An optimal discrimination of two mixed qubit states with a fixed rate of inconclusive

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In this paper we consider the optimal discrimination of two mixed qubit states for a measurement that allows a fixed rate of inconclusive results. Our strategy is to transform the problem of two qubit states into a minimum error discrimination for three qubit states by adding a specific quantum state ρ_0 and a prior probability q_0 , which behaves as an inconclusive degree. First, we introduce the beginning and the end of practical interval of inconclusive result, $q_0^{(0)}$ and $q_0^{(1)}$, which are key ingredients in investigating our problem. Then we obtain the analytic form of them. Next, we show that our problem can be classified into two cases $q_0 = q_0^{(0)}$ (or $q_0 = q_0^{(1)}$) and $q_0^{(0)} < q_0 < q_0^{(1)}$. In fact, by maximum confidences of two qubit states and non-diagonal element of ρ_0 , the our problem is completely understood. We provide an analytic solution of our problem when $q_0 = q_0^{(0)}$ (or $q_0 = q_0^{(1)}$). However, when $q_0^{(0)} < q_0 < q_0^{(1)}$, we rather supply the numerical method to find the solution, because of the complex relation between inconclusive degree and corresponding failure probability. Finally we confirm our results using previously known examples.

INTRODUCTION

The information encoded in the quantum state by a sender can be delivered to a receiver, who performs a measurement to extract this information. A proper measurement strategy is required when the receiver wants to obtain information from nonorthogonal quantum states because those states cannot be perfectly discriminated [1-4]. Measurement strategies can be classified by the constraints on conclusive or inconclusive results. In quantum state discrimination, inconclusive results indicate that the given quantum state cannot be definitely discriminated. Minimum-error discrimination(MD)[5–18] is able to minimize the average error of conclusive results without inconclusive results. Unambiguous discrimination(UD)[19-29] and maximum-confidence discrimination(MC)[30] strategies permit inconclusive results and minimize individual errors associated with the conclusive results.

In addition to these strategies, there is a scheme for minimizing the average error of conclusive results while maintaining a fixed rate of inconclusive results (FRIR) [31–38]. FRIR is actually a generalization of other known strategies. For example, when the fixed rate is zero, the FRIR is equivalent to the MD. If the fixed rate is sufficiently large, the FRIR becomes equivalent to MC(or UD). For the MD, the solution of two mixed quantum states is explicitly known[5, 10], however it is not known for the FRIR. The solution of the FRIR of two qubit states with identical maximal confidences exists[35] but that of the general case does not.

Recently, Bagan et al. [36] changed the FRIR of N quantum states into the MD of N quantum states by modifying prior probabilities and the quantum states. This approach can be useful for obtaining a solution to symmetric states but it cannot be used for arbitrary quantum states and prior probabilities because it requires solving complicated equations. On the other hand, Nakahira et al. [37] and Herzog [38] provided another method to transform the FRIR into the MD. Their method does not modify the given quantum states or the prior probabilities. Instead, this method only adds an appropriate density operator ρ_0 with a suitable probability q_0 (which we will call an inconclusive degree) for a given quantum system. Then, based on the FRIR of N quantum states, one can form the MD of N+1 quantum states by using a measurement operator that provides inconclusive results. In order to transform the problem of optimal discrimination of N quantum states with a fixed rate of inconclusive results into that of minimum error discrimination of N+1 quantum states, one must deal with special inconclusive degrees $q_0^{(0)}$ and $q_0^{(1)}$. Even though they mentioned the relation between failure probability of original problem and inconclusive degree of modified problem, they could not find special inconclusive degrees $q_0^{(0)}$ and $q_0^{(1)}$ in an analytic form, which appear naturally in modified problem. Even more they could not solve even the simplest FRIR problem for two qubit mixed states. Here special inconclusive degrees $q_0^{(0)}$ and $q_0^{(1)}$ are the beginning and the end of practical interval of inconclusive degree. In fact $q_0^{(0)}$ and $q_0^{(1)}$ are the key to solve FRIR of two qubit states.

In fact, Nakahira et al. [37] and Herzog [38] could not give a solution to the FRIR of two mixed qubit states. In

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this paper we provide a solution to the FRIR of two mixed qubit states. In Section II we derive the detailed relation between original FRIR problem and modified FRIR problem, which is given by MD of three qubit states. Furthermore we introduce special inconclusive degrees $q_0^{(0)}$ and $q_0^{(1)}$ and investigate their feature. In Section III we divide FRIR problem of two qubit states into two cases of $q_0 = q_0^{(0)}$ (or $q_0 = q_0^{(1)}$) and $q_0^{(0)} < q_0 < q_0^{(1)}$, and specify that the problem can be solved by maximum confidences of two qubit states and the non-diagonal element of ρ_0 . Using complementarity problem, we find the analytic form of $q_0^{(0)}$ and $q_0^{(1)}$, and provide the complete understanding of modified FRIR problem in $q_0^{(0)} \le q_0 \le q_0^{(1)}$. That is, we provide an analytic solution of original FRIR problem in case of $q_0 = q_0^{(0)}$ (or $q_0 = q_0^{(1)}$). If $q_0^{(0)} < q_0 < q_0^{(1)}$, because of complex relation between inconclusive degree and corresponding failure probability, we provide the method to solve original problem numerically. Finally, we confirm our results by providing the correct solutions to known examples [35]. In Section IV we summarize our result.

II. FRIR

We consider the quantum state ensemble $\{q_i, \rho_i\}_{i=1}^N$. This ensemble suggests that with the prior probability q_i , one prepares the quantum state corresponding to the density operator ρ_i on a d-dimensional complex Hilbert space \mathcal{H}_d . Without loss of generality, we assume that the eigenvectors of $\rho_0 \equiv \sum_{i=1}^N q_i \rho_i$ (with nonzero eigenvalues) span \mathcal{H}_d . The quantum state of the system may be discriminated by the positive operator valued measure (POVM) $\{M_i\}_{i=0}^N$. The POVM consists of N+1 positive semidefinite Hermitian operators on \mathcal{H}_d and satisfies $\sum_{i=0}^N M_i = I_d$. I_d is the identity operator on \mathcal{H}_d . Here M_0 provides inconclusive results, while $M_{i(\neq 0)}$ gives conclusive results. The probability that the quantum state ρ_i can be guessed to be ρ_j is $\mathrm{tr}[\rho_i M_j]$ by the Born rule. Therefore, the probability for conclusive results P_C turns out to be $\sum_{i=1}^N \mathrm{tr}[\rho_0 M_i]$, and the probability for inconclusive results P_I becomes $\mathrm{tr}[\rho_0 M_0]$. The probability of correctly guessing the quantum state and the error probability are $P_{\mathrm{cor}} = \sum_{i=1}^N q_i \mathrm{tr}[\rho_i M_i]$ and $P_{\mathrm{err}} = P_C - P_{\mathrm{cor}}$ respectively. We use $R_{\mathrm{cor(err)}}$ to denote the probability of correctly guessing when we succeed in guessing the quantum state. That is, $R_{\mathrm{cor(err)}} = P_{\mathrm{cor(err)}}/P_C$.

A. Original FRIR problem

Our discrimination strategy is to maximize(or minimize) $R_{\text{cor(err)}}$ with fixed $P_{\text{I}} = Q(0 \leq Q < 1)$. Because $P_{\text{C}} + P_{\text{I}} = 1$, this is equivalent to maximizing(or minimizing) $P_{\text{cor(err)}}$ with fixed $P_{\text{I}} = Q$, which can be reformulated into the following optimization problem:

$$\max \qquad P_{\text{cor}} = \sum_{i=1}^{N} q_i \text{tr}[\rho_i M_i]$$
 subject to
$$M_i \ge 0 \ \forall i, \ \sum_{i=0}^{N} M_i = I_d, \ \text{tr}[\rho_0 M_0] = Q. \tag{1}$$

In this paper, we use the superscript "opt" to denote the optimized value or variable. For example, $P_{\text{cor}}^{\text{opt}}(Q)$ and $R_{\text{cor}}^{\text{opt}}(Q)$ indicate the maximum of P_{cor} and R_{cor} when $P_{\text{I}} = Q$, respectively.

B. Modified FRIR problem

Instead of simply attacking the problem as described above, we can modify it as follows. Here, we introduce a positive number q_0 (called an inconclusive degree) which corresponds to the a priori probability of ρ_0 . Further, M_0 denotes the measurement operator of guessing ρ_0 in the system:

$$\max \qquad \bar{P}_{cor} \equiv \sum_{i=0}^{N} q_i tr[\rho_i M_i]$$
 subject to
$$M_i \ge 0 \ \forall i, \ \sum_{i=0}^{N} M_i = I_d, \ q_0 = q.$$
 (2)

We use $\bar{P}_{\text{cor}}^{\text{opt}}(q)$ to denote the maximum value of \bar{P}_{cor} when $q_0 = q$. The following relation[37] between $P_{\text{cor}}^{\text{opt}}(Q)$ and $\bar{P}_{\text{cor}}^{\text{opt}}(q)$ implies that when the MD of $\{q_i, \rho_i\}_{i=0}^N$ can be completely analyzed, $R_{\text{cor}}^{\text{opt}}(Q)$ can be found in the FRIR of $\{q_i, \rho_i\}_{i=1}^N$.

$$\textbf{Lemma II.1} \ \textit{If} \ P_{\rm I} = Q \ \textit{and} \ \bar{P}_{\rm cor} = \bar{P}_{\rm cor}^{\rm opt}(q) \ \textit{for some POVM}, \ P_{\rm cor}^{\rm opt}(Q) = \bar{P}_{\rm cor}^{\rm opt}(q) - qQ.$$

The proof is given in Appendix A.

Equation (2) represents a convex optimization problem[39] (or semidefinite program) to minimum-error discrimination problem for $\{q_i, \rho_i\}_{i=0}^N$ with non-normalized priori probabilities. For investigating the analytic structure of POVM for an optimal solution of (2), we consider Karush-Kuhn-Tucker(KKT) optimality conditions, composed of constraints of primal and dual problem and complementary conditions, instead of necessary and sufficient conditions [5–7].

C. Optimality conditions of modified FRIR problem

The optimization problem (2) is equivalent to MD of $\{q_i, \rho_i\}_{i=0}^N$ with non-normalized priori probabilities. Since the semidefinite programming of MD[40] hold regardless of the normalization condition, we can apply the results into this modified FRIR problem. First, the modified problem (3) has the following Lagrange dual problem.

min tr[K] subject to
$$K = q_i \rho_i + r_i \tau_i \ \forall i, \ q_0 = q.$$
 (3)

K is a Hermitian operator on \mathcal{H}_d , which is a Lagrange multiplier of an equality constraint $\sum_{i=0}^{N} M_i = I_d$. $r_i \tau_i$ is a Lagrange multiplier of an inequality constraint $M_i \geq 0$, where r_i and τ_i are a non-negative real number and a density operator on \mathcal{H}_d , respectively. $r_i\tau_i$ is separated into r_i and τ_i , for geometric understanding of qubit state discrimination. Second, the optimized values of two problems (2),(3) of $q_0 = q$ coincide. Finally, the complementary slackness condition $r_i \operatorname{tr}[\tau_i M_i] = 0 (\forall i)$ is a necessary and sufficient condition for optimizing the feasible variables in two problems(primal and dual problems)(2),(3) of $q_0 = q$. We summarize the KKT optimality condition for modified FRIR problem of $q_0 = q$ as follows:

$$\begin{split} \text{(i)} \quad & M_i \geq 0 \ \forall i, \ \sum_{i=0}^N M_i = I_d, \\ \text{(ii)} \quad & q\rho_0 + r_0\tau_0 = q_i\rho_i + r_i\tau_i \ \forall i, \\ \text{(iii)} \quad & r_i \text{tr}[\tau_i M_i] = 0 \ \forall i. \end{split}$$

In order to express KKT optimality condition (4) in a form that we can deal with, we define the following variables.

$$\bar{M}_i = \rho_0^{1/2} M_i \rho_0^{1/2}, \ \bar{\rho}_i = \rho_0^{-1/2} q_i \rho_i \rho_0^{-1/2}, \ \bar{\tau}_i = \rho_0^{-1/2} r_i \tau_i \rho_0^{-1/2}.$$

$$(5)$$

In terms of these newly defined variables, the KKT condition can be rewritten as:

(i)
$$\bar{M}_i \geq 0 \ \forall i, \ \sum_{i=0}^N \bar{M}_i = \rho_0,$$

(ii) $qI_d + \bar{\tau}_0 = \bar{\rho}_i + \bar{\tau}_i \ \forall i,$
(iii) $\operatorname{tr}[\bar{\tau}_i \bar{M}_i] = 0 \ \forall i.$ (6)

We denote C_i and $|\nu_i\rangle$ as the largest eigenvalue of $\bar{\rho}_i$ and the corresponding eigenvector, respectively. C_i physically represents the maximum achievable confidence of ρ_i in terms of MC[30]. Note that the product $r_i^{\text{opt}} \tau_i^{\text{opt}}$ of r_i^{opt} and τ_i^{opt} satisfying optimality condition (4) is unique, but optimal POVM elements M_i^{opt} is not always unique[16]. $\bar{\tau}_i^{\text{opt}}$ fulfilling another optimality condition (6) is unique. However \bar{M}_i^{opt} can be unique or non-unique. We will use the fact to find the analytic expression of optimal POVM element M_i^{opt} or \bar{M}_i^{opt} .

When d=2, by introducing a real number p_i and Bloch vectors u_i , v_i , and w_i , we can express POVM elements M_i and density operators ρ_i, τ_i as:

$$M_i = p_i(I_2 + \boldsymbol{u}_i \cdot \boldsymbol{\sigma}), \ \rho_i = \frac{1}{2}(I_2 + \boldsymbol{v}_i \cdot \boldsymbol{\sigma}), \ \tau_i = \frac{1}{2}(I_2 + \boldsymbol{w}_i \cdot \boldsymbol{\sigma}).$$
 (7)

Then, the KKT optimality condition (4) can be described as:

(i)
$$p_i \ge 0 \ \forall i, \ \sum_{i=0}^{N} p_i = 1, \ \sum_{i=0}^{N} p_i \boldsymbol{u}_i = 0,$$

(ii)
$$q + r_0 = q_i + r_i$$
, $q\mathbf{v}_0 + r_0\mathbf{w}_0 = q_i\mathbf{v}_i + r_i\mathbf{w}_i \ \forall i$,
(iii) $p_i r_i (1 + \mathbf{u}_i \cdot \mathbf{w}_i) = 0 \ \forall i$. (8)

In Section III we investigate optimal variables of primal problem (2) and dual problem (3), using two optimality conditions (6) and (8). The approach is called complementarity problem [15, 16] in semidefinite programming.

D. Special inconclusive degrees

The fact that optimal measurement may not be unique in the MD leads us to introduce the following definition.

Definition II.1 When q is a positive number, we define $P_{\rm I}(q)$ as follows:

$$P_{I}(q) = \left\{ \operatorname{tr}[\rho_{0} M_{0}] : M_{i} \geq 0 \ \forall i, \sum_{i=0}^{N} M_{i} = I_{d}, \ q \operatorname{tr}[\rho_{0} M_{0}] + \sum_{i=1}^{N} q_{i} \operatorname{tr}[\rho_{i} M_{i}] = \bar{P}_{\operatorname{cor}}^{\operatorname{opt}}(q) \right\}.$$
(9)

The case of $Q \ge (=)a$ for any $Q \in P_{\mathrm{I}}(q)$ will be denoted as $P_{\mathrm{I}}(q) \ge (=)a$, whereas that of $Q \ge (=)Q'$ for any $Q \in P_{\mathrm{I}}(q)$ and $Q' \in P_{\mathrm{I}}(q')$ will be written as $P_{\mathrm{I}}(q) \ge (=)P_{\mathrm{I}}(q')$. Note that $0 \le P_{\mathrm{I}}(q) \le 1$ for any q.

The following lemma shows how $P_{\rm I}(q)$ behaves as q increases.

Lemma II.2 $P_{I}(q)$ is a convex set for any q, and $P_{I}(q) \leq P_{I}(q')$ for any q, q' with q < q'.

The proof is given in Appendix A. Through Lemma II.2, $P_{\rm I}(q)$ is generally an interval. However, when optimal measurement of modified FRIR problem $\{q_i, \rho_i\}_{i=0}^N (q_0 = q)$ is unique, $P_{\rm I}(q)$ becomes a point.

Lemma II.2 enables us to define the following special inconclusive degrees.

Definition II.2 (special inconclusive degrees) We define $q_0^{(0)}, q_0^{(1)}$ as follows:

$$q_0^{(0)} = \max\{q > 0 : 0 \in P_{\mathcal{I}}(q)\},$$

$$q_0^{(1)} = \min\{q > 0 : 1 \in P_{\mathcal{I}}(q)\}.$$
(10)

This implies that a proper inconclusive degree q, which satisfies $0 < P_{\rm I}(q) < 1$, exists in the region $[q_0^{(0)}, q_0^{(1)}]$. Therefore, $R_{\rm cor}^{\rm opt}(Q)$ in $0 \le Q < 1$ can be found from $P_{\rm I}(q)$ and $\bar{P}_{\rm cor}^{\rm opt}(q)$ in $q_0^{(0)} \le q \le q_0^{(1)}$. That is,

$$R_{\text{cor}}^{\text{opt}}(Q) = \frac{\bar{P}_{\text{cor}}^{\text{opt}}(q) - qQ}{1 - Q} \quad \forall Q \in P_{\text{I}}(q).$$

$$\tag{11}$$

The following lemma provides the lower bound of $q_0^{(0)}$ and the upper bound of $q_0^{(1)}$.

Lemma II.3 $q_0^{(0)} \ge 1/N$ and $q_0^{(1)} \le \max_i C_i$.

The proof is given in Appendix A.

III. MAIN RESULT: FRIR OF TWO QUBIT MIXED STATES

In this section we analyze the FRIR of two qubit-mixed states (d = N = 2), using the transformed KKT optimality condition which has two different forms. The first KKT optimality condition (6) is obtained by \bar{M}_i , $\bar{\rho}_i$, $\bar{\tau}_i$ of Eq. (5). The second one (8) is expressed by Bloch vectors \boldsymbol{u}_i , \boldsymbol{v}_i , \boldsymbol{w}_i defined in Eq. (7). In certain situations, (6) or (8) is used. For two qubit-mixed states, $\bar{\rho}_1$, $\bar{\rho}_2$ are two positive semidefinite Hermitian operators on two-dimensional Hilbert space and they satisfy $\bar{\rho}_1 + \bar{\rho}_2 = I_2$. $(1 - C_1)$ and $|\nu_1\rangle((1 - C_2))$ and $|\nu_2\rangle$ are the smallest eigenvalue and the corresponding eigenvector of $\bar{\rho}_2(\bar{\rho}_1)$, which implies $\langle \nu_1|\nu_2\rangle = 0$. Therefore $\bar{\rho}_1$ and $\bar{\rho}_2$ become:

$$\bar{\rho}_1 = C_1 |\nu_1\rangle \langle \nu_1| + (1 - C_2) |\nu_2\rangle \langle \nu_2|,
\bar{\rho}_2 = (1 - C_1) |\nu_1\rangle \langle \nu_1| + C_2 |\nu_2\rangle \langle \nu_2|.$$
(12)

We assume $C_1 \leq C_2$, which does not spoil generality of the problem. Here e denotes the difference between q_1 and q_2 , and l expresses the distance between two weighted Bloch vectors $q_1 \mathbf{v}_1$ and $q_2 \mathbf{v}_2$.

$$e = |q_1 - q_2|, \ l = ||q_1 \mathbf{v}_1 - q_2 \mathbf{v}_2||_2.$$
 (13)

In this section, we divide our problem into two cases, by using two special inconclusive degrees $q_0^{(0)}, q_0^{(1)}$. In Subsection III A and III B, when fixed rate Q of inconclusive results belongs to $P_I(q_0^{(1)})$ or $P_I(q_0^{(0)})$, we obtain what are optimal value $R_{\text{cor}}^{\text{opt}}$ and optimal measurement operators M_i^{opt} (or \bar{M}_i^{opt}). In Subsection III C, when fixed rate lies in the other region $Q \in P_I(q_0^{(0)})^{\mathsf{C}} \cap P_I(q_0^{(1)})^{\mathsf{C}}$, we explain how complex optimal solution can be found. The result obtained from KKT optimality condition (6) or (8) is classified, according to the relation of two maximum confidences C_1, C_2 and that of $\rho_{11}, \rho_{12}, \rho_{22}$ of ρ_0 .

More specifically, in Subsection III A, Corollary III.1, which is the result of the section, is expressed in two cases, according to the equality between C_1 and C_2 . Specially, the result of the case of $C_1 = C_2$ is shown in three types, according the magnitude of three nonnegative numbers $\rho_{11}, \rho_{22}, |\rho_{12}|$. In Subsection III B, Theorem III.1, which is the final result of the section, is obtained in two cases, by comparision between $\frac{1}{2}$ and C_1 . When $\frac{1}{2} < C_1$, the result is classified into two cases, by the existence of non-diagonal element ρ_{12} of ρ_0 . In Subsection III C, the case of $\rho_{12} = 0$ provides Theorem III.2 and that of $\rho_{12} \neq 0$ gives Theorem III.3. The former one is the result corresponding to the total range of fixed rate $Q(\text{that is, } 0 \leq Q \leq 1)$. The latter one is the case of $Q \in P_1(q_0^{(0)})^{\mathsf{C}} \cap P_1(q_0^{(1)})^{\mathsf{C}}$.

A. FRIR at
$$P_{\mathbf{I}} = Q$$
 for all $Q \in P_{\mathbf{I}}(q_0^{(1)})$

In the following lemma modified FRIR problem to the case of $q_0 = C_2$ is completely analyzed.

Lemma III.1 $\bar{P}_{cor}^{opt}(C_2)$ is C_2 . When $C_1 = C_2$, \bar{M}_i^{opt} to $q_0 = C_2$ is expressed as

$$\bar{M}_0^{\text{opt}} = \rho_0 - \alpha |\nu_1\rangle\langle\nu_1| - \beta |\nu_2\rangle\langle\nu_2|,
\bar{M}_1^{\text{opt}} = \alpha |\nu_1\rangle\langle\nu_1|, \quad 0 \le \alpha \le \rho_{11}, \quad 0 \le \beta \le \rho_{22},
\bar{M}_2^{\text{opt}} = \beta |\nu_2\rangle\langle\nu_2|, \quad (\rho_{11} - \alpha)(\rho_{22} - \beta) \ge |\rho_{12}|^2,$$
(14)

where

$$\rho_{ij} = \langle \nu_i | \rho_0 | \nu_j \rangle. \tag{15}$$

Then $P_{\rm I}(C_2)$ becomes

$$P_{\rm I}(C_2) = \begin{cases} [Q_1, 1] & \text{if } \rho_{11} < |\rho_{12}| \le \rho_{22}, \\ [Q_2, 1] & \text{if } \rho_{22} < |\rho_{12}| \le \rho_{11}, \\ [2|\rho_{12}|, 1] & \text{if } |\rho_{12}| \le \rho_{11}, \rho_{22}, \end{cases}$$
(16)

where

$$Q_i = \rho_{ii} + \frac{|\rho_{12}|^2}{\rho_{ii}}. (17)$$

However when $C_1 < C_2$, $P_I(C_2)$ becomes $[Q_1, 1]$, and \bar{M}_i^{opt} for $q_0 = C_2$ is expressed as

$$\bar{M}_0^{\text{opt}} = \rho_0 - \beta |\nu_2\rangle\langle\nu_2|, \ \bar{M}_1^{\text{opt}} = 0,
\bar{M}_2^{\text{opt}} = \beta |\nu_2\rangle\langle\nu_2|, \ 0 \le \beta \le 1 - Q_1.$$
(18)

The proof is given in Appendix B. Note that three real numbers Q_1, Q_2 , and $2|\rho_{12}|$ are less than 1. By Lemma II.2, $1 \in P_{\rm I}(q)$ implies $q_0^{(1)} = q$ and $1 \in P_{\rm I}(C_2)$ derived by Lemma III.1 means $q_0^{(1)} = C_2$. Therefore, when d = N = 2, inequality $q_0^{(1)} \le \max_i C_i$ of Lemma II.3 becomes an equality.

Lemma III.1 tells how the analytic solution of original FRIR problem is changed according to $Q \in P_{\rm I}(q_0^{(1)})$. $\bar{P}_{\rm cor}^{\rm opt}(C_2)$ is C_2 in any case and $R_{\rm cor}^{\rm opt}(Q)$ is C_2 for any $Q \in P_{\rm I}(C_2)$, because of Eq. (11). In case of $C_1 = C_2$, modified FRIR measurement is represented by two variables α and β . However, in case of $C_1 < C_2$, modified FRIR measurement is expressed only by β . This can be understood in terms of uniqueness of FRIR measurement. In case of $C_1 = C_2$, since ${\rm tr}[\bar{M}_0^{\rm opt}] = 1 - \alpha - \beta$, there may exist different (α, β) providing the same value of ${\rm tr}[\bar{M}_0^{\rm opt}]$, which implies that there are different forms of FRIR measurement in a fixed $P_{\rm I}$. However, in case of $C_1 < C_2$, because of ${\rm tr}[\bar{M}_0^{\rm opt}] = 1 - \beta$, different β provides different value of ${\rm tr}[\bar{M}_0^{\rm opt}]$. Therefore, according to $Q \in P_{\rm I}(q_0^{(1)})$, FRIR measurement uniquely exists. In other words, FRIR measurement of $C_1 < C_2$ becomes FRIR measurement of $C_1 = C_2$ with $\alpha = 0$. Then, FRIR measurements of two cases can be represented by a variable $\epsilon = \rho_{11} - \alpha$. The following corollary summarizes the result.

Corollary III.1 (FRIR of $Q \in P_{\mathbf{I}}(q_0^{(1)})$) $q_0^{(1)}$ is C_2 , and $P_{\mathbf{I}}(q_0^{(1)})$ can be classified into

$$P_{\rm I}(q_0^{(1)}) = \begin{cases} [Q_1, 1] & \text{if } C_1 < C_2, \\ [Q_1, 1] & \text{if } C_1 = C_2, \ \rho_{11} < |\rho_{12}| \le \rho_{22}, \\ [Q_2, 1] & \text{if } C_1 = C_2, \ \rho_{22} < |\rho_{12}| \le \rho_{11}, \\ [2|\rho_{12}|, 1] & \text{if } C_1 = C_2, \ |\rho_{12}| \le \rho_{11}, \rho_{22}. \end{cases}$$

$$(19)$$

 $R_{\rm cor}^{\rm opt}(Q)$ is C_2 , and $\bar{M}_i^{\rm opt}$ of $P_{\rm I}=Q$ can be expressed as

$$\bar{M}_0^{\text{opt}} = \epsilon |\nu_1\rangle\langle\nu_1| + \rho_{12}|\nu_1\rangle\langle\nu_2| + \rho_{21}|\nu_2\rangle\langle\nu_1| + (Q - \epsilon)|\nu_2\rangle\langle\nu_2|,
\bar{M}_1^{\text{opt}} = (\rho_{11} - \epsilon)|\nu_1\rangle\langle\nu_1|, \ \bar{M}_2^{\text{opt}} = (\rho_{22} - Q + \epsilon)|\nu_2\rangle\langle\nu_2|,$$
(20)

where

$$\max \left\{ Q - \rho_{22}, \frac{Q}{2} - \sqrt{\frac{Q^2}{4} - |\rho_{12}|^2} \right\} \le \epsilon \le \min \left\{ \rho_{11}, \frac{Q}{2} + \sqrt{\frac{Q^2}{4} - |\rho_{12}|^2} \right\} \text{ if } C_1 < C_2,$$

The result implies that when $Q \in P_{\rm I}(q_0^{(1)})$, if $C_1 < C_2$, FRIR measurement to $P_{\rm I} = Q$ is unique, but when $C_1 = C_2$, it is not unique. In the region of $P_{\rm I}(q_0^{(1)})$, though fixed rate Q increases, $R_{\rm cor}^{\rm opt}(Q)$ is fixed as C_2 and the FRIR measurement has a unique form. The FRIR can be regarded as a MC. In the case of $C_1 = C_2$, the FRIR, corresponding to the left-bound of $P_{\rm I}(q_0^{(1)})$, is equivalent to an optimal MC. In other words, when $C_1 = C_2$, one of Q_1, Q_2 , and $2|\rho_{12}|$ becomes the minimum failure probability of MC, according to the relation of $\rho_{11}, \rho_{22}, \rho_{12}$ which are the component of ρ_0 . However, since our strategy is to maximize average confidence at fixed failure probability, in case of $C_1 < C_2$, the relation does not hold when $\rho_{11} < |\rho_{12}| \le \rho_{22}$ is not satisfied.

B. FRIR at
$$P_1 = Q$$
 for all $Q \in P_1(q_0^{(0)})$

When $q_0 = q_0^{(0)}$, we classify modified FRIR problem into three cases, using two maximum confidences and the non-diagonal element ρ_{12} of ρ_0 . Then we analyze the three cases completely. The first case is $C_1 \leq \frac{1}{2} < C_2$, and the second one $\frac{1}{2} < C_1 \leq C_2$, $\rho_{12} = 0$. The third case is $\frac{1}{2} < C_1 \leq C_2$ and $\rho_{12} \neq 0$. The following lemma shows the complete analysis to modified FRIR problem in $C_1 \leq \frac{1}{2} < C_2$ and $q_0 = 1 - C_1$.

Lemma III.2 When $C_1 \leq \frac{1}{2} < C_2$, $\bar{P}_{cor}^{opt}(1 - C_1)$ is q_2 , and $P_I(1 - C_1)$ is $[0, 1 - Q_2]$. If $C_1 < 1/2$, \bar{M}_i^{opt} to $q_0 = 1 - C_1$ becomes

$$\bar{M}_0^{\text{opt}} = \alpha |\nu_1\rangle\langle\nu_1|, \ \bar{M}_1^{\text{opt}} = 0,
\bar{M}_2^{\text{opt}} = \rho_0 - \alpha |\nu_1\rangle\langle\nu_1|, \ 0 \le \alpha \le 1 - Q_2.$$
(21)

However if $C_1 = 1/2$, they can be expressed as

$$\bar{M}_0^{\text{opt}} = \alpha |\nu_1\rangle\langle\nu_1|, \ \bar{M}_1^{\text{opt}} = \beta |\nu_1\rangle\langle\nu_1|,
\bar{M}_2^{\text{opt}} = \rho_0 - (\alpha + \beta)|\nu_1\rangle\langle\nu_1|, \ \alpha, \beta \ge 0, \ \alpha + \beta \le 1 - Q_2.$$
(22)

The proof is given in Appendix B. Note that $1-Q_2$ is larger than zero. By Lemma II.2, $0 \in P_{\rm I}(q)$ implies $q_0^{(0)}=q$ and by Lemma III.2, $0 \in P_{\rm I}(1-C_1)$ means $q_0^{(0)}=1-C_1$. Therefore, when $C_1 \le \frac{1}{2} < C_2$, if $C_1 < \frac{1}{2}$, inequality $q_0^{(0)} \ge \frac{1}{2}$ of Lemma II.3 becomes strictly an inequality, but if $C_1 = \frac{1}{2}$, it becomes equality.

Lemma III.2 shows how the analytic solution of original FRIR problem in the case of $C_1 \leq \frac{1}{2} < C_2$ can be varied in terms of $Q \in P_1(q_0^{(0)})$. In this case, $\bar{P}_{\text{cor}}^{\text{opt}}(1-C_1)$ is q_2 and $R_{\text{cor}}^{\text{opt}}(Q)$ is $1-C_1+\frac{C_1-q_1}{1-Q}$ for any $Q \in P_1(1-C_1)$, because of Eq. (11). However, the optimal measurements have different forms according to the case of $C_1 < \frac{1}{2}$ or $C_1 = \frac{1}{2}$. It is because in the case of $C_1 < \frac{1}{2}$ the modified FRIR measurement is expressed only by a variable α but in the case of $C_1 = \frac{1}{2}$ the modified FRIR measurement is given by α and β . In fact, FRIR measurement of the case of $C_1 < \frac{1}{2}$ is equivalent to FRIR measurement of the case of $C_1 = \frac{1}{2}$ with $\beta = 0$. Therefore, by introducing a variable $\epsilon = \beta$, one can find the following corollary.

Corollary III.2 When $C_1 \leq \frac{1}{2} < C_2$, $q_0^{(0)}$ is $1 - C_1$, and $P_I(q_0^{(0)})$ is $[0, 1 - Q_2]$. \bar{M}_i^{opt} for $P_I = Q(\in P_I(q_0^{(0)}))$ can be expressed as

$$\bar{M}_0^{\text{opt}} = Q|\nu_1\rangle\langle\nu_1|, \ \bar{M}_1^{\text{opt}} = \epsilon|\nu_1\rangle\langle\nu_1|,
\bar{M}_2^{\text{opt}} = (\rho_{11} - Q - \epsilon)|\nu_1\rangle\langle\nu_1| + \rho_{12}|\nu_1\rangle\langle\nu_2| + \rho_{21}|\nu_2\rangle\langle\nu_1| + \rho_{22}|\nu_2\rangle\langle\nu_2|,$$
(23)

where

$$\epsilon = 0 if C_1 < \frac{1}{2} < C_2,
0 \le \epsilon \le 1 - Q_2 - Q if C_1 = \frac{1}{2} < C_2.$$
(24)

The result tells that when $Q \in P_{\rm I}(q_0^{(0)})$, if $C_1 < 1/2 < C_2$, FRIR measurement for $P_{\rm I} = Q$ is unique, but when $C_1 = 1/2 < C_2$ it is not unique.

The following lemma shows the solution to modified FRIR problem of $\frac{1}{2} < C_1 < C_2, \rho_{12} = 0, q_0 = C_1$.

Lemma III.3 When $\frac{1}{2} < C_1 \le C_2$ and $\rho_{12} = 0$, $\bar{P}^{\rm opt}_{\rm cor}(C_1)$ becomes $\rho_{11}C_1 + \rho_{22}C_2$, and $P_{\rm I}(C_1)$ is $[0, \rho_{11} + \rho_{22}\delta_{C_1, C_2}]$. If $C_1 < C_2$, $\bar{M}^{\rm opt}_i$ for $q_0 = C_1$ is expressed as

$$\bar{M}_0^{\text{opt}} = \alpha |\nu_1 \rangle \langle \nu_1|,
\bar{M}_1^{\text{opt}} = (\rho_{11} - \alpha) |\nu_1 \rangle \langle \nu_1|, \ 0 \le \alpha \le \rho_{11}
\bar{M}_2^{\text{opt}} = \rho_{22} |\nu_2 \rangle \langle \nu_2|.$$
(25)

However, if $C_1 = C_2$, \bar{M}_i^{opt} for $q_0 = C_1$ is given by

$$\bar{M}_0^{\text{opt}} = \alpha |\nu_1\rangle\langle\nu_1| + \beta |\nu_2\rangle\langle\nu_2|,
\bar{M}_1^{\text{opt}} = (\rho_{11} - \alpha)|\nu_1\rangle\langle\nu_1|, \ 0 \le \alpha \le \rho_{11},
\bar{M}_2^{\text{opt}} = (\rho_{22} - \beta)|\nu_2\rangle\langle\nu_2|, \ 0 \le \beta \le \rho_{22}.$$
(26)

The proof is given in Appendix B. Note that ρ_{11} is larger than zero. $0 \in P_{\rm I}(C_1)$ in Lemma III.3 includes $q_0^{(0)} = C_1$ by Lemma II.2. Therefore, when $\frac{1}{2} < C_1 \le C_2$, inequality $q_0^{(0)} \ge \frac{1}{2}$ in Lemma II.3 becomes strict. From Lemma III.3, one can understand the behavior of analytic solution of original FRIR problem according to

From Lemma III.3, one can understand the behavior of analytic solution of original FRIR problem according to $Q \in P_{\rm I}(q_0^{(0)})$ when $\frac{1}{2} < C_1 \le C_2$ and $\rho_{12} = 0$. In this case, $\bar{P}_{\rm cor}^{\rm opt}(C_1)$ is $\rho_{11}C_1 + \rho_{22}C_2$ and $R_{\rm cor}^{\rm opt}(Q)$ becomes $C_1 + \frac{\rho_{22}(C_2 - C_1)}{1 - Q}$ for any $Q \in P_{\rm I}(C_1)$, because of Eq. (11). When $C_1 < C_2$, the modified FRIR measurement is expressed only by α . However, when $C_1 = C_2$, the modified FRIR measurement is given by α and β . In case of $C_1 < C_2$, because of ${\rm tr}[\bar{M}_0^{\rm opt}] = \alpha$, the FRIR measurement is uniquely determined at a fixed Q. In case of $C_1 = C_2$, because of ${\rm tr}[\bar{M}_0^{\rm opt}] = \alpha + \beta$, a fixed Q cannot uniquely determine α and β . Therefore, it implies that FRIR measurement of $P_1 = Q$ may not be unique. The FRIR measurement of the case of $C_1 < C_2$ may not be unique. FRIR measurement of the case of $C_1 < C_2$ with $\beta = 0$. Then, the following corollary can be obtained by introducing the variable $\epsilon = \beta$.

Corollary III.3 When $\frac{1}{2} < C_1 \le C_2$ and $\rho_{12} = 0$, $q_0^{(0)}$ is C_1 , and $P_1(q_0^{(0)})$ becomes $[0, \rho_{11} + \rho_{22}\delta_{C_1, C_2}]$. \bar{M}_i^{opt} of $P_1 = Q(\in P_1(q_0^{(0)}))$ is expressed as

$$\bar{M}_{0}^{\text{opt}} = (Q - \epsilon)|\nu_{1}\rangle\langle\nu_{1}| + \epsilon|\nu_{2}\rangle\langle\nu_{2}|,
\bar{M}_{1}^{\text{opt}} = (\rho_{11} - Q + \epsilon)|\nu_{1}\rangle\langle\nu_{1}|,
\bar{M}_{2}^{\text{opt}} = (\rho_{22} - \epsilon)|\nu_{2}\rangle\langle\nu_{2}|,$$
(27)

where

$$\epsilon = 0 if \frac{1}{2} < C_1 < C_2, \ \rho_{12} = 0,
\max\{0, Q - \rho_{11}\} \le \epsilon \le \min\{\rho_{22}, Q\} if \frac{1}{2} < C_1 = C_2, \ \rho_{12} = 0.$$
(28)

This result implies that when $Q \in P_{\rm I}(q_0^{(0)})$, if $1/2 < C_1 < C_2$ and $\rho_{12} = 0$, the FRIR measurement to $P_{\rm I} = Q$ is unique. However, if $1/2 < C_1 = C_2$ and $\rho_{12} = 0$, it is not unique.

From lemma III.4, $q_0^{(0)}$ and $P_I(q_0^{(0)})$ can be found in modified FRIR problem to $q_0 = \chi$ when $1/2 < C_1 \le C_2$ and $\rho_{12} \ne 0$. Here χ is as follows:

$$\chi = \frac{\chi_1 + \chi_2 - \sqrt{(\chi_1 - \chi_2)^2 + 4|\gamma_{12}|^2}}{2},\tag{29}$$

where

$$\gamma_{ij} = \frac{l^2 - e^2}{4l} \langle \nu_i | \rho_0^{-1} | \nu_j \rangle, \ \chi_i = \frac{1}{2} + \gamma_{ii} + \frac{(2q_i - 1)(2C_i - 1)}{2l}.$$
 (30)

Lemma III.4 When $\frac{1}{2} < C_1 \le C_2$ and $\rho_{12} \ne 0$, we find $q_0^{(0)} = \chi$, $\bar{P}_{cor}^{opt}(\chi) = \frac{1+l}{2}$, and $P_{\rm I}(\chi) = 0$. Then $M_i^{\rm opt}$ to $P_{\rm I} = 0$ is expressed as

$$M_0^{\text{opt}} = 0,$$

$$M_1^{\text{opt}} = \frac{1}{2} \left[I_2 + \frac{(q_1 \mathbf{v}_1 - q_2 \mathbf{v}_2) \cdot \boldsymbol{\sigma}}{\|q_1 \mathbf{v}_1 - q_2 \mathbf{v}_2\|_2} \right],$$

$$M_2^{\text{opt}} = \frac{1}{2} \left[I_2 - \frac{(q_1 \mathbf{v}_1 - q_2 \mathbf{v}_2) \cdot \boldsymbol{\sigma}}{\|q_1 \mathbf{v}_1 - q_2 \mathbf{v}_2\|_2} \right].$$
(31)

The proof is given in Appendix B. In Lemma III.4, optimal POVM of $\{q_i, \rho_i\}_{i=0}^N (q_0 = \chi)$ is unique and we consider $P_I(\chi)$ not as a set $\{0\}$ but as a value 0.

The following theorem summarizes the previous results.

Theorem III.1 (FRIR of $Q \in P_{\mathbf{I}}(q_0^{(0)})$) $q_0^{(0)}$ and $P_{\mathbf{I}}(q_0^{(0)})$ can be classified as follows:

$$q_0^{(0)} = 1 - C_1, \ P_1(q_0^{(0)}) = [0, 1 - Q_2] \quad \text{if } C_1 \le \frac{1}{2} < C_2,$$

$$q_0^{(0)} = C_1, \quad P_1(q_0^{(0)}) = [0, \rho_{11} + \rho_{22}\delta_{C_1, C_2}] \quad \text{if } \frac{1}{2} < C_1 \le C_2, \ \rho_{12} = 0,$$

$$q_0^{(0)} = \chi, \quad P_1(q_0^{(0)}) = 0 \quad \text{if } \frac{1}{2} < C_1 \le C_2, \ \rho_{12} \ne 0.$$

$$(32)$$

 $R_{\rm cor}^{\rm opt}(Q)$ becomes

$$R_{\text{cor}}^{\text{opt}}(Q) = \begin{cases} 1 - C_1 + \frac{C_1 - q_1}{1 - Q} & \text{if } C_1 \le \frac{1}{2} < C_2, \\ C_1 + \frac{\rho_{22}(C_2 - C_1)}{1 - Q} & \text{if } \frac{1}{2} < C_1 \le C_2, \ \rho_{12} = 0, \\ \frac{1 + l}{2} & \text{if } \frac{1}{2} < C_1 \le C_2, \ \rho_{12} \ne 0. \end{cases}$$

$$(33)$$

FRIR measurement of $P_1 = Q$ becomes, if $C_1 \leq \frac{1}{2} < C_2$, (23), and if $\frac{1}{2} < C_1 \leq C_2$ and $\rho_{12} = 0$, is (27), and if $\frac{1}{2} < C_1 \leq C_2$ and $\rho_{12} \neq 0$, becomes (31).

When Q, corresponding to $P_1=Q$, exists in $P_1(q_0^{(0)})$, $R_{\rm cor}^{\rm opt}(Q)$ and the FRIR measurement have different forms in certain situations. In the case of $C_1 \leq \frac{1}{2} < C_2$, $P_1(q_0^{(0)})$ is neither $\{0\}$ nor [0,1], because of $0 < Q_2 < 1$. P_1 , corresponding to $q_0^{(0)}$ or $q_0^{(1)}$, exists not as a point but as an separate interval. It is not true in the case of $\frac{1}{2} < C_1 \leq C_2$. If $\rho_{12} \neq 0$, $P_1(q_0^{(0)}) = \{0\}$. If $\rho_{12} = 0$ and $C_1 = C_2$, $P_1(q_0^{(0)}) = [0,1]$. This implies that $q_0^{(0)} = q_0^{(1)}$. Then, the left-bound of $P_1(q_0^{(1)})$ becomes 0. Since FRIR of $P_1 = 0$ is MD, MD is an optimal MC when $\rho_{12} = 0$ and $C_1 = C_2$.

C. FRIR at
$$P_1 = Q$$
 for all $Q \in P_1(q_0^{(0)})^{\mathsf{C}} \cap P_1(q_0^{(1)})^{\mathsf{C}}$

In the previous section we considered the case that the failure probability $P_{\rm I}$ is fixed as $Q \in P_{\rm I}(q_0^{(0)}) \cap P_{\rm I}(q_0^{(1)})$. In this section, to investigate FRIR in the other region, we classify modified FRIR problem of $q_0^{(0)} < q_0 < q_0^{(1)}$ into two cases. The first case is $C_1 \leq \frac{1}{2} < C_2$ and $\rho_{12} \neq 0$, and the second one is $\frac{1}{2} < C_1 \leq C_2$ and $\rho_{12} \neq 0$. Note that $\rho_{12} = 0$ is not included. It is because, from corollary III.1 and theorem III.1, when $C_1 = C_2$ and $\rho_{12} = 0$, we find $q_0^{(0)} = q_0^{(1)}$. When $C_1 < C_2$ and $\rho_{12} = 0$, we have $q_0^{(0)} < q_0^{(1)}$. However, $P_{\rm I}(q_0^{(0)}) = [0, \rho_{11}]$ and $P_{\rm I}(q_0^{(1)}) = [\rho_{11}, 1]$ implies $P_{\rm I}(q) = \rho_{11}(q_0^{(0)} < q < q_0^{(1)})$, which includes $R_{\rm cor}^{\rm opt}(\rho_{11}) = C_2$. In addition, the following lemma tells that FRIR measurement to $P_{\rm I} = \rho_{11}$ is unique.

Lemma III.5 When $C_1 < C_2$ and $\rho_{12} = 0$, if $q_0^{(0)} < q < q_0^{(1)}$, \bar{M}_i^{opt} at $q_0 = q$ can be expressed as

$$\bar{M}_0^{\text{opt}} = \rho_{11} |\nu_1\rangle\langle\nu_1|, \ \bar{M}_1^{\text{opt}} = 0, \ \bar{M}_2^{\text{opt}} = \rho_{22} |\nu_2\rangle\langle\nu_2|.$$
 (34)

The proof is given in Appendix B. The following theorem summarizes FRIR to $\rho_{12} = 0$.

Theorem III.2 (FRIR of $\rho_{12} = 0$) If $C_1 = C_2$, $R_{\text{cor}}^{\text{opt}}(Q)$ becomes C_2 at any Q, and \bar{M}_i^{opt} to $P_1 = Q$ can be representes as

$$\bar{M}_0^{\text{opt}} = \epsilon_1 |\nu_1\rangle\langle\nu_1| + (Q - \epsilon_1)|\nu_2\rangle\langle\nu_2|,
\bar{M}_1^{\text{opt}} = (\rho_{11} - \epsilon_1)|\nu_1\rangle\langle\nu_1|,
\bar{M}_2^{\text{opt}} = (\rho_{22} - Q + \epsilon_1)|\nu_2\rangle\langle\nu_2|,$$
(35)

where

$$\max\{0, Q - \rho_{22}\} \le \epsilon_1 \le \min\{\rho_{11}, Q\}. \tag{36}$$

When $C_1 < C_2$, if $0 \le Q \le \rho_{11}$, $R_{cor}^{opt}(Q)$ becomes

$$R_{\text{cor}}^{\text{opt}}(Q) = \begin{cases} 1 - C_1 + \frac{\rho_{22}(C_1 + C_2 - 1)}{1 - Q} & \text{if } C_1 \le \frac{1}{2}, \\ C_1 + \frac{\rho_{22}(C_2 - C_1)}{1 - Q} & \text{if } \frac{1}{2} < C_1. \end{cases}$$
(37)

Then \bar{M}_i^{opt} for $P_{\mathrm{I}} = Q$ is expressed as

$$\bar{M}_0^{\text{opt}} = Q|\nu_1\rangle\langle\nu_1|, \ \bar{M}_1^{\text{opt}} = \epsilon_2|\nu_1\rangle\langle\nu_1|,
\bar{M}_2^{\text{opt}} = (\rho_{11} - Q - \epsilon_2)|\nu_1\rangle\langle\nu_1| + \rho_{22}|\nu_2\rangle\langle\nu_2|,$$
(38)

where

$$\epsilon_{2} = 0 \quad \text{if } C_{1} < \frac{1}{2} < C_{2},
0 \le \epsilon_{2} \le \rho_{11} - Q \quad \text{if } C_{1} = \frac{1}{2} < C_{2},
\epsilon_{2} = \rho_{11} - Q \quad \text{if } \frac{1}{2} < C_{1} < C_{2}.$$
(39)

If $\rho_{11} \leq Q < 1$, $R_{cor}^{opt}(Q)$ is always C_2 , and \bar{M}_i^{opt} to $P_I = Q$ becomes

$$\bar{M}_{0}^{\text{opt}} = \rho_{11}|\nu_{1}\rangle\langle\nu_{1}| + (Q - \rho_{11})|\nu_{2}\rangle\langle\nu_{2}|,
\bar{M}_{1}^{\text{opt}} = 0, \quad \bar{M}_{2}^{\text{opt}} = (1 - Q)|\nu_{2}\rangle\langle\nu_{2}|.$$
(40)

When $\rho_{12} = 0$, if $C_1 = C_2$, MD becomes an optimal MC. When $C_1 < C_2$, the right-bound of $P_I(q_0^{(0)})$ is the same as the left-bound of $P_I(q_0^{(1)})$, which happens only when $\rho_{12} = 0$.

The following theorem describes modified FRIR to $\rho_{12} \neq 0$ in $q_0 \in (q_0^{(0)}, q_0^{(1)})$.

Theorem III.3 When $\rho_{12} \neq 0$ and $q_0^{(0)} < q < q_0^{(1)}$, the optimal POVM to $q_0 = q$ is unique. Then at least one of M_1^{opt} and M_2^{opt} is nonzero. In the case of $M_x^{\text{opt}} \neq 0$, $M_y^{\text{opt}} = 0(\{x,y\} = \{1,2\})$, the index x turns out to be the index i in $\max_{i \in 1,2} [q_i + ||q\mathbf{v}_0 - q_i\mathbf{v}_i||_2]$. In this case $\bar{P}_{\text{cor}}^{\text{opt}}(q)$ and $P_{\text{I}}(q)$ are given as

$$\bar{P}_{\text{cor}}^{\text{opt}}(q) = \frac{1}{2} \left(q + q_x + \|q \boldsymbol{v}_0 - q_x \boldsymbol{v}_x\|_2 \right),
P_{\text{I}}(q) = \frac{1}{2} \left[1 + \frac{(q \boldsymbol{v}_0 - q_x \boldsymbol{v}_x) \cdot \boldsymbol{v}_0}{\|q \boldsymbol{v}_0 - q_x \boldsymbol{v}_x\|_2} \right].$$
(41)

The optimal POVM elements is represented as

$$M_0^{\text{opt}} = \frac{1}{2} \left[I_2 + \frac{(q \mathbf{v}_0 - q_x \mathbf{v}_x) \cdot \boldsymbol{\sigma}}{\|q \mathbf{v}_0 - q_x \mathbf{v}_x\|_2} \right],$$

$$M_x^{\text{opt}} = \frac{1}{2} \left[I_2 + \frac{(q_x \mathbf{v}_x - q \mathbf{v}_0) \cdot \boldsymbol{\sigma}}{\|q \mathbf{v}_0 - q_x \mathbf{v}_x\|_2} \right], M_y^{\text{opt}} = 0.$$
(42)

If $M_i^{\text{opt}} \neq 0(\forall i)$, $\bar{P}_{\text{cor}}^{\text{opt}}(q)$ and $P_{\text{I}}(q)$ become

$$\bar{P}_{\text{cor}}^{\text{opt}}(q) = q + \rho_{11}\lambda_{1} + \rho_{22}\lambda_{2} - 2|\rho_{12}|\sqrt{\lambda_{1}\lambda_{2}},
P_{\text{I}}(q) = \left[\frac{(2C_{1}-1)(2C_{2}-1)-(2q-1)^{2}}{2(2q-1)^{2}(C_{1}+C_{2}-1)}\right] \cdot \left[1 - 2\rho_{11}C_{1} - 2\rho_{22}C_{2} + \frac{|\rho_{12}|(2C_{1}-1)(2C_{2}-1)[(C_{1}-q)(C_{2}-q)+(q-1+C_{1})(q-1+C_{2})]}{\sqrt{(2C_{1}-1)(2C_{2}-1)(C_{1}-q)(C_{2}-q)(q-1+C_{1})(q-1+C_{2})}}\right],$$
(43)

where

$$\lambda_{1} = \frac{(2C_{2} - 1)(C_{1} - q)(q - 1 + C_{1})}{(2q - 1)(C_{1} + C_{2} - 1)},$$

$$\lambda_{2} = \frac{(2C_{1} - 1)(C_{2} - q)(q - 1 + C_{2})}{(2q - 1)(C_{1} + C_{2} - 1)}.$$
(44)

Then $\{\bar{M}_i^{\text{opt}}\}_{i=0}^2$ is expressed as

$$\bar{M}_{0}^{\text{opt}} = \frac{\eta_{0}}{\lambda_{1} + \lambda_{2}} \left[\lambda_{2} |\nu_{1}\rangle\langle\nu_{1}| + \frac{\rho_{12}\sqrt{\lambda_{1}\lambda_{2}}}{|\rho_{12}|} |\nu_{1}\rangle\langle\nu_{2}| + \frac{\rho_{21}\sqrt{\lambda_{1}\lambda_{2}}}{|\rho_{21}|} |\nu_{2}\rangle\langle\nu_{1}| + \lambda_{1}|\nu_{2}\rangle\langle\nu_{2}| \right],$$

$$\bar{M}_{1}^{\text{opt}} = \frac{\eta_{1}}{\lambda_{1} + \lambda_{2} + 2q - 1 - C_{1} + C_{2}} \left[(\lambda_{2} + q - 1 + C_{2})|\nu_{1}\rangle\langle\nu_{1}| + \frac{\rho_{12}\sqrt{\lambda_{1}\lambda_{2}}}{|\rho_{12}|} |\nu_{1}\rangle\langle\nu_{2}| + \frac{\rho_{21}\sqrt{\lambda_{1}\lambda_{2}}}{|\rho_{21}|} |\nu_{2}\rangle\langle\nu_{1}| + (\lambda_{1} + q - C_{1})|\nu_{2}\rangle\langle\nu_{2}| \right],$$

$$\bar{M}_{2}^{\text{opt}} = \frac{\eta_{2}}{\lambda_{1} + \lambda_{2} + 2q - 1 + C_{1} - C_{2}} \left[(\lambda_{2} + q - C_{2})|\nu_{1}\rangle\langle\nu_{1}| + \frac{\rho_{12}\sqrt{\lambda_{1}\lambda_{2}}}{|\rho_{12}|} |\nu_{1}\rangle\langle\nu_{2}| + \frac{\rho_{21}\sqrt{\lambda_{1}\lambda_{2}}}{|\rho_{21}|} |\nu_{2}\rangle\langle\nu_{1}| + (\lambda_{1} + q - 1 + C_{1})|\nu_{2}\rangle\langle\nu_{2}| \right].$$

$$(45)$$

Here η_0 becomes $P_{\rm I}(q)$ of (43), and η_1 and η_2 are given by

$$\eta_{1} = \left[\frac{(2C_{1}-1)(2C_{2}-1)+2(C_{2}-C_{1})(2q-1)+(2q-1)^{2}}{2(2q-1)^{2}(C_{1}+C_{2}-1)} \right] \cdot \left[\rho_{11}(q-1+C_{1})+\rho_{22}(C_{2}-q) - \frac{|\rho_{12}|(C_{2}-q)(q-1+C_{1})[(2C_{1}-1)(q-1+C_{2})+(2C_{2}-1)(C_{1}-q)]}{\sqrt{(2C_{1}-1)(2C_{2}-1)(C_{1}-q)(C_{2}-q)(q-1+C_{1})(q-1+C_{2})}} \right],$$

$$\eta_{2} = \left[\frac{(2C_{1}-1)(2C_{2}-1)-2(C_{2}-C_{1})(2q-1)+(2q-1)^{2}}{2(2q-1)^{2}(C_{1}+C_{2}-1)} \right] \cdot \left[\rho_{11}(C_{1}-q)+\rho_{22}(q-1+C_{2}) - \frac{|\rho_{12}|(C_{1}-q)(q-1+C_{2})[(2C_{2}-1)(q-1+C_{1})+(2C_{1}-1)(C_{2}-q)]}{\sqrt{(2C_{1}-1)(2C_{2}-1)(C_{1}-q)(C_{2}-q)(q-1+C_{1})(q-1+C_{2})}} \right].$$
(46)

Two cases are distinguished by the signs of $\{\lambda_i\}_{i=1}^2$ and $\{\eta_i\}_{i=1}^3$. If $\lambda_1, \lambda_2 \geq 0$ and $\eta_1, \eta_2, \eta_3 > 0$, we have $M_i^{\mathrm{opt}} \neq 0 (\forall i)$. Otherwise, we obtain $M_1^{\mathrm{opt}} = 0$ or $M_2^{\mathrm{opt}} = 0$.

The proof is given in Appendix C. In Theorem III.3, optimal POVM corresponding to $q_0 = q$ is always unique and $P_{\rm I}(q)$ becomes a set with only one element. In this case, we consider $P_{\rm I}(q)$ as a value corresponding to the element of the set, like (41) and (43).

The following lemma shows the result related with inconclusive degrees satisfying $M_1^{\text{opt}} = 0$.

Lemma III.6 When $\rho_{12} \neq 0$, if $C_1 \leq \frac{1}{2} < C_2$ and $q_0^{(0)} < q_0 < q_0^{(1)}$ (or $\frac{1}{2} < C_1 \leq C_2$ and $C_1 < q_0 < q_0^{(1)}$) can be satisfied, the optimal POVM element M_1^{opt} of modified FRIR problem becomes 0.

The proof is given in Appendix B.

To obtain $R_{\text{cor}}^{\text{opt}}(Q)$, we need to express the inconclusive degree as a function of the failure probability. This task is not easy since the relation between inconclusive degree and failure probability is very complex; see $P_{\text{I}}(q)$ of (41) and (43). However, it should be noted that the relation between q and $P_{\text{I}}(q)$ is one-to-one, which implies that we can obtain $R_{\text{cor}}^{\text{opt}}(Q)$ numerically using Eq. (11) and theorem III.3.

For example, let us consider the following case of $\{q_i, \rho_i\}_{i=1}^2$.

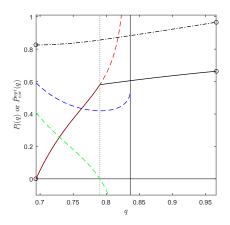
$$q_{1} = 0.4, \, \rho_{1} = \begin{pmatrix} 0.15 & -0.30 + 0.10i \\ -0.30 - 0.10i & 0.85 \end{pmatrix},$$

$$q_{2} = 0.6, \, \rho_{2} = \begin{pmatrix} 0.80 & -0.30 + 0.05i \\ -0.30 - 0.05i & 0.20 \end{pmatrix}.$$

$$(47)$$

The Bloch vectors of the two qubit states are

$$\mathbf{v}_1 = (-0.6, -0.2, -0.7)$$
 and $\mathbf{v}_2 = (-0.6, -0.1, 0.6)$.



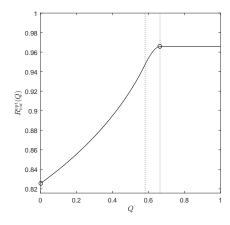


FIG. 1: The behavior of $P_1(q)$, $\bar{P}_{\text{cor}}^{\text{opt}}(q)$, and $R_{\text{cor}}^{\text{opt}}(Q)$ at $\boldsymbol{v}_1 = (-0.6, -0.2, -0.7)$ and $\boldsymbol{v}_2 = (-0.6, -0.1, 0.6)$ with $q_1 = 0.4$ and $q_2 = 0.6$. We have $|\rho_{12}| = 0.3075$, $C_1 = 0.8361$, $C_2 = 0.9657$, $q_0^{(0)} = \chi = 0.6940$, and $q_0^{(1)} = C_2$. The left figure shows $P_1(q)$ (solid line) and $\bar{P}_{\text{cor}}^{\text{opt}}(q)$ (dashed-dotted line) in $q_0^{(0)} < q < q_0^{(1)}$. η_0 (red dashed line) and η_2 (blue dashed line) are always positive in $q_0^{(0)} < q \le C_1$. However, η_1 (green dashed line) is equal to or less than 0 in the region $0.7902 \le q \le C_1$. In $\chi < q_0 < 0.7902$ we get $M_i^{\text{opt}} \ne 0 (\forall i)$, and in $0.7902 \le q_0 < C_2$ we have $M_0^{\text{opt}} \ne 0$, $M_1^{\text{opt}} = 0$, and $M_2^{\text{opt}} \ne 0$. The right figure displays the behavior of $R_{\text{cor}}^{\text{opt}}(Q)$, obtained from $P_1(q)$ and $\bar{P}_{\text{cor}}^{\text{opt}}(q)$ of the left figure. In 0 < Q < 0.5805 we have $M_i^{\text{opt}} \ne 0 (\forall i)$, and in $0.5805 \le Q < Q_1 = 0.6635$ we get $M_0^{\text{opt}} \ne 0$, $M_1^{\text{opt}} = 0$, and $M_2^{\text{opt}} \ne 0$.

Then, $|\rho_{12}|$, C_1 , and C_2 become 0.3075, 0.8361, and 0.9657, respectively. From corollary III.1 and theorem III.1, $q_0^{(0)}$ Then, $|\rho_{12}|$, C_1 , and C_2 become 0.3075, 0.8501, and 0.9057, respectively. From coronary III.1 and theorem III.1, q_0 and $q_0^{(1)}$ are $\chi = 0.6940$ and C_2 , respectively. In the region of $\chi < q \le C_1$, λ_1 and λ_2 are non-negative, and η_0 and η_2 are positive, but η_1 is not. In $\chi < q < 0.7902$, η_1 is positive. However, in 0.7902 $\le q \le C_1$, η_1 is negative or equal to zero. Therefore, in $\chi < q_0 < 0.7902$, we find $M_i^{\text{opt}} \neq 0 (\forall i)$. However, in 0.7902 $\le q_0 \le C_1$, because of $q_1 + \|q_0 v_0 - q_1 v_1\|_2 < q_2 + \|q_0 v_0 - q_2 v_2\|_2$, we have $M_1^{\text{opt}} = 0$. In addition, by lemma III.6, in $C_1 < q_0 < C_2$, M_1^{opt} becomes 0. Therefore if fixed failure probability P_1 is $0 < P_1 < 0.5805$, we find $M_i^{\text{opt}} \neq 0$. However, if $0.5805 \le P_1 < Q_1 = 0.6635$, we have $M_0^{\text{opt}} \neq 0$, $M_1^{\text{opt}} = 0$, and $M_2^{\text{opt}} \neq 0$. Figure 1 shows, in this example, the behavior of $P_1(q_0)$ and $\bar{P}_{\text{cor}}^{\text{opt}}(q)(R_{\text{cor}}^{\text{opt}}(Q))$ in $q_0^{(0)} < q_0 < q_0^{(1)}(0 < Q < 1)$.

To confirm the effectiveness of our results, we consider the case of $C_1 = C_2 (= C)$ and $\rho_{12} \neq 0$. By corollary III.1 and theorem III.1 in this case, $q_1^{(0)} = \chi$ and $q_2^{(1)} = C$. If $C_1 = C_2$ is applied to χ_1 , χ_2 and η_3 , we have the

and theorem III.1, in this case, $q_0^{(0)} = \chi$ and $q_0^{(1)} = C$. If $C_1 = C_2$ is applied to $\lambda_1, \lambda_2, \eta_0, \eta_1, \eta_2$, we have the following expresssion.

$$\lambda_1 = \lambda_2 = \frac{(C - q)(q - 1 + C)}{2q - 1},\tag{48}$$

and

$$\eta_{0} = \frac{2}{(2q-1)^{2}} \left[|\rho_{12}|(C-q)^{2} + |\rho_{12}|(q-1+C)^{2} - (C-q)(q-1+C) \right],
\eta_{1} = \frac{(2C-1)^{2} + (2q-1)^{2}}{2(2q-1)^{2}(2C-1)} \left[\rho_{11}(q-1+C) + \rho_{22}(C-q) - |\rho_{12}|(2C-1) \right],
\eta_{2} = \frac{(2C-1)^{2} + (2q-1)^{2}}{2(2q-1)^{2}(2C-1)} \left[\rho_{11}(C-q) + \rho_{22}(q-1+C) - |\rho_{12}|(2C-1) \right].$$
(49)

When $\rho_{11}, \rho_{22} \ge |\rho_{12}|$, in $\chi < q < C$, these are all positive, and we have $M_i^{\text{opt}} \ne 0 (\forall i)$. Then $q_0 (=q)$ can be expressed in terms of the failure probability Q:

$$q = \frac{1}{2} + \frac{2C - 1}{2} \sqrt{\frac{1 - 2|\rho_{12}|}{1 + 2|\rho_{12}| - 2Q}}.$$
 (50)

Applying this q to $\bar{P}_{\text{cor}}^{\text{opt}}(q) - qQ$, we find $P_{\text{cor}}^{\text{opt}}(Q)$ in $0 < Q < 2|\rho_{12}|$, which agrees with the previous result:

$$P_{\text{cor}}^{\text{opt}}(Q) = \frac{1-Q}{2} + \frac{2C-1}{2}\sqrt{(1-2|\rho_{12}|)(1+2|\rho_{12}|-2Q)}.$$
 (51)

When $\rho_{11} < |\rho_{12}| \le \rho_{22}$, then $\lambda_1, \lambda_2, \eta_0$, and η_2 are always positive in the region of $\chi < q < C$; however, η_1 can be found only in the following case

$$\chi < q < \frac{1}{2} + \frac{(1 - 2|\rho_{12}|)(2C - 1)}{2(\rho_{22} - \rho_{11})}.$$
(52)

In this region, like (50), q can be expressed by Q. Therefore, in the following region of the failure probability, $P_{\text{cor}}^{\text{opt}}(Q)$ is the same as (51), and we have $M_i^{\text{opt}} \neq 0 (\forall i)$.

$$0 < Q < \frac{2(\rho_{11}\rho_{22} - |\rho_{12}|^2)}{1 - 2|\rho_{12}|}. (53)$$

In the other region, we get $M_0^{\text{opt}} \neq 0$, $M_1^{\text{opt}} = 0$, and $M_2^{\text{opt}} \neq 0$, which coincides with the previous result[35].

IV. CONCLUSION

In this paper, we provided a solution to the FRIR of two mixed qubit states. The solution was obtaind by considering the modified FRIR problem(MD of three qubit states). In fact, since the added specific quantum state ρ_0 with the prior probability q_0 (called an conclusive degree) was obtained from the given two qubit states, the structure of the modified problem is more complex than that of the MD of three qubit states with no constraint[17]. First, we introduced special inconclusive degrees $q_0^{(0)}$ and $q_0^{(1)}$, which are the beginning and the end of proper inconclusive degrees. Using this, we divided the problem into the two cases of $q_0 = q_0^{(0)}$ (or $q_0 = q_0^{(1)}$) and $q_0^{(0)} < q_0 < q_0^{(1)}$. By maximum confidences of two qubit states and non-diagonal element of ρ_0 , we solved each case. We obtained $q_0^{(0)}$ and $q_0^{(1)}$ in the analytic form, and completely understood modified FRIR problem when $q_0^{(0)} \le q_0 \le q_0^{(1)}$. Finally, we verified that our results also provide the same solutions as known examples in the literature.

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Appendix A: Proofs of Lemmas in Section II

Proof of Lemma II.1 Suppose that when $\{q_i, \rho_i\}_{i=1}^N$ is given, POVM $\{M_i\}_{i=0}^N$ can cause $P_{\rm I} = Q$ and $\bar{P}_{\rm cor} = \bar{P}_{\rm cor}^{\rm opt}(q)$. It follows that the POVM can make $P_{\rm cor}(Q) = \bar{P}_{\rm cor}^{\rm opt}(q) - qQ$. If there exists a POVM that can build $P_{\rm I} = Q$ and $P_{\rm cor}(Q) = P > \bar{P}_{\rm cor}^{\rm opt}(q) - qQ$, it can also construct $\bar{P}_{\rm cor}(q) = P + qQ$. However since P + qQ is larger than $\bar{P}_{\rm cor}^{\rm opt}(q)$, this is contradictory. Therefore $\{M_i\}_{i=0}^N$ should produce $P_{\rm cor}^{\rm opt}(Q)$, which means $P_{\rm cor}^{\rm opt}(Q) = \bar{P}_{\rm cor}^{\rm opt}(q) - qQ$.

Proof of Lemma II.2 Assume that when $\{q_i, \rho_i\}_{i=1}^N$ is given, the POVM $\{M_i\}_{i=0}^N(\{M_i'\}_{i=0}^N)$ can produce $P_{\rm I}=Q$ and $\bar{P}_{\rm cor}=\bar{P}_{\rm cor}^{\rm opt}(q)(P_{\rm I}=Q')$ and $\bar{P}_{\rm cor}=\bar{P}_{\rm cor}^{\rm opt}(q')$. If q=q' and Q< Q', the POVM $\{M_i''\}_{i=0}^N$ composed of $M_i''=pM_i+(1-p)M_i'(0\leq p\leq 1)$ will build $\bar{P}_{\rm cor}=\bar{P}_{\rm cor}^{\rm opt}(q)$ and $P_{\rm I}=pQ+(1-p)Q'$. Therefore $P_{\rm I}(q)$ becomes a convex set.

Now suppose that q < q'. $\{M_i\}_{i=0}^N$ constructs $\bar{P}_{cor} = (q'-q)Q + \bar{P}_{cor}^{opt}(q)$ when $q_0 = q'$, and the value should be equal to or less than $\bar{P}_{cor}^{opt}(q') = (q'-q)Q' + \bar{P}$, where \bar{P} is \bar{P}_{cor} corresponding to $\{M_i'\}_{i=0}^N$ when $q_0 = q$. This means that $(Q'-Q) \ge (\bar{P}_{cor}^{opt}(q) - \bar{P})/(q'-q)$. Therefore we have $Q \le Q'$. This means that $P_I(q) \le P_I(q')$.

Proof of Lemma II.3. When $q_0 < 1/N$, we get $\bar{\tau}_0^{\text{opt}} = (1/N - q_0)I_d + (1/N)\sum_{i=1}^N \bar{\tau}_i^{\text{opt}}$ by (ii) of (6). If we multiply \bar{M}_0^{opt} to both sides of the equation and take the trace of the result, we obtain $(1/N - q_0)\text{tr}[\bar{M}_0^{\text{opt}}] \leq 0$ by (iii) and the positivity of $\bar{\tau}_i^{\text{opt}}(\forall i)$. From the assumption on q_0 , \bar{M}_0^{opt} should be zero and we find $P_{\text{I}}(q) = 0(\forall q < 1/N)$. Therefore using lemma II.2, we have $q_0^{(0)} \geq 1/N$.

using lemma II.2, we have $q_0^{(0)} \ge 1/N$. Therefore when $C = \max_i C_i$, $\bar{M}_i = \delta_{i0}\rho_0$ and $\bar{\tau}_i = CI_d - \bar{\rho}_i$ satisfy the optimality condition (6) of $q_0 = C$, and $1 \in P_I(C)$. Therefore we get $q_0^{(1)} \le C$ by lemma II.2.

Appendix B: Proofs of Lemmas in Section III

Proof of Lemma III.1. When $q_0 = C_2$, since $\{\bar{M}_i = \delta_{i0}\rho_0\}_{i=0}^2$ and $\{\bar{\tau}_i^* = C_2I_2 - \bar{\rho}_i\}_{i=0}^2$ satisfy the KKT optimality condition (6), $\bar{P}_{\text{cor}}^{\text{opt}}(C_2) = C_2$. This means that $\bar{\tau}_i^{\text{opt}} = \bar{\tau}_i^*(\forall i)$. Since the rank of $\bar{\tau}_2^{\text{opt}}$ should be one by $C_1 + C_2 > 1$, we have $\bar{M}_2^{\text{opt}} = \beta |\nu_2\rangle\langle\nu_2|$ from (iii). The form of \bar{M}_0^{opt} and \bar{M}_1^{opt} can be classified into the cases of $C_1 = C_2$ and $C_1 < C_2$.

If $C_1 = C_2$, the rank of $\bar{\tau}_1^{\text{opt}}$ becomes 1, and we get $\bar{M}_1^{\text{opt}} = \alpha |\nu_1\rangle\langle\nu_1|$ from (iii). Furthermore (i) indicates that $\bar{M}_0^{\text{opt}} = \rho_0 - \alpha |\nu_1\rangle\langle\nu_1| - \beta |\nu_2\rangle\langle\nu_2|$, $0 \le \alpha \le \rho_{11}$, $0 \le \beta \le \rho_{22}$, and $(\rho_{11} - \alpha)(\rho_{22} - \beta) \ge |\rho_{12}|^2$. Since $\text{tr}[\bar{M}_0^{\text{opt}}] = 1 - \alpha - \beta$, the maximum can be found at $\alpha = \beta = 0$. α and β corresponding to its minimum can be different. When $\rho_{11} < |\rho_{12}| \le \rho_{22}$, we have $\alpha = 0$ and $\beta = 1 - Q_1$. When $\rho_{22} < |\rho_{12}| \le \rho_{11}$, we obtain $\alpha = 1 - Q_2$ and $\beta = 0$. When $|\rho_{12}| \le \rho_{11}$, ρ_{22} , we have $\alpha = \rho_{11} - |\rho_{12}|$ and $\beta = \rho_{22} - |\rho_{12}|$. Therefore $P_1(C_2)$ becomes (16). If $C_1 < C_2$, the rank of $\bar{\tau}_1^{\text{opt}}$ becomes 2, and (iii) implies that $\bar{M}_1^{\text{opt}} = 0$. Then (i) means $\bar{M}_0^{\text{opt}} = \rho_0 - \beta |\nu_2\rangle\langle\nu_2|$ and

If $C_1 < C_2$, the rank of $\bar{\tau}_1^{\text{opt}}$ becomes 2, and (iii) implies that $M_1^{\text{opt}} = 0$. Then (i) means $M_0^{\text{opt}} = \rho_0 - \beta |\nu_2\rangle\langle\nu_2|$ and $0 \le \beta \le 1 - Q_1$. Since $\text{tr}[\bar{M}_0^{\text{opt}}] = 1 - \beta$, the minimum(maximum) can be found at $\beta = 1 - Q_1(\beta = 0)$. Therefore we get $P_1(C_2) = [Q_1, 1]$.

Proof of Lemma III.2. When $C_1 \leq 1/2 < C_2$, since $\{\bar{M}_i = \delta_{i2}\rho_0\}_{i=0}^2$ and the following $\{\bar{\tau}_i^{\star}\}_{i=0}^2$ satisfies the optimality condition (6) of $q_0 = 1 - C_1$, $\bar{P}_{\text{cor}}^{\text{opt}}(1 - C_1) = q_2$.

$$\bar{\tau}_0^{\star} = (C_1 + C_2 - 1)|\nu_2\rangle\langle\nu_2|, \ \bar{\tau}_1^{\star} = (1 - 2C_1)|\nu_1\rangle\langle\nu_1| + (2C_2 - 1)|\nu_2\rangle\langle\nu_2|, \ \bar{\tau}_2^{\star} = 0. \tag{B1}$$

This implies that $\bar{\tau}_i^{\text{opt}} = \bar{\tau}_i^{\star}(\forall i)$ when $q_0 = 1 - C_1$. Since the rank of $\bar{\tau}_0^{\text{opt}}$ is 1 given that $C_1 + C_2 > 1$, (iii) implies $\bar{M}_0^{\text{opt}} = \alpha |\nu_1\rangle\langle\nu_1|$. However \bar{M}_1^{opt} and \bar{M}_2^{opt} are classified into cases where $C_1 < 1/2$ and $C_1 = 1/2$.

When $C_1 < 1/2$, since the rank of $\bar{\tau}_1^{\text{opt}}$ becomes 2, (iii) gives $\bar{M}_1^{\text{opt}} = 0$ and (i) means $\bar{M}_2^{\text{opt}} = \rho_0 - \alpha |\nu_1\rangle\langle\nu_1|$ and $0 \le \alpha \le 1 - Q_2$. Then, since $\text{tr}[\bar{M}_0^{\text{opt}}] = \alpha$ shows a minimum at $\alpha = 0$ and a maximum at $\alpha = 1 - Q_2$, we have $P_1(1 - C_1) = [0, 1 - Q_2]$. However, when $C_1 = 1/2$, the rank of $\bar{\tau}_1^{\text{opt}}$ is 1 and (iii) implies $\bar{M}_1^{\text{opt}} = \beta |\nu_1\rangle\langle\nu_1|$. Therefore (i) means $\bar{M}_2^{\text{opt}} = \rho_0 - (\alpha + \beta)|\nu_1\rangle\langle\nu_1|$ and we have $\alpha, \beta \ge 0$, and $\alpha + \beta \le 1 - Q_2$. $\text{tr}[\bar{M}_0^{\text{opt}}] = \alpha$ has a minimum(maximum) at $\alpha = 0(\alpha = 1 - Q_2)$ and $\beta = 0$. Therefore, we obtain $P_1(1 - C_1) = [0, 1 - Q_2]$.

Proof of Lemma III.3. When $1/2 < C_1 \le C_2, \rho_{12} = 0$, since the following $\{\bar{M}_i\}_{i=0}^2, \{\bar{\tau}_i^{\star}\}_{i=0}^2$ satisfies the optimality condition (6) of $q_0 = C_1$, $\bar{P}_{\text{cor}}^{\text{opt}}(C_1) = \rho_{11}C_1 + \rho_{22}C_2$.

$$\bar{M}_0 = 0, \qquad \bar{M}_1 = \rho_{11} |\nu_1\rangle\langle\nu_1|, \qquad \bar{M}_2 = \rho_{22} |\nu_2\rangle\langle\nu_2|,
\bar{\tau}_0^* = (C_2 - C_1) |\nu_2\rangle\langle\nu_2|, \quad \bar{\tau}_1^* = (2C_2 - 1) |\nu_2\rangle\langle\nu_2|, \quad \bar{\tau}_2^* = (2C_1 - 1) |\nu_1\rangle\langle\nu_1|.$$
(B2)

This means that $\bar{\tau}_i^{\text{opt}} = \bar{\tau}_i^{\star}(\forall i)$ when $q_0 = C_1$. Since, if $C_1 < C_2$, the rank of $\bar{\tau}_i^{\text{opt}}$ is one, (iii) tells that \bar{M}_0^{opt} and \bar{M}_1^{opt} are proportional to $|\nu_1\rangle\langle\nu_1|$, and \bar{M}_2^{opt} is proportional to $|\nu_2\rangle\langle\nu_2|$. By (i), \bar{M}_i^{opt} can be expressed as (25). However, since $C_1 = C_2$ implies $\bar{\tau}_0^{\text{opt}} = 0$, \bar{M}_i^{opt} becomes (26). Therefore $P_{\text{I}}(C_1)$ can be written as $[0, \rho_{11} + \rho_{22}\delta_{C_1, C_2}]$. \square

Proof of Lemma III.4. When $1/2 < C_1 \le C_2$ and $\rho_{12} \ne 0$, lemma II.2 and corollary III.1 reveal that $q_0^{(0)} < C_2$. By (ii) of optimality condition (6) and the nonnegativity of $\bar{\tau}_i(\forall i)$, $\bar{\tau}_0^{\text{opt}} = 0$ includes $q_0 \ge C_2$, and $\bar{\tau}_{1(2)}^{\text{opt}} = 0$ contains $C_{2(1)} \le 1/2$. This implies that $\bar{\tau}_i^{\text{opt}} \ne 0(\forall i)$ if $q_0 < q_0^{(1)}$. Then $\bar{M}_0^{\text{opt}} = 0$ means that \bar{M}_1^{opt} , $\bar{M}_2^{\text{opt}} \ne 0$ because $\bar{M}_0^{\text{opt}} = \bar{M}_1^{\text{opt}} = 0$ implies $\bar{\tau}_2^{\text{opt}} = 0$ and $\bar{M}_0^{\text{opt}} = \bar{M}_2^{\text{opt}} = 0$ contains $\bar{\tau}_1^{\text{opt}} = 0$. When $\bar{M}_0^{\text{opt}} = 0$, in order to obtain the explicit form of $\{\bar{M}_i^{\text{opt}}, \bar{\tau}_i^{\text{opt}}\}_{i=1}^2$, we use the optimality condition (8). Since $\bar{M}_0 = 0$ includes $p_0 = 0$, it has no effect on r_0 and w_0 . $\bar{M}_i, \bar{\tau}_i \ne 0 (i = 1, 2)$ implies $p_i, r_i \ne 0 (i = 1, 2)$, and by (iii) we have $\|w_i\|_2 = 1$, $u_i = -w_i(i = 1, 2)$. Since $r_1 + r_2 = l$ and $r_2 - r_1 = q_1 - q_2$ should be satisfied by (ii), $\{p_i^{\text{opt}}, u_i^{\text{opt}}\}_{i=1}^2$ and $\{r_i^{\text{opt}}, w_i^{\text{opt}}\}_{i=1}^2$ can be obtained as follows:

$$p_{1}^{\text{opt}} = p_{2}^{\text{opt}} = \frac{1}{2}, r_{1}^{\text{opt}} = \frac{1+l}{2} - q_{1}, r_{2}^{\text{opt}} = \frac{1+l}{2} - q_{2},$$

$$\boldsymbol{u}_{1}^{\text{opt}} = \boldsymbol{w}_{2}^{\text{opt}} = \frac{q_{1}\boldsymbol{v}_{1} - q_{2}\boldsymbol{v}_{2}}{\|q_{1}\boldsymbol{v}_{1} - q_{2}\boldsymbol{v}_{2}\|_{2}}, \boldsymbol{u}_{2}^{\text{opt}} = \boldsymbol{w}_{1}^{\text{opt}} = \frac{q_{2}\boldsymbol{v}_{2} - q_{1}\boldsymbol{v}_{1}}{\|q_{1}\boldsymbol{v}_{1} - q_{2}\boldsymbol{v}_{2}\|_{2}}.$$
(B3)

From these, we find $\bar{P}_{\text{cor}}^{\text{opt}}(q) = (1+l)/2 (\forall q \leq q_0^{(0)})$, and can decide the explicit form of $\{M_i^{\text{opt}}\}_{i=0}^2$ and $\{\tau_i^{\text{opt}}\}_{i=1}^2$. However r_0^{opt} and $\boldsymbol{w}_0^{\text{opt}}$ are not decided yet. These are affected only by (ii). The triangle made of $\{q_i\boldsymbol{v}_i\}_{i=0}^2$ lies in the plane with the origin, and the triangle consisting of $\{-r_i^{\text{opt}}\boldsymbol{w}_i^{\text{opt}}\}_{i=0}^2$ should be located in the same plane. Since the two triangles are congruent, then as $\|r_0^{\text{opt}}\boldsymbol{w}_0^{\text{opt}}\|_2$ grows larger $\|q_0\boldsymbol{v}_0\|_2$ becomes larger. Since $q_0 + r_0^{\text{opt}}$ is fixed as

(1+l)/2, when $\|\boldsymbol{w}_0^{\text{opt}}\|_2$ reaches the maximum (that is, when $\|\boldsymbol{w}_0^{\text{opt}}\|_2 = 1$), q_0 reaches the maximum. Therefore the determinant of $\bar{\tau}_0^{\text{opt}}$ is 0 when $q_0 = q_0^{(0)}$. From (ii), we have $(\chi_1 - q_0^{(0)})(\chi_2 - q_0^{(0)}) = |\gamma_{12}|^2$. Though there are two roots of this equation, the nonnegativity of $\bar{\tau}_0^{\text{opt}}$ implies that $q_0^{(0)} \leq \min\{\chi_1, \chi_2\}$, and the analytic form of $q_0^{(0)}$ can be obtained as χ of (29). The optimal POVM of $q_0 = \chi$ is unique since $r_i^{\text{opt}} \neq 0 (\forall i)$ and $\{q_i v_i\}_{i=0}^2$ forms a triangle; see the Appendix D. This means that $P_{\rm I}(\chi) = 0$.

Proof of Lemma III.5. When $C_1 < C_2$ and $\rho_{12} = 0$, if $q_0^{(0)} < q < q_0^{(1)}$, POVM, defined as (34), and $\{\tau_i^*\}_{i=0}^2$ satisfies KKT optimality condition (6) to $q_0 = q$:

$$\bar{\tau}_{0}^{\star} = (C_{2} - q)|\nu_{2}\rangle\langle\nu_{2}|,
\bar{\tau}_{1}^{\star} = (q - C_{1})|\nu_{1}\rangle\langle\nu_{1}| + (2C_{2} - 1)|\nu_{2}\rangle\langle\nu_{2}|,
\bar{\tau}_{2}^{\star} = (q - 1 + C_{1})|\nu_{1}\rangle\langle\nu_{1}|.$$
(B4)

This means that $\bar{\tau}_i^{\text{opt}} = \bar{\tau}_i^{\star}(\forall i)$. Since the rank of $\bar{\tau}_1^{\text{opt}}$ is two, (iii) implies $\bar{M}_1^{\text{opt}} = 0$. However, since the rank of $\bar{\tau}_0^{\text{opt}}$ and $\bar{\tau}_2^{\text{opt}}$ are one, \bar{M}_0^{opt} and \bar{M}_2^{opt} are proportional to $|\nu_1\rangle\langle\nu_1|$ and $|\nu_2\rangle\langle\nu_2|$, respectively. Therefore (i) means that \bar{M}_i^{opt} is unique as (34).

Proof of Lemma III.6. First of all, let us consider the case of $C_1 \leq \frac{1}{2} < C_2$ and $\rho_{12} \neq 0$. In the region of $q_0^{(0)} < q_0 < q_0^{(1)}$, since λ_1 of (44) is less than 0, we find $M_1^{\text{opt}} = 0$ or $M_2^{\text{opt}} = 0$ by theorem III.3. If $M_2^{\text{opt}} = 0$, since optimality condition (6) means $\bar{\tau}_1^{\text{opt}} = \bar{\tau}_0^{\text{opt}} + q_0 I_2 - \bar{\rho}_1$ and $\det(\bar{\tau}_0^{\text{opt}}) = \det(\bar{\tau}_1^{\text{opt}}) = 0$, $t_i = \langle \nu_i | \bar{\tau}_0^{\text{opt}} | \nu_i \rangle$ satisfies $t_1 t_2 = (t_1 + q_0 - C_1)(t_2 + q_0 - 1 + C_2)$. However, this result is contradictory because $(t_1 + q_0 - C_1)(t_2 + q_0 - 1 + C_2)$ is greater than t_1t_2 in the region of $(C_1, 1 - C_2 \le 1 - C_1 =)q_0^{(0)} < q_0 < q_0^{(1)}$. Therefore we get $M_1^{\text{opt}} = 0$. Next, let us consider the case of $\frac{1}{2} < C_1 \le C_2$ and $\rho_{12} \ne 0$. Here $(q_0^{(0)}, q_0^{(1)})$ is divided into two cases: $(\chi, C_1]$ and

 (C_1, C_2) . In latter case, because of $\lambda_1 < 0$, we can obtain $M_1^{\text{opt}} = 0$.

Appendix C: Proof of Theorem III.3

Proof. When $\rho_{xy} \neq 0$ and $q_0 = q \in (q_0^{(0)}, q_0^{(1)})$, the line intersecting v_1 and v_2 does not contain the origin, and $\{q_i v_i\}_{i=0}^2$ forms a triangle. $r_0^{\text{opt}} = 0$ implies that $\{M_i = \delta_{i0} I_2\}_{i=0}^2$ provide an optimal POVM, which includes $q_0^{(1)} \leq q$. Since $r_k^{\text{opt}} = 0 (k \in \{1, 2\})$ indicates that $\{M_i = \delta_{ik} I_2\}_{i=0}^2$ yields the optimal POVM, this means $q \leq q_0^{(0)}$. Therefore the element of $\{r_i^{\text{opt}}\}_{i=0}^2$ are all nonzero. In this case, the optimal POVM is unique; see the Appendix D. In addition, M_0^{opt} is nonzero, and at least one of M_1^{opt} and M_2^{opt} is nonzero. In the case of $M_x^{\text{opt}} \neq 0$, $M_y^{\text{opt}} = 0(\{x,y\} = \{1,2\})$, the index x turns out to be the index i in $\max_{i \in 1,2} [q_i + ||qv_0 - q_i|]$

 $q_i v_i \|_2$] because $\bar{P}_{\text{cor}}^{\text{opt}}(q) = \max_{i \in 1,2} [q + q_i + \|q\rho_0 - q_i\rho_i\|_1]/2$. The optimal POVM, by the optimality condition (8), can be expressed as (42).

In the case of $M_i^{\text{opt}} \neq 0 (\forall i)$, by the optimality condition (6), $\{\bar{M}_i^{\text{opt}}, \bar{\tau}_i^{\text{opt}}\}_{i=0}^2$ can be found explicitly. From condition (ii), $\{\bar{\tau}_i^{\text{opt}}\}_{i=0}^2$ are given as follows.

$$\bar{\tau}_{0}^{\text{opt}} = \tau_{11}|\nu_{1}\rangle\langle\nu_{1}| + \tau_{12}|\nu_{1}\rangle\langle\nu_{2}| + \tau_{21}|\nu_{2}\rangle\langle\nu_{1}| + \tau_{22}|\nu_{2}\rangle\langle\nu_{2}|,
\bar{\tau}_{1}^{\text{opt}} = (\tau_{11} + q - C_{1})|\nu_{1}\rangle\langle\nu_{1}| + \tau_{12}|\nu_{1}\rangle\langle\nu_{2}| + \tau_{21}|\nu_{2}\rangle\langle\nu_{1}| + (\tau_{22} + q - 1 + C_{2})|\nu_{2}\rangle\langle\nu_{2}|,
\bar{\tau}_{2}^{\text{opt}} = (\tau_{11} + q - 1 + C_{1})|\nu_{1}\rangle\langle\nu_{1}| + \tau_{12}|\nu_{1}\rangle\langle\nu_{2}| + \tau_{21}|\nu_{2}\rangle\langle\nu_{1}| + (\tau_{22} + q - C_{2})|\nu_{2}\rangle\langle\nu_{2}|.$$
(C1)

By the complementary slackness condition (iii) of (6), the every rank of $\{\bar{M}_i^{\text{opt}}, \bar{\tau}_i^{\text{opt}}\}_{i=0}^2$ is one. Therefore their determinants become 0, which means

$$|\tau_{12}| = \sqrt{\tau_{11}\tau_{22}}$$
 and $\bar{M}_i^{\text{opt}} = \text{tr}[\bar{M}_i^{\text{opt}}] \cdot \left[I_2 - \frac{\bar{\tau}_i^{\text{opt}}}{\text{tr}[\bar{\tau}_i^{\text{opt}}]} \right] \quad \forall i.$ (C2)

Then, we have $\tau_{11} = \lambda_1$ and $\tau_{22} = \lambda_2$. Since $\operatorname{tr}[\bar{M}_i^{\text{opt}}]$ is the probability that M_i^{opt} may be detected, $\operatorname{tr}[\bar{M}_0^{\text{opt}}]$ becomes $P_{\rm I}(q)$. The phase of τ_{12} and the form of ${\rm tr}[\bar{M}_i^{\rm opt}]$ can be obtained by condition (i). The completeness condition of the POVM is represented as

$$\frac{\operatorname{tr}[\bar{M}_0^{\text{opt}}]}{\operatorname{tr}[\bar{\tau}_0^{\text{opt}}]} \cdot \bar{\tau}_0^{\text{opt}} + \frac{\operatorname{tr}[\bar{M}_1^{\text{opt}}]}{\operatorname{tr}[\bar{\tau}_1^{\text{opt}}]} \cdot \bar{\tau}_1^{\text{opt}} + \frac{\operatorname{tr}[\bar{M}_2^{\text{opt}}]}{\operatorname{tr}[\bar{\tau}_2^{\text{opt}}]} \cdot \bar{\tau}_2^{\text{opt}} = I_2 - \rho_0.$$
(C3)

 ρ_{12} and τ_{12} have the relation of $\rho_{12}/\tau_{12} = -\sum_{i=0}^{2} (\text{tr}[\bar{M}_{i}^{\text{opt}}]/\text{tr}[\bar{\tau}_{i}^{\text{opt}}])$. By $M_{i}^{\text{opt}} \neq 0 (\forall i)$ and the non-negativity of POVM, the right hand side of the equation is always negative, and we get $\rho_{12}/\tau_{12} = -|\rho_{12}|/|\tau_{12}|$. That is, $\tau_{12} = -(\rho_{12}/|\rho_{12}|)\sqrt{\lambda_{1}\lambda_{2}}$. And $\bar{P}_{\text{cor}}^{\text{opt}}(q)$ is found as (43). Then we have $\text{tr}[\bar{M}_{i}^{\text{opt}}] = \eta_{i}$ by the following relation:

$$\begin{pmatrix} \operatorname{tr}[\bar{M}_{0}^{\text{opt}}] \\ \operatorname{tr}[\bar{M}_{1}^{\text{opt}}] \\ \operatorname{tr}[\bar{M}_{2}^{\text{opt}}] \end{pmatrix} = \begin{pmatrix} \frac{1}{\operatorname{tr}[\bar{\tau}_{0}^{\text{opt}}]} & \frac{1}{\operatorname{tr}[\bar{\tau}_{1}^{\text{opt}}]} & \frac{1}{\operatorname{tr}[\bar{\tau}_{2}^{\text{opt}}]} \\ \frac{\lambda_{1}}{\operatorname{tr}[\bar{\tau}_{0}^{\text{opt}}]} & \frac{\lambda_{1} + q - C_{1}}{\operatorname{tr}[\bar{\tau}_{1}^{\text{opt}}]} & \frac{\lambda_{1} + q - 1 + C_{1}}{\operatorname{tr}[\bar{\tau}_{2}^{\text{opt}}]} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \frac{|\rho_{12}|}{\sqrt{\lambda_{1}\lambda_{2}}} \\ \rho_{22} \end{pmatrix}.$$
(C4)

Therefore \bar{M}_i^{opt} is represented as (45). The result implies the following. If $\lambda_i \geq 0 (\forall i)$ and $\eta_i > 0 (\forall i)$, we have $M_i^{\text{opt}} \neq 0 (\forall i)$. Otherwise, we find $M_1^{\text{opt}} = 0$ or $M_2^{\text{opt}} = 0$.

Appendix D: Proof of Uniqueness of Optimal POVM

Here we prove the following fact: When $\{q_i \mathbf{v}_i\}_{i=0}^2$ forms a triangle, if $r_i^{\text{opt}} \neq 0(\forall i)$, then the POVM $\{M_i = p_i(I_2 + \mathbf{u}_i \cdot \boldsymbol{\sigma})\}_{i=0}^2$ fulfilling the optimality condition (8) is unique. For the proof, we use \mathbf{v}_0 as an arbitrary Bloch vector extrinsic to $\mathbf{v}_1, \mathbf{v}_2$. Since $M_k^{\text{opt}} = I_2$ implies $r_k^{\text{opt}} = 0$, at least two of $\{M_i^{\text{opt}}\}_{i=0}^2$ are nonzero. First, we consider the case that there exists $\{p_i \neq 0, \mathbf{u}_i\}_{i=0}^2$ and $\{r_i \neq 0, \mathbf{w}_i\}_{i=0}^2$ fulfilling optimality condition (8). Without loss of generality, we can set $q_0 \geq q_1, q_2$. Then (iii) becomes $\mathbf{u}_i \cdot \mathbf{w}_i = -1(\forall i)$. This can be rewritten

First, we consider the case that there exists $\{p_i \neq 0, \boldsymbol{u}_i\}_{i=0}^2$ and $\{r_i \neq 0, \boldsymbol{w}_i\}_{i=0}^2$ fulfilling optimality condition (8). Without loss of generality, we can set $q_0 \geq q_1, q_2$. Then (iii) becomes $\boldsymbol{u}_i \cdot \boldsymbol{w}_i = -1(\forall i)$. This can be rewritten as $\|\boldsymbol{u}_i\|_2 = 1$, $\boldsymbol{w}_i = -\boldsymbol{u}_i(\forall i)$, and (ii) is as follows: $r_i - r_0 = e_i$, $\mathbf{R} \equiv q_i \boldsymbol{v}_i - r_i \boldsymbol{u}_i (i = 0, 1, 2)$. e_i is the difference between two prior probabilities q_0 and q_i . This condition means the following; $\{r_i \boldsymbol{u}_i\}_{i=0}^2$ forms a triangle congruent to a triangle $\{q_i \boldsymbol{v}_i\}_{i=0}^2$, and $\{r_i \boldsymbol{u}_i\}_{i=0}^2$ coincides with $\{q_i \boldsymbol{v}_i\}_{i=0}^2$ by parallel transport \mathbf{R} . Then (i) contains the following statement. \mathbf{R} lies in the interior of the triangle $\{q_i \boldsymbol{v}_i\}_{i=0}^2$, and the distance from this point to the vertex of the triangle $q_i \boldsymbol{v}_i$ is r_i . The points fulfilling $r_i - r_0 = e_i$ satisfy the following hyperbolic equation:

$$r_0 = \frac{l_i^2 - e_i^2}{2(l_i \cos \theta_i + e_i)}.$$
(D1)

Above l_i is the distance between two vectors $q_0 \mathbf{v}_0$ and $q_i \mathbf{v}_i$, and θ_i is the angle between two sides $\{\mathbf{R}, q_0 \mathbf{v}_0\}$ and $\{q_0 \mathbf{v}_0, q_i \mathbf{v}_i\}$. As θ_i increases, r_0 also increases, and inside the triangle $\{q_i \mathbf{v}_i\}_{i=0}^2$ the position of \mathbf{R} is unique. This means that the $\{p_i, \mathbf{u}_i\}_{i=0}^2$ are unique. Therefore, the optimal POVM in which every element is nonzero is unique. To make a distinction, we denote this POVM as $\{M_i'\}_{i=0}^2$. Suppose that there exists another POVM satisfying the optimality condition and denote it as $\{M_i''\}_{i=0}^2$. Then the POVM consisting of $M_i = \epsilon M_i' + (1 - \epsilon)M_i''(0 < \epsilon < 1)$ is optimal, and we have $M_i \neq 0 (\forall i)$. This is contradictory, and therefore the optimal POVM is unique.

Next, we consider the case that there exist $\{p_i, u_i\}_{i=0}^2$ and $\{r_i \neq 0, w_i\}_{i=0}^2$ fulfilling optimality condition (8) and one of $\{p_i\}_{i=0}^2$ is zero and the others are nonzero. Without loss of generality, we can set $p_0 = 0$. Then (iii) becomes $u_i \cdot w_i = -1 (i = 1, 2)$. This can turn into $||u_i||_2 = 1$, $w_i = -u_i (i = 1, 2)$, and (ii) can be expressed in the following way: $r_1 - r_2 = q_2 - q_1$, $\mathbf{R} \equiv q_1 v_1 - r_1 u_1 = q_2 v_2 - r_2 u_2$. This condition implies that $\{r_i u_i\}_{i=1}^2$ coincides with the line segment $\{q_i v_i\}_{i=1}^2$ by parallel translation \mathbf{R} . (i) means that \mathbf{R} lies in the interior of $\{q_i v_i\}_{i=1}^2$ and the distance from the point to $q_i v_i$ is r_i . That is, we have $r_1 + r_2 = l_{12}$. l_{12} is the distance between two vectors $q_1 v_1$ and $q_2 v_2$. Then r_1 and r_2 satisfying $r_1 - r_2 = q_2 - q_1$ are apparently unique. This implies that $\{p_i, u_i\}_{i=1}^2$ are unique. Therefore the optimal POVM satisfing $M_0 = 0, M_1 \neq 0, M_2 \neq 0$ is unique. To differentiate from the other POVM, we represent this POVM as $\{M_i'\}_{i=0}^2$. We assume that there exists a POVM satisfying $M_0 \neq 0$ and the optimality condition, and denote it as $\{M_i''\}_{i=0}^2$. Then the POVM consisting of $M_i = \epsilon M_i' + (1 - \epsilon) M_i''(0 < \epsilon < 1)$ is optimal. The result is that POVM fulfilling $M_i \neq 0 (\forall i)$ and geometric optimality condition is not unique. This contradicts the previous result, and the optimal POVM is unique.

In conclusion, when $\{q_i v_i\}_{i=0}^2$ forms a triangle and $r_i^{\text{opt}} \neq 0 (\forall i)$, the optimal POVM is unique.

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