma}(|j|)\right)\right) \\ &- \frac{1}{J_n} \sum_{j=1}^{J_n} S\left(\frac{1}{|\Gamma|} \sum_{i=1}^{|\Gamma|} V_{t^{\nu(n)}}\left(c_{\mathbf{x}^{\nu(n)}, \gamma}^{\nu(n)}\right) \otimes V_{t^n}\left(E_{\gamma}(|j|)\right)\right) \\ &- \frac{1}{|\Gamma|} \sum_{i=1}^{|\Gamma|} S\left(\frac{1}{J_n} \sum_{j=1}^{J_n} V_{t^{\nu(n)}}\left(c_{\mathbf{x}^{\nu(n)}, \gamma}^{\nu(n)}\right) \otimes V_{t^n}\left(E_{\gamma}(|j|)\right)\right) \\ &+ \frac{1}{J_n} \frac{1}{|\Gamma|} \sum_{i=1}^{J_n} \sum_{i=1}^{|\Gamma|} S\left(V_{t^{\nu(n)}}\left(c_{\mathbf{x}^{\nu(n)}, \gamma}^{\nu(n)}\right) \otimes V_{t^n}\left(E_{\gamma}(|j|)\right)\right) \\ &+ \lambda \leq \lambda . \end{split}$$

By (24), for any  $t^{\nu(n)+n} \in \theta^{\nu(n)+n}$ ,

$$\sum_{\mathbf{x}^{\nu(n)+n}} \sum_{\mathbf{y}^{\nu(n)+n}} p(\mathbf{x}^{\nu(n)+n}, \mathbf{y}^{\nu(n)+n}) P_e\left(\mathcal{C}(\mathbf{x}^{\nu(n)+n}, \mathbf{y}^{\nu(n)+n}), t^{\nu(n)+n}\right)$$

$$=1-\sum_{\mathbf{x}^{\nu(n)+n}}\sum_{\mathbf{y}^{\nu(n)+n}}p\left(\mathbf{x}^{\nu(n)+n},\mathbf{y}^{\nu(n)+n}\right)\frac{1}{J_{n}}\sum_{j=1}^{J_{n}}\operatorname{tr}\left(\left[\frac{1}{|\Gamma|}\sum_{\gamma=1}^{|\Gamma|}\sum_{\gamma=1}^{|\Gamma|}\right]\right)$$

$$\leq 1-\sum_{\mathbf{x}^{\nu(n)+n}}\sum_{\mathbf{y}^{\nu(n)+n}}p(\mathbf{x}^{\nu(n)+n},\mathbf{y}^{\nu(n)+n})\frac{1}{J_{n}}\sum_{j=1}^{J_{n}}\operatorname{tr}\left(\frac{1}{|\Gamma|}\sum_{\gamma=1}^{|\Gamma|}\sum_{\gamma=1}^{|\Gamma|}\sum_{\gamma=1}^{|\Gamma|}\left[V_{t^{\nu(n)}}(c_{\mathbf{x}^{\nu(n)},\gamma}^{\nu(n)})\otimes V_{t^{n}}(E_{\gamma}(|j))\right]\cdot\left[D_{(\mathbf{y}^{\nu(n)}),\gamma}^{\nu(n)}\otimes D_{\gamma,j}^{n}\right]\right)$$

$$=1-\sum_{\mathbf{x}^{\nu(n)}}\sum_{\mathbf{y}^{\nu(n)}}p(\mathbf{x}^{\nu(n)},\mathbf{y}^{\nu(n)})\frac{1}{J_{n}}\sum_{j=1}^{J_{n}}\operatorname{tr}\left(\frac{1}{|\Gamma|}\sum_{\gamma=1}^{|\Gamma|}\left[V_{t^{\nu(n)}}(c_{\mathbf{x}^{\nu(n)},\gamma}^{\nu(n)})D_{(\mathbf{y}^{\nu(n)}),\gamma}^{\nu(n)}\right]\otimes\left[V_{t^{n}}(E_{\gamma}(|j))D_{\gamma,j}^{n}\right]\right)$$

$$=1-\sum_{\mathbf{x}^{\nu(n)}}\sum_{\mathbf{y}^{\nu(n)}}p(\mathbf{x}^{\nu(n)},\mathbf{y}^{\nu(n)})\frac{1}{|\Gamma|}\sum_{\gamma=1}^{|\Gamma|}\operatorname{tr}\left(V_{t^{\nu(n)}}(c_{\mathbf{x}^{\nu(n)},\gamma}^{\nu(n)})D_{(\mathbf{y}^{\nu(n)}),\gamma}^{\nu(n)}\right)$$

$$\cdot\left(\frac{1}{J_{n}}\sum_{j=1}^{J_{n}}\operatorname{tr}(V_{t^{n}}(E_{\gamma}(|j))D_{\gamma,j}^{n})\right)$$

$$\leq\lambda+\vartheta.$$
(27)

We now combine (27) and (26) and obtain the following result.

If I(X,Y) and the m - a - (X,Y) secrecy capacity of  $\{(W_t, V_t) : t \in \theta\}$ are positive, we define  $\lambda := \min\{\epsilon, \zeta\} + \vartheta$  and the following statement is valid. For any  $n \in \mathbb{N}$  and positive  $\lambda$ , if there is an  $(n, J_n)$  randomness-assisted code  $(\{\mathcal{C}^{\gamma} : \gamma \in \Lambda\}, G)$  for  $\{(W_t, V_t) : t \in \theta\}$  such that

$$\max_{t^n\in\theta^n}\int_{\Lambda}P_e(\mathcal{C}^\gamma,t^n)dG(\gamma)<\epsilon\ ,$$

and

$$\max_{t^n \in \theta^n} \int_{\Lambda} \chi\left(R_{\mathrm{uni}}, Z_{\mathcal{C}^{\gamma}, t^n}\right) dG(\gamma) < \zeta \ ,$$

then there is also a  $(\nu(n) + n, J_n)$  common randomness-assisted code  $\mathcal{C}(X, Y) = \left\{ \left( E_{\mathbf{x}^{\nu(n)+n}}, D_j^{\mathbf{y}^{\nu(n)+n}} \right) : j \in \{1, \dots, J_n\}, \mathbf{x}^{\nu(n)+n} \in \mathbf{X}^{\nu(n)+n} \mathbf{y}^{\nu(n)+n} \in \mathbf{Y}^{\nu(n)+n} \right\}$  such that

$$\max_{t^{\nu(n)+n} \in \theta^{\nu(n)+n}} \sum_{\mathbf{x}^{\nu(n)+n} \in \mathbf{X}^{\nu(n)+n} \mathbf{y}^{\nu(n)+n} \in \mathbf{Y}^{\nu(n)+n}} p(\mathbf{x}^{\nu(n)+n}, \mathbf{y}^{\nu(n)+n}) 
P_e(C(\mathbf{x}^{\nu(n)+n}, \mathbf{y}^{\nu(n)+n}), t^{\nu(n)+n}) < \lambda ,$$
(28)

and

$$\max_{t^{\nu(n)+n}\in\theta^{\nu(n)+n}}\chi\left(R_{\mathrm{uni}};Z_{t^{\nu(n)+n},\mathbf{x}^{\nu(n)+n}}\mid X\right)<\lambda.$$
(29)

(28) and (29) mean that

$$C_s(\{(W_t, V_t) : t \in \theta\}; corr(X, Y)) \ge C_s(\{(W_t, V_t) : t \in \theta\}; r) - \frac{1}{n} \cdot \log J_n + \frac{1}{\nu(n) + n} \log J_n.$$

We know that  $2^{\nu(n)}$  is in polynomial order of n. For any positive  $\varepsilon$ , if n is large enough we have  $\frac{1}{n} \log J_n - \frac{1}{\log n+n} \log J_n \leq \varepsilon$ . Therefore, if I(X,Y) and  $C_s(\{(W_t, V_t) : t \in \theta\}; corr(X, Y))$  are both positive, we have

$$C_s(\{(W_t, V_t) : t \in \theta\}; corr(X, Y)) \ge C_s(\{(W_t, V_t) : t \in \theta\}; r) - \varepsilon.$$
(30)

This and the fact that

$$C_s(\{(W_t, V_t) : t \in \theta\}; corr(X, Y)) \le C_s(\{(W_t, V_t) : t \in \theta\}; r)$$

prove Theorem 4.1 for the case that  $C_s((W_t, V_t)_{t \in \theta}; corr(X, Y))$  is positive.

# 4.2 When the randomness-assisted code has zero secrecy capacity

If the randomness-assisted secrecy capacity of  $(W_t, V_t)_{t \in \theta}$  is equal to zero, with a similar technique as the techniques in [8] and [18] we show that the (X, Y)correlation-assisted secrecy capacity of  $(W_t, V_t)_{t \in \theta}$  is also equal to zero.

Now we assume that the m-a-(X, Y) secrecy capacity of  $\{(W_t, V_t) : t \in \theta\}$  is equal to zero. If  $C_s(\{(W_t, V_t) : t \in \theta\}; r)$  is also equal to zero, then there is nothing to prove. Thus let us assume that  $C_s(\{(W_t, V_t) : t \in \theta\}; r)$  is positive.

Assume that there is an  $(n, J_n)$  randomness-assisted code  $(\{\mathcal{C}^{\gamma} : \gamma \in \Lambda\}, G)$  for  $\{(W_t, V_t) : t \in \theta\}$  such that

$$\begin{split} \max_{t^n \in \theta^n} \int_{\Lambda} P_e(\mathcal{C}^{\gamma}, t^n) dG(\gamma) < \lambda \ , \\ \max_{n \in \theta^n} \int_{\Lambda} \chi \left( R_{\mathrm{uni}}, Z_{\mathcal{C}^{\gamma}, t^n} \right) dG(\gamma) < \lambda \end{split}$$

We denote  $\mathfrak{F}$  and the arbitrarily varying classical-quantum channel  $(\tilde{U}_t)_{t\in\theta}: \mathfrak{F} \to \mathcal{S}(H^{n|\mathbf{Y}|})$  as above. If the deterministic capacity of  $(\tilde{U}_t)_{t\in\theta}$  is positive, we can build, as above, a  $(\nu(n) + n, J_n)$  common randomness-assisted code  $\mathcal{C}(X, Y) = \left\{ \left( E_{\mathbf{x}^{\nu(n)+n}}, \{D_j^{\mathbf{y}^{\nu(n)+n}}: j \in \{1, \dots, J_n\}\} \right) : \mathbf{x}^{\nu(n)+n} \in \mathbf{X}^{\nu(n)+n}, \mathbf{y}^{\nu(n)+n} \in \mathbf{Y}^{\nu(n)+n} \right\}$  such that

$$\max_{t^{\nu(n)+n} \in \theta^{\nu(n)+n}} \sum_{\mathbf{x}^{\nu(n)+n} \in \mathbf{X}^{\nu(n)+n}, \mathbf{y}^{\nu(n)+n} \in \mathbf{Y}^{\nu(n)+n}} p(\mathbf{x}^{\nu(n)+n}, \mathbf{y}^{\nu(n)+n})$$

$$P_e(C(\mathbf{x}^{\nu(n)+n}, \mathbf{y}^{\nu(n)+n}), t^{\nu(n)+n}) < \epsilon ,$$

$$\max_{t^{\nu(n)+n} \in \theta^{\nu(n)+n}} \sum_{\mathbf{y}^{\nu(n)+n} \in \mathbf{Y}^{\nu(n)+n}} p\left(\mathbf{x}^{\nu(n)+n}, \mathbf{y}^{\nu(n)+n}\right) \chi\left(R_{\text{uni}}; Z_{t^{\nu(n)+n}, \mathbf{x}^{\nu(n)+n}}\right) < \zeta \ .$$

But this would mean

$$C_{s}(\{(W_{t}, V_{t}) : t \in \theta\}; corr(X, Y)) = C_{s}(\{(W_{t}, V_{t}) : t \in \theta\}; r) ,$$

and there is nothing to prove.

Thus we may assume that the deterministic capacity of  $(\tilde{U}_t)_{t\in\theta}$  is equal to zero. This implies that  $(\tilde{U}_t)_{t\in\theta}$  is symmetrizable (cf. [7]), i.e., there is a parametrized set of distributions  $\{\tau(\cdot \mid f) : f \in \mathfrak{F}\}$  on  $\theta$  such that for all f,  $f' \in \mathfrak{F}$  we have

$$\sum_{t\in\theta} \tau(t\mid f') \sum_{\mathbf{x}} \sum_{\mathbf{y}} P\left(\mathbf{X} \times \mathbf{Y}\right) \breve{\kappa}_{y} \otimes W_{t}\left(f(\mathbf{x})\right) = \sum_{t\in\theta} \tau(t\mid f) \sum_{\mathbf{x}} \sum_{\mathbf{y}} P\left(\mathbf{X} \times \mathbf{Y}\right) \breve{\kappa}_{y} \otimes W_{t}\left(f'(\mathbf{x})\right)$$
$$\Rightarrow \sum_{t\in\theta} \tau(t\mid f') \sum_{\mathbf{x}} P\left(\mathbf{X} \times \mathbf{Y}\right) W_{t}\left(f(\mathbf{x})\right) = \sum_{t\in\theta} \tau(t\mid f) \sum_{\mathbf{x}} P\left(\mathbf{X} \times \mathbf{Y}\right) W_{t}\left(f'(\mathbf{x})\right)$$
(31)

for all  $\mathbf{y} \in \mathbf{Y}$ .

Our approach is similar to the technique of [8]. Let  $\mathbf{A} = \{0, 1, \dots, |\mathbf{A}| - 1\}$ ,  $\mathbf{X} = \mathbf{Y} = \{0, 1\}$ . We define functions  $g^*$  and  $g_i \in \mathfrak{F}$  for  $i = 1, \dots, a - 1$  such that  $g^*(0) = g^*(1) = 0$  and  $g_i(u) := i + u \mod |\mathbf{A}|$  for  $u \in \{0, 1\}$ . Since  $(\tilde{U}_t)_{t\in\theta}$  is symmetrizable, by (31) there is a parametrized set of distributions  $\{\tau(t \mid f) : f \in \mathfrak{F}\}$  on  $\theta$  such that for all  $a \in \mathbf{A}$ , the following two equalities are valid

$$\begin{split} &\sum_{t \in \theta} p(0,0)\tau(t \mid g^*)W_t(a) \\ &+ \sum_{t \in \theta} p(1,0)\tau(t \mid g^*)W_t(a+1 \bmod |\mathbf{A}|) \\ &= \sum_{t \in \theta} p(0,0)\tau(t \mid g_i)W_t(a) \\ &+ \sum_{t \in \theta} p(1,0)\tau(t \mid g_i)W_t(a) \\ &= \sum_{t \in \theta} \tau(t \mid g_i)W_t(a) ; \\ &\sum_{t \in \theta} p(0,1)\tau(t \mid g^*)W_t(a) \\ &+ \sum_{t \in \theta} p(1,1)\tau(t \mid g^*)W_t(a+1 \bmod |\mathbf{A}|) \\ &= \sum_{t \in \theta} p(0,1)\tau(t \mid g_i)W_t(a) \\ &+ \sum_{t \in \theta} p(1,1)\tau(t \mid g_i)W_t(a) \\ &= \sum_{t \in \theta} \tau(t \mid g_i)W_t(a) . \end{split}$$

If we choose an arbitrary orthonormal basis on  ${\cal H}$  to write the following quantum states in form of matrices

$$(m_{k,l})_{k,l=1,...,\dim H} = \sum_{t\in\theta} \tau(t \mid g^*) W_t(a) ,$$
  
$$(m'_{k,l})_{k,l=1,...,\dim H} = \sum_{t\in\theta} \tau(t \mid g^*) W_t(a+1 \mod |\mathbf{A}|) ,$$
  
$$(m_{k,l})_{k,l=1,...,\dim H} = \sum_{t\in\theta} \tau(t \mid g_i) W_t(a) ,$$

for all  $k, l \in \{1, \ldots, \dim H\}$  we have

$$p(0,0)m_{k,l} + p(1,0)m'_{k,l} = m_{k,l} ,$$
  
$$p(0,1)m_{k,l} + p(1,1)m'_{k,l} = m_{k,l} .$$

Since I(X, Y) is positive,  $p(0,0) \neq p(1,0)$  and  $p(0,1) \neq p(1,1)$ , therefore det  $\begin{pmatrix} p(0,0) & p(1,0) \\ p(0,1) & p(1,1) \end{pmatrix} \neq 0$ . Thus  $m_{k,l} = m'_{k,l} = m_{k,l}$  for all  $k, l \in \{1, \ldots, \dim H\}$ , this means

$$\sum_{t \in \theta} \tau(t \mid g^*) W_t(a) = \sum_{t \in \theta} \tau(t \mid g^*) W_t(a + 1 \mod |\mathbf{A}|)$$

for all  $a \in \mathbf{A}$ .

Therefore, for any  $n \in \mathbb{N}$  and any given  $(n, J_n)$  code  $\mathcal{C}^{\gamma} = \left(E^{\gamma}, \{D_j^{\gamma} : j = 1, \ldots, J_n\}\right)$ , the following statement is valid. Let  $a'^n$  be an arbitrary sequence in  $\mathbf{A}^n$ , we have

$$\begin{split} &\sum_{t^n \in \theta^n} \tau(t^n \mid g^*) P_e(\mathcal{C}^{\gamma}, t^n) \\ &= \sum_{t^n \in \theta^n} \tau(t^n \mid g^*) \left[ 1 - \frac{1}{J_n} \sum_{j=1}^{J_n} \operatorname{tr}(W_{t^n}(E_{\gamma}(\mid j)) D_j^{\gamma}) \right] \\ &= \sum_{t^n \in \theta^n} \tau(t^n \mid g^*) \left[ 1 - \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{a^n \in \mathbf{A}^n} E^{\gamma}(a^n \mid j) \operatorname{tr}(W_{t^n}(a^n) D_j^{\gamma}) \right] \\ &= 1 - \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{a^n \in \mathbf{A}^n} E^{\gamma}(a^n \mid j) \operatorname{tr}\left( \sum_{t^n \in \theta^n} \tau(t^n \mid g^*) W_{t^n}(a^n) D_j^{\gamma} \right) \\ &= 1 - \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{a^n \in \mathbf{A}^n} E^{\gamma}(a^n \mid j) \operatorname{tr}\left( \sum_{t^n \in \theta^n} \tau(t^n \mid g^*) W_{t^n}(a'^n) D_j^{\gamma} \right) \\ &= 1 - \frac{1}{J_n} \sum_{j=1}^{J_n} \operatorname{tr}\left( \sum_{t^n \in \theta^n} \tau(t^n \mid g^*) W_{t^n}(a'^n) D_j^{\gamma} \right) \\ &= 1 - \frac{1}{J_n} \sum_{t^n \in \theta^n} \tau(t^n \mid g^*) \operatorname{tr}\left( W_{t^n}(a'^n) \sum_{j=1}^{J_n} D_j^{\gamma} \right) \end{split}$$

$$= 1 - \frac{1}{J_n} \sum_{t^n \in \theta^n} \tau(t^n \mid g^*) \operatorname{tr} \left( W_{t^n}(a'^n) \right)$$
  
=  $1 - \frac{1}{J_n}$ , (32)

where  $\tau(t^n \mid g^*) := \tau(t_1 \mid g^*) \tau(t_2 \mid g^*) \dots \tau(t_n \mid g^*)$  for  $t^n = (t_1, t_2, \dots, t_n)$ . The second and the fifth equations hold because the trace function and matrices? multiplication are linear. The first, the fourth, and the last equations hold because  $\sum_{t^n \in \theta^n} \tau(t^n \mid g^*) = \sum_{a^n \in \mathbf{A}^n} E^{\gamma}(a^n \mid j) = \operatorname{tr} \left( W_{t^n}(a'^n) \right) = 1$  for all  $g^*, j$ , and  $a'^n$ . The sixth equation holds because  $\sum_{j=1}^{J_n} D_j^{\gamma} = id$ . Thus for any  $n \in \mathbb{N}$ , any  $J_n \in \mathbb{N} \setminus \{1\}$ , and any  $(n, J_n)$  randomness-assisted

quantum code  $({\mathcal{C}^{\gamma} : \gamma \in \Lambda}, G)$  we have

$$\frac{J_n - 1}{J_n} = \int_{\Lambda} \sum_{t^n \in \theta^n} \tau(t^n \mid g^*) P_e(\mathcal{C}^{\gamma}, t^n) dG(\gamma) \\
= \sum_{t^n \in \theta^n} \tau(t^n \mid g^*) \int_{\Lambda} P_e(\mathcal{C}^{\gamma}, t^n) dG(\gamma) \\
= \mathbb{E} \left( \int_{\Lambda} P_e(\mathcal{C}^{\gamma}, \mathfrak{T}^n) dG(\gamma) \right) ,$$
(33)

where  $\mathfrak{T}^n$  is a random variable on  $\theta^n$  such that  $Pr(\mathfrak{T}^n = t^n) = \tau(t^n \mid g^*)$  for all  $t^n \in \theta^n$ .

By (33) for any  $n \in \mathbb{N}$ , any  $J_n \in \mathbb{N} \setminus \{1\}$  and any  $(n, J_n)$  random-assisted quantum code ({ $\mathcal{C}^{\gamma} : \gamma \in \Lambda$ }, G), there exists at least one  $t^n \in \theta^n$  such that

$$\int_{\Lambda} P_e(\mathcal{C}^{\gamma}, t^n) dG(\gamma) \ge \frac{J_n - 1}{J_n} .$$
(34)

By (34) for any  $n \in \mathbb{N}$ , any  $J_n > 1$ , there is no  $(n, J_n)$  randomness-assisted code  $(\{\mathcal{C}^{\gamma} : \gamma \in \Lambda\}, G)$  for  $\{(W_t, V_t) : t \in \theta\}$  such that

$$\max_{t^n \in \theta^n} \int_{\Lambda} P_e(\mathcal{C}^{\gamma}, t^n) dG(\gamma) < \frac{1}{2} ,$$

therefore if the m - a - (X, Y) secrecy capacity of  $\{(W_t, V_t) : t \in \theta\}$  is equal to zero and I(X,Y) is positive, the randomness-assisted secrecy capacity of  $\{(W_t, V_t) : t \in \theta\}$  is equal to  $\log 1 = 0$ . But this is a contradiction to our assumption that  $C_s(\{(W_t, V_t) : t \in \theta\}; r)$  is positive.

This result and the result for the case when the m - a - (X, Y) secrecy capacity of  $\{(W_t, V_t) : t \in \theta\}$  is positive complete our proof for Theorem 4.1.

Theorem 4.1 shows that the correlation is a very helpful resource for the secure message transmission through an arbitrarily varying classical-quantum wiretap channel. As Example 5.1 shows, there are indeed arbitrarily varying classical-quantum wiretap channels which have zero deterministic secrecy capacity, but at the same time positive random secrecy capacity. Theorem 4.1 shows that if we have a m - a - (X, Y) correlation as a resource, even when it is insecure and very weak (i.e. I(X, Y) needs only to be slightly larger than zero), these channels will have a positive m - a - (X, Y) secrecy capacity.

# 5 Applications and Further Notes

In Subsection 5.1 we will discuss the importance of the Ahlswede dichotomy for arbitrarily varying classical-quantum wiretap channels. We will show that it can occur that the deterministic capacity of an arbitrarily varying classical-quantum wiretap channel is not equal to its randomness-assisted capacity.

In Subsection 5.1.1 we will show that the research in quantum channels not only sets limitations, but also offers new fascinating possibilities. Applying the Ahlswede dichotomy, we can prove that two arbitrarily varying classicalquantum wiretap channels, both with zero security capacity, allow perfect secure transmission, if we use them together. This is a phenomenon called "superactivation" which appears in quantum information theory (cf. [31]).

#### 5.1 Further Notes on Resource Theory

In this subsection, we give some notes on resource theory and the Ahlswede dichotomy.

1) The Ahlswede dichotomy states that either the deterministic security capacity of an arbitrarily varying classical-quantum wiretap channel is zero or it equals its randomness-assisted security capacity. There are actually arbitrarily varying classical-quantum wiretap channels which have zero deterministic security capacity, but achieve a positive security capacity if the sender and the legal receiver can use a resource, as the following example shows. This shows that the Ahlswede dichotomy is indeed a "dichotomy", and how helpful a resource can be for the robust and secure message transmission.

**Example 5.1.** Let  $\{(W_t, V_t) : t \in \theta\}$  be an arbitrarily varying classical-quantum wiretap channel. By Theorem 3.1. 1,  $C_s(\{(W_t, V_t) : t \in \theta\})$  is equal to  $C_s(\{(W_t, V_t) : t \in \theta\})$  is not symmetrizable, and equal to zero if  $\{W_t : t \in \theta\}$  is symmetrizable. If  $\{W_t : t \in \theta\}$  is symmetrizable, it can actually occur that  $C_s(\{(W_t, V_t) : t \in \theta\})$  is zero, but  $C_s(\{(W_t, V_t) : t \in \theta\}; r)$  is positive, as following example shows (c.f. [7] for the case of an arbitrarily varying classical-quantum channel without wiretap).

Let  $\theta := \{1, 2\}$ . Let  $\mathbf{A} = \{0, 1\}$ . Let  $H^{\mathfrak{B}} = \mathbb{C}^3$ . Let  $\{|0\rangle^{\mathfrak{B}}, |1\rangle^{\mathfrak{B}}, |2\rangle^{\mathfrak{B}}$  be a set of orthonormal vectors on  $H^{\mathfrak{B}}$ .

For  $r \in [0,1]$  let  $P_r$  be the probability distribution on  $\mathbf{A}$  such that  $P_r(0) = r$ and  $P_r(1) = 1 - r$ . We define a channel  $W_1 : P(\mathbf{A}) \to \mathcal{S}(H^{\mathfrak{B}})$  by

$$W_1(P_r) = r|0\rangle \langle 0|^{\mathfrak{B}} + (1-r)|1\rangle \langle 1|^{\mathfrak{B}}$$
,

and a channel  $W_2: P(\mathbf{A}) \to \mathcal{S}(H^{\mathfrak{B}})$  by

$$W_1(P_r) = r|1\rangle\langle 1|^{\mathfrak{B}} + (1-r)|2\rangle\langle 2|^{\mathfrak{B}}.$$

In other word

$$W_1(0) = |0\rangle \langle 0|^{\mathfrak{B}} ,$$
  

$$W_1(1) = |1\rangle \langle 1|^{\mathfrak{B}} ,$$
  

$$W_2(0) = |1\rangle \langle 1|^{\mathfrak{B}} ,$$

$$W_2(1) = |2\rangle \langle 2|^{\mathfrak{B}}$$
.

Let  $H^{\mathfrak{E}} = \mathbb{C}^2$ . Let  $\{|3\rangle^{\mathfrak{E}}, |4\rangle^{\mathfrak{E}}$  be a set of orthonormal vectors on  $H^{\mathfrak{E}}$ . We define a channel  $V_1 : P(\mathbf{A}) \to \mathcal{S}(H^{\mathfrak{E}})$  by

$$V_1(P_r) = |3\rangle\langle 3|^{\mathfrak{E}}$$
,

and a channel  $V_2: P(\mathbf{A}) \to \mathcal{S}(H^{\mathfrak{E}})$  by

$$V_1(P_r) = |4\rangle \langle 4|^{\mathfrak{E}}$$
.

 $\{(W_t, V_t) : t \in \theta\}$  defines an arbitrarily varying classical-quantum wiretap channel.

We set

$$\begin{split} \tau(1 \mid 0) &= 0 \ ; \quad \tau(2 \mid 0) = 1 \ ; \\ \tau(1 \mid 1) &= 1 \ ; \quad \tau(2 \mid 1) = 0 \ . \end{split}$$

It holds

$$\sum_{t \in \theta} \tau(t \mid 0) W_t(0) = \sum_{t \in \theta} \tau(t \mid 0) W_t(0) \ , \ \sum_{t \in \theta} \tau(t \mid 1) W_t(1) = \sum_{t \in \theta} \tau(t \mid 1) W_t(1) \ ,$$

and

$$\sum_{t\in\theta}\tau(t\mid 0)W_t(1)=|1\rangle\langle 1|^{\mathfrak{E}}=\sum_{t\in\theta}\tau(t\mid 1)W_t(0)\ .$$

 $\{(W_t): t \in \theta\}$  is therefore symmetrizable. By Theorem 3.1. 1, we have

$$C_s(\{(W_t, V_t) : t \in \theta\}) = 0 .$$
(35)

By [15], for any arbitrarily varying classical-quantum wiretap channel  $\{(W_t, V_t) :$  $t \in \theta$ , we have

$$C_{s}(\{(W_{t}, V_{t}) : t \in \theta\}; r)$$

$$\geq \max_{P \in \mathcal{P}} \left( \min_{Q \in \mathcal{Q}} \chi \left( P, \{U^{Q}(a) : a \in \mathbf{A}\} \right) - \lim_{n \to \infty} \max_{t^{n} \in \theta^{n}} \frac{1}{n} \chi(P^{n}, \{V_{t^{n}}(a^{n}) : a^{n} \in \mathbf{A}^{n}\}) \right), \qquad (36)$$

where  $\mathcal{P}$  is the set of distributions on  $\mathbf{A}$ ,  $\mathcal{Q}$  is the set of distributions on  $\theta$ , and

 $U^{Q}(a) = \sum_{t \in \Theta} Q(t)W_{t}(a) \text{ for } Q \in \mathcal{Q}.$ For all  $n \in \mathbb{N}$ ,  $t^{n} \in \theta^{n}$ , and  $P^{n} \in \mathcal{P}^{n}$ , we have  $\chi(P^{n}, \{V_{t^{n}}(a^{n}) : a^{n} \in \mathbf{A}^{n}\}) =$  $1\log 1 - 1\log 1 = 0$  and therefore

$$C_s(\{(W_t, V_t) : t \in \theta\}; r) \ge \max_{P \in \mathcal{P}} \min_{Q \in \mathcal{Q}} \chi\left(P, \{U^Q(a) : a \in \mathbf{A}\}\right) .$$

We denote by  $p' \in P(A)$  the distribution on A such that  $p'(1) = p'(2) = \frac{1}{2}$ . Let  $q \in [0,1]$ . We define Q(1) = q, Q(2) = 1 - q. We have

$$\begin{split} \chi\left(p', \{W_Q^0(a) : a \in \mathbf{A}\}\right) \\ &= -\frac{1}{2}q\log\frac{1}{2}q + \frac{1}{2}(1-q)\log\frac{1}{2}(1-q) - \frac{1}{2}\log\frac{1}{2} \\ &+ q\log q + (1-q)\log(1-q) \ . \end{split}$$

By the differentiation by q, we obtain

$$\frac{1}{\log e} \left( -\frac{1}{2} \log \frac{1}{2}q - \frac{1}{2} + \frac{1}{2} \log \frac{1}{2}(1-q) + \frac{1}{2} + \log q + 1 - \log(1-q) - 1 \right)$$
$$= \frac{1}{2 \log e} \left( \log q - \log(1-q) \right) .$$

This term is equal to zero if and only if  $q = \frac{1}{2}$ . By further calculation, one can show that  $\chi\left(p', \{W_Q^0(a) : a \in \mathbf{A}\}\right)$  achieves its minimum when  $q = \frac{1}{2}$ . This minimum is equal to  $-\frac{1}{2}\log\frac{1}{4} + \frac{1}{2}\log\frac{1}{2} = \frac{1}{2} > 0$ . Thus

$$\max_{p} \min_{q} \chi\left(p, B_{q}^{0}\right) \geq \frac{1}{2}$$

For all  $t \in \theta$ , it holds  $V_t^0(0) = V_t^0(1)$  and therefore for all  $t^n \in \theta^n$  and any  $p^n \in P(A^n)$ , we have

$$\begin{split} \chi(p; Z_{t^n}^0) \\ &= S(V_{t^n}^0(p^n)) - \sum_{a^n \in A^n} p^n(a^n) S(V_{t^n}^0(a^n)) \\ &= 0 \ . \end{split}$$

By (36),

$$C_s(\{(W_t^0, V_t^0) : t \in \theta\}, cr) \ge \frac{1}{2} - 0 > 0 .$$
(37)

This shows an example of an arbitrarily varying classical-quantum channel such that its deterministic capacity is zero, but its random capacity is positive.

Thus, a "useless" arbitrarily varying classical-quantum channel, i.e., with zero deterministic secrecy capacity, allows secure transmission if the sender and the legal receiver have the possibility to use a resource, either randomness, common randomness, or even a "cheap", insecure, and weak correlation. Here we say "cheap" and "weak" in the sense of the discussion in Section 4.

#### 5.1.1 Super-Activation

One of the properties of classical channels is that in the majority of cases, if we have a channel system where two sub-channels are used together, the capacity of this channel system is the sum of the two sub-channels' capacities. Particularly, a system consisting of two orthogonal classical channels, where both are "useless" in the sense that they both have zero capacity for message transmission, the capacity for message transmission of the whole system is zero as well ("0 + 0 = 0"). For the definition of "two orthogonal channels" in classical systems, please see [25].

In contrast to the classical information theory, it is known that the capacities of quantum channels can be super-additive, i.e., there are cases in which the capacity of the product  $W_1 \otimes W_2$  of two quantum channels  $W_1$  and  $W_2$  is larger than the sum of the capacity of  $W_1$  and the capacity of  $W_2$  (cf. [31] and [26]). "The whole is greater than the sum of its parts" - Aristotle.

Particularly in quantum information theory, there are examples of two quantum channels,  $W_1$  and  $W_2$ , with zero capacity, which allow perfect transmission if they are used together, i.e., the capacity of their product  $W_1 \otimes W_2$  is positive, (cf. [36], [35], [33] and also [19] for a rare case result when this phenomenon occurs using two classical arbitrarily varying wiretap channels). This is due to the fact that there are different reasons why a quantum channel can have zero capacity. We call this phenomenon "super-activation" ("0 + 0 > 0").

It is known that arbitrarily varying classical-quantum wiretap channels with positive secrecy capacities are super-additive. This means that the product  $W_1 \otimes W_2$  of two arbitrarily varying classical-quantum wiretap channels  $W_1$  and  $W_2$ , both with positive secrecy capacities, can have a capacity which is larger than the sum of the capacity of  $W_1$  and the capacity of  $W_2$  (cf. [31]).

Using Theorem 3.1. 3.1, we can demonstrate the following Theorem,

# **Theorem 5.2.** Super-activation occurs for arbitrarily varying classical-quantum wiretap channels.

Please note that the results of [31] (super-additivity of arbitrarily varying classical-quantum wiretap channels with positive secrecy capacities) do not imply super-activation of arbitrarily varying classical-quantum wiretap channels, since here we consider channels with zero secrecy capacity.

We will prove Theorem 5.2 by giving an example (Example 5.3) in which two arbitrarily varying classical-quantum wiretap channels, which are themselves "useless" in the sense that they have both zero secrecy capacity, acquire positive secrecy capacity when used together. This is due the following.

Suppose we have an arbitrarily varying classical-quantum wiretap channel with positive randomness-assisted secrecy capacity. By Theorem 3.1. 2, the randomness-assisted secrecy capacity is equal to the common randomnessassisted secrecy capacity. But the problem for the sender and the legal receiver is that each party does not know which code is used in the particular transmission if the channel that connects them has zero deterministic capacity for message transmission. However, suppose we have another arbitrarily varying classical-quantum wiretap channel which has a positive deterministic capacity for message transmission. Then the sender and the legal receiver can use it to transmit which code is used in the particular transmission. This is possible even when the second arbitrarily varying classical-quantum wiretap channel has zero randomness-assisted secrecy capacity, since we allow the wiretapper to know which specific code is used.

We may see it in the following way. If we have two arbitrarily varying classical-quantum wiretap channels, one of them is relatively secure, but not very robust against jamming, while the other one is relatively robust, but not very secure against eavesdropping. We can achieve that they "remove" their weaknesses from each other, or, in other words, "activate" each other.

We now give an example of super-activation for arbitrarily varying classicalquantum wiretap channels.

**Example 5.3.** Let  $\theta = \{1, 2\}$ ,  $\mathbf{A} = \{0, 1\}$ , and let H = H' be spanned by the orthonormal vectors  $|0\rangle$  and  $|1\rangle$ . We define  $\{(W_t, V_t) : t \in \theta\}$  as in Example 5.1. We define  $\{(W'_t, V'_t) : t \in \theta\}$  by

$$W'_{1}(0) = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$$

$$W'_{1}(1) = \frac{1}{4}|0\rangle\langle 0| + \frac{3}{4}|1\rangle\langle 1|$$

$$W'_{2}(0) = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$$

$$W'_{2}(1) = \frac{1}{4}|0\rangle\langle 0| + \frac{3}{4}|1\rangle\langle 1|$$

$$V'_{1}(0) = |0\rangle\langle 0|$$

$$V'_{1}(1) = |1\rangle\langle 1|$$

$$V'_{2}(0) = |0\rangle\langle 0|$$

$$V'_{2}(1) = |1\rangle\langle 1|$$
(38)

We denote the uniform distribution on **A** by *P*. We have  $P(0) = P(1) = \frac{1}{2}$ . By [7] the capacity of  $\{W'_t : t \in \theta\}$  is larger or equal to  $\min_{Q \in \mathcal{Q}} \chi\left(P, \{U^Q(a) : a \in \mathbf{A}\}\right) = \frac{1}{2} - \frac{3}{4}\log \frac{3}{4} > 0$ .

However, for all  $(n, J_n)$  code  $(E^n, \{D_j^n : j = 1, ..., J_n\})$  the wiretapper can define a set of decoding operators  $\{D_{j,wiretap}^n : j = 1, ..., J_n\}$  by  $D_{j,wiretap}^n := \sum_{a^n} E^n (a^n \mid j) (\bigotimes_i \mid a_i)) (\bigotimes_i \langle a_i \mid)$ . For any probability distribution  $Q^n$  on  $\mathbf{A}^n$ , denote the wiretapper's random output using  $\{D_{j,wiretap}^n : j = 1, ..., J_n\}$  at channel state  $t^n$  by  $C_{t^n}$ , then  $\chi(Q^n, Z_{t^n}) \geq I(Q^n, C_{t^n}) = H(Q^n)$ , where  $I(\cdot, \cdot)$  is the mutual information, and  $H(\cdot)$  is the Shannon entropy (please cf. [21] for the definitions of the mutual information and the Shannon entropy for classical random variables). If  $\chi(R_{uni}, Z_{t^n}) < \frac{1}{2}$  holds, we also have  $\log J_n = H(R_{uni}) < \frac{1}{2}$ , but this implies  $J_n = 1$ . Thus

$$C_s(\{(W'_t, V'_t) : t \in \theta\}) = 0 .$$
(39)

Let us now consider the arbitrarily varying classical-quantum wiretap channel  $\left\{ \left( W_{t_1} \otimes W'_{t_2}, V_{t_1} \otimes V'_{t_2} \right) : (t_1, t_2) \in \theta^2 \right\}, \text{ where } \left\{ \left( W_{t_1} \otimes W'_{t_2} \right) : (t_1, t_2) \in \theta^2 \right\} \text{ is } an \text{ arbitrarily varying classical-quantum channel } \{(00), (01), (10), (11)\} \rightarrow H^{\otimes 2}, (a, a') \rightarrow W_{t_1}(a) \otimes W'_{t_2}(a'), \text{ and } \left\{ \left( V_{t_1} \otimes V'_{t_2} \right) : (t_1, t_2) \in \theta^2 \right\} \text{ is an arbitrarily varying classical-quantum channel } \{(00), (01), (10), (11)\} \rightarrow H^{\otimes 2}, (a, a') \rightarrow V_{t_1}(a) \otimes V'_{t_2}(a'), \text{ if the channel state is } (t_1, t_2).$ 

We have

$$C_{s}\left(\left\{\left(W_{t_{1}}\otimes W'_{t_{2}}, V_{t_{1}}\otimes V'_{t_{2}}\right) : (t_{1}, t_{2}) \in \theta^{2}\right\}; r\right) \geq \frac{1}{2} > 0 .$$
 (40)

Assume  $\left\{ \left( W_{t_1} \otimes W'_{t_2} \right) : (t_1, t_2) \in \theta^2 \right\}$  is symmetrizable, then there exists a parametrized set of distributions  $\{ \tau(\cdot \mid (a, a')) : (a, a') \in \{(00), (01), (10), (11)\} \}$  on  $\theta^2$  such that for all  $(a, a'), (b, b') \in \{(00), (01), (10), (11)\}$  it holds

$$\sum_{\substack{(t_1,t_2)\in\theta^2}} \tau((t_1,t_2) \mid (b,b')) W_{t_1}(a) \otimes W'_{t_2}(a')$$
  
= 
$$\sum_{\substack{(t_1,t_2)\in\theta^2}} \tau((t_1,t_2) \mid (a,a')) W_{t_1}(b) \otimes W'_{t_2}(b') .$$
(41)

(41) implies that

$$\begin{split} \sum_{(t_1,t_2)\in\theta^2} \tau((t_1,t_2)\mid (0,0))W_{t_1}(0)\otimes W'_{t_2}(1) \\ &= \sum_{(t_1,t_2)\in\theta^2} \tau((t_1,t_2)\mid (0,1))W_{t_1}(0)\otimes W'_{t_2}(0) \\ &\Rightarrow (\tau((1,1)\mid (0,0)) + \tau((1,2)\mid (0,0)))\mid 0\rangle\langle 0|\otimes \left(\frac{1}{4}|0\rangle\langle 0| + \frac{3}{4}|1\rangle\langle 1|\right) \\ &+ (\tau((2,1)\mid (0,0)) + \tau((2,2)\mid (0,0)))\mid 1\rangle\langle 1|\otimes \left(\frac{1}{4}|0\rangle\langle 0| + \frac{3}{4}|1\rangle\langle 1|\right) \\ &= (\tau((1,1)\mid (0,1)) + \tau((1,2)\mid (0,1)))\mid 0\rangle\langle 0|\otimes \left(\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|\right) \\ &+ (\tau((2,1)\mid (0,1)) + \tau((2,2)\mid (0,1)))\mid 1\rangle\langle 1|\otimes \left(\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|\right) \\ &\Rightarrow (\tau((1,1)\mid (0,0)) + \tau((1,2)\mid (0,0))) = 9\left(\tau((1,1)\mid (0,0)) + \tau((1,2)\mid (0,0))\right) \quad and \\ (\tau((2,1)\mid (0,0)) + \tau((2,2)\mid (0,0))) = 9\left(\tau((2,1)\mid (0,0)) + \tau((2,2)\mid (0,0))\right) \\ &\Rightarrow \notin . \end{split}$$

Therefore  $\left\{ \left( W_{t_1} \otimes W'_{t_2} \right) : (t_1, t_2) \in \theta^2 \right\}$  is not symmetrizable, and by Theorem 3.1. 1,

$$C_{s}\left(\left\{\left(W_{t_{1}}\otimes W'_{t_{2}}, V_{t_{1}}\otimes V'_{t_{2}}\right) : (t_{1}, t_{2}) \in \theta^{2}\right\}\right)$$
$$= C_{s}\left(\left\{\left(W_{t_{1}}\otimes W'_{t_{2}}, V_{t_{1}}\otimes V'_{t_{2}}\right) : (t_{1}, t_{2}) \in \theta^{2}\right\}; r\right)$$
$$> 0.$$
(43)

This example shows that although both  $\{(W_t, V_t) : t \in \theta\}$  and  $\{(W'_t, V'_t) : t \in \theta\}$  are themselves useless, they allow secure transmission using together ("0 + 0 > 0"). Thus Theorem 5.2 is proven. This shows that the research in quantum channels with channel uncertainty and eavesdropping can lead to some promising applications.

# 6 Conclusion

In this paper, we studied message transmission over a classical-quantum channel with both a jammer and an eavesdropper, which is called an arbitrarily varying classical-quantum wiretap channel. We also studied how helpful various resources can be.

The Ahlswede dichotomy for classical arbitrarily varying channels was introduced in [3]. The Ahlswede dichotomy for arbitrarily varying classical-quantum channels was established in [6]. In our paper, we have generalized the result of [15] by establishing the Ahlswede dichotomy for arbitrarily varying classicalquantum wiretap channels: Either the deterministic secrecy capacity of an arbitrarily varying classical-quantum wiretap channel is zero, or it equals its randomness-assisted secrecy capacity. Interestingly, the Ahlswede dichotomy shows that the deterministic capacity for secure message transmission is, in general, not specified by entropy quantities. This is a new behavior in communication due to active wiretap attacks.

Dealing with channel uncertainty and eavesdropping is one of the main tasks in modern communication systems, caused, for example, by hardware imperfection. For practical implementation, a reasonable assistance for the transmitters is to share resources. For example, in wireless communication, the communication service may send some signals via satellite to its users. Hence, we analyzed the secrecy capacities of various coding schemes with resource assistance. A surprising and promising result of this paper is that the resources do not have to be secure themselves to be helpful for secure message transmission considering channel uncertainty. Another interesting fact is that in [18], it has been shown that the correlation is a much "cheaper" resource than randomness and common randomness. However, the results in this paper show that for secure message transmission considering channel uncertainty. correlation is as helpful as randomness and common randomness. Furthermore, a correlation (X, Y)does not have to be "very good" to be helpful in achieving a positive secrecy capacity, since (X, Y) is a helpful resource even if I(X, Y) is only slightly larger than zero. We also gave an example that shows not only theoretically, but also physically, how helpful a resource can be. In this example, an arbitrarily varying classical-quantum wiretap channel has zero deterministic secrecy capacity, but as soon as the sender and the receiver can use a resource, either randomness, common randomness, or correlation, we can achieve positive secrecy capacity. This example shows that for communication in practice, having weak public signals will be very useful.

In [36] and [35], it has been shown that the phenomenon "super-activation" can occur for certain quantum channels ("0 + 0 > 0"). In this paper, we have proved that "super-activation" occurs for arbitrarily varying classical-quantum wiretap channels. In classical information theory, adding a telegraph wire that relays no information to a system does not help in the majority of cases. Our result shows that for message transmission over classical-quantum channels with both a jammer and an eavesdropper, adding a fiber-optic cable that relays non-secure information can be really useful. This result sets a new challenging task for the design of media access control, which is an important topic for standardization and certification. Unlike in classical communication, for quantum media access control, we have to consider that we can lose security if we have two orthogonal useless arbitrarily varying classical-quantum wiretap channels. To

provide security, we therefore need a more sophisticated design of media access control than in the classical case. For example, we have to avoid two useless channels to be orthogonal.

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