Classical-Quantum Arbitrarily Varying Wiretap Channel: Common Randomness Assisted Code and Continuity

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Abstract We determine the secrecy capacities under common randomness assisted coding of arbitrarily varying classical-quantum wiretap channels. Furthermore, we determine the secrecy capacity of a mixed channel model which is compound from the sender to the legitimate receiver and varies arbitrarily from the sender to the eavesdropper. We examine when the secrecy capacity is a continuous function of the system parameters as an application and show that resources, e.g., having access to a perfect copy of the outcome of a random experiment, can guarantee continuity of the capacity function of arbitrarily varying classical-quantum wiretap channels.

Contents

1	Introduction	2
2	Preliminaries	6
3	Compound-Arbitrarily Varying Wiretap Classical-Quantum Channel $\ . \ . \ . \ .$	13
4	Secrecy Capacity of Arbitrarily Varying Classical-Quantum Wiretap Channel $\ . \ . \ .$	31
5	Investigation of Secrecy Capacity's Continuity	40
6	Conclusion	46

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1 Introduction

In the last few years, the developments in modern communication systems have produced many results in a short amount of time. Quantum communication systems, especially, have developed into a very active field, setting new properties and limits. Our goal is to deliver a general theory considering both channel robustness against jamming and security against eavesdropping in quantum information theory, since many modern communication systems are often not perfect, but are vulnerable to jamming and eavesdropping. The transmitters have to solve two main problems. First, the message (a secret key or a secure message) has to be encoded robustly, i.e., despite channel uncertainty, it can be decoded correctly by the legitimate receiver. Second, the message has to be encoded in such a way that the wiretapper's knowledge of the transmitted classical message can be kept arbitrarily small. This work is an extension of our previous paper [22].

In our earlier work [22], we investigated the transmission of messages from a sending party to a receiving party. The messages were kept secret from an eavesdropper. Communication took place over a quantum channel which was, in addition to noise from the environment, subjected to the action of a jammer which actively manipulated the states. The Ahlswede Dichotomy for arbitrarily varying classical-quantum wiretap channels has been established, i.e. either the deterministic capacity of an arbitrarily varying channel was zero or equal to its shared randomness assisted capacity. We also analyzed the secrecy capacity of arbitrarily varying classical-quantum wiretap channels when the sender and the receiver used various resources and studied the helpfulness of certain resources for robust and secure information transmission. We found out that even using the weakest non-secure resource (the correlation), one could achieve the same security capacity using a strong resource as the common randomness. But, nonetheless, a capacity formula was not given in [22].

In this paper, we carry on our investigation of arbitrarily varying classicalquantum wiretap channels and shared randomness. We deliver a capacity formula for secure information transmission through an arbitrarily varying classical-quantum wiretap channel using correlation as a resource. Together with the result of [22], it yields a formula for deterministic secrecy capacity of the arbitrarily varying classical-quantum wiretap channel. Using this formula, we analyze the stability of secrecy capacity, i.e., we ask under which condition, it is discontinuous as a function of channel parameters, in other words, when small variations in the underlying model dramatically change the effect of the jammer's actions.

To determine our capacity formula, we follow the idea of [13] and [39] in the classical cases: At first, we consider a mixed channel model that is called the arbitrarily varying classical-quantum wiretap channel. Then, we apply Ahlswede's robustification technique to establish the common randomness assisted secrecy capacity of an arbitrarily varying classical-quantum wiretap channel.

Quantum mechanics differs significantly from classical mechanics; it has its own laws. A quantum channel is a communication channel which can transmit quantum information. In this paper, we consider the classical-quantum channels, i.e., the sender's inputs are classical data and the receiver's outputs are quantum systems. The capacity of classical-quantum channels has been determined in [32] and [38].

In the model of an *arbitrarily varying channel*, we consider channel uncertainty, i.e. transmission over a channel which is not stationary, but can change with every use of the channel. We interpret it as a channel with a jammer who may change his input with every channel use and is not restricted to using a repetitive probabilistic strategy. It is understood that the sender and the receiver have to select their coding scheme first. After that, the jammer makes his choice of the channel state to sabotage the message transmission. However, due to the physical properties, we consider that the jammer's changes only take place in a known set. The arbitrarily varying channel was first introduced in [17].

As was already mentioned in our earlier work [22], we are interested in the role that shared randomness plays for the arbitrarily varying classicalquantum wiretap channel. This is used in [2], [3], and [4] for the determination of the random capacity. [2] showed a surprising result which is now known as the Ahlswede Dichotomy: Either the capacity of an arbitrarily varying channel is zero, or it equals its shared randomness assisted capacity. After this discovery, it has remained an open question as to exactly when the deterministic capacity is positive. In [29], a sufficient condition for this has been given, and in [26] it is proved that this condition is also necessary. In [1] it has also been shown that the capacity of certain arbitrarily varying channels can be equated to the zero-error capacity of related discrete memoryless channels. The Ahlswede Dichotomy demonstrates the importance of shared randomness for communication in a very clear form.

A classical-quantum channel with a jammer is called an arbitrarily varying classical-quantum channel. The arbitrarily varying classical-quantum channel was introduced in [6]. A lower bound for its capacity has been given. An alternative proof and a proof of the strong converse are given in [13]. In [5], the Ahlswede Dichotomy for the arbitrarily varying classical-quantum channels is established, and a sufficient and necessary condition for the zero deterministic capacity is given. In [23], a simplification of this condition for the arbitrarily varying classical-quantum channels is given.

In the model of a wiretap channel, we consider communication with security. This was first introduced in [42] (in this paper, we will use a stronger security criterion than [42]'s security criterion, cf. Remark 2). We interpret the wiretap channel as a channel with an eavesdropper. The relation of the different security criteria is discussed, for example, in [19] with some generality and in [39] with respect to arbitrarily varying channels.

A classical-quantum channel with an eavesdropper is called a classicalquantum wiretap channel, its secrecy capacity has been determined in [27] and [25]. In the model of an arbitrarily varying wiretap channel, we consider transmission with both a jammer and an eavesdropper. Its secrecy capacity has been analyzed in [16]. A lower bound of the randomness assisted secrecy capacity has been given.

A classical-quantum channel with both a jammer and an eavesdropper is called an arbitrarily varying classical-quantum wiretap channel. It is defined as a family of pairs of indexed channels $\{(W_t, V_t) : t = 1, \dots, T\}$ with a common input alphabet and possible different output alphabets and connects a sender with two receivers, a legitimate one and a wiretapper, where t is called a channel state of the channel pair. The legitimate receiver accesses the output of the first part of the pair, i.e., the first channel W_t in the pair, and the wiretapper observes the output of the second part, i.e., the second channel V_t , respectively. A channel state t, which varies from symbol to symbol in an arbitrary manner, governs both the legitimate receiver's channel and the wiretap channel. A code for the channel conveys information to the legitimate receiver such that the wiretapper knows nothing about the transmitted information in the sense of the stronger security criterion (cf. Remark 2). This is a generalization of compound classical-quantum wiretap channels in [21], when the channel states are not stationary, but can change over time.

The secrecy capacity of the arbitrarily varying classical-quantum wiretap channels has been analyzed in [18]. A lower bound of the randomness assisted capacity has been given, and it has been shown that this bound is either a lower bound for the deterministic capacity, or else the deterministic capacity is equal to zero.

References [11] and [10] are two well-known examples for secure quantum information transmission using quantum key distributions. Good one-shot results for quantum channels with a wiretapper who is limited in his actions have been obtained. But our goal is to have a more general theory for channel security in quantum information theory, i.e., message transmission should be secure against every possible kind of eavesdropping. Furthermore, we are interested in asymptotic behavior when we deliver a large volume of messages by many channel uses. Therefore, we consider a new paradigm for the design of quantum channel systems, which is called *embedded security*. Instead of the standard approach in secret communication, i.e. first ensuring a successful transmission of messages and then implementing a cryptographic protocol, here we embed protocols with a guaranteed security right from the start into the physical layer, which is the bottom layer of the model of communications systems. The concept covers both secure message transmission and secure key generation.

In [23], a classification of various resources is given. A distinction is made between two extremal cases: randomness and correlation. Randomness is the strongest resource, and it requires a perfect copy of the outcome of a random experiment, and thus, we should assume an additional perfect channel. On the other hand, correlation is the weakest resource. The work [23] also puts emphasis on the quantification of the differences between correlation and common randomness and used the arbitrarily varying classical-quantum channel as a method of proof. It can be shown that common randomness is a stronger resource than correlation in the following sense: An example is given where not even a finite amount of common randomness can be extracted from a given correlation. On the contrary, a sufficiently large amount of common randomness allows the sender and receiver to asymptotically simulate the statistics of any correlation.

In view of the aforementioned importance of shared randomness for robustness, it is clear that the shared randomness is not allowed to be known by the jammer (In stark contrast to this, we assume the eavesdropper has access to the outcomes of the shared random experiment). Therefore, backward communication from the eavesdropper to the jammer would render the shared randomness completely useless. Thus we concentrate our analysis on the case without feedback, i.e. the eavesdropper cannot send messages toward the jammer. The communication from the jammer to the eavesdropper is explicitly possible, i.e. the eavesdropper could know the jammer's strategy. It is a challenging task for future studies when the resource is secure against eavesdropping and two-way communication between the jammer and the eavesdropper is allowed. In this case, we have to build a code in such a way that the transmission of both the message and the randomization is secure.

As an application of our results, we turn to the question: when the secrecy capacity is a continuous function of the system parameters? The analysis of the continuity of capacities of quantum channels is raised from the question whether small changes in the channel system are able to cause dramatic losses in the performance. The continuity of the message and entanglement transmission capacity of a stationary memoryless quantum channel has been listed as an open problem in [43] and was solved in [33]. Considering channels with active jamming faces an especially new difficulty. The reason is that the capacity in this case is, in general, not specified by entropy quantities. In [24] it has been shown when the message transmission capacity of an arbitrarily varying quantum channels is continuous. The condition for continuity of message transmission capacity of a classical arbitrarily varying wiretap channel has been given in [39].

As a direct consequence of our capacity formula, we show in this paper that a sharing resource is very helpful for the channel stability in the sense that it provides continuity of secrecy capacities.

This paper is organized as follows:

The main definitions are given in Section 2.

In Section 3 we determine a capacity formula for a mixed channel model, i.e. the enhanced secrecy capacity of compound-arbitrarily varying wiretap classical-quantum channels. This formula will be used for our result in Section 4.

In Section 4 our main result is presented. In this section we determine the secrecy capacities under common randomness assisted coding of arbitrarily varying classical-quantum wiretap channels.

As an application of our main result, in Section 5 we discuss when the secrecy capacity of an arbitrarily varying classical-quantum wiretap channel is a continuous quantity of the system parameters.

2 Preliminaries

2.1 Basic Notations

For a finite set A, we denote the set of probability distributions on A by P(A). Let ρ_1 and ρ_2 be Hermitian operators on a finite-dimensional complex Hilbert space G. We say $\rho_1 \ge \rho_2$ and $\rho_2 \le \rho_1$ if $\rho_1 - \rho_2$ is positive semidefinite. For a finite-dimensional complex Hilbert space G, we denote the set of density operators on G by

$$\mathcal{S}(G) := \{ \rho \in \mathcal{L}(G) : \rho \text{ is Hermitian, } \rho \ge 0_G , \operatorname{tr}(\rho) = 1 \},\$$

where $\mathcal{L}(G)$ is the set of linear operators on G, and 0_G is the null matrix on G. Note that any operator in $\mathcal{S}(G)$ is bounded.

For finite-dimensional complex Hilbert spaces G and G', a quantum channel $N: \mathcal{S}(G) \to \mathcal{S}(G'), \mathcal{S}(G) \ni \rho \to N(\rho) \in \mathcal{S}(G')$ is represented by a completely positive trace-preserving map which accepts input quantum states in $\mathcal{S}(G)$ and produces output quantum states in $\mathcal{S}(G')$.

If the sender wants to transmit a classical message of a finite set A to the receiver using a quantum channel N, his encoding procedure will include a classical-to-quantum encoder to prepare a quantum message state $\rho \in \mathcal{S}(G)$ suitable as an input for the channel. If the sender's encoding is restricted to transmitting an indexed finite set of quantum states $\{\rho_x : x \in A\} \subset \mathcal{S}(G)$, then we can consider the choice of the signal quantum states ρ_x as a component of the channel. Thus, we obtain a channel $\sigma_x := N(\rho_x)$ with classical inputs $x \in A$ and quantum outputs, which we call a classical-quantum channel. This is a map $\mathbf{N}: A \to \mathcal{S}(G'), A \ni x \to \mathbf{N}(x) \in \mathcal{S}(G')$ which is represented by the set of |A| possible output quantum states $\{\sigma_x = \mathbf{N}(x) := N(\rho_x) : x \in A\} \subset \mathcal{S}(G')$, meaning that each classical input of $x \in A$ leads to a distinct quantum output $\sigma_x \in \mathcal{S}(G')$. In view of this, we have the following definition.

Let A be a finite set and H be a finite-dimensional complex Hilbert space. A classical-quantum channel is a linear map $W : \mathsf{P}(\mathsf{A}) \to \mathcal{S}(H), \ \mathsf{P}(\mathsf{A}) \ni P \to W(P) \in \mathcal{S}(H)$. Let $a \in \mathsf{A}$. For a $P_a \in \mathsf{P}(\mathsf{A})$, defined by $P_a(a') = \begin{cases} 1 & \text{if } a' = a \\ 0 & \text{if } a' \neq a \end{cases}$, we write W(a) instead of $W(P_a)$.

Remark 1 In much literature, a classical-quantum channel is defined as a map $A \to S(H)$, $A \ni a \to W(a) \in S(H)$. This is a special case when the input is limited on the set $\{P_a : a \in A\}$.

For a probability distribution P on a finite set A and a positive constant δ , we denote the set of typical sequences by

$$\mathsf{T}^n_{P,\delta} := \left\{ a^n \in \mathsf{A}^n : |\frac{1}{n} N(a' \mid a^n) - P(a')| \le \frac{\delta}{n} \forall a' \in \mathsf{A} \right\} \;,$$

where $N(a' \mid a^n)$ is the number of occurrences of the symbol a' in the sequence a^n .

Let $n \in \mathbb{N}$. we define $A^n := \{(a_1, \cdots, a_n) : a_i \in A \ \forall i \in \{1, \cdots, n\}\}$. The space which the vectors $\{v_1 \otimes \cdots \otimes v_n : v_i \in H \ \forall i \in \{1, \cdots, n\}\}$ span is denoted by $H^{\otimes n}$. We also write a^n for the elements of A^n .

Associated to W is the channel map on the n-block $W^{\otimes n}$: $P(A^n) \to \mathcal{S}(H^{\otimes n})$, such that $W^{\otimes n}(P^n) = W(P_1) \otimes \cdots \otimes W(P_n)$ if $P^n \in \mathsf{P}(A^n)$ can be given by $P^n(a^n) = \prod_j P_j(a_j)$ for every $a^n = (a_1, \cdots, a_n) \in \mathsf{A}^n$. Let $\theta := \{1, \cdots, T\}$ be a finite set. Let $\{W_t : t \in \theta\}$ be a set of classical-quantum channels. For $t^n = (t_1, \cdots, t_n), t_i \in \theta$ we define the n-block W_{t^n} such that for $W_{t^n}(P^n) = W_{t_1}(P_1) \otimes \cdots \otimes W_{t_n}(P_n)$ if $P^n \in \mathsf{P}(\mathsf{A}^n)$ can be given by $P^n(a^n) = \prod_j P_j(a_j)$ for every $a^n \in \mathsf{A}^n$.

For a quantum state $\rho \in \mathcal{S}(G)$ we denote the von Neumann entropy of ρ by

$$S(\rho) = -\operatorname{tr}(\rho \log \rho)$$
.

Let \mathfrak{P} and \mathfrak{Q} be quantum systems. We denote the Hilbert space of \mathfrak{P} and \mathfrak{Q} by $G^{\mathfrak{P}}$ and $G^{\mathfrak{Q}}$, respectively. Let $\phi^{\mathfrak{P}\mathfrak{Q}}$ be a bipartite quantum state in $\mathcal{S}(G^{\mathfrak{P}\mathfrak{Q}})$. We denote the partial trace over $G^{\mathfrak{P}}$ by

$$\mathrm{tr}_{\mathfrak{P}}(\phi^{\mathfrak{P}\mathfrak{Q}}) := \sum_{l} \langle l |_{\mathfrak{P}} \phi^{\mathfrak{P}\mathfrak{Q}} | l \rangle_{\mathfrak{P}}$$

where $\{|l\rangle_{\mathfrak{P}} : l\}$ is an orthonormal basis of $G^{\mathfrak{P}}$. We denote the conditional entropy by

$$S(\mathfrak{P} \mid \mathfrak{Q})_{\phi} := S(\phi^{\mathfrak{PQ}}) - S(\phi^{\mathfrak{Q}})$$
 .

The quantum mutual information is denoted by

$$I(\mathfrak{P};\mathfrak{Q})_{\phi} = S(\phi^{\mathfrak{P}}) + S(\phi^{\mathfrak{Q}}) - S(\phi^{\mathfrak{PQ}})$$
.

Here $\phi^{\mathfrak{Q}} = \operatorname{tr}_{\mathfrak{P}}(\phi^{\mathfrak{PQ}})$ and $\phi^{\mathfrak{P}} = \operatorname{tr}_{\mathfrak{Q}}(\phi^{\mathfrak{PQ}})$. Let $\mathbb{V}: \mathbb{A} \to \mathcal{S}(G)$ be a classicalquantum channel. Following [7], for $P \in P(\mathbb{A})$ the conditional entropy of the channel for \mathbb{V} with input distribution P is denoted by

$$S(\mathbf{V}|P) := \sum_{x \in \mathbf{A}} P(x) S(\mathbf{V}(x)) \ .$$

Let $\Phi := \{\rho_a : a \in \mathsf{A}\}$ be a set of quantum states labeled by elements of A . For a probability distribution P on A , the Holevo χ quantity is defined as

$$\chi(P;\Phi) := S\left(\sum_{a \in \mathsf{A}} P(a)\rho_a\right) - \sum_{a \in \mathsf{A}} P(a)S\left(\rho_a\right) \;.$$

For a set **A** and a Hilbert space G, let **V**: $A \to S(G)$ be a classical-quantum channel. For a probability distribution P on A, the Holevo χ quantity of the channel for **V** with input distribution P is defined as

$$\chi(\mathbf{V}; \Phi) := S\left(\mathbf{V}(P)\right) - S\left(\mathbf{V}|P\right) \ .$$

Let G be a finite-dimensional complex Hilbert space. Let $n \in \mathbb{N}$ and $\alpha > 0$. We suppose $\rho \in \mathcal{S}(G)$ has the spectral decomposition $\rho = \sum_{x} P(x)|x\rangle\langle x|$. Notice that by definition of the spectral decomposition, the eigenvectors $\{|x\rangle : x\}$ form an orthonormal system (sometimes also called the "computational basis"). Its α -typical subspace is the subspace spanned by $\{|x^n\rangle : x^n \in \mathbb{T}_{P,\alpha}^n\}$, where $|x^n\rangle := \bigotimes_{i=1}^n |x_i\rangle$. The orthogonal subspace projector onto the typical subspace is

$$\Pi_{\rho,\alpha} = \sum_{x^n \in \mathsf{T}^n_{P,\alpha}} |x^n\rangle \langle x^n| \ .$$

Similarly let A be a finite set, and G be a finite-dimensional complex Hilbert space. Let V: A $\rightarrow S(G)$ be a classical-quantum channel. For $a \in A$, suppose V(a) has the spectral decomposition $V(a) = \sum_{j} V(j|a)|j\rangle_a \langle j|_a$ for a stochastic matrix $V(\cdot|\cdot)$. In an effort to enhance readability, we will typically suppress the subscript a in the above decomposition and typically write $\sum_{j} V(j|a)|j\rangle\langle j|$ whenever this causes no ambiguity. The same reasoning applies to the next definition. The α -conditional typical subspace of V for a typical sequence a^n is the subspace spanned by $\left\{\bigotimes_{a\in A} |j^{I}\rangle_a : j^{I_a} \in \mathsf{T}_{V(\cdot|a),\delta}^{I_a}\right\}$. Here $I_a := \{i \in \{1, \cdots, n\} : a_i = a\}$ is an indicator set that selects the indices i in the sequence $a^n = (a_1, \cdots, a_n)$ for which the *i*th symbol a_i is equal to $a \in A$. The subspace is often referred to as the α -conditional typical subspace of the state $\mathsf{V}^{\otimes n}(a^n)$. The orthogonal subspace projector onto it is defined as

$$\varPi_{\mathbf{V},\alpha}(a^n) = \bigotimes_{a \in \mathbf{A}} \sum_{j^{\mathbf{I}_a} \in \mathsf{T}^{\mathbf{I}_a}_{\mathbf{V}(\cdot \mid a^n),\alpha}} |j^{\mathbf{I}_a}\rangle \langle j^{\mathbf{I}_a}| \ .$$

The typical subspace has following properties:

For $\sigma \in \mathcal{S}(G^{\otimes n})$ and $\alpha > 0$, there are positive constants $\beta(\alpha)$, $\gamma(\alpha)$, and $\delta(\alpha)$, depending on α , such that

$$\operatorname{tr}\left(\sigma\Pi_{\sigma,\alpha}\right) > 1 - 2^{-n\beta(\alpha)} , \qquad (1)$$

$$2^{n(S(\sigma)-\delta(\alpha))} \le \operatorname{tr}\left(\Pi_{\sigma,\alpha}\right) \le 2^{n(S(\sigma)+\delta(\alpha))} , \qquad (2)$$

$$2^{-n(S(\sigma)+\gamma(\alpha))}\Pi_{\sigma,\alpha} \le \Pi_{\sigma,\alpha}\sigma\Pi_{\sigma,\alpha} \le 2^{-n(S(\sigma)-\gamma(\alpha))}\Pi_{\sigma,\alpha} .$$
(3)

For $a^n \in \mathsf{T}^n_{P,\alpha}$, there are positive constants $\beta(\alpha)'$, $\gamma(\alpha)'$, and $\delta(\alpha)'$, depending on α such that

$$\operatorname{tr}\left(\mathbf{V}^{\otimes n}(a^{n})\Pi_{\mathbf{V},\alpha}(a^{n})\right) > 1 - 2^{-n\beta(\alpha)'}, \qquad (4)$$

$$2^{-n(S(\mathbf{V}|P)+\gamma(\alpha)')}\Pi_{\mathbf{V},\alpha}(a^n) \leq \Pi_{\mathbf{V},\alpha}(a^n)\mathbf{V}^{\otimes n}(a^n)\Pi_{\mathbf{V},\alpha}(a^n)$$
$$\leq 2^{-n(S(\mathbf{V}|P)-\gamma(\alpha)')}\Pi_{\mathbf{V},\alpha}(a^n) , \qquad (5)$$

$$2^{n(S(\mathbf{V}|P) - \delta(\alpha)')} \le \operatorname{tr}\left(\Pi_{\mathbf{V},\alpha}(a^n)\right) \le 2^{n(S(\mathbf{V}|P) + \delta(\alpha)')} .$$
(6)

For the classical-quantum channel $\mathbb{V} : \mathbb{P}(\mathbb{A}) \to \mathcal{S}(G)$ and a probability distribution P on \mathbb{A} , we define a quantum state $P\mathbb{V} := \mathbb{V}(P)$ on $\mathcal{S}(G)$. For $\alpha > 0$, we define an orthogonal subspace projector $\Pi_{P\mathbb{V},\alpha}$ fulfilling (1), (2), and (3). Let $x^n \in \mathsf{T}_{P,\alpha}^n$. For $\Pi_{P\mathbb{V},\alpha}$, there is a positive constant $\beta(\alpha)''$ such that following inequality holds:

$$\operatorname{tr}\left(\mathbf{V}^{\otimes n}(x^{n})\cdot\Pi_{P\mathbf{V},\alpha}\right) \ge 1 - 2^{-n\beta(\alpha)^{\prime\prime}} . \tag{7}$$

We give here a sketch of the proof. For a detailed proof, please see [40].

Proof (1) holds because tr $(\sigma \Pi_{\sigma,\alpha}) = \text{tr}(\Pi_{\sigma,\alpha}\sigma\Pi_{\sigma,\alpha}) = P(\mathsf{T}_{P,\alpha}^n)$. (2) holds because tr $(\Pi_{\sigma,\alpha}) = |\mathsf{T}_{P,\alpha}^n|$. (3) holds because $2^{-n(S(\sigma)+\gamma(\alpha))} \leq P^n(x^n) \leq 2^{-n(S(\sigma)-\gamma(\alpha))}$ for $x \in \mathsf{T}_{P,\alpha}^n$ and a positive $\gamma(\alpha)$. (4), (5), and (6) can be obtained in similar way. (7) follows from the permutation-invariance of $\Pi_{P\mathsf{V},\alpha}$.

2.2 Communication Scenarios and Code Concepts

Definition 1 Let A be a finite set, let H be a finite-dimensional complex Hilbert space, and $\theta := \{1, \dots, T\}$ be an index set. For every $t \in \theta$, let W_t be a classical-quantum channel $\mathsf{P}(\mathsf{A}) \to \mathcal{S}(H)$. We call the set of the classicalquantum channels $\{W_t : t \in \theta\}$ an **arbitrarily varying classical-quantum channel** when the channel state t varies from symbol to symbol in an arbitrary manner.

When the sender inputs a sequence $a^n \in A^n$ into the channel, the receiver receives the output $W_t^n(a^n) \in \mathcal{S}(H^{\otimes n})$, where $t^n = (t_1, t_2, \cdots, t_n) \in \theta^n$ is the channel state of W_t^n .

Definition 2 We say that the arbitrarily varying classical-quantum channel $\{W_t : t \in \theta\}$ is **symmetrizable** if there exists a parametrized set of distributions $\{\tau(\cdot \mid a) : a \in \mathsf{A}\}$ on θ such that for all $a, a' \in \mathsf{A}$,

$$\sum_{t \in \theta} \tau(t \mid a) W_t(a') = \sum_{t \in \theta} \tau(t \mid a') W_t(a) .$$

When the sender inputs a sequence $a^n \in \mathsf{A}^n$ into the channel, the receiver receives the output $W_t^{\otimes n}(a^n) \in \mathcal{S}(H^{\otimes n})$, where $t^n = (t_1, t_2, \cdots, t_n) \in \theta^n$ is the channel state, while the wiretapper receives an output quantum state $V_t^{\otimes n}(a^n) \in \mathcal{S}(H'^{\otimes n})$.

Definition 3 Let A be a finite set. Let H and H' be finite-dimensional complex Hilbert spaces. Let $\theta := \{1, 2, \dots\}$ be an index set. For every $t \in \theta$, let W_t be a classical-quantum channel $\mathsf{P}(\mathsf{A}) \to \mathcal{S}(H)$ and V_t be a classicalquantum channel $\mathsf{P}(\mathsf{A}) \to \mathcal{S}(H')$. We call the set of the classical-quantum channel pairs $\{(W_t, V_t) : t \in \theta\}$ an **arbitrarily varying classical-quantum wiretap channel** when the state t varies from symbol to symbol in an arbitrary manner, while the legitimate receiver accesses the output of the first channel, i.e., W_t in the pair (W_t, V_t) , and the wiretapper observes the output of the second channel, i.e., V_t in the pair (W_t, V_t) , respectively.

Definition 4 Let A be a finite set. Let H and H' be finite-dimensional complex Hilbert spaces. Let $\overline{\theta} := \{1, 2, \dots\}$ and $\theta := \{1, 2, \dots\}$ be index sets. For every $s \in \overline{\theta}$ let \overline{W}_s be a classical-quantum channel $\mathsf{P}(\mathsf{A}) \to \mathcal{S}(H)$. For every $t \in \theta$ let V_t be a classical-quantum channel $\mathsf{P}(\mathsf{A}) \to \mathcal{S}(H')$. We call the set of the classical-quantum channel pairs $\{(\overline{W}_s, V_t) : s \in \overline{\theta}, t \in \theta\}$ a **compound-arbitrarily varying wiretap classical-quantum channel**, when the channel state s remains constant over time, but the legitimate users can not control which s in the set $\overline{\theta}$ will be used and the state t varies from symbol to symbol in an arbitrary manner, while the legitimate receiver accesses the output of the first channel, i.e., \overline{W}_s in the pair (\overline{W}_s, V_t) and the wiretapper observes the output of the second channel, i.e., V_t in the pair (\overline{W}_s, V_t) , respectively.

Definition 5 An (n, J_n) (deterministic) code \mathcal{C} for a classical-quantum channel consists of a stochastic encoder $E : \{1, \dots, J_n\} \to \mathsf{P}(\mathsf{A}^n)$, specified by a matrix of conditional probabilities $E(\cdot|\cdot)$ and a collection of positive-semidefinite operators $\{D_j : j \in \{1, \dots, J_n\}\} \subset \mathcal{S}(H^{\otimes n})$, which is a partition of the identity, i.e., $\sum_{j=1}^{J_n} D_j = \mathrm{id}_{H^{\otimes n}}$. We call these operators the decoder operators.

A code is created by the sender and the legitimate receiver before the message transmission starts. The sender uses the encoder to encode the message that he wants to send, while the legitimate receiver uses the decoder operators on the channel output to decode the message.

Definition 6 An (n, J_n) randomness assisted quantum code for the arbitrarily varying classical-quantum wiretap channel $\{(W_t, V_t) : t \in \theta\}$ is a distribution G on (Λ, σ) , where we denote the set of (n, J_n) deterministic codes by Λ and σ is a sigma-algebra so chosen such that the functions $\gamma \to P_e(\mathcal{C}^{\gamma}, t^n)$ and $\gamma \to \chi(R_{uni}; \mathbb{Z}_{\mathcal{C}^{\gamma}, t^n})$ are both G-measurable with respect to σ for every $t^n \in \theta^n$, here $\mathbb{Z}_{\mathcal{C}^{\gamma}, t^n} := \{V_{t^n}(E^{\gamma}(\cdot \mid 1)), V_{t^n}(E^{\gamma}(\cdot \mid 2)), \cdots, V_{t^n}(E^{\gamma}(\cdot \mid J_n))\}.$

Definition 7 A non-negative number R is an achievable (deterministic) **secrecy rate** for the arbitrarily varying classical-quantum wiretap channel $\{(W_t, V_t) : t \in \theta\}$ if for every $\epsilon > 0$, $\delta > 0$, $\zeta > 0$ and sufficiently large n there exists an (n, J_n) code $C = (E, \{D_j^n : j = 1, \dots, J_n\})$ such that $\frac{\log J_n}{n} > R - \delta$, and

$$\max_{t^n \in \theta^n} P_e(\mathcal{C}, t^n) < \epsilon , \qquad (8)$$

$$\max_{t^n \in \theta^n} \chi\left(R_{uni}; Z_{t^n}\right) < \zeta , \qquad (9)$$

where R_{uni} is the uniform distribution on $\{1, \dots, J_n\}$. Here $P_e(\mathcal{C}, t^n)$ (the average probability of the decoding error of a deterministic code \mathcal{C} , when the channel state of the arbitrarily varying classical-quantum wiretap channel $\{(W_t, V_t) : t \in \theta\}$ is $t^n = (t_1, t_2, \dots, t_n)$), is defined as

$$P_e(\mathcal{C}, t^n) := 1 - \frac{1}{J_n} \sum_{j=1}^{J_n} \operatorname{tr}(W_{t^n}(E(|j))D_j)$$

and $Z_{t^n} = \{V_{t^n}(E(|i)) : i \in \{1, \dots, J_n\}\}$ is the set of the resulting quantum state at the output of the wiretap channel when the channel state of $\{(W_t, V_t) : t \in \theta\}$ is t^n .

Remark 2 A weaker and widely used security criterion is obtained if we replace (9) with $\max_{t\in\theta} \frac{1}{n}\chi(R_{uni}; Z_{t^n}) < \zeta$. In this paper, we will follow [15] and use (9).

Definition 8 An (n, J_n) common randomness assisted quantum code for the arbitrarily varying classical-quantum wiretap channel $\{(W_t, V_t) : t \in \theta\}$ is a finite subset $\{C^{\gamma} = \{(E^{\gamma}, D_j^{\gamma}) : j = 1, \dots, J_n\} : \gamma \in \Gamma\}$ of the set of (n, J_n) deterministic codes, labeled by a finite set Γ .

Definition 9 A non-negative number R is an achievable **enhanced secrecy** rate for the compound-arbitrarily varying wiretap classical-quantum channel $\{(\overline{W}_s, V_t) : s \in \overline{\theta}, t \in \theta\}$ if for every $\epsilon > 0, \delta > 0, \zeta > 0$ and sufficiently large n there exists an (n, J_n) code $\mathcal{C} = (E^n, \{D_j^n : j = 1, \cdots, J_n\})$ such that $\frac{\log J_n}{n} > R - \delta$, and

$$\max_{s\in\overline{\theta}} P_e(\mathcal{C}, s, n) < \epsilon , \qquad (10)$$

$$\max_{t^n \in \theta^n} \max_{\pi \in \Pi_n} \chi\left(R_{uni}; Z_{t^n, \pi}\right) < \zeta , \qquad (11)$$

where R_{uni} is the uniform distribution on $\{1, \dots, J_n\}$. Here $P_e(\mathcal{C}, s, n)$ is defined as follows

$$P_e(\mathcal{C}, s, n) := 1 - \frac{1}{J_n} \sum_{j=1}^{J_n} \operatorname{tr}(\overline{W}_s^{\otimes n}(E^n(|j))D_j^n) ,$$

and $Z_{t^n,\pi} = \left\{ \sum_{a^n \in \mathsf{A}^n} E^n(\pi(a^n)|1) V^{t^n}(\pi(a^n)), \sum_{a^n \in \mathsf{A}^n} E^n(\pi(a^n)|2) V^{t^n}(\pi(a^n)), \cdots, \sum_{a^n \in \mathsf{A}^n} E^n(\pi(a^n)|J_n) V^{t^n}(\pi(a^n)) \right\}.$

Definition 10 A non-negative number R is an achievable secrecy rate for $\{(\overline{W}_s, V_t) : s \in \overline{\theta}, t \in \theta\}$ if for every $\epsilon > 0, \delta > 0, \zeta > 0$ and sufficiently large n there exists an (n, J_n) code $\mathcal{C} = (E^n, \{D_j^n : j = 1, \cdots, J_n\})$ such that $\frac{\log J_n}{n} > R - \delta$, and

$$\max_{s\in\overline{\theta}} P_e(\mathcal{C}, s, n) < \epsilon ,$$
$$\max_{t^n\in\theta^n} \max_{\pi\in\Pi_n} \chi\left(R_{uni}; Z_{t^n}\right) < \zeta$$

Definition 11 A non-negative number R is an achievable secrecy rate for the arbitrarily varying classical-quantum wiretap channel $\{(W_t, V_t) : t \in \theta\}$ under common randomness assisted quantum coding if for every $\delta > 0$, $\zeta > 0$, and $\epsilon > 0$, if n is sufficiently large there is an (n, J_n) common randomness assisted quantum code $(\{C^{\gamma} : \gamma \in \Gamma\})$ such that $\frac{\log J_n}{n} > R - \delta$, and

$$\max_{t^n \in \theta^n} \frac{1}{|\Gamma|} \sum_{\gamma=1}^{|\Gamma|} P_e(\mathcal{C}^{\gamma}, t^n) < \epsilon ,$$
$$\max_{n \in \theta^n} \frac{1}{|\Gamma|} \sum_{\gamma=1}^{|\Gamma|} \chi(R_{uni}, Z_{\mathcal{C}^{\gamma}, t^n}) < \zeta .$$

This means that we do not require common randomness to be secure against eavesdropping.

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Definition 12 Let X and Y be finite sets. We denote the sets of joint probability distributions on X and Y by P(X, Y). Let (X, Y) be a random variable distributed to a joint probability distribution $p \in P(X, Y)$. An (X, Y)-correlation assisted (n, J_n) code C(X, Y) for the arbitrarily varying classicalquantum wiretap channel $(W_t, V_t)_{t\in\theta}$ consists of a set of stochastic encoders $\{E_{\mathbf{x}^n} : \{1, \cdots, J_n\} \to \mathsf{P}(\mathsf{A}^n) : \mathbf{x}^n \in \mathsf{X}^n\}$, and a set of collections of positive semidefinite operators $\{\{D_j^{(y^n)} : j = 1, \cdots, J_n\} : \mathbf{y}^n \in \mathsf{Y}^n\}$ on $\mathcal{S}(H^{\otimes n})$ which fulfills $\sum_{j=1}^{J_n} D_j^{(y^n)} = \mathrm{id}_{H^{\otimes n}}$ for every $\mathbf{y}^n \in \mathsf{Y}^n$. R is an achievable (X, Y) secrecy rate for $(W_t, V_t)_{t\in\theta}$ if for every positive

R is an achievable (X, Y) secrecy rate for $(W_t, V_t)_{t \in \theta}$ if for every positive ϵ, δ, ζ and sufficiently large *n* there exist an (X, Y)-correlation assisted (n, J_n) code $C(X, Y) = \left\{ \left(E_{\mathsf{x}^n}, D_j^{(\mathsf{y}^n)} \right) : j \in \{1, \cdots, J_n\}, \, \mathsf{x}^n \in \mathsf{X}^n, \, \mathsf{y}^n \in \mathsf{Y}^n \right\}$ such that $\frac{\log J_n}{n} > R - \delta$, and

$$\begin{aligned} \max_{t^n \in \theta^n} \sum_{\mathbf{x}^n \in \mathbf{X}^n} \sum_{\mathbf{y}^n \in \mathbf{Y}^n} p(\mathbf{x}^n, \mathbf{y}^n) P_e(C(\mathbf{x}^n, \mathbf{y}^n), t^n) < \epsilon \\ \max_{t^n \in \theta^n} \sum_{x^n \in \mathbf{X}^n} p_{\mathbf{X}}^{\otimes n}(x^n) \chi\left(R_{uni}; Z_{t^n, \mathbf{x}^n}\right) < \zeta , \end{aligned}$$

where $P_e(C(\mathbf{x}^n, \mathbf{y}^n), t^n) := 1 - \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{a^n \in \mathbf{A}^n} E_{\mathbf{x}^n}(a^n|j) \operatorname{tr}(W_{t^n}(a^n)D_j^{(\mathbf{y}^n)}).$

Definition 13 The supremum of all achievable (deterministic) secrecy rates of $\{(W_t, V_t) : t \in \theta\}$ is called the (deterministic) secrecy capacity of $\{(W_t, V_t) : t \in \theta\}$, denoted by $C_s(\{(W_t, V_t) : t \in \theta\})$.

The supremum of all achievable secrecy rates under common randomness assisted quantum coding of $\{(W_t, V_t) : t \in \theta\}$ is called the common randomness assisted secrecy capacity of $\{(W_t, V_t) : t \in \theta\}$, denoted by $C_s(\{(W_t, V_t) : t \in \theta\}; cr)$.

The supremum of all achievable enhanced secrecy rates of $\{(\overline{W}_s, V_t) : s \in \overline{\theta}, t \in \theta\}$ is called the enhanced secrecy capacity of $\{(\overline{W}_s, V_t) : s \in \overline{\theta}, t \in \theta\}$, denoted by $\hat{C}_s(\{(\overline{W}_s, V_t) : s \in \overline{\theta}, t \in \theta\})$.

The supremum of all achievable (X, Y) secrecy rate of $\{(W_t, V_t) : t \in \theta\}$ is called the (X, Y) secrecy capacity of $\{(W_t, V_t) : t \in \theta\}$.

3 Compound-Arbitrarily Varying Wiretap Classical-Quantum Channel

Let A, H, H', θ , and $(W_t, V_t)_{t \in \theta}$ be defined as in Section 2.

Following the idea of [39], we first prove the following Theorem.

Theorem 1 Let $\overline{\theta} := \{1, \dots, \overline{T}\}$ and $\theta := \{1, \dots, T\}$ be finite index sets. Let $\{(\overline{W}_s, V_t) : s \in \overline{\theta}, t \in \theta\}$ be a compound-arbitrarily varying wiretap classicalquantum channel. We have

$$\hat{C}_{s}(\{(\overline{W}_{s}, V_{t}) : s \in \overline{\theta}, t \in \theta\}) = \lim_{n \to \infty} \frac{1}{n} \max_{\Lambda_{n}} \left(\min_{s \in \overline{\theta}} \chi(p_{U}; B_{s}^{\otimes n}) - \max_{t^{n} \in \theta^{n}} \chi(p_{U}; Z_{t^{n}}) \right) , \qquad (12)$$

where B_s are the resulting quantum states at the output of the legitimate receiver's channels. Z_{t^n} are the resulting quantum states at the output of wiretap channels. By \max_{A_n} , we mean that the maximum is taken over all ensembles that arise from taking an arbitrary finite set U and defining ensembles $\{p_U(u), \sum_{a^n \in A^n} p_{A^n|U}(a^n|u)W_s^{\otimes n}(a^n)\}_{u \in U}$ and $\{p_U(u), \sum_{a^n \in A^n} p_{A^n|U}(a^n|u)V_{t^n}(a^n)\}_{u \in U}$ for every $s \in \overline{\theta}$, and $t^n \in \theta^n$ to calculate the respective Holevo quantities. A^n is here a random variable taking values on A^n , U a random variable taking values on U with probability distribution p_U , and $p_{A^n|U} \in P(A^n)$ the conditional distribution of A^n given U.

Proof We fix a probability distribution $p \in P(A)$. Let

$$J_n = \lfloor 2^{n \min_{s \in \overline{\theta}} \chi(p; B_s) - \log L_n - 2n\mu} \rfloor .$$

Let $p'(x^n) := \begin{cases} \frac{p^n(x^n)}{p^n(\mathsf{T}^n_{p,\delta})} , & \text{if } x^n \in \mathsf{T}^n_{p,\delta} ; \\ 0 , & \text{else }. \end{cases}$

Let $X^n := \{X_{j,l}\}_{j \in \{1,...,J_n\}, l \in \{1,...,L_n\}}$ be a family of random variables taking value according to p', i.e., with the uniform distribution over $\mathsf{T}_{P,\delta}^n$. Here L_n is a natural number which will be specified later. We fix a $t^n \in \theta^n$ and define a map $V : \mathsf{P}(\theta) \times \mathsf{P}(\mathsf{A}) \to \mathcal{S}(H)$ by

$$\mathsf{V}(t,p):=V_t(p) \ .$$

For $t \in \theta$ we define $q(t) := \frac{N(t|t^n)}{n}$. t^n is trivially a typical sequence of q. For $p \in \mathsf{P}(\mathsf{A})$, V defines a map $\mathsf{V}(\cdot, p) : \mathsf{P}(\theta) \to \mathcal{S}(H)$. Let

$$Q_{t^n}(x^n) := \Pi_{\mathsf{V}(\cdot,p),\alpha}(t^n) \Pi_{\mathsf{V},\alpha}(t^n,x^n) \cdot V_{t^n}(x^n) \cdot \Pi_{\mathsf{V},\alpha}(t^n,x^n) \Pi_{\mathsf{V}(\cdot,p),\alpha}(t^n) \ .$$

Lemma 1 (Gentle Operator, cf. [41] and [36]) Let ρ be a quantum state and X be a positive operator with $X \leq I$ and $1 - tr(\rho X) \leq \lambda \leq 1$. Then

$$\|\rho - \sqrt{X}\rho\sqrt{X}\|_1 \le \sqrt{2\lambda} . \tag{13}$$

The Gentle Operator was first introduced in [41], where it has been shown that $\|\rho - \sqrt{X}\rho\sqrt{X}\|_1 \leq \sqrt{8\lambda}$. In [36], the result of [41] has been improved, and (13) has been proved.

In view of the fact that $\Pi_{\mathsf{V}(\cdot,p),\alpha}(t^n)$ and $\Pi_{\mathsf{V},\alpha}(t^n,x^n)$ are both projection matrices, by (1), (7), and Lemma 1 for any t and x^n , it holds that

$$\|Q_{t^n}(x^n) - V_{t^n}(x^n)\|_1 \le \sqrt{2^{-n\beta(\alpha)} + 2^{-n\beta(\alpha)''}} .$$
(14)

The following Lemma was first given in [7]. Here we cite the lemma as it was formulated in [40].

Lemma 2 (Covering Lemma) Let \mathcal{V} be a finite-dimensional Hilbert space. Let M be a finite set. Suppose we have an ensemble $\{\rho_m : m \in M\} \subset \mathcal{S}(\mathcal{V})$ of quantum states. Let p be a probability distribution on M.

Suppose a total subspace projector Π and codeword subspace projectors $\{\Pi_m : m \in \mathsf{M}\}\$ exist which project onto subspaces of the Hilbert space in which the states exist, and for all $m \in \mathsf{M}$ there are positive constants $\epsilon \in]0, 1[$, D, d such that the following conditions hold:

$$\begin{split} & \operatorname{tr}(\rho_m \Pi) \geq 1 - \epsilon \ , \\ & \operatorname{tr}(\rho_m \Pi_m) \geq 1 - \epsilon \ , \\ & \operatorname{tr}(\Pi) \leq D \ , \\ & \Pi_m \rho_m \Pi_m \leq \frac{1}{d} \Pi_m \ . \end{split}$$

We denote $\rho := \sum_{m} p(m)\rho_m$. We define a sequence of i.i.d. random variables X_1, \ldots, X_L , taking values in $\{\rho_m : m \in \mathsf{M}\}$. If $L \gg \frac{d}{D}$, then

$$Pr\left(\|L^{-1}\sum_{l=1}^{L}\Pi\cdot\Pi_{X_{l}}\cdot X_{l}\cdot\Pi_{X_{l}}\cdot\Pi-\rho\|_{1}\leq\epsilon+4\sqrt{\epsilon}+24\sqrt[4]{\epsilon}\right)$$

$$\geq 1-2D\exp\left(-\frac{\epsilon^{3}Ld}{2\ln 2D}\right) . \tag{15}$$

For our result we use an alternative Covering Lemma.

Lemma 3 Let \mathcal{V} be a finite-dimensional Hilbert space. Let M and $\mathsf{M}' \subset \mathsf{M}$ be finite sets. Suppose we have an ensemble $\{\rho_m : m \in \mathsf{M}\} \subset \mathcal{S}(\mathcal{V})$ of quantum states. Let p be a probability distribution on M .

Suppose a total subspace projector Π and codeword subspace projectors $\{\Pi_m : m \in \mathsf{M}\}\$ exist which project onto subspaces of the Hilbert space in which the states exist, and for all $m \in \mathsf{M}'$ there are positive constants $\epsilon \in]0, 1[$, D, d such that the following conditions hold:

$$\begin{aligned} &\operatorname{tr}(\rho_m \Pi) \geq 1 - \epsilon \ , \\ &\operatorname{tr}(\rho_m \Pi_m) \geq 1 - \epsilon \ , \\ &\operatorname{tr}(\Pi) \leq D \ , \end{aligned}$$

and

$$\Pi_m \rho_m \Pi_m \le \frac{1}{d} \Pi_m \; .$$

We denote $\omega := \sum_{m \in \mathsf{M}'} p(m)\rho_m$. Notice that ω is not a density operator in general. We define a sequence of i.i.d. random variables X_1, \ldots, X_L , taking values in $\{\rho_m : m \in \mathsf{M}\}$. If $L \gg \frac{d}{D}$ then

$$Pr\left(\|L^{-1}\sum_{i=1}^{L}\Pi\cdot\Pi_{X_{i}}\cdot X_{i}\cdot\Pi_{X_{i}}\cdot\Pi-\omega\|_{1}\right)$$

$$\leq 1-p(\mathsf{M}')+4\sqrt{1-p(\mathsf{M}')}+42\sqrt[8]{\epsilon}\right)$$

$$\geq 1-2D\exp\left(-p(\mathsf{M}')\frac{\epsilon^{3}Ld}{2\ln 2D}\right).$$
(16)

Proof We define a function $\mathbb{1}_{\mathsf{M}'}:\mathsf{M}\to\mathsf{M}'\cup\{0^{\mathcal{V}}\}$ by

$$\mathbb{1}_{\mathsf{M}'}(\rho_m) := \begin{cases} \rho_m \ , & \text{if } m \in \mathsf{M}' \\ 0^{\mathcal{V}} \ , & \text{if } m \in \mathsf{M}' \end{cases}$$

where $0^{\mathcal{V}}$ is the zero operator on \mathcal{V} , i.e., $\langle j|0^{\mathcal{V}}|j\rangle = 0$ for all $j \in \mathcal{V}$. Notice that $0^{\mathcal{V}}$ is not a density operator.

We have

$$\operatorname{tr}\left(\sum_{m\in\mathsf{M}}p(m)\mathbb{1}_{\mathsf{M}'}(\rho_m)\right)$$
$$=\operatorname{tr}\left(\sum_{m\in\mathsf{M}'}p(m)\rho_m\right)$$
$$=\sum_{m\in\mathsf{M}'}p(m)\operatorname{tr}(\rho_m)$$
$$=p(\mathsf{M}') . \tag{17}$$

Let $\hat{\Pi}$ be the projector onto the subspace spanned by the eigenvectors of $\sum_{m \in \mathsf{M}'} p(m) \Pi \Pi_m \rho_m \Pi_m \Pi$ whose corresponding eigenvalues are greater than $p(\mathsf{M}') \frac{\epsilon}{D}$.

The following three inequalities can be shown by the same arguments as in the proof of Lemma 2 in [40]:

$$\sum_{m \in \mathsf{M}} p(m) d \cdot \hat{\Pi} \Pi \Pi_m \mathbb{1}_{\mathsf{M}'}(\rho_m) \Pi_m \Pi \hat{\Pi} \ge p(\mathsf{M}') \frac{d\epsilon}{D} \hat{\Pi} .$$
 (18)

$$\begin{split} \|\sum_{m\in\mathsf{M}} p(m)\Pi\cdot\Pi_{m}\cdot\mathbb{1}_{\mathsf{M}'}(\rho_{m})\cdot\Pi_{m}\cdot\Pi-\sum_{m\in\mathsf{M}} p(m)\cdot\mathbb{1}_{\mathsf{M}'}(\rho_{m})\|_{1} \\ &\leq \sum_{m\in\mathsf{M}'} p(m)\|\Pi\cdot\Pi_{m}\rho_{m}\cdot\Pi_{m}\cdot\Pi-\rho_{m}\|_{1} \\ &\leq \sum_{m\in\mathsf{M}'} p(m)\left(2\sqrt{\epsilon}+2\sqrt{\epsilon}+2\sqrt{\epsilon}\right) \\ &= p(\mathsf{M}')\left(2\sqrt{\epsilon}+2\sqrt{\epsilon}+2\sqrt{\epsilon}\right) \\ &\leq 2\sqrt{\epsilon}+2\sqrt{\epsilon+2\sqrt{\epsilon}} \\ &\leq 6\sqrt[4]{\epsilon} \ . \end{split}$$
(19)

The last inequality holds because $\sqrt{\epsilon + 2\sqrt{\epsilon}} \leq 2\sqrt[4]{\epsilon}$ for $0 \leq \epsilon \leq 1$. When $\{\rho_1, \cdots, \rho_L\}$ fulfills

$$\begin{split} &(1-\epsilon)\sum_{m\in\mathsf{M}}p(m)\hat{\Pi}\Pi\cdot\Pi_{m}\cdot\mathbbm{1}_{\mathsf{M}'}(\rho_{m})\cdot\Pi_{m}\cdot\Pi\hat{\Pi}\\ &\leq L^{-1}\sum_{i=1}^{L}\hat{\Pi}\Pi\cdot\Pi_{\rho_{i}}\cdot(\mathbbm{1}_{\mathsf{M}'}(\rho_{i}))\cdot\Pi_{\rho_{i}}\cdot\Pi\hat{\Pi}\\ &\leq (1+\epsilon)\sum_{m\in\mathsf{M}}p(m)\hat{\Pi}\Pi\cdot\Pi_{m}\cdot\mathbbm{1}_{\mathsf{M}'}(\rho_{m})\cdot\Pi_{m}\cdot\Pi\hat{\Pi}\ , \end{split}$$

(i.e. we assume the event considered in (22) below),

then

$$\|L^{-1}\sum_{i=1}^{L}\hat{\Pi}\Pi\cdot\Pi_{\rho_{i}}\cdot(\mathbb{1}_{\mathsf{M}'}(\rho_{i}))\cdot\Pi_{\rho_{i}}\cdot\Pi\hat{\Pi} -\sum_{m\in\mathsf{M}}p(m)\hat{\Pi}\Pi\cdot\Pi_{m}\cdot\mathbb{1}_{\mathsf{M}'}(\rho_{m})\cdot\Pi_{m}\cdot\Pi\hat{\Pi}\|_{1} \leq\epsilon.$$
(20)

i) Application of the Operator Chernoff Bound

For all $m \in \mathsf{M}'$ we have

$$d \cdot \hat{\Pi} \Pi \Pi_m \mathbb{1}_{\mathsf{M}'}(\rho_m) \Pi_m \Pi \hat{\Pi}$$

= $d \cdot \hat{\Pi} \Pi \Pi_m \rho_m \Pi_m \Pi \hat{\Pi}$
 $\leq \hat{\Pi}$ (21)

as a consequence of the inequality $A^{\dagger}BA \leq A^{\dagger}A$ which is valid whenever $B \leq id$.

By (21) and the fact that $d \cdot 0^{\mathcal{V}} \leq \hat{\Pi}$, we have for all $m \in \mathsf{M}$.

$$0^{\mathcal{V}} \leq d \cdot \hat{\Pi} \Pi \Pi_m \mathbb{1}_{\mathsf{M}'}(\rho_m) \Pi_m \Pi \hat{\Pi} \leq \hat{\Pi} .$$

Now we apply the Operator Chernoff Bound (cf. [40]) on the set of operator $\{d\mathbb{1}_{\mathsf{M}'}(\rho_m) : m \in \mathsf{M}\}\)$ and the subspace spanned by the eigenvectors of $\sum_{m\in\mathsf{M}'} p(m)\Pi\Pi_m\rho_m\Pi\Pi_m$ whose corresponding eigenvalues are greater than $p(\mathsf{M}')\frac{\epsilon}{D}$; here $\hat{\Pi}$ acts as the identity on the subspace.

By (18) we obtain

$$\begin{aligned} Pr\bigg((1-\epsilon)\sum_{m\in\mathsf{M}}p(m)\hat{\Pi}\Pi\cdot\Pi_{m}\cdot\mathbbm{1}_{\mathsf{M}'}(\rho_{m})\cdot\Pi_{m}\cdot\Pi\hat{\Pi}\\ &\leq L^{-1}\sum_{i=1}^{L}\hat{\Pi}\Pi\cdot\Pi_{X_{i}}\cdot(\mathbbm{1}_{\mathsf{M}'}(X_{i}))\cdot\Pi_{X_{i}}\cdot\Pi\hat{\Pi}\\ &\leq (1+\epsilon)L^{-1}\sum_{i=1}^{L}\hat{\Pi}\Pi\cdot\Pi_{X_{i}}\cdot(\mathbbm{1}_{\mathsf{M}'}(X_{i}))\cdot\Pi_{X_{i}}\cdot\Pi\hat{\Pi}\\ &= Pr\bigg(d(1-\epsilon)\sum_{m\in\mathsf{M}}p(m)\hat{\Pi}\Pi\cdot\Pi_{m}\cdot\mathbbm{1}_{\mathsf{M}'}(\rho_{m})\cdot\Pi_{m}\cdot\Pi\hat{\Pi}\\ &\leq dL^{-1}\sum_{i=1}^{L}\hat{\Pi}\Pi\cdot\Pi_{X_{i}}\cdot(\mathbbm{1}_{\mathsf{M}'}(X_{i}))\cdot\Pi_{X_{i}}\cdot\Pi\hat{\Pi}\\ &\leq d(1+\epsilon)\sum_{m\in\mathsf{M}}p(m)\hat{\Pi}\Pi\cdot\Pi_{m}\cdot\mathbbm{1}_{\mathsf{M}'}(\rho_{m})\cdot\Pi_{m}\cdot\Pi\hat{\Pi}\bigg)\end{aligned}$$

Holger Boche et al.

$$\geq 1 - 2D \exp\left(-p(\mathsf{M}')\frac{\epsilon^3 L d}{2\ln 2D}\right) \ . \tag{22}$$

 $ii) \ Upper \ Bound \ for \ \|\sum_{m \in \mathsf{M}} p(m) \mathbbm{1}_{\mathsf{M}'}(\rho_m) - \sum_{m \in \mathsf{M}} p(m) \hat{\Pi} \Pi \Pi_m \mathbbm{1}_{\mathsf{M}'}(\rho_m) \Pi_m \Pi \hat{\Pi} \hat{\Pi} \|_1$

Let $\sum_i \lambda_i |i\rangle \langle i|$ be a spectral decomposition of $\sum_{m \in M'} \frac{p(m)}{p(M')} \Pi \Pi_m \rho_m \Pi_m \Pi$. In view of the fact that $\hat{\Pi}$ is the projector onto the subspace spanned by the eigenvectors of the density operator $\sum_{m \in M'} \frac{p(m)}{p(M')} \Pi \Pi_m \rho_m \Pi_m \Pi$ whose corresponding eigenvalues are greater than $\frac{\epsilon}{D}$, we have

$$\operatorname{tr}\left(\sum_{m\in\mathsf{M}}\frac{p(m)}{p(\mathsf{M}')}\Pi\cdot\Pi_{m}\cdot\mathbb{1}_{\mathsf{M}'}(\rho_{m})\cdot\Pi_{m}\cdot\Pi\right)$$
$$-\operatorname{tr}\left(\sum_{m\in\mathsf{M}}\frac{p(m)}{p(\mathsf{M}')}\hat{\Pi}\Pi\cdot\Pi_{m}\cdot\mathbb{1}_{\mathsf{M}'}(\rho_{m})\cdot\Pi_{m}\cdot\Pi\hat{\Pi}\right)$$
$$=\sum_{\lambda_{i}\geq\frac{\epsilon}{D}}\lambda_{i}$$
$$<\epsilon.$$

We apply the gentle operator lemma (cf. [40]) to obtain

$$\begin{aligned} \|\sum_{m\in\mathsf{M}} p(m)\Pi\cdot\Pi_{m}\cdot\mathbbm{1}_{\mathsf{M}'}(\rho_{m})\cdot\Pi_{m}\cdot\Pi - \sum_{m\in\mathsf{M}} p(m)\hat{\Pi}\Pi\cdot\Pi_{m}\cdot\mathbbm{1}_{\mathsf{M}'}(\rho_{m})\cdot\Pi_{m}\cdot\Pi\hat{\Pi}\|_{1} \\ &= p(\mathsf{M}')\|\sum_{m\in\mathsf{M}} \frac{p(m)}{p(\mathsf{M}')}\Pi\cdot\Pi_{m}\cdot\mathbbm{1}_{\mathsf{M}'}(\rho_{m})\cdot\Pi_{m}\cdot\Pi - \sum_{m\in\mathsf{M}} \frac{p(m)}{p(\mathsf{M}')}\hat{\Pi}\Pi\cdot\Pi_{m}\cdot\mathbbm{1}_{\mathsf{M}'}(\rho_{m})\cdot\Pi_{m}\cdot\Pi\hat{\Pi}\|_{1} \\ &\leq 2\sqrt{\epsilon+2\sqrt{\epsilon}} \\ &\leq 4\sqrt[4]{\epsilon} \ . \end{aligned}$$

$$(23)$$

When $\{\rho_1, \cdots, \rho_L\}$ fulfills

$$\begin{split} \|L^{-1} \sum_{i=1}^{L} \hat{\Pi} \Pi \cdot \Pi_{i} \cdot (\mathbb{1}_{\mathsf{M}'}(\rho_{i})) \cdot \Pi_{i} \cdot \Pi \hat{\Pi} \\ - \sum_{m \in \mathsf{M}} p(m) \hat{\Pi} \Pi \cdot \Pi_{m} \cdot \mathbb{1}_{\mathsf{M}'}(\rho_{m}) \cdot \Pi_{m} \cdot \Pi \hat{\Pi} \|_{1} \\ \leq \epsilon \end{split}$$

(i.e. we assume the event considered in (22) occurs and thus (20) holds), then by (19) and (23) it holds that

$$\|L^{-1}\sum_{i=1}^{L}\hat{\Pi}\Pi\cdot\Pi_{i}\cdot(\mathbb{1}_{\mathsf{M}'}(\rho_{i}))\cdot\Pi_{i}\cdot\Pi\hat{\Pi}-\sum_{m\in\mathsf{M}}p(m)\mathbb{1}_{\mathsf{M}'}(\rho_{m})\|_{1}$$

$$\leq \epsilon + 10 \sqrt[4]{\epsilon}$$

$$\leq 11 \sqrt[4]{\epsilon} . \tag{24}$$

iii) Upper Bound for $\|L^{-1} \sum_{i=1}^{L} \Pi \Pi_i(\mathbb{1}_{\mathsf{M}'}(\rho_i)) \Pi_i \Pi - L^{-1} \sum_{i=1}^{L} \hat{\Pi} \Pi \Pi_i(\mathbb{1}_{\mathsf{M}'}(\rho_i)) \Pi_i \Pi \hat{\Pi} \hat{\Pi}\|_1$

When the event considered in (22) is true, i.e., when (24) holds, then by (17)

$$\operatorname{tr}\left(L^{-1}\sum_{i=1}^{L}\widehat{\Pi}\Pi\cdot\Pi_{i}\cdot(\mathbb{1}_{\mathsf{M}'}(\rho_{i}))\cdot\Pi_{i}\cdot\Pi\widehat{\Pi}\right)$$

$$\geq p(\mathsf{M}')-11\sqrt[4]{\epsilon}.$$

We apply the gentle operator lemma (cf. [40]) to obtain

$$\|L^{-1}\sum_{i=1}^{L}\hat{\Pi}\Pi\cdot\Pi_{i}\cdot(\mathbb{1}_{\mathsf{M}'}(\rho_{i}))\cdot\Pi_{i}\cdot\Pi\hat{\Pi}-L^{-1}\sum_{i=1}^{L}\Pi\cdot\Pi_{i}\cdot(\mathbb{1}_{\mathsf{M}'}(\rho_{i}))\cdot\Pi_{i}\cdot\Pi\|_{1}$$

$$\leq 2\sqrt{1-p(\mathsf{M}')+11\sqrt[4]{\epsilon}}$$

$$\leq 2\sqrt{1-p(\mathsf{M}')}+22\sqrt[8]{\epsilon}.$$
(25)

The last inequality holds because $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for positive a and b.

iv) Upper Bound for $\|L^{-1} \sum_{i=1}^{L} \Pi \Pi_i \rho_i \Pi_i \Pi - L^{-1} \sum_{i=1}^{L} \Pi \Pi_i (\mathbb{1}_{\mathsf{M}'}(\rho_i)) \Pi_i \Pi\|_1$

In view of the fact that Π and Π_i are projection matrices for every $\rho_i \in \{\rho_1, \cdots, \rho_L\}$ it holds that

$$\operatorname{tr}(\Pi_i \rho_l \Pi_i) \le \operatorname{tr}(\rho_l) = 1$$

and

$$\operatorname{tr}(L^{-1}\sum_{i=1}^{L}\Pi\Pi_{i}\rho_{l}\Pi_{i}\Pi)$$
$$\leq \operatorname{tr}(L^{-1}\sum_{i=1}^{L}\Pi_{i}\rho_{l}\Pi_{i})$$
$$\leq 1.$$

When $\{\rho_1, \cdots, \rho_L\}$ fulfills

$$\begin{split} \|L^{-1} \sum_{i=1}^{L} \Pi \cdot \Pi_{i} \cdot (\mathbb{1}_{\mathsf{M}'}(\rho_{i})) \cdot \Pi_{i} \cdot \Pi - \sum_{m \in \mathsf{M}} p(m) \mathbb{1}_{\mathsf{M}'}(\rho_{m})\|_{1} \\ \leq 2\sqrt{1 - p(\mathsf{M}')} + 20 \sqrt[8]{\epsilon} , \end{split}$$

i.e., we assume that the event considered in (22) is true, and then by (17) and the triangle inequality, we have

$$\operatorname{tr}\left(\Pi \cdot \Pi_{i} \cdot (\mathbb{1}_{\mathsf{M}'}(\rho_{i})) \cdot \Pi_{i} \cdot \Pi\right)$$

$$\geq p(\mathsf{M}') - 2\sqrt{1 - p(\mathsf{M}')} - 20\sqrt[8]{\epsilon} . \tag{26}$$

Since

$$L^{-1} \sum_{i=1}^{L} \Pi \cdot \Pi_{i} \cdot \rho_{i} \cdot \Pi_{i} \cdot \Pi$$
$$= L^{-1} \sum_{i=1}^{L} \Pi \cdot \Pi_{i} \cdot \mathbb{1}_{\mathsf{M}'}(\rho_{i}) \cdot \Pi_{i} \cdot \Pi$$
$$+ L^{-1} \sum_{i \notin \mathsf{M}'} \Pi \cdot \Pi_{i} \cdot \rho_{i} \cdot \Pi_{i} \cdot \Pi , \qquad (27)$$

we have

$$\|L^{-1} \sum_{i \notin \mathsf{M}'} \Pi \cdot \Pi_i \cdot \rho_i \cdot \Pi_i \cdot \Pi\|_1$$

= tr $\left(L^{-1} \sum_{i \notin \mathsf{M}'} \Pi \cdot \Pi_i \cdot \rho_i \cdot \Pi_i \cdot \Pi \right)$
 $\leq 1 - p(\mathsf{M}') + 2\sqrt{1 - p(\mathsf{M}')} + 20 \sqrt[8]{\epsilon},$ (28)

which implies

$$\|L^{-1} \sum_{i=1}^{L} \Pi \cdot \Pi_{i} \cdot \rho_{i} \cdot \Pi_{i} \cdot \Pi - \sum_{m \in \mathsf{M}} p(m) \mathbb{1}_{\mathsf{M}'}(\rho_{m})\|_{1}$$

$$\leq 1 - p(\mathsf{M}') + 4\sqrt{1 - p(\mathsf{M}')} + 42\sqrt[8]{\epsilon} .$$
(29)

By (29), we have

$$Pr\left(\|L^{-1}\sum_{i=1}^{L}\Pi\cdot\Pi_{X_{i}}\cdot X_{i}\cdot\Pi_{X_{i}}\cdot\Pi-\sum_{m\in\mathsf{M}}p(m)\cdot\mathbb{1}_{\mathsf{M}'}(\rho_{m})\|_{1}\right)$$

$$\leq 1-p(\mathsf{M}')+4\sqrt{1-p(\mathsf{M}')}+42\sqrt[s]{\epsilon}\right)$$

$$\geq 1-2D\exp\left(-p(\mathsf{M}')\frac{\epsilon^{3}Ld}{2\ln 2D}\right).$$
(30)

By (2), we have

$$\operatorname{tr}(\Pi_{\mathsf{V}(\cdot,p),\alpha}(t^n))$$

$$\leq 2^{n(S(\mathsf{V}(\cdot,p)|q)+\delta(\alpha))}$$

= $2^{n(\sum_{t} q(t)\mathsf{V}(t,p)+\delta(\alpha))}$
= $2^{n(\sum_{t} q(t)S(V_t(p))+\delta(\alpha))}$. (31)

Furthermore, for all x^n it holds that

$$\Pi_{\mathbf{V},\alpha}(t^{n},x^{n})V_{t^{n}}(x^{n})\Pi_{\mathbf{V},\alpha}(t^{n},x^{n})$$

$$\leq 2^{-n(S(\mathbf{V}|r)+\delta(\alpha)')}\Pi_{\mathbf{V},\alpha}(t^{n},x^{n})$$

$$= 2^{-n(\sum_{t,x}r(t,x)S(\mathbf{V}(t,x))+\delta(\alpha)')}\Pi_{\mathbf{V},\alpha}(t^{n},x^{n}).$$
(32)

We define

$$\theta' := \left\{ t \in \theta : nq(t) \ge \sqrt{n} \right\}$$

By properties of classical typical set (cf. [41]), there is a positive $\hat{\beta}(\alpha)$ such that

$$\Pr_{p'}\left(x^n \in \left\{x^n \in \mathsf{A}^n : (x_{\mathsf{I}_t}) \in \mathsf{T}_{p,\delta}^{nq(t)} \,\forall t \in \theta'\right\}\right) \ge \left(1 - 2^{-\sqrt{n}\hat{\beta}(\alpha)}\right)^{|\theta|} \ge 1 - 2^{-\sqrt{n}\frac{1}{2}\hat{\beta}(\alpha)}$$

$$(33)$$

where $I_t := \{i \in \{1, \cdots, n\} : t_i = t\}$ is an indicator set that selects the indices *i* in the sequence $t^n = (t_1, \cdots, t_n)$.

We denote the set $\{x^n : (x_{\mathbf{I}_t}) \in \mathsf{T}_{p,\delta}^{nq(t)} \ \forall t \in \theta'\} \subset \mathsf{A}^n$ by M_{t^n} . For all $x^n \in \mathsf{M}_{t^n}$, if n is sufficiently large, we have

$$\begin{aligned} \left| \sum_{t,x} r(t,x) S(\mathsf{V}(t,x)) - \sum_{t} q(t) S(V_{t}|p) \right| \\ &\leq \left| \sum_{t \in \theta',x} r(t,x) S(\mathsf{V}(t,x)) - \sum_{t \in \theta'} q(t) S(V_{t}|p) \right| \\ &+ \left| \sum_{t \notin \theta',x} r(t,x) S(\mathsf{V}(t,x)) - \sum_{t \notin \theta'} q(t) S(V_{t}|p) \right| \\ &\leq \sum_{t \in \theta'} \left| \sum_{x} r(t,x) S(\mathsf{V}(t,x)) - q(t) S(V_{t}|p) \right| + 2|\theta| \frac{1}{\sqrt{n}} C \\ &\leq 2|\theta| \frac{\delta}{n} C + 2|\theta| \frac{1}{\sqrt{n}} C , \end{aligned}$$
(34)

where $C := \max_{t \in \theta} \max_{x \in A} (S(\mathsf{V}(t, x)) + S(V_t|p)).$ We set $\Theta_{t^n} := \sum_{x^n \in \mathsf{M}_{t^n}} p(x^n)Q_{t^n}(x^n)$. For given $z^n \in \mathsf{M}_{t^n}$ and $t^n \in \theta^n$, $\langle z^n | \Theta_{t^n} | z^n \rangle$ is the expected value of $\langle z^n | Q_{t^n}(x^n) | z^n \rangle$ under the condition $x^n \in$ M_{t^n} .

We choose a positive $\bar{\beta}(\alpha)$ such that $\bar{\beta}(\alpha) \leq \min(2^{-n\beta(\alpha)}, 2^{-n\beta(\alpha)'})$, and set $\epsilon := 2^{-n\bar{\beta}(\alpha)}$. In view of (32), we now apply Lemma 3, where we consider the set $\mathsf{M}_{t^n} \subset \mathsf{A}^n$: If *n* is sufficiently large, for all *j* we have

$$Pr\left(\left\|\sum_{l=1}^{L_{n}}\frac{1}{L_{n}}Q_{t^{n}}(X_{j,l}) - \Theta_{t^{n}}\right\|_{1} > 2^{-\sqrt{n}\frac{1}{8}\hat{\beta}(\alpha)} + 40\sqrt[8]{\epsilon}\right)$$

$$\leq 2^{n(\sum_{t,x}r(t,x)S(\mathsf{V}(t,x)) + \delta(\alpha))}$$

$$\cdot \exp\left(-L_{n}\frac{\epsilon^{3}}{2\ln 2}(1 - 2^{-\sqrt{n}\frac{1}{2}\hat{\beta}(\alpha)}) \cdot 2^{n(\sum_{t}q(t)S(V_{t}(p)) - \sum_{t}q(t)S(V_{t}|p)) + \delta(\alpha) + \delta(\alpha)' + 2|\theta|\frac{\delta}{n}C + 2|\theta|\frac{1}{\sqrt{n}}C}\right)$$

$$= 2^{n(\sum_{t,x}r(t,x)S(\mathsf{V}(t,x)) + \delta(\alpha)}$$

$$\cdot \exp\left(-L_{n}\frac{\epsilon^{3}}{2\ln 2} \cdot (1 - 2^{-\sqrt{n}\frac{1}{2}\hat{\beta}(\alpha)})2^{n(-\sum_{t}q(t)\chi(p;Z_{t}) + \delta(\alpha) + \delta(\alpha)' + 2|\theta|\frac{\delta}{n}C + 2|\theta|\frac{1}{\sqrt{n}}C}\right).$$
(35)

The equality holds since $S(V_t(p)) - S(V_t|p) = \chi(p; Z_t)$. Furthermore,

$$Pr\left(\|\sum_{l=1}^{L_{n}} \frac{1}{L_{n}} Q_{t^{n}}(X_{j,l}) - \Theta_{t^{n}} \|_{1} > 2^{-\sqrt{n}\frac{1}{8}\hat{\beta}(\alpha)} + 40\sqrt[8]{\epsilon} \forall t^{n} \forall j \right)$$

$$\leq J_{n} |\theta|^{n} 2^{n(\sum_{t,x} r(t,x)S(\mathsf{V}(t,x)) + \delta(\alpha))} \cdot \exp\left(-L_{n} \frac{\epsilon^{3}}{2\ln 2} (1 - 2^{-\sqrt{n}\frac{1}{2}\hat{\beta}(\alpha)}) 2^{n(-\sum_{t} q(t)\chi(p;Z_{t}) + \delta(\alpha) + \delta(\alpha)' + 2|\theta|\frac{\delta}{n}C + 2|\theta|\frac{1}{\sqrt{n}}C)} \right)$$
(36)

Let ϕ_t^j be the quantum state at the output of wiretapper's channel when the channel state is t and j has been sent. We have

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$$\begin{split} &\sum_{t\in\theta} q(t)\chi\left(p;Z_{t}\right) - \chi\left(p;\sum_{t}q(t)Z_{t}\right) \\ &= \sum_{t\in\theta} q(t)S\left(\sum_{j=1}^{J_{n}}\frac{1}{J_{n}}\phi_{t}^{j}\right) - \sum_{t\in\theta}\sum_{j=1}^{J_{n}}q(t)\frac{1}{J_{n}}S\left(\phi_{t}^{j}\right) \\ &- S\left(\frac{1}{J_{n}}\sum_{t\in\theta}\sum_{j=1}^{J_{n}}q(t)\phi_{t}^{j}\right) + \sum_{j=1}^{J_{n}}\frac{1}{J_{n}}S\left(\sum_{t\in\theta}q(t)\phi_{t}^{j}\right) \ . \end{split}$$

Let $H^{\mathfrak{T}}$ be a $|\theta|$ -dimensional Hilbert space spanned by an orthonormal basis $\{|t\rangle : t = 1, \cdots, |\theta|\}$. Let $H^{\mathfrak{J}}$ be a J_n dimensional Hilbert space spanned by an orthonormal basis $\{|j\rangle : j = 1, \cdots, J_n\}$. We define

$$\varphi^{\mathfrak{IT}H^n} := \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{t \in \theta} q(t) |j\rangle \langle j| \otimes |t\rangle \langle t| \otimes \phi_t^j \ .$$

We have

$$\begin{split} \varphi^{\mathfrak{J}H^n} &= \operatorname{tr}_{\mathfrak{T}} \left(\varphi^{\mathfrak{J}\mathfrak{T}H^n} \right) = \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{t \in \theta} q(t) |j\rangle \langle j| \otimes \phi_t^j ; \\ \varphi^{\mathfrak{T}H^n} &= \operatorname{tr}_{\mathfrak{J}} \left(\varphi^{\mathfrak{J}\mathfrak{T}H^n} \right) = \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{t \in \theta} q(t) |t\rangle \langle t| \otimes \phi_t^j ; \\ \varphi^{H^n} &= \operatorname{tr}_{\mathfrak{J}} \mathfrak{T} \left(\varphi^{\mathfrak{J}\mathfrak{T}H^n} \right) = \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{t \in \theta} q(t) \phi_t^j . \end{split}$$

Thus,

$$\begin{split} S(\varphi^{\Im H^n}) &= H(R_{uni}) + \frac{1}{J_n} \sum_{j=1}^{J_n} S\left(\sum_{t \in \theta} q(t)\phi_t^j\right) \;; \\ S(\varphi^{\Im H^n}) &= H(Y_q) + \sum_{t \in \theta} q(t) S\left(\frac{1}{J_n} \sum_{j=1}^{J_n} \phi_t^j\right) \;; \\ S(\varphi^{\Im \Pi^n}) &= H(R_{uni}) + H(Y_q) + \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{t \in \theta} q(t) S\left(\phi_t^j\right) \;, \end{split}$$

where Y_q is a random variable on θ with distribution q(t).

By strong subadditivity of von Neumann entropy, it holds that $S(\varphi^{\mathfrak{I}H^n}) + S(\varphi^{\mathfrak{I}H^n}) \geq S(\varphi^{H^n}) + S(\varphi^{\mathfrak{I}\mathfrak{I}^n})$, therefore

$$\sum_{t} q(t)\chi(p; Z_t) - \chi\left(p; \sum_{t} q(t)Z_t\right) \ge 0.$$
(37)

For an arbitrary ζ , we define $L_n = \lceil 2^{\max_{t^n} \chi(p; Z_{t^n}) + n\zeta} \rceil$, and choose a suitable α , $\bar{\beta}(\alpha)$, and sufficiently large n such that $6\bar{\beta}(\alpha) + 2\delta(\alpha) + 2\delta(\alpha)' + 2|\theta|\frac{\delta}{n}C + 2|\theta|\frac{1}{\sqrt{n}}C \leq \zeta$. By (37), if n is sufficiently large, we have $L_n \geq \lceil 2^{n(\sum_t q(t)\chi(p; Z_t) + \zeta)} \rceil$ and

$$L_n \frac{\epsilon^3}{2\ln 2} (1 - 2^{-\sqrt{n}\frac{1}{2}\hat{\beta}(\alpha)}) 2^{n(-\sum_t q(t)\chi(p;Z_t) + \delta(\alpha) + \delta(\alpha)' + 2|\theta|\frac{\delta}{n}C + 2|\theta|\frac{1}{\sqrt{n}}C)} > 2^{\frac{1}{2}n\zeta} .$$

When n is sufficiently large for any positive ϑ it holds that

$$J_{n}|\theta|^{n}2^{n(\sum_{t,x}r(t,x)S(\mathsf{V}(t,x))+\delta(\alpha)}\exp(-2^{\frac{1}{4}n\zeta})$$

$$\leq 2^{-n\vartheta}$$

and

$$2^{-\sqrt{n}\frac{1}{8}\hat{\beta}(\alpha)} + 40\sqrt[8]{\epsilon} \le 2^{-\sqrt{n}\frac{1}{16}\hat{\beta}(\alpha)} .$$

Thus for sufficiently large n we have

$$Pr\left(\|\sum_{l=1}^{L_n} \frac{1}{L_n} Q_{t^n}(X_{j,l}) - \Theta_{t^n}\|_1 \le 2^{-\sqrt{n} \frac{1}{16}\hat{\beta}(\alpha)} \ \forall t^n \ \forall j\right)$$
$$\ge 1 - 2^{n\vartheta} \tag{38}$$

for any positive ρ .

Now we have $J_n \cdot L_n < 2^{n(\min_s \chi(p; B_s) - \mu)}$.

In [12] and [14], the following was shown (using results of [31]). Let $\{X_{j,l}\}_{j \in \{1,...,J_n\}, l \in \{1,...,L_n\}}$ be a family of random variables taking value according to $\dot{p} \in \mathsf{P}(\mathsf{A}^n)$. If n is sufficiently large, and if $J_n \cdot L_n \leq 2^{\min_s n(\chi(\dot{p}; B_s) - \mu)}$ for an arbitrary positive μ there exists a projection q_{x^n} on H for every $x^n \in A^n$ and positive constants β and γ , such that for any $(s, j, l) \in \theta \times \{1, \dots, J_n\} \times \{1, \dots, L_n\}$, it holds that

$$P_{\dot{p}}^{r}\left[\operatorname{tr}\left(\overline{W}_{s}^{\otimes n}(\dot{X}_{j,l})D_{\dot{X}_{j,l}}\right) \geq 1 - |\overline{\theta}|2^{-n^{1/16}\beta}\right] > 1 - 2^{-n\gamma} , \qquad (39)$$

where for $j \in \{1, ..., J_n\}, l \in \{1, ..., L_n\}$, we have

$$D_{\dot{X}_{j,l}} := \left(\sum_{j',l'} q_{\dot{X}_{j',l'}}\right)^{-\frac{1}{2}} q_{\dot{X}_{j,l}} \left(\sum_{j',l'} q_{\dot{X}_{j',l'}}\right)^{-\frac{1}{2}}$$

Notice that by this definition, for any realization $\{\dot{x}_{j,l}: j, l\}$ of $\{\dot{X}_{j,l}: j, l\}$ it

holds that $\sum_{j=1}^{J_n} \sum_{l=1}^{L_n} D_{\dot{x}_{j,l}} \leq \operatorname{id}_{H^{\otimes n}}$. (Actually in [20], it was shown that there exists a collection of positive semidefinite operators $\{D_{s,\dot{X}_{j,l}}: s \in \overline{\theta}, j \in \{1,\ldots,J_n\}, l \in \{1,\ldots,L_n\}\}$ such that for any s, j, and l it holds that

$$Pr\left[\operatorname{tr}\left(\overline{W}_{s}^{\otimes n}(\dot{X}_{j,l})D_{s,\dot{X}_{j,l}}\right) \geq 1 - 2^{|\overline{\theta}|}2^{-n\beta}\right] > 1 - 2^{-n\gamma} ,$$

and for any realization $\{\dot{x}_{j,l}: j, l\}$ of $\{\dot{X}_{j,l}: j, l\}$ it holds that $\sum_{s \in \overline{\theta}} \sum_{j=1}^{J_n} \sum_{l=1}^{L_n} D_{s, \dot{x}_{j,l}} \leq C_{s, j, l}$ $\operatorname{id}_{H^{\otimes n}}$.)

For any given $s \in \theta$, it holds that

$$\overline{W}_{s}^{\otimes n}(p^{n}) - \overline{W}_{s}^{\otimes n}(p'^{n}) = \left(1 - \frac{1}{P(\mathsf{T}_{p,\delta}^{n})}\right) \sum_{a^{n} \in \mathsf{T}_{p,\delta}^{n}} p^{n}(a^{n}) \overline{W}_{s}^{\otimes n}(a^{n}) + \sum_{a^{n} \notin \mathsf{T}_{p,\delta}^{n}} p^{n}(a^{n}) \overline{W}_{s}^{\otimes n}(a^{n}) \ .$$

Thus, we have $\left| \operatorname{tr} \left(\overline{W}_s^{\otimes n}(p^n) - \overline{W}_s^{\otimes n}(p'^n) \right) \right| \leq 2P(\mathsf{T}_{p,\delta}^n) \leq 2^{-n\eta(\delta)}$ for a positive $\eta(\delta)$.

Lemma 4 (Fannes-Audenaert Ineq., cf. [30], [9]) Let Φ and Ψ be two quantum states in a d-dimensional complex Hilbert space and $\|\Phi - \Psi\|_1 \leq \mu < \frac{1}{e}$, then

$$|S(\Phi) - S(\Psi)| \le \mu \log(d - 1) + h(\mu) , \qquad (40)$$

where $h(\nu) := -\nu \log \nu - (1 - \nu) \log(1 - \nu)$ for $\nu \in [0, 1]$.

The Fannes Inequality was first introduced in [30] where it has been shown that $|S(\Phi) - S(\Psi)| \leq \mu \log d - \mu \log \mu$. In [9], the result of [30] has been improved, and (40) has been proved.

By Lemma 4 for any positive ω , if n is sufficiently large, we have

$$S\left(\overline{W}_{s}^{\otimes n}(p^{n})\right) - S\left(\overline{W}_{s}^{\otimes n}(p'^{n})\right) \\\leq 2^{-n\eta(\delta)}\log(d^{n}-1) + h(2^{-n\eta(\delta)}) \\\leq \omega .$$

Furthermore, we have

$$\begin{split} & \left| \sum_{a^n \in \mathsf{T}^n_{p,\delta}} p'^n(a^n) S\left(\overline{W}_s^{\otimes n}(a^n)\right) - \sum_{a^n \in \mathsf{T}^n_{p,\delta}} p'^n(a^n) S\left(\overline{W}_s^{\otimes n}(a^n)\right) \right| \\ &= \left| \left(1 - \frac{1}{P(\mathsf{T}^n_{p,\delta})} \right) \sum_{a^n \in \mathsf{T}^n_{p,\delta}} p^n(a^n) S\left(\overline{W}_s^{\otimes n}(a^n)\right) + \sum_{a^n \notin \mathsf{T}^n_{p,\delta}} p^n(a^n) S\left(\overline{W}_s^{\otimes n}(a^n)\right) \right| \\ &\leq 2P(\mathsf{T}^n_{p,\delta}) \max_{a^n \in \mathsf{A}^n} S\left(\overline{W}_s^{\otimes n}(a^n)\right) \\ &\leq \omega \end{split}$$

for any positive ω , if n is sufficiently large.

We now have

$$\begin{split} & \left| \chi(p; B_s^{\otimes n}) - \chi(p'; B_s^{\otimes n}) \right| \\ & \leq \left| S\left(\overline{W}_s^{\otimes n}(p^n) \right) - S\left(\overline{W}_s^{\otimes n}(p'^n) \right) \right| \\ & + \left| \sum_{a^n \in \mathsf{T}_{p,\delta}^n} p'^n(a^n) S\left(\overline{W}_s^{\otimes n}(a^n) \right) - \sum_{a^n \in \mathsf{T}_{p,\delta}^n} p'^n(a^n) S\left(\overline{W}_s^{\otimes n}(a^n) \right) \right| \\ & \leq 2\omega \end{split}$$

for any positive ω , if n is sufficiently large.

Thus, when $J_n \cdot L_n < 2^{\min_s n\chi(p;B_s)-\mu}$ holds, we also have

$$J_n \cdot L_n < 2^{\min_s n\chi(p';B_s)-\mu} \tag{41}$$

if n is sufficiently large.

By (41), we can apply (39) to $X_{j,l}$. We have: If n is sufficiently large, the event

$$\left(\bigcap_{s} \left\{ \max_{j \in \{1,\dots,J_n\}} \max_{l \in \{1,\dots,L_n\}} \operatorname{tr}\left(\overline{W}_{s}^{\otimes n}(X_{j,l})D_{X_{j,l}}\right) \ge 1 - |\overline{\theta}| 2^{-n^{1/16}\beta} \right\} \right)$$
$$\cap \left(\|\sum_{l=1}^{L_n} \frac{1}{L_n} Q_{t^n}(X_{j,l}) - \Theta_{t^n}\|_1 \le 2^{-\sqrt{n}\frac{1}{16}\hat{\beta}(\alpha)} \ \forall t^n \ \forall j \right)$$

has a positive probability with respect to p'.

This means that for any $\epsilon > 0$, if n is sufficiently large we can find a realization $x_{j,l}$ of $X_{j,l}$ with a positive probability such that for all $s \in \overline{\theta}$, $t^n \in \theta^n, \pi \in \Pi_n, \text{ and } j \in \{1, \dots, J_n\}, \text{ we have }$

$$\sum_{l=1}^{L_n} \operatorname{tr}\left(\overline{W}_s^{\otimes n}(x_{j,l}) D_{x_{j,l}}\right) \ge 1 - \epsilon , \qquad (42)$$

and

$$\|\sum_{l=1}^{L_n} \frac{1}{L_n} Q_{t^n}(x_{j,l}) - \Theta_{t^n}\|_1 \le 2^{-\sqrt{n} \frac{1}{16}\hat{\beta}(\alpha)} .$$
(43)

We define for $\pi \in \Pi_n$ its permutation matrix on $H^{\otimes n}$ by P_{π} . We have $V_{t^n}(\pi(x^n)) = P_{\pi}V_{\pi^{-1}(t^n)}(x^n)P_{\pi}^{\dagger}$. For $\pi \in \Pi_n$, we define $\Theta_{t^n,\pi} := \sum_{x^n \in \mathsf{T}_{p,\delta}} p'(x^n)Q_{t^n}(\pi(x^n))$. We have $\Theta_{t^n,\pi} = P_{\pi} \left(\sum_{x^n \in \mathsf{T}_{p,\delta}} p'(x^n) Q_{\pi^{-1}(t^n)}(x^n) \right) P_{\pi}^{\dagger} = P_{\pi} \Theta_{\pi(t^n)} P_{\pi}^{\dagger}.$ We choose a suitable positive α . For any given $j' \in \{1, \ldots, J_n\}$, we have

$$\begin{split} \left\| \sum_{l=1}^{L_{n}} \frac{1}{L_{n}} V_{t^{n}}(\pi(x_{j',l})) - \Theta_{t^{n},\pi} \right\|_{1} \\ \leq \left\| \sum_{l=1}^{L_{n}} \frac{1}{L_{n}} V_{t^{n}}(\pi(x_{j',l})) - \sum_{l=1}^{L_{n}} \frac{1}{L_{n}} Q_{t^{n}}(\pi(x_{j',l})) \right\|_{1} \\ + \left\| \sum_{l=1}^{L_{n}} \frac{1}{L_{n}} Q_{t^{n}}(\pi(x_{j',l})) - \Theta_{t^{n},\pi} \right\|_{1} \\ \leq \sum_{l=1}^{L_{n}} 2^{-\sqrt{n} \frac{1}{16} \hat{\beta}(\alpha)} + \left\| P_{\pi} Q_{\pi^{-1}(t^{n})}(x_{j',l}) P_{\pi}^{\dagger} - P_{\pi} \Theta_{\pi(t^{n})} P_{\pi}^{\dagger} \right\|_{1} \\ = 2^{-\sqrt{n} \frac{1}{16} \hat{\beta}(\alpha)} + \left\| \sum_{l=1}^{L_{n}} \frac{1}{L_{n}} Q_{\pi^{-1}(t^{n})}(x_{j',l}) - \Theta_{\pi^{-1}(t^{n})} \right\|_{1} \\ \leq 2^{-\sqrt{n} \frac{1}{16} \hat{\beta}(\alpha)} + \sqrt{2^{-\frac{1}{2}n\beta(\alpha)} + 2^{-\frac{1}{2}n\beta(\alpha)''}} \\ \leq 2^{-\sqrt{n} \frac{1}{32} \hat{\beta}(\alpha)} , \end{split}$$

$$(44)$$

where the first inequality is an application of the triangle inequality and the second is again the triangle inequality combined with (14). The following equality follows because $||U \cdot A \cdot U^{\dagger}||_1 = ||A||_1$ for all $A \in \mathcal{B}(H^{\otimes n})$ and unitary matrices $U \in \mathcal{B}(H^{\otimes n})$. At last, we use (43).

By (44), we have

$$\begin{split} & \| \frac{1}{J_n \cdot L_n} \sum_{j=1}^{J_n} \sum_{l=1}^{L_n} V_{t^n}(\pi(x_{j,l})) - \Theta_{t^n,\pi} \|_1 \\ & \leq 2^{-\sqrt{n} \frac{1}{32} \hat{\beta}(\alpha)} . \end{split}$$

By Lemma 4 and the inequality (44), for a uniformly distributed random variable R_{uni} with values in $\{1, \ldots, J_n\}$ and all $\pi \in \Pi_n$ and $t^n \in \theta^n$, we have

$$\chi(R_{uni}; Z_{t^{n}, \pi}) = S\left(\sum_{j=1}^{J_{n}} \frac{1}{J_{n}} \sum_{l=1}^{L_{n}} \frac{1}{L_{n}} V_{t^{n}}(\pi(x_{j,l}))\right) - \sum_{j=1}^{J_{n}} \frac{1}{J_{n}} S\left(\sum_{l=1}^{L_{n}} \frac{1}{L_{n}} V_{t^{n}}(\pi(x_{j,l}))\right) \\ \leq \left|S\left(\sum_{j=1}^{J_{n}} \frac{1}{J_{n}} \sum_{l=1}^{L_{n}} \frac{1}{L_{n}} V_{t^{n}}(\pi(x_{j,l}))\right) - S\left(\Theta_{t^{n},\pi}\right)\right| \\ + \left|S(\Theta_{t^{n},\pi}) - \sum_{j=1}^{J_{n}} \frac{1}{J_{n}} S\left(\sum_{l=1}^{L_{n}} \frac{1}{L_{n}} V_{t^{n}}(\pi(x_{j,l}))\right)\right| \\ \leq 2 \cdot 2^{-\sqrt{n} \frac{1}{32}\hat{\beta}(\alpha)} \log(nd-1) + 2h(2^{-\sqrt{n} \frac{1}{32}\hat{\beta}(\alpha)}) .$$
(45)

By (45), for any positive λ if n is sufficiently large, we have

$$\max_{t^n \in \theta^n} \chi(R_{uni}; Z_{t^n, \pi}) \le \lambda .$$
(46)

For an arbitrary positive δ , let

 $J_n := 2^{n \min_{s \in \overline{\theta}} \chi(p; B_s) - \max_{t^n \in \theta^n} \chi(p; Z_{t^n}) - n\delta} .$

Now we define a code $(E, \{D_j : j = 1, \ldots, J_n\})$, by $E(x^n \mid j) = \frac{1}{L_n}$ if $x^n \in \{x_{j,l} : l \in \{1, \ldots, L_n\}$, and $E(x^n \mid j) = 0$ if $x \notin \{x_{j,l} : l \in \{1, \ldots, L_n\}$, and $D_j := \frac{1}{L_n} \sum_{l=1}^{L_n} D_{x_{j,l}}$. For any positive λ and ϵ if n is sufficiently large, by (42) and (46), it holds that

$$\max_{s\in\overline{\theta}} \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{a^n\in A^n} E^n(a^n|j) \operatorname{tr}\left(\overline{W}_s^{\otimes n}(a^n)D_j\right) \ge 1-\epsilon$$
$$\max_{t^n\in\theta^n} \max_{\pi\in\Pi_n} \chi\left(R_{uni}; Z_{t^n,\pi}\right) \le \epsilon \;.$$

We obtain

$$\hat{C}_s(\{(\overline{W}_s, V_t) : s \in \overline{\theta}, t \in \theta\}) \ge \min_{s \in \overline{\theta}} \chi(p; B_s) - \lim_{n \to \infty} \frac{1}{n} \max_{t^n \in \theta^n} \chi(p; Z_{t^n}) .$$
(47)

The achievability of $\lim_{n\to\infty} \frac{1}{n} \left(\min_{s\in\overline{\theta}} \chi(p_U; B_s^{\otimes n}) - \max_{t^n\in\theta^n} \chi(p_U; Z_{t^n}) \right)$ is then shown via standard arguments (cf. [27]).

Now we are going to prove the converse.

Let $(\mathcal{C}_n) = (E^{(n)}, \{D_j^{(n)}: j\})$ be a sequence of (n, J_n) code such that

$$\max_{s \in \overline{\theta}} P_e(\mathcal{C}_n, s, n) \le \lambda_n ,$$
$$\max_{s \in \overline{\theta}} \max_{x \in \overline{\theta}} \chi(R_{uni}; Z_{t^n, \pi}) \le \epsilon_n ,$$

$$t^n \in \theta^n \pi \in \Pi_n$$

 $\lambda = 0$ and $\lim_{n \to \infty} c_n = 0$ where B_{n-1} is the t

where $\lim_{n\to\infty} \lambda_n = 0$ and $\lim_{n\to\infty} \epsilon_n = 0$, where R_{uni} is the uniform distribution on $\{1, \cdots, J_n\}$.

It is known (cf. ([41])) that the capacity of a classical-quantum channel \overline{W} cannot exceed $I(R_{uni}; B)$. Since the capacity of a compound classical-quantum channel $(\overline{W}_s)_{s\in\overline{\theta}}$ cannot exceed the worst channel in $\{\overline{W}_s : s\in\overline{\theta}\}$, its capacity is bounded by $\frac{1}{n}(\min_{s\in\overline{\theta}}\chi(p_U;B_s))$. The enhanced achievable secrecy rate for the compound-arbitrarily varying wiretap classical-quantum channel cannot exceed the capacity without a wiretapper; thus for any $\xi > 0$ let us choose $\epsilon_n = \frac{1}{2}\xi$, if n is sufficiently large, the secrecy rate of (\mathcal{C}_n) cannot be greater than

$$\min_{s\in\overline{\theta}} I(R_{uni}; B_s) - \xi$$

$$\leq \min_{s\in\overline{\theta}} I(R_{uni}; B_s) - \frac{1}{n} \max_{t^n\in\theta^n} \chi(R_{uni}; Z_{t^n}) - \xi + \frac{1}{n} \epsilon_n$$

$$\leq \min_{s\in\overline{\theta}} I(R_{uni}; B_s) - \frac{1}{n} \max_{t^n\in\theta^n} \chi(R_{uni}; Z_{t^n}) - \frac{1}{2}\xi$$

$$= \min_{s\in\overline{\theta}} H(R_{uni}) + H(B_s) - H(R_{uni}B_s) - \frac{1}{n} \max_{t^n\in\theta^n} \chi(R_{uni}; Z_{t^n}) - \frac{1}{2}\xi$$

$$\leq \min_{s\in\overline{\theta}} \chi(R_{uni}; B_s) - \frac{1}{n} \max_{t^n\in\theta^n} \chi(R_{uni}; Z_{t^n}) - \frac{1}{2}\xi$$

$$\leq \frac{1}{n} \max_{A_n} (\min_{s\in\overline{\theta}} \chi(p_U; B_s^n) - \max_{t^n\in\theta^n} \chi(p_U; Z_{t^n}) - \frac{1}{2}\xi$$
(48)

The third inequality holds because $R_{uni} \to A \to \{B_s^{\otimes n}, Z_{t^n} : s, t_n\}$ is always a Markov chain.

This and (47) prove Theorem 1.

Corollary 1 Let $\theta := \{1, \dots, T\}$ be a finite index set. Let $\overline{\theta} := \{1, 2\dots\}$ be an infinite index set. Let $\{(\overline{W}_s, V_t) : s \in \overline{\theta}, t \in \theta\}$ be a compound-arbitrarily varying wiretap classical-quantum channel. We have

$$\hat{C}_s(\{(\overline{W}_s, V_t) : s \in \overline{\theta}, t \in \theta\}) = \lim_{n \to \infty} \frac{1}{n} \max_{\Lambda_n} \left(\inf_{s \in \overline{\theta}} \chi(p_U; B_s^{\otimes n}) - \max_{t^n \in \theta^n} \chi(p_U; Z_{t^n}) \right) \,.$$

Proof For a linear map $W: \mathcal{S}(H') \to \mathcal{S}(H'')$ let

$$\|W\|_{\Diamond} := \sup_{n \in \mathbb{N}} \max_{a \in S(\mathbb{C}^n \otimes H'), \|a\|_1 = 1} \|(\mathbf{I}_n \otimes W)(a)\|_1 .$$

$$\tag{49}$$

It is known [37] that this norm is multiplicative, i.e., $||W \otimes W'||_{\diamond} = ||W||_{\diamond} \cdot ||W'||_{\diamond}$.

A τ -net in the space of the completely positive trace-preserving maps $\mathcal{S}(H') \to \mathcal{S}(H'')$ is a finite set $(W^{(k)})_{k=1}^K$ of completely positive trace-preserving maps $\mathcal{S}(H') \to \mathcal{S}(H'')$ with the property that for each completely positive trace-preserving map $W : \mathcal{S}(H') \to \mathcal{S}(H'')$, there is at least one $k \in \{1, \ldots, K\}$ with $||W - W^{(k)}||_{\Diamond} < \tau$.

Lemma 5 $(\tau-\text{net }[34])$ Let H' and H'' be finite-dimensional complex Hilbert spaces. For any $\tau \in (0, 1]$, there is a τ -net of quantum channels $(W^{(k)})_{k=1}^{K}$ in the space of the completely positive trace-preserving maps $\mathcal{S}(H') \to \mathcal{S}(H'')$ with $K \leq (\frac{3}{\tau})^{2d'^4}$, where $d' = \dim H'$.

We now consider a $\overline{\theta}$ such that $|\overline{\theta}|$ is not finite. For $n \in \mathbb{N}$ we define $\tau_n := n^2$. $\{\tau_n : n \in \mathbb{N}\}$ is a series of positive constants such that $(\frac{3}{\tau_n})^{2d'^4} < 2^{\frac{1}{2}n^{1/16}\beta}$ and $\lim_{n\to\infty} n\tau_n = 0$. By Lemma 5, there exists a finite set $\overline{\theta}_{\tau_n}'$ with $|\overline{\theta}_{\tau_n}'| \leq (\frac{3}{\tau_n})^{2d'^4}$ and τ_n -nets $(\overline{W}_{s'})_{s'\in\overline{\theta}_{\tau_n}'}$, $(V_{s'})_{s'\in\overline{\theta}_{\tau_n}'}$ such that for every $t\in\overline{\theta}$ we can find a $s'\in\overline{\theta}_{\tau_n}'$ with $|\overline{W}_s-\overline{W}_{s'}|_{\diamond} \leq \tau_n$.

We assume that the sender's encoding is restricted to transmitting an indexed finite set of quantum states $\{\rho_x : x \in \mathsf{A}\} \subset \mathcal{S}(H'^{\otimes n})$.

By Theorem 1, the legitimate transmitters are able to build a code $C_2 = \{E, \{D_j : j\}\}$ such that for all $s'' \in \overline{\theta_{\tau_n}}'$, $t \in \theta$, and $\pi \in \Pi_n$, it holds that

$$\frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{x^n \in \mathsf{A}^n} E(x^n \mid j) \operatorname{tr} \left(\overline{W}_{s''}^{\otimes n} \left(\rho_{x^n} \right) D_j^n \right) \ge 1 - \left(\frac{3}{\tau_n}\right)^{2d'^4} 2^{-n^{1/16}\beta} \ge 1 - 2^{-\frac{1}{2}n^{1/16}\beta} ,$$
(50)

$$\chi(R_{uni}; Z_{t,\pi}^n) \le 2^{-n\upsilon} . \tag{51}$$

 $\begin{array}{l} \operatorname{Let} |\psi_{x^n}\rangle\langle\psi_{x^n}| \in \mathcal{S}({H'}^{\otimes n} \otimes {H'}^{\otimes n}) \text{ be an arbitrary purification of the quantum state } \rho_{x^n}, \operatorname{then} \operatorname{tr}\left[\left(\overline{W}_s^{\otimes n} - \overline{W}_{s'}^{\otimes n}\right)(\rho_{x^n})\right] = \operatorname{tr}\left(\operatorname{tr}_{H'^{\otimes n}}\left[\operatorname{I}_{H'}^{\otimes n} \otimes (\overline{W}_s^{\otimes n} - \overline{W}_{s'}^{\otimes n})\left(|\psi_{x^n}\rangle\langle\psi_{x^n}|\right)\right]\right). \end{array}$

We have

1

$$\begin{aligned} \operatorname{tr} \left| \sum_{x^{n} \in \mathsf{A}^{n}} E(x^{n} \mid j) \left(\overline{W}_{s}^{\otimes n} - \overline{W}_{s'}^{\otimes n} \right) (\rho_{x^{n}}) \right| \\ &= \operatorname{tr} \left(\sum_{x^{n} \in \mathsf{A}^{n}} E(x^{n} \mid j) \operatorname{tr}_{H'^{\otimes n}} \left| \mathrm{I}_{H'}^{\otimes N} \otimes \left(\overline{W}_{s}^{\otimes n} - \overline{W}_{s'}^{\otimes n} \right) \left(|\psi_{x^{n}}\rangle\langle\psi_{x^{n}}| \right) \right| \right) \\ &= \operatorname{tr} \left| \sum_{x^{n} \in \mathsf{A}^{n}} E(x^{n} \mid j) \mathrm{I}_{H'}^{\otimes n} \otimes \left(\overline{W}_{s}^{\otimes n} - \overline{W}_{s'}^{\otimes n} \right) \left(|\psi_{x^{n}}\rangle\langle\psi_{x^{n}}| \right) \right| \\ &= \sum_{x^{n} \in \mathsf{A}^{n}} E(x^{n} \mid j) \left\| \mathrm{I}_{H'}^{\otimes n} \otimes \left(\overline{W}_{s}^{\otimes n} - \overline{W}_{s'}^{\otimes N} \right) \left(|\psi_{x^{n}}\rangle\langle\psi_{x^{n}}| \right) \right\|_{1} \\ &\leq \sum_{x^{n} \in \mathsf{A}^{n}} E(x^{n} \mid j) \| \overline{W}_{s}^{\otimes n} - \overline{W}_{s'}^{\otimes n} \|_{\diamond} \cdot \| \left(|\psi_{x^{n}}\rangle\langle\psi_{x^{n}}| \right) \|_{1} \\ &\leq N\tau_{n} . \end{aligned}$$

The second equality follows from the definition of trace. The third inequality follows by the definition of $\|\cdot\|_{\diamond}$. The second inequality follows from the facts that $\|(|\psi_{x^n}\rangle\langle\psi_{x^n}|)\|_1 = 1$ and $\left\|\overline{W}_s^{\otimes n} - \overline{W}_{s'}^{\otimes N}\right\|_{\diamond} = \left\|\left(\overline{W}_s - \overline{W}_{s'}\right)^{\otimes N}\right\|_{\diamond} = N \cdot \left\|\overline{W}_s - \overline{W}_{s'}\right\|_{\diamond}$, since $\|\cdot\|_{\diamond}$ is multiplicative.

It follows that

$$\begin{aligned} \left| \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{x^n \in \mathsf{A}^n} E(x^n \mid j) \operatorname{tr} \left(\overline{W}_s^{\otimes n} \left(\rho_{x^n} \right) D_j^n \right) \right| \\ &- \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{x^n \in \mathsf{A}^n} E(x^n \mid j) \operatorname{tr} \left(\overline{W}_{s'}^{\otimes n} \left(\rho_{x^n} \right) D_j^n \right) \right| \\ &\leq \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{x^n \in \mathsf{A}^n} E(x^n \mid j) \left| \operatorname{tr} \left[\left(\overline{W}_s^{\otimes n} - \overline{W}_{s'}^{\otimes n} \right) \left(\rho_{x^n} \right) D_j^n \right] \right| \\ &\leq \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{x^n \in \mathsf{A}^n} E(x^n \mid j) \operatorname{tr} \left[\left(\overline{W}_s^{\otimes n} - \overline{W}_{s'}^{\otimes n} \right) \left(\rho_{x^n} \right) D_j^n \right] \\ &\leq \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{x^n \in \mathsf{A}^n} E(x^n \mid j) \operatorname{tr} \left[\left(\overline{W}_s^{\otimes n} - \overline{W}_{s'}^{\otimes n} \right) \left(\rho_{x^n} \right) D_j^n \right] \\ &\leq \frac{1}{J_n} n\tau_n \\ &= n\tau_n . \end{aligned}$$
(52)

By (52), we have

$$\sup_{s\in\overline{\theta}}\frac{1}{J_n}\sum_{j=1}^{J_n}\sum_{x^n\in\mathsf{A}^n}E(x^n\mid j)\operatorname{tr}\left(\overline{W}_s^{\otimes n}\left(\rho_{x^n}\right)D_j^n\right)\geq 1-\lambda_{\tau_n}-n\tau_n\;.$$

Thus,

$$\hat{C}_s(\{(\overline{W}_s, V_t) : s \in \overline{\theta}, t \in \theta\}) \ge \lim_{n \to \infty} \frac{1}{n} (\inf_{s \in \overline{\theta}} \chi(p; B_s^{\otimes n}) - \max_{t^n \in \theta^n} \chi(p; Z_{t^n})) .$$
(53)

The achievability of $\lim_{n\to\infty} \frac{1}{n} \left(\min_{s\in\overline{\theta}} \chi(p_U; B_s) - \max_{t^n\in\theta^n} \chi(p_U; Z_{t^n}) \right)$ is then shown via standard arguments.

The proof of the converse is similar to those given in the proof of Theorem 1. $\hfill \Box$

Corollary 2 Let $\overline{\theta}$ and θ be finite index sets. Let $\{(\overline{W}_s, V_t) : s \in \overline{\theta}, t \in \theta\}$ be a compound-arbitrarily varying wiretap classical-quantum channel. The secrecy capacity of $\{(\overline{W}_s, V_t) : s \in \overline{\theta}, t \in \theta\}$ is equal to

$$\lim_{n \to \infty} \frac{1}{n} \max_{\Lambda_n} \left(\min_{s \in \overline{\theta}} \chi(p_U; B_s^{\otimes n}) - \max_{t^n \in \theta^n} \chi(p_U; Z_{t^n}) \right) \,.$$

Proof The corollary follows immediately from the fact that the enhanced secrecy capacity of a compound-arbitrarily varying wiretap classical-quantum channel is less or equal to its secrecy capacity.

4 Secrecy Capacity of Arbitrarily Varying Classical-Quantum Wiretap Channel

In this section, we use the results of Section 3 to prove our main result: the formula for the secrecy capacities under common randomness assisted coding of arbitrarily varying classical-quantum wiretap channels.

Theorem 2 Let $\theta := \{1, \dots, T\}$ be a finite index set. Let $(W_t, V_t)_{t \in \theta}$ be an arbitrarily varying classical-quantum wiretap channel. We have

$$C_{s}(\{(W_{t}, V_{t}) : t \in \theta\}; cr)$$

$$= \lim_{n \to \infty} \frac{1}{n} \max_{A_{n}} \left(\inf_{B_{q} \in Conv((B_{t})_{t \in \theta})} \chi(p_{U}; B_{q}^{\otimes n}) - \max_{t^{n} \in \theta^{n}} \chi(p_{U}; Z_{t^{n}}) \right).$$
(54)

Here $Conv((B_t)_{t\in\theta})$ is the convex hull of $\{B_t : t\in\theta\}$.

Proof i) Achievement

Our idea is similar to the results for classical arbitrarily varying wiretap channel in [39]: Applying Ahlswede's robustification technique (cf. [13]), we use the results of Section 3 to show the existence of a common randomness assisted quantum code. Additionally, we have to consider the security.

We denote the set of distribution function on θ by $\mathsf{P}(\theta)$. For every $q \in \mathsf{P}(\theta)$, we define a classical-quantum channel $\overline{W}_q := \sum_{s \in \theta} q(s) W_s$. We now define a compound-arbitrarily varying wiretap classical-quantum channel by

$$\{(\overline{W}_q, V_t); q \in \mathsf{P}(\theta), t \in \theta\}$$

We fix a probability distribution $p\in\mathsf{A}.$ We choose arbitrarily $\epsilon>0,\,\delta>0,$ and $\zeta>0.$ Let

$$J_n = \left\lfloor 2^{n \inf_{B_q \in Conv((B_s)_{s \in \theta})} \chi(p; B_q) - \max_{t^n \in \theta^n} \chi(p; Z_{t^n}) - n\delta} \right\rfloor \,.$$

By Corollary 1, if n is sufficiently large, there exists an (n, J_n) code $C = (E^n, \{D_j^n : j = 1, \dots, J_n\})$ such that

$$\max_{q \in \mathsf{P}(\theta)} 1 - \frac{1}{J_n} \sum_{j=1}^{J_n} \operatorname{tr}(\overline{W}_q(E^n(|j))D_j^n) < \epsilon ,$$
$$\max_{t^n \in \theta^n} \max_{\pi \in \Pi_n} \chi\left(R_{uni}; Z_{t^n, \pi}\right) < \zeta .$$

Similar to the proofs in [13], we now apply Ahlswede's robustification technique.

Lemma 6 (cf. [3], [4], and [5]) Let S be a finite set and $n \in \mathbb{N}$. If a function $f: S^n \to [0,1]$ satisfies

$$\sum_{s^n \in S^n} f(s^n)q(s_1)q(s_2)\cdots q(s_n) \ge 1 - \epsilon ,$$

for all $q \in \mathsf{P}(\theta)$ and a positive $\epsilon \in [0, 1]$, then

$$\frac{1}{n!} \sum_{\pi \in \Pi_n} f(\pi(s^n)) \ge 1 - 3(n+1)^{|S|} \epsilon .$$
(55)

We define a function $f: \theta^n \to [0,1]$ by

$$f(t^n) := \frac{1}{J_n} \sum_{j=1}^{J_n} \operatorname{tr}(W_{t^n}(E^n(|j))D_j^n) \ .$$

For every $q \in \mathsf{P}(\theta)$ we have

$$\begin{split} &\sum_{t^n \in \theta^n} f(t^n) q(t_1) \cdots q(t_n) \\ &= \sum_{t^n \in \theta^n} \frac{1}{J_n} \sum_{j=1}^{J_n} \operatorname{tr}(W_{t^n}(E^n(|j|))D_j^n) q(t_1) \cdots q(t_n) \\ &= \frac{1}{J_n} \sum_{j=1}^{J_n} \operatorname{tr}\left(\sum_{t^n \in \theta^n} q(t_1) \cdots q(t_n) W_{t^n}(E^n(|j|))D_j^n\right) \\ &= \frac{1}{J_n} \sum_{j=1}^{J_n} \operatorname{tr}(\overline{W}_q(E^n(|j|))D_j^n) \\ &> 1 - 2^{-n\beta/2} \,. \end{split}$$

Applying Lemma 6, we have

$$1 - 3(n+1)^{|\theta|} 2^{-n\beta/2}$$

$$\leq \frac{1}{n!} \sum_{\pi \in \Pi_n} f(\pi(t^n))$$

$$= \frac{1}{n!} \sum_{\pi \in \Pi_n} \frac{1}{J_n} \sum_{j=1}^{J_n} \operatorname{tr}(W_{\pi(t^n)}(E^n(|j|))D_j^n)$$

$$= \frac{1}{n!} \sum_{\pi \in \Pi_n} \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{a^n \in \mathsf{A}^n} E^n(a^n|j)\operatorname{tr}(W_{\pi(t^n)}(a^n)D_j^n)$$

$$= \frac{1}{n!} \sum_{\pi \in \Pi_n} \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{a^n \in \mathsf{A}^n} E^n(a^n|j)\operatorname{tr}(W_{t^n}(\pi^{-1}(a^n))P_{\pi}^{\dagger}D_j^n P_{\pi}), \quad (56)$$

where for $\pi \in \Pi_n$, P_{π} is its permutation matrix on $H^{\otimes n}$.

We now define our common randomness assisted quantum code by

$$\left\{ \left(\pi \circ E^n, \{ P_{\pi} D_j^n P_{\pi}^{\dagger}, j \in \{1, \cdots, J_n\} \} \right) : \pi \in \Pi_n \right\}$$

 $P_{\pi}D_{j}^{n}P_{\pi}^{\dagger}$ is Hermitian and positive semidefinite. Furthermore, it holds that $\sum_{j=1}^{J_{n}}P_{\pi}D_{j}^{n}P_{\pi}^{\dagger}=\sum_{j=1}^{J_{n}}P_{\pi}id_{H^{\otimes n}}P_{\pi}^{\dagger}=id_{H^{\otimes n}}.$

By (56), and by the fact that

$$\frac{1}{n!} \sum_{\pi \in \Pi_n} \max_{t^n \in \theta^n} \chi\left(R_{uni}; Z_{t^n, \pi}\right)$$

$$\leq \max_{t^n \in \theta^n} \max_{\pi \in \Pi_n} \chi\left(R_{uni}; Z_{t^n, \pi}\right)$$

$$< \zeta ,$$

for any positive ε when n is sufficiently large, it holds that:

$$C_s(\{(W_t, V_t) : t \in \theta\}; cr) \ge \inf_{B_q \in Conv((B_s)_{s \in \theta})} \chi(p; B_q) - \lim_{n \to \infty} \frac{1}{n} \max_{t^n \in \theta^n} \chi(p; Z_{t^n}) - \varepsilon .$$
(57)

The achievability of $\lim_{n\to\infty} \frac{1}{n} \left(\min_{B_q \in Conv((B_s)_{s \in \theta})} \chi(p_U; B_q^{\otimes n}) - \max_{t^n \in \theta^n} \chi(p_U; Z_{t^n}) \right)$ is then shown via standard arguments (cf. [27]).

ii) Converse

Now we are going to prove the converse. Similar to the results for classical arbitrarily varying wiretap channel in [39], we limit the amount of common randomness.

Let $(\{C_n^{\gamma} : \gamma \in \Gamma\})$ be a sequence of (n, J_n) common randomness assisted codes such that

$$\max_{s \in \theta} \frac{1}{|\Gamma|} \sum_{\gamma=1}^{|\Gamma|} P_e(\mathcal{C}_n^{\gamma}, t^n) \le \lambda_n , \qquad (58)$$

$$\max_{t^n \in \theta^n} \frac{1}{|\Gamma|} \sum_{\gamma=1}^{|\Gamma|} \chi\left(R_{uni}; Z_{\mathcal{C}^{\gamma}, t^n}\right) \le \epsilon_n , \qquad (59)$$

where $\lim_{n\to\infty} \lambda_n = 0$ and $\lim_{n\to\infty} \epsilon_n = 0$.

We consider a $|\Gamma|$ -long sequence of outputs $(1, \dots, |\Gamma|)$ has been given by the common randomness and a $n |\Gamma|$ -long block has been sent. The legitimate receiver obtains the quantum states $\{B_q^{\gamma} : \gamma \in \Gamma\}$. By (58), he is able to decode $2^{n|\Gamma|\log J_n}$ messages. By [13], for every $B_q \in Conv((B_s)_{s \in \theta})$ we have

$$\log J_n \leq \frac{1}{|\Gamma|} \frac{1}{n} \sum_{\gamma=1}^{|\Gamma|} \chi(R_{uni}; B_q^{\gamma \otimes n}) ,$$

and by (59), for and every $t^n \in \theta^n$, we have

$$\frac{1}{n}\log J_n \leq \frac{1}{|\Gamma|} \frac{1}{n} \sum_{\gamma=1}^{|\Gamma|} (\chi(R_{uni}; B_q^{\gamma \otimes n}) - \chi(R_{uni}; Z_{t^n}^{\gamma})) + \epsilon_n .$$

Lemma 7 Let c > 0. For every $q \in \mathsf{P}(\theta)$ and $s^n \in \theta^n$, let a function I_{q,s^n} : $\Gamma \to [0,c]$ be given. We assume that these functions satisfy the following: for every $\gamma \in \Gamma$ and $s^n \in \theta^n$

$$|I_{q,s^n}(\gamma) - I_{q',s^n}(\gamma)| \le f(\delta) ,$$

if $q, q' \in \mathsf{P}(\theta)$ satisfy $||q - q'||_1 \leq \delta$ for some $f(\delta)$ which tends to 0 as δ tends to 0. We write $\mu(I_{q,s^n}) := \sum_{\gamma \in \Gamma} \mu(\gamma)I_{q,s^n}(\gamma)$, where $\mu(\gamma)$ is the probability of γ . Then, for every $\varepsilon > 0$ and sufficiently large n, there are $L = n^2$ realizations $\gamma_1, \dots, \gamma_L$ such that

$$\frac{1}{L}\sum_{l=1}^{L} I_{q,s^n}(\gamma_l) \ge (1-\varepsilon)\mu(I_{q,s^n}) - \varepsilon$$

for every $q \in \mathsf{P}(\theta)$ and $s^n \in \theta^n$.

Proof Let $0 < \delta < \frac{1}{2}$ and K be a positive integer. We denote the set of possible types of sequences of length K by $P_0^K(\theta)$. As in the approximation argument in [17], one can show that every $q \in \mathsf{P}(\theta)$ is at most a distance δ away from some $q' \in P_0^K(\theta)$ if $K \ge 2\frac{|\theta|-1}{\delta}$.

Let $K := \lceil 2 \frac{|\theta|-1}{\delta} \rceil$. Then, $|P_0^K(\theta)| \leq \left(2 \frac{|\theta|}{\delta}\right)^{|\theta|}$. This approximating set is used to handle the infinite set $\mathsf{P}(\theta)$.

Now let G_1, \dots, G_L be i.i.d. random variables with values in Γ and distributed according to μ . Set $\mu_* := \min_{q \in \mathsf{P}(\theta)} \min_{s^n \in \theta^n} \mu(I_{q,s^n})$. Using the union bound and the Chernoff bound (cf. [28]), we obtain

$$Pr\left\{\frac{1}{L}\sum_{l=1}^{L}I_{q,s^{n}}(G_{l}) < \mu(I_{q,s^{n}}) \;\forall q \in P_{0}^{K}(\theta) \;\forall s^{n} \in \theta^{n}\right\}$$
$$\leq \exp\left(|\theta|\log\left(\frac{2|\theta|}{\delta}\right) + n\log|\theta| - \frac{L\epsilon^{2}\mu_{*}}{3c}\right) \;.$$

This, probability is smaller than 1 if L tends to infinity faster than n, e.g., if $L = n^2$.

Thus we have proved the existence of $\gamma_1, \cdots, \gamma_L$ which satisfies

$$\frac{1}{L}\sum_{l=1}^{L} I_{q,s^n}(\gamma_l) \ge (1-\epsilon)\mu(I_{q,s^n})$$

for every $q \in P_0^K(\theta)$ and $s^n \in \theta^n$. Now let $q \in \mathsf{P}(\theta)$ be arbitrary and let $q' \in P_0^K(\theta)$ satisfy $||q - q'||_1 \leq \delta$. Then

$$\frac{1}{L} \sum_{l=1}^{L} I_{q,s^n}(\gamma_l)$$

$$\geq \frac{1}{L} \sum_{l=1}^{L} I_{q',s^n}(\gamma_l) - f(\delta)$$

$$\geq (1-\epsilon)\mu(I_{q',s^n}) - f(\delta)$$

$$\geq (1-\epsilon)\mu(I_{q,s^n}) - (2-\epsilon)f(\delta)$$

Choosing δ sufficiently small proves the claim of the lemma.

For $q \in Conv(\{s : s \in \theta\})$, we define

$$I_{q,s^n}(\gamma) := \frac{1}{n} \left(\chi(R_{uni}; B_q^{\gamma \otimes n}) - \chi(R_{uni}; Z_{t^n}^{\gamma}) \right) \ .$$

In [24], the continuity of $q \to \frac{1}{n}\chi(R_{uni}; B_q^{\gamma\otimes n})$ has been shown; thus, there is a $f(\delta)$ such that $|I_{q,s^n}(\gamma) - I_{q',s^n}(\gamma)| \frac{1}{n} \frac{1}{|\Gamma|} \sum_{\gamma=1}^{|\Gamma|} (\chi(R_{uni}; B_q^{\gamma\otimes n}) - \frac{1}{n} \frac{1}{|\Gamma|} \sum_{\gamma=1}^{|\Gamma|} (\chi(R_{uni}; B_{q'}^{\gamma\otimes n}) \leq f(\delta)$ for a $f(\delta)$ that fulfills $f(\delta) \to 0$ when $||q-q'||_1 = \delta \to 0$. By Lemma 7, there is a set $\Gamma' \subset \Gamma$ such that $|\Gamma'| = n^2$ and

$$\frac{1}{|\Gamma'|} \frac{1}{n} \sum_{\gamma' \in \Gamma'} \left(\chi(R_{uni}; B_q^{\gamma' \otimes n}) - \chi(R_{uni}; Z_{t^n}^{\gamma'}) \right)$$

$$\geq (1 - \varepsilon) \frac{1}{n} \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \left(\chi(R_{uni}; B_q^{\gamma \otimes n}) - \chi(R_{uni}; Z_{t^n}^{\gamma}) \right)$$

where $B_q^{\gamma'}$ and $Z_{t^n}^{\gamma}$ are the quantum states at the output of legitimate receiver channel and the wiretapper's channel, respectively, when the output of the common randomness is γ' .

Thus,

$$\frac{1}{n}\log J_n \le \frac{1}{1-\varepsilon} \frac{1}{n} \frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} \left(\chi(R_{uni}; B_q^{\gamma \otimes n}) - \chi(R_{uni}; Z_{t^n}^{\gamma}) + \epsilon_n \right) .$$
(60)

To prove the converse, we now consider

$$\frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} \frac{1}{n} \left(\chi(R_{uni}; B_q^{\gamma \otimes n}) - \chi(R_{uni}; Z_{t^n}^{\gamma}) \right) - \frac{1}{n} \left(\chi(R_{uni}; B_q^{\otimes n}) - \chi(R_{uni}; Z_{t^n}) \right) \\
= \frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} \frac{1}{n} \left(\chi(R_{uni}; B_q^{\gamma \otimes n}) - \chi(R_{uni}; Z_{t^n}^{\gamma}) \right) \\
- \frac{1}{n} \left(\chi(R_{uni}; \frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} B_q^{\gamma \otimes n}) - \chi(R_{uni}; \frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} Z_{t^n}^{\gamma}) \right) \\
= \frac{1}{n} \frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} \left(\chi(R_{uni}; B_q^{\gamma \otimes n}) - \chi(R_{uni}; \frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} B_q^{\gamma \otimes n}) \right) \\
- \frac{1}{n} \frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} \left(\chi(R_{uni}; Z_{t^n}^{\gamma}) + \frac{1}{n} \chi(R_{uni}; \frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} Z_{t^n}^{\gamma}) \right).$$

Let G_{uni} be the uniformly distributed random variable with value in Γ' . We have

$$\frac{1}{n} \frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} \chi(R_{uni}; B_q^{\gamma \otimes n})$$

$$= \frac{1}{n} \frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} I(R_{uni}; B_q^{\gamma \otimes n})$$

$$= \frac{1}{n} \frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} (H(R_{uni}) - H(R_{uni}|B_q^{\gamma \otimes n}))$$

$$= \frac{1}{n} H(R_{uni}) - \frac{1}{n} H(R_{uni}|B_q^{\gamma \otimes n}, \Gamma')$$

$$\leq \frac{1}{n} H(R_{uni}) - \frac{1}{n} H(R_{uni}|B_q) + H(G_{uni})$$

$$= \frac{1}{n} I(R_{uni}; B_q^{\otimes n}) + H(G_{uni})$$

$$= \frac{1}{n} \chi\left(R_{uni}; \frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} B_q^{\gamma \otimes n}\right) + H(G_{uni})$$

$$= \frac{1}{n} \chi \left(R_{uni}; \frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} B_q^{\gamma \otimes n} \right) + 2 \log n .$$
 (61)

Let $\phi_{t^n}^{j,\gamma}$ be the quantum state at the output of the wiretapper's channel when the channel state is t^n , the output of the common randomness is γ , and j has been sent.

We have

$$\frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} \chi \left(R_{uni}; Z_{t^n}^{\gamma} \right) - \chi \left(R_{uni}; \frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} Z_{t^n}^{\gamma} \right) \\
= \frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} S \left(\frac{1}{J_n} \sum_{j=1}^{J_n} \phi_{t^n}^{j,\gamma} \right) - \frac{1}{|\Gamma'|} \frac{1}{J_n} \sum_{\gamma \in \Gamma'} \sum_{j=1}^{J_n} S \left(\phi_{t^n}^{j,\gamma} \right) \\
- S \left(\frac{1}{|\Gamma'|} \frac{1}{J_n} \sum_{\gamma \in \Gamma'} \sum_{j=1}^{J_n} \phi_{t^n}^{j,\gamma} \right) + \frac{1}{J_n} \sum_{j=1}^{J_n} S \left(\frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} \phi_{t^n}^{j,\gamma} \right) .$$
(62)

Let $H^{\mathfrak{G}}$ be a $|\Gamma'|$ -dimensional Hilbert space, spanned by an orthonormal basis $\{|i\rangle : i = 1, \dots, |\Gamma'|\}$. Let $H^{\mathfrak{I}}$ be a J_n -dimensional Hilbert space, spanned by an orthonormal basis $\{|j\rangle : j = 1, \dots, J_n\}$. Similar to (37), we define

$$\varphi^{\mathfrak{IGH}^n} := \frac{1}{J_n} \frac{1}{|\Gamma'|} \sum_{j=1}^{J_n} \sum_{\gamma \in \Gamma'} |j\rangle \langle j| \otimes |i\rangle \langle i| \otimes \phi_{t^n}^{j,\gamma}$$

By strong subadditivity of von Neumann entropy, it holds that $S(\varphi^{\mathfrak{J}H^n}) + S(\varphi^{\mathfrak{G}H^n}) \geq S(\varphi^{H^n}) + S(\varphi^{\mathfrak{J}\mathfrak{G}H^n})$, therefore

$$\frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} \chi\left(R_{uni}; Z_{t^n}^{\gamma}\right) - \chi\left(R_{uni}; \frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} Z_{t^n}^{\gamma}\right) \ge 0.$$
(63)

By (61) and (63), we have

$$\chi(R_{uni}; B_q) - \frac{1}{n} \chi(R_{uni}; Z_{t^n}) + 2\log n \ge \frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} \frac{1}{n} \left(\chi(R_{uni}; B_q^{\gamma \otimes n}) - \chi(R_{uni}; Z_{t^n}^{\gamma}) \right)$$

Thus for every $B_q \in Conv((B_s)_{s \in \theta})$ and every $t^n \in \theta^n$ we have

$$\frac{1}{n}\log J_n \le \frac{1}{1-\varepsilon} \frac{1}{n} \left(\chi(R_{uni}; B_q^{\otimes n}) - \chi(R_{uni}; Z_{t^n}) + \epsilon_n + 2\frac{1}{n}\log n \right) .$$
(64)

Similar to the proof of Theorem 1, we have $\frac{1}{n} \left(\inf_{B_q \in Conv((B_t)_{t \in \theta})} \chi(R_{uni}; B_q^{\otimes n}) - \max_{t^n \in \theta^n} \chi(R_{uni}; Z_{t^n}) \right) \leq \frac{1}{n} \max_{\Lambda_n} \left(\inf_{B_q \in Conv((B_t)_{t \in \theta})} \chi(p_U; B_q^{\otimes n}) - \max_{t^n \in \theta^n} \chi(p_U; Z_{t^n}) \right).$ The converse has been shown. (57) and (64) prove Theorem 2.

Corollary 3 Let $\{(W_t, V_t) : t \in \theta\}$ be an arbitrarily varying classical-quantum wiretap channel.

1) Let X and Y be finite sets. If I(X, Y) > 0 holds for a random variable (X, Y) which is distributed to a joint probability distribution $p \in P(X, Y)$, then the (X, Y) correlation assisted secrecy capacity of $\{(W_t, V_t) : t \in \theta\}$ is equal to

$$\lim_{n \to \infty} \frac{1}{n} \max_{\Lambda_n} \left(\inf_{B_q \in Conv((B_t)_{t \in \theta})} \chi(p_U; B_q^{\otimes n}) - \max_{t^n \in \theta^n} \chi(p_U; Z_{t^n}) \right)$$

2) If the arbitrarily varying classical-quantum channel $\{W_t : t \in \theta\}$ is not symmetrizable, then the deterministic secrecy capacity of $\{(W_t, V_t) : t \in \theta\}$ is equal to

$$\lim_{n \to \infty} \frac{1}{n} \max_{\Lambda_n} \left(\inf_{B_q \in Conv((B_t)_{t \in \theta})} \chi(p_U; B_q^{\otimes n}) - \max_{t^n \in \theta^n} \chi(p_U; Z_{t^n}) \right) \,.$$

Proof 1) follows immediately from Theorem 2 and the results of [22].

To show 2) we use a technique similar to the proof of Theorem 3.1 in [22]: We build a two-part code word which consists of a non-secure code word and a common randomness assisted secure code word. The first part is used to create the common randomness for the sender and the legitimate receiver. The second part is a common randomness assisted secure code word transmitting the message to the legitimate receiver.

We consider the Markov chain $U \to A \to \{B_q^{\otimes n}, Z_{t^n} : q, t_n\}$, where we define the classical channel $U \to A$ by T_U . Let

$$J_n = \left\lfloor 2^{n \inf_{B_q \in Conv((B_s)_{s \in \theta})} \chi(p_U; B_q) - \max_{t^n \in \theta^n} \chi(p_U; Z_{t^n}) - n\delta} \right\rfloor \,.$$

By Theorem 2, for any positive ϵ if n is sufficiently large, there is an (n, J_n) code $(E^n, \{D_j^n : j = 1, \dots, J_n\})$ for the arbitrarily varying classical-quantum wiretap channel $\{(W_t \circ T_U, V_t \circ T_U) : t \in \theta\}$ such that

$$\frac{1}{n!} \sum_{\pi \in \Pi_n} \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{a^n \in \mathsf{A}^n} E^n(a^n | j) \operatorname{tr}(W_{t^n}(\pi^{-1}(a^n)) P_{\pi}^{\dagger} D_j^n P_{\pi}) \ge 1 - \epsilon$$

and

$$\frac{1}{n!} \sum_{\pi \in \Pi_n} \max_{t^n \in \theta^n} \chi\left(R_{uni}; Z_{t^n, \pi}\right) \le \epsilon \; .$$

By Theorem 3.1.2 in [22], for any positive λ if n is sufficiently large, there is an (n, J_n) common randomness assisted code $\{C_1, C_2, \dots, C_{n^3}\}$ for the arbitrarily varying classical-quantum wiretap channel $\{(W_t \circ T_U, V_t \circ T_U) : t \in \theta\}$ such that

$$\max_{t^n \in \theta^n} \frac{1}{n^3} \sum_{i=1}^{n^3} P_e(\mathcal{C}_i, t^n) < \lambda ,$$

and

$$\max_{t^{n}\in\theta^{n}}\frac{1}{n^{3}}\sum_{i=1}^{n^{3}}\chi\left(R_{uni},Z_{\mathcal{C}_{i},t^{n}}\right)<\lambda$$

Similar to the proof of Theorem 3.1.1 in [22], for any positive ϑ if $\{W_t : t \in \theta\}$ is not symmetrizable and n is sufficiently large, there is a code $\left(\begin{pmatrix} c_i^{\mu(n)} \end{pmatrix}_{i \in \{1, \dots, n^3\}}, \{D_i^{\mu(n)} : i \in \{1, \dots, n^3\}\} \right)$ with deterministic encoder of length $\mu(n)$, where $2^{\mu(n)} = o(n)$ for the arbitrarily varying classical-quantum wiretap channel $\{(W_t, V_t) : t \in \theta\}$ such that

$$1 - \frac{1}{n^3} \sum_{i=1}^{n^3} \operatorname{tr}(W_{t^n}(c_i^{\mu(n)}) D_i^{\mu(n)}) \le \vartheta \ .$$

We now can construct a code $\mathcal{C}^{det} = \left(E^{\mu(n)+n}, \left\{D_j^{\mu(n)+n}: j=1,\cdots,J_n\right\}\right)$, where for $a^{\mu(n)+n} = (a^{\mu(n)}, a^n) \in \mathsf{A}^{\mu(n)+n}$

$$E^{\mu(n)+n}(a^{\mu(n)+n}|j) = \begin{cases} \frac{1}{n^3} E_i^n(a^n|j) \text{ if } a^{\mu(n)} = c_i^{\mu(n)} \\ 0 \text{ else} \end{cases}$$

and

$$D_j^{\mu(n)+n} := \sum_{i=1}^{n^3} D_i^{\mu(n)} \otimes D_{i,j}^n$$
.

Similar to the proof of Theorem 3.1.1 in [22], for any positive λ if n is sufficiently large, we have

$$\max_{\substack{t^{\mu(n)+n}\in\theta^{\mu(n)+n}}} P_e(\mathcal{C}^{det}, t^{\mu(n)+n}) < \lambda ,$$
$$\max_{t^{\mu(n)+n}\in\theta^{\mu(n)+n}} \chi \left(R_{uni}, Z_{\mathcal{C}^{det}, t^{\mu(n)+n}} \right) < \lambda .$$

Remark 3 For the proof of Corollary 3, 2), it is important to assume that $\left(\binom{c_i^{\mu(n)}}{i \in \{1, \dots, n^3\}}, \{D_i^{\mu(n)} : i \in \{1, \dots, n^3\}\}\right)$ is a code for the channel $\{(W_t, V_t) : t \in \theta\}$ and not for $\{(W_t \circ T_U, V_t \circ T_U) : t \in \theta\}$, since it may happen that $\{W_t \circ T_U : t \in \theta\}$ is symmetrizable although $\{W_t : t \in \theta\}$ is not symmetrizable, as the following example shows:

We assume that $\{W_t : t \in \theta\} : P(\mathsf{A}) \to \mathcal{S}(H)$ is not symmetrizable, but there is a subset $\mathsf{A}' \subset \mathsf{A}$ such that $\{W_t : t \in \theta\}$ limited on A' is symmetrizable. We choose a T_U such that for every $u \in \mathsf{U}$ there is $a \in \mathsf{A}'$ such that $T_U(a \mid u) = 1$, and $T_U(a \mid u) = 0$ for all $a \in \mathsf{A} \setminus \mathsf{A}'$ and $u \in \mathsf{U}$. It is clear that $\{W_t \circ T_U : t \in \theta\}$ is symmetrizable (cf. also [35] for an example for classical channels).

5 Investigation of Secrecy Capacity's Continuity

In this section we show that the secrecy capacity of an arbitrarily varying classical-quantum wiretap channel under common randomness assisted quantum coding is continuous in the following sense:

Corollary 4 For an arbitrarily varying classical-quantum wiretap channel $\{(W_t, V_t) : t \in \theta\}$, where $W_t : \mathsf{P}(\mathsf{A}) \to \mathcal{S}(H)$ and $V_t : \mathsf{P}(\mathsf{A}) \to \mathcal{S}(H')$ and a positive δ , let C_{δ} be the set of all arbitrarily varying classical-quantum wiretap channels $\{(W'_t, V'_t) : t \in \theta\}$, where $W'_t : \mathsf{P}(\mathsf{A}) \to \mathcal{S}(H)$ and $V'_t : \mathsf{P}(\mathsf{A}) \to \mathcal{S}(H')$, such that

$$\max_{a \in \mathsf{A}} \| W_t(a) - {W'}_t(a) \|_1 < \delta$$

and

$$\max_{a \in \mathsf{A}} \| V_t(a) - V'_t(a) \|_1 < \delta$$

for all $t \in \theta$.

For any positive ϵ there is a positive δ such that for all $\{(W'_t, V'_t) : t \in \theta\} \in \mathsf{C}_{\delta}$ we have

$$|C_s(\{(W_t, V_t) : t \in \theta\}; cr) - C_s(\{((W'_t, V'_t) : t \in \theta\}; cr)| \le \epsilon .$$
(65)

Proof By Corollary 3, the secrecy capacity of $\{(W_t, V_t) : t \in \theta\}$ is

$$\lim_{n \to \infty} \frac{1}{n} \max_{\Lambda_n} \left(\inf_{B_q \in Conv((B_t)_{t \in \theta})} \chi(p_U; B_q^{\otimes n}) - \max_{t^n \in \theta^n} \chi(p_U; Z_{t^n}) \right) \,,$$

and for every $\{(W'_t, V'_t) : t \in \theta\} \in C_{\delta}$ the secrecy capacity of $\{(W'_t, V'_t) : t \in \theta\}$ is

$$\lim_{n \to \infty} \frac{1}{n} \max_{\Lambda_n} \left(\inf_{B'_q \in Conv((B'_t)_{t \in \theta})} \chi(p_U; B'_q^{\otimes n}) - \max_{t^n \in \theta^n} \chi(p_U; Z'_{t^n}) \right) ,$$

where B'_t is the resulting quantum state at the output of W'_t and Z'_t is the resulting quantum state at the output of V'_t .

To analyze $|\chi(p; Z_{t^n}) - \chi(p; Z'_{t^n})|$, we use the technique introduced in [33] and apply the following lemma given in [8].

Lemma 8 (Alicki-Fannes Inequality) Suppose we have a composite system \mathfrak{PQ} with components \mathfrak{P} and \mathfrak{Q} . Let $G^{\mathfrak{P}}$ and $G^{\mathfrak{Q}}$ be Hilbert space of \mathfrak{P} and \mathfrak{Q} , respectively. Suppose we have two bipartite quantum states $\phi^{\mathfrak{PQ}}$ and $\sigma^{\mathfrak{PQ}}$ in $\mathcal{S}(G^{\mathfrak{PQ}})$ such that $\|\phi^{\mathfrak{PQ}} - \sigma^{\mathfrak{PQ}}\|_1 = \epsilon < 1$, it holds that

$$S(\mathfrak{P} \mid \mathfrak{Q})_{\rho} - S(\mathfrak{P} \mid \mathfrak{Q})_{\sigma} \le 4\epsilon \log(d-1) - 2h(\epsilon) , \qquad (66)$$

where d is the dimension of $G^{\mathfrak{P}}$ and $h(\epsilon)$ is defined as in Lemma 4.

In contrast to [8], we consider here classical-quantum channels instead of quantum-quantum channels.

We fix an $n \in \mathbb{N}$ and a $t^n = (t_1, \cdots t_n) \in \theta^n$. For any $a^n \in \mathsf{A}^n$ we have

$$\begin{aligned} &|S\left(V_{t^{n}}(a^{n})\right) - S\left(V'_{t^{n}}(a^{n})\right)| \\ &= \left|\sum_{k=1}^{n} S\left(V_{(t_{1},\cdots t_{k-1})} \otimes V'_{(t_{k},\cdots t_{n})}(a^{n})\right) - S\left(V_{(t_{1},\cdots t_{k})} \otimes V'_{(t_{k+1},\cdots t_{n})}(a^{n})\right)\right| \\ &\leq \sum_{k=1}^{n} \left|S\left(V_{(t_{1},\cdots t_{k-1})} \otimes V'_{(t_{k},\cdots t_{n})}(a^{n})\right) - S\left(V_{(t_{1},\cdots t_{k})} \otimes V'_{(t_{k+1},\cdots t_{n})}(a^{n})\right)\right| \,. \end{aligned}$$

For a $k \in \{1, \cdots, n\}$ and $a^n = (a_1, \cdots a_n) \in \mathsf{A}^n$ by Lemma 8 we have

$$\begin{aligned} \left| S\left(V_{(t_1,\cdots t_{k+1})} \otimes V'_{(t_k,\cdots t_n)}(a^n) \right) - S\left(V_{(t_1,\cdots t_{k+1})} \otimes V'_{(t_{k+1},\cdots t_n)}(a^n) \right) \right| \\ &= \left| S\left(V_{(t_1,\cdots t_k)} \otimes V'_{(t_k,\cdots t_n)}(a^n) \right) - S\left(V_{(t_1,\cdots t_{k-1})} \otimes V'_{(t_{k+1},\cdots t_n)}((a_1,\cdots a_{k-1},a_{k+1},\cdots a_n)) \right) \right| \\ &- S\left(V_{(t_1,\cdots t_k)} \otimes V'_{(t_{k+1},\cdots t_n)}(a^n) \right) + S\left(V_{(t_1,\cdots t_{k-1})} \otimes V'_{(t_{k+1},\cdots t_n)}((a_1,\cdots a_{k-1},a_{k+1},\cdots a_n)) \right) \right| \\ &= \left| S\left(V'_{t_k}(a_k) \mid V_{(t_1,\cdots t_{k-1})} \otimes V'_{(t_{k+1},\cdots t_n)}((a_1,\cdots a_{k-1},a_{k+1},\cdots a_n)) \right) \right| \\ &- S\left(V_{t_k}(a_k) \mid V_{(t_1,\cdots t_{k-1})} \otimes V'_{(t_{k+1},\cdots t_n)}((a_1,\cdots a_{k-1},a_{k+1},\cdots a_n)) \right) \right| \\ &\leq 4\delta \log(d_E - 1) - 2 \cdot h(\delta) \;, \end{aligned}$$

where d_E is the dimension of $H^{\mathfrak{E}}$.

Thus,

$$|S(V_{t^n}(a^n)) - S(V'_{t^n}(a^n))| \le 4n\delta \log(d_E - 1) - 2n \cdot h(\delta) .$$
 (67)

For any probability distribution $p \in \mathsf{P}(\mathsf{A})$, $n \in \mathbb{N}$, and $t^n \in \theta^n$, we have

$$\begin{aligned} |\chi(p; Z_{t^n}) - \chi(p; Z'_{t^n})| \\ &= \left| S(\sum_a p(a)V_{t^n}(a)) - \sum_a p(a)S(V_{t^n}(a)) \right| \\ &- S(\sum_a p(a)V'_{t^n}(a)) + S(\sum_a p(a)V'_{t^n}(a)) \right| \\ &\leq \left| S(\sum_a p(a)V_{t^n}(a)) - S(\sum_a p(a)V'_{t^n}(a)) \right| \\ &+ \left| \sum_a p(a)S(V'_{t^n}(a)) - \sum_a p(a)S(V'_{t^n}(a)) \right| \\ &\leq 8n\delta \log(d_E - 1) - 4n \cdot h(\delta) . \end{aligned}$$
(68)

We fix a probability distribution q on θ , a probability distribution $p \in P(A)$, and an $n \in \mathbb{N}$. By Lemma 4 we have

$$\begin{aligned} |\chi(p; B_q) - \chi(p; B'_q)| \\ &= \left| \sum_t q(t) S(\sum_a p(a) W_t(a)) - \sum_t \sum_a q(t) p(a) S(W_t(a)) \right| \\ &- \sum_t q(t) S(\sum_a p(a) W'_t(a)) + S(\sum_t \sum_a q(t) p(a) W'_t(a)) \right| \\ &\leq \left| \sum_t q(t) S(\sum_a p(a) W_t(a)) - \sum_t q(t) S(\sum_a p(a) W'_t(a)) \right| \\ &+ \left| \sum_t \sum_a q(t) p(a) S(W_t(a)) - S(\sum_t \sum_a q(t) p(a) W'_t(a)) \right| \\ &\leq 8\delta \log(d_B - 1) - 4 \cdot h(\delta) , \end{aligned}$$
(69)

where d_B is the dimension of $H^{\mathfrak{B}}$.

Thus, for any probability distribution q on θ , $n \in \mathbb{N}$, $p \in P(A)$, and $t^n \in \theta^n$, we have for all $\{(W'_t, V'_t) : t \in \theta\} \in \mathsf{C}_{\delta}$

$$\left| (\chi(p; B_q) - \frac{1}{n} \chi(p; Z_{t^n})) - (\chi(p; B'_q) - \frac{1}{n} \chi(p; Z'_{t^n})) \right| \\ \leq 8\delta \log(d_B - 1) + 8\delta \log(d_E - 1) - 8 \cdot h(\delta) .$$
(70)

For any positive ϵ we can find a positive δ such that $8\delta \log(d_B - 1) +$ $8\delta \log(d_E - 1) - 8 \cdot h(\delta) \le \epsilon.$

Thus for all $n \in \mathbb{N}$ and any positive ϵ we can find a positive δ such that for all $\{(W'_t, V'_t) : t \in \theta\} \in \mathsf{C}_{\delta}$

$$\left| (\max_{p} \inf_{\substack{B_q \in Conv((B_t)_{t \in \theta})}} \chi(p; B_q) - \max_{t^n \in \theta^n} \chi(p; Z_{t^n})) - (\max_{p} \inf_{\substack{B'_q \in Conv((B'_t)_{t \in \theta})}} \chi(p; B'_q) - \frac{1}{n} \max_{t^n \in \theta^n} \chi(p; Z'_{t^n})) \right|$$

$$\leq \epsilon .$$
(71)
nows Corollary 4.

(71) shows Corollary 4.

Corollary 5 The deterministic secrecy capacity of an arbitrarily varying classicalquantum wiretap channel is in general not continuous.

Proof We show Corollary 5 by giving an example.

Let $\theta := \{1, 2\}$. Let $\mathsf{A} = \{0, 1\}$. Let $H^{\mathfrak{B}} = \mathbb{C}^5$. Let $\{|0\rangle^{\mathfrak{B}}, |1\rangle^{\mathfrak{B}}, |2\rangle^{\mathfrak{B}}, |3\rangle^{\mathfrak{B}}, |4\rangle^{\mathfrak{B}}\}$ be a set of orthonormal vectors on $H^{\mathfrak{B}}$. Let λ be $\in [0, 1]$.

For $r \in [0, 1]$, let P_r be the probability distribution on A such that $P_r(0) = r$ and $P_r(1) = 1 - r$. We define a channel $W_1^{\lambda} : \mathsf{P}(\mathsf{A}) \to \mathcal{S}(H^{\mathfrak{B}})$ by

$$W_1^{\lambda}(P_r) = (1-\lambda)r|0\rangle\langle 0|^{\mathfrak{B}} + (1-\lambda)(1-r)|1\rangle\langle 1|^{\mathfrak{B}} + \lambda|3\rangle\langle 3|^{\mathfrak{B}} ,$$

and a channel $W_2^{\lambda}:\mathsf{P}(\mathsf{A})\to\mathcal{S}(H^{\mathfrak{B}})$ by

$$W_2^{\lambda}(P_r) = (1-\lambda)r|1\rangle\langle 1|^{\mathfrak{B}} + (1-\lambda)(1-r)|2\rangle\langle 2|^{\mathfrak{B}} + \lambda|4\rangle\langle 4|^{\mathfrak{B}}.$$

In other words:

$$\begin{split} W_1^{\lambda}(0) &= (1-\lambda)|0\rangle\langle 0|^{\mathfrak{B}} + \lambda|3\rangle\langle 3|^{\mathfrak{B}} ,\\ W_1^{\lambda}(1) &= (1-\lambda)|1\rangle\langle 1|^{\mathfrak{B}} + \lambda|3\rangle\langle 3|^{\mathfrak{B}} ,\\ W_2^{\lambda}(0) &= (1-\lambda)|1\rangle\langle 1|^{\mathfrak{B}} + \lambda|4\rangle\langle 4|^{\mathfrak{B}} ,\\ W_2^{\lambda}(1) &= (1-\lambda)|2\rangle\langle 2|^{\mathfrak{B}} + \lambda|4\rangle\langle 4|^{\mathfrak{B}} .\end{split}$$

Let $H^{\mathfrak{E}} = \mathbb{C}^5$. Let $\{|0\rangle^{\mathfrak{E}}, |1\rangle^{\mathfrak{E}}, |2\rangle^{\mathfrak{E}}, |3\rangle^{\mathfrak{E}}, |4\rangle^{\mathfrak{E}}$ be a set of orthonormal vectors on $H^{\mathfrak{E}}$.

We define a channel $V_1^\lambda:\mathsf{P}(\mathsf{A})\to\mathcal{S}(H^\mathfrak{E})$ by

$$V_1^{\lambda}(P_r) = \lambda r |0\rangle \langle 0|^{\mathfrak{E}} + \lambda (1-r) |1\rangle \langle 1|^{\mathfrak{E}} + (1-\lambda) |3\rangle \langle 3|^{\mathfrak{E}} ,$$

and a channel $V_2^\lambda:\mathsf{P}(\mathsf{A})\to\mathcal{S}(H^\mathfrak{E})$ by

$$V_2^{\lambda}(P_r) = \lambda r |1\rangle \langle 1|^{\mathfrak{E}} + \lambda (1-r) |2\rangle \langle 2|^{\mathfrak{E}} + (1-\lambda) |4\rangle \langle 4|^{\mathfrak{E}} .$$

In other words:

$$\begin{split} V_1^{\lambda}(0) &= \lambda |0\rangle \langle 0|^{\mathfrak{E}} + (1-\lambda) |3\rangle \langle 3|^{\mathfrak{E}} \ ,\\ V_1^{\lambda}(1) &= \lambda |1\rangle \langle 1|^{\mathfrak{E}} + (1-\lambda) |3\rangle \langle 3|^{\mathfrak{E}} \ ,\\ V_2^{\lambda}(0) &= \lambda |1\rangle \langle 1|^{\mathfrak{E}} + (1-\lambda) |4\rangle \langle 4|^{\mathfrak{E}} \ ,\\ V_2^{\lambda}(1) &= \lambda |2\rangle \langle 2|^{\mathfrak{E}} + (1-\lambda) |4\rangle \langle 4|^{\mathfrak{E}} \ . \end{split}$$

For every $a \in \mathsf{A}$ and $t \in \theta$ we have

$$\begin{split} \|W_t^0(a) - W_t^\lambda(a)\|_1 \\ &= \|\lambda|t + a - 1\rangle\langle t + a - 1|^{\mathfrak{B}} - \lambda|t + 2\rangle\langle t + 2|^{\mathfrak{B}}\|_1 \\ &= 2\lambda \end{split}$$

and

$$\begin{aligned} \|V_t^0(a) - V_t^\lambda(a)\|_1 \\ &= \|-\lambda|t+a-1\rangle\langle t+a-1|^{\mathfrak{E}} + \lambda|t+2\rangle\langle t+2|^{\mathfrak{E}}\|_1 \\ &= 2\lambda \ . \end{aligned}$$

 $\{(W_t^{\lambda}, V_t^{\lambda}) : t \in \theta\}$ defines an arbitrarily varying classical-quantum wiretap channel for every $\lambda \in [0, 1]$.

At first, we consider $\{(W_t^0, V_t^0) : t \in \theta\}$.

i) The deterministic secrecy capacity of $\{(W_t^0, V_t^0) : t \in \theta\}$ is equal to zero.

We set

$$\begin{split} \tau(1\mid 0) &= 0 \ ; \quad \tau(2\mid 0) = 1 \ ; \\ \tau(1\mid 1) &= 1 \ ; \quad \tau(2\mid 1) = 0 \ . \end{split}$$

It holds that

$$\sum_{t\in\theta}\tau(t\mid 0)W^0_t(1)=|1\rangle\langle 1|^{\mathfrak{E}}=\sum_{t\in\theta}\tau(t\mid 1)W^0_t(0)+\sum_{t\in\theta}\tau(t\mid 1)W^0_t(0)+\sum_{t\in$$

and of course for every $a \in A$

$$\sum_{t \in \theta} \tau(t \mid a) W_t^0(a) = \sum_{t \in \theta} \tau(t \mid a) W_t^0(a)$$

 $\{(W_t^0): t \in \theta\}$ is therefore symmetrizable. By [22], we have

$$C_s(\{(W_t^0, V_t^0) : t \in \theta\}) = 0.$$
(72)

ii) The secrecy capacity of $\{(W_t^0, V_t^0) : t \in \theta\}$ under common randomness assisted quantum coding is positive.

We denote by $p' \in \mathsf{P}(\mathsf{A})$ the distribution on A such that $p'(1) = p'(2) = \frac{1}{2}$. Let $q \in [0, 1]$. We define Q(1) = q, Q(2) = 1 - q. We have

$$\begin{split} \chi \left(p', \{ W_Q^0(a) : a \in \mathsf{A} \} \right) \\ &= -\frac{1}{2}q\log\frac{1}{2}q + \frac{1}{2}(1-q)\log\frac{1}{2}(1-q) - \frac{1}{2}\log\frac{1}{2} \\ &+ q\log q + (1-q)\log(1-q) \ . \end{split}$$

When we differentiate this term by q, we obtain

$$\begin{aligned} &\frac{1}{\log e} \left(-\frac{1}{2} \log \frac{1}{2}q - \frac{1}{2} + \frac{1}{2} \log \frac{1}{2}(1-q) + \frac{1}{2} + \log q + 1 - \log(1-q) - 1 \right) \\ &= \frac{1}{2 \log e} \left(\log q - \log(1-q) \right) \;. \end{aligned}$$

 $\log q - \log(1-q)$ is equal to zero if and only if $q = \frac{1}{2}$. By further calculation, one can show that $\chi(p', \{W_Q^0(a) : a \in \mathsf{A}\})$ achieves its minimum when $q = \frac{1}{2}$. This minimum is equal to $-\frac{1}{2}\log \frac{1}{4} + \frac{1}{2}\log \frac{1}{2} = \frac{1}{2} > 0$. Thus,

$$\max_{p} \min_{q} \chi\left(p, B_{q}^{0}\right) \geq \frac{1}{2} .$$

For all $t \in \theta$ it holds that $V_t^0(0) = V_t^0(1)$; therefore for all $t^n \in \theta^n$ and any $p^n \in \mathsf{P}(\mathsf{A}^n)$ we have

$$\begin{split} &\chi(p; Z_{t^n}^0) \\ &= S(V_{t^n}^0(p^n)) - \sum_{a^n \in \mathsf{A}^n} p^n(a^n) S(V_{t^n}^0(a^n)) \\ &= S(V_{t^n}^0(0^n)) - \sum_{a^n \in \mathsf{A}^n} p^n(a^n) S(V_{t^n}^0(0^n)) \\ &= 0 \; . \end{split}$$

Thus,

$$C_s(\{(W_t^0, V_t^0) : t \in \theta\}, cr) \ge \frac{1}{2} - 0 > 0.$$
(73)

Now we consider $\{(W_t^{\lambda}, V_t^{\lambda}) : t \in \theta\}$ when $\lambda \neq 0$.

iii) When $\lambda \neq 0$, the deterministic secrecy capacity of $\{(W_t^{\lambda}, V_t^{\lambda}) : t \in \theta\}$ is equal to its secrecy capacity of under common randomness assisted quantum coding.

We suppose that for any $a, a' \in A$ there are two distributions $\tau(\cdot \mid a)$ and $\tau(\cdot \mid a')$ on θ such that

$$\begin{split} \sum_{t\in\theta} \tau(t\mid a') \cdot W_t^{\lambda}(a) &= \sum_{t\in\theta} \tau(t\mid a) \cdot W_t^{\lambda}(a') \\ \Rightarrow (1-\lambda) \sum_{t\in\theta} \tau(t\mid a') |t+a-1\rangle \langle t+a-1|^{\mathfrak{B}} + \lambda\tau(1\mid a') |3\rangle \langle 3|^{\mathfrak{E}} + \lambda\tau(2\mid a') |4\rangle \langle 4|^{\mathfrak{E}} \\ &= (1-\lambda) \sum_{t\in\theta} \tau(t\mid a) |t+a'-1\rangle \langle t+a'-1|^{\mathfrak{B}} + \lambda\tau(1\mid a) |3\rangle \langle 3|^{\mathfrak{E}} + \lambda\tau(2\mid a) |4\rangle \langle 4|^{\mathfrak{E}} . \end{split}$$

$$(74)$$

Since $|t + a - 1\rangle\langle t + a - 1|^{\mathfrak{B}} \in \left\{ |0\rangle\langle 0|^{\mathfrak{E}}, |1\rangle\langle 1|^{\mathfrak{E}}, |2\rangle\langle 2|^{\mathfrak{E}} \right\}$ for all t and a, if $\lambda \neq 0, (74)$ implies that $\tau(t \mid a') = \tau(t \mid a)$

for all
$$t \in \theta$$
. This means we have a distribution p on θ such that $p(t) = \tau(t \mid a)$ for all $a \in A$.

But there is clearly no such distribution p' such that $\sum_{t \in \theta} p(t) W_t^{\lambda}(0) = \sum_{t \in \theta} p(t) W_t^{\lambda}(1)$, because then we would have

$$\begin{split} \dot{p}(1)|0\rangle\langle 0|^{\mathfrak{B}} + \dot{p}(2)|1\rangle\langle 1|^{\mathfrak{B}} \\ &= \dot{p}(1)|1\rangle\langle 1|^{\mathfrak{B}} + \dot{p}(2)|2\rangle\langle 2|^{\mathfrak{B}} \end{split}$$

This would mean $\dot{p}(1) = \dot{p}(2) = 0$, which obviously cannot be true. Thus, $(W_t^{\lambda})_{t \in \theta}$ is not symmetric.

By [22], if $\lambda \neq 0$

$$C_s(\{(W_t^{\lambda}, V_t^{\lambda}) : t \in \theta\}) = C_s(\{(W_t^{\lambda}, V_t^{\lambda}) : t \in \theta\}, cr) .$$

$$(75)$$

When $\lambda \searrow 0$ for every $a \in \mathsf{A}$ and $t \in \theta$ we have $||W_t^0(a) - W_t^\lambda(a)||_1 = ||V_t^0(a) - V_t^\lambda(a)||_1 = 2\lambda \searrow 0.$

By Corollary 4, the secrecy capacity of $\{(W_t^{\lambda}, V_t^{\lambda}) : t \in \theta\}$ under common randomness assisted quantum coding is continues. Thus for any positive ε there is a δ , such that for all $\lambda \in]0, \delta[$, we have

$$C_s(\{(W_t^{\lambda}, V_t^{\lambda}) : t \in \theta\}) \ge C_s(\{(W_t^0, V_t^0) : t \in \theta\}, cr) - \varepsilon \ge \frac{1}{2} - \varepsilon .$$
(76)

In other words, when $\lambda \neq 0$ tends to zero, the deterministic secrecy capacity of $\{(W_t^{\lambda}, V_t^{\lambda}) : t \in \theta\}$ tends to the secrecy capacity of $\{(W_t^0, V_t^0) : t \in \theta\}$ under common randomness assisted quantum coding, which is positive, but the deterministic secrecy capacity of $\{(W_t^0, V_t^0) : t \in \theta\}$ is equal to zero. Hence, the deterministic secrecy capacity of $\{(W_t^{\lambda}, V_t^{\lambda}) : t \in \theta\}$ is not continues at zero.

Corollary 5 shows that small errors in the description of an arbitrarily varying classical-quantum wiretap channel may have severe consequences on the secrecy capacity. Corollary 4 shows that resources are very helpful to protect these consequences.

6 Conclusion

In this paper, we deliver the formula for the secrecy capacities under common randomness assisted coding of arbitrarily varying classical-quantum wiretap channels. In our previous paper [22], we established the Ahlswede Dichotomy for arbitrarily varying classical-quantum wiretap channels: Either the deterministic secrecy capacity of an arbitrarily varying classical-quantum wiretap channel is zero or it equals its randomness assisted secrecy capacity, depending on the status whether the legitimate receiver's channel is symmetrizable or not. When we combine the results of these two works we can now completely characterize the secrecy capacity formulas for arbitrarily varying classical-quantum wiretap channels (cf. Corollary 3).

As an application of these results, we turn to the general question: When is secure message transmission through arbitrarily varying classical-quantum wiretap channels continuous? Our results show the discontinuity in general and demonstrate the importance of shared randomness: it stabilizes the secure message transmission through arbitrarily varying classical-quantum wiretap channels.

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