Rényi and Tsallis formulations of separability conditions in finite dimensions

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Separability conditions for a bipartite quantum system of finite-dimensional subsystems are formulated in terms of Rényi and Tsallis entropies. Entropic uncertainty relations often lead to entanglement criteria. We propose new approach based on the convolution of discrete probability distributions. Measurements on a total system are constructed of local ones according to the convolution scheme. Separability conditions are derived on the base of uncertainty relations of the Maassen–Uffink type as well as majorization relations. On each of subsystems, we use a pair of sets of subnormalized vectors that form rank-one POVMs. We also obtain entropic separability conditions for local measurements with a special structure, such as mutually unbiased bases and symmetric informationally complete measurements. The relevance of the derived separability conditions is demonstrated with several examples.

Keywords: entropic uncertainty principle, convolution, majorization, separable states

I. INTRODUCTION

Quantum entanglement stands among fundamentals of the quantum world. This quantum-mechanical feature was concerned by founders in the Schrödinger "cat paradox" paper [1] and in the Einstein–Podolsky–Rosen paper [2]. Entanglement is central to all questions of the emerging technologies of quantum information. Features of quantum entanglement are currently the subject of active research (see, e.g., the review [3] and references therein). Due to progress in quantum information processing, both the detection and quantification of entanglement are very important. In the case of discrete variables, the positive partial transpose (PPT) criterion [4] and the reduction criterion [5] are very powerful. On the other hand, no universal criteria are known even for discrete variables. Say, the PPT criterion is necessary and sufficient for 2×2 and 2×3 systems, but ceases to be so in higher dimensions [6]. Separability conditions can be derived from uncertainty relations of various forms [7–13]. The author of [14] proposed a unifying formalism that reveals links between certain classes of criteria.

Since the Heisenberg principle appeared [15], many formulations and scenarios were studied to understand uncertainty of complementary observables [16, 17]. One of approaches to quantifying uncertainty in quantum measurements is based on the use of entropies [18–20]. The first entropic uncertainty relation for position and momentum was derived in [21] and later improved in [22, 23]. Entropic uncertainty relations are currently the subject of active research [24–26]. Entropic bounds cannot distinguish the uncertainty inherent in obtaining a particular selection of the outcomes [27]. Fine-grained uncertainty relations were studied for several scenarios [28, 29]. Majorization approach provides an alternative way to express the uncertainty principle in terms of probabilities *per se* [30]. Majorization relations in finite dimensions were formulated in [31–33]. The author of [34] derived coarse-grained counterparts of discrete uncertainty relations based on the concept of majorization. Majorization uncertainty relations for quantum operations were examined in [35]. Majorization-based entropic bounds are sometimes better than bounds of the Maassen–Uffink type [31, 33].

The aim of the present work is to derive separability conditions on the base of local entropic bounds of various types. To build total measurement operators, we propose a new unifying scheme based on the convolution operation. In particular, this approach allows us to compare majorization uncertainty relations with relations of the Maassen–Uffink type in the context of their application in entanglement detection. The paper is organized as follows. In Sect. II, we review required material of matrix analysis and several facts concerning the convolution operation with discrete indices. In Sect. III, some known forms of entropic uncertainty relations are recalled. We also give a reformulation of the majorization approach for two rank-one POVMs. Separability conditions in terms of Rényi and Tsallis entropies are formulated in Sect. IV. These conditions correspond to certain classes of uncertainty relations in combination with the convolution scheme. In Sect. V, the relevance of derived criteria is reasoned with several examples.

II. PRELIMINARIES

In this section, the required definitions will be given. We shall also prove two preliminary results concerning the convolution of discrete probability distributions. For two integers $m,n\geq 1$, the symbol $\mathbb{M}_{m\times n}(\mathbb{C})$ denotes the space of all $m\times n$ complex matrices. In the case of square matrices, when $m=n$, we write $\mathbb{M}_n(\mathbb{C})$. Each matrix $\mathsf{M}\in\mathbb{M}_{m\times n}(\mathbb{C})$

can be represented in terms of the singular value decomposition [36],

$$\mathsf{M}=\mathsf{U}_m\boldsymbol{\Sigma}\,\mathsf{U}_n^{\dagger}\,, \tag{2.1}$$

where $\mathsf{U}_m\in\mathbb{M}_m(\mathbb{C})$ and $\mathsf{U}_n\in\mathbb{M}_n(\mathbb{C})$ are unitary. The $m\times n$ matrix $\boldsymbol{\Sigma}=[[\varsigma_{ij}]]$ has real entries with $\varsigma_{ij}=0$ for all $i\neq j$. If the given matrix M has rank r, then diagonal entries of $\boldsymbol{\Sigma}$ can be chosen so that

$$\varsigma_{11}\geq\cdots\geq\varsigma_{rr}>0=\varsigma_{r+1,r+1}=\cdots=\varsigma_{\ell\ell}\,,$$

where $\ell=\min\{m,n\}$. Following [37], we denote the Schatten ∞-norm as

$$\|\mathsf{M}\|_{\infty}=\max\bigl\{\varsigma_{jj}(\mathsf{M}):\,1\leq j\leq\ell\bigr\}\,. \tag{2.2}$$

Treated as an operator norm, the norm (2.2) is induced by the Euclidean norm of vectors [36]. In this sense, it is sometimes designated with subscript 2 instead of ∞.

The space of linear operators on d-dimensional Hilbert space $\mathcal{H}$ will be denoted as $\mathcal{L}(\mathcal{H})$. By $\mathcal{L}_{+}(\mathcal{H})$, we mean the set of positive operators. The state of a d-level quantum system is described by density matrix $\boldsymbol{\rho}\in\mathcal{L}_{+}(\mathcal{H})$ normalized as $\mathrm{Tr}(\boldsymbol{\rho})=1$. With respect to the prescribed basis, vectors of $\mathcal{H}$ are represented by elements of $\mathbb{M}_{d\times1}(\mathbb{C})$, whereas operators of $\mathcal{L}(\mathcal{H})$ are represented by elements of $\mathbb{M}_d(\mathbb{C})$.

To formulate schemes for detecting entanglement, we have to build total-system measurements of local ones. Local measurements are assumed to be formed by sets of subnormalized vectors. In the simplest case, we take an orthonormal basis $\mathcal{E}=\{|e_i\rangle\}$ with $i=0,1,\ldots,d-1$. For the pre-measurement state $\boldsymbol{\rho}$, i-th outcome appears with the probability $p_i(\mathcal{E}|\boldsymbol{\rho})=\langle e_i|\boldsymbol{\rho}|e_i\rangle$. Generalized quantum measurement are typically described in terms of POVMs. Let $\mathcal{N}=\{\mathsf{N}_i\}$ be a set of elements of $\mathcal{L}_{+}(\mathcal{H})$, satisfying the completeness relation

$$\sum_{i=0}^{D-1}\mathsf{N}_i=\mathbb{1}_d\,.$$

Such operators form a positive operator-valued measure (POVM). The probability of i-th outcome is expressed as

$$p_i(\mathcal{N}|\boldsymbol{\rho})=\mathrm{Tr}(\mathsf{N}_i\boldsymbol{\rho})\,. \tag{2.3}$$

In opposite to von Neumann measurements, the number D of different outcomes can exceed the dimensionality of the Hilbert space.

Let $p=\{p_i\}$ be a probability distribution. For $0<\alpha\neq1$, the Rényi α-entropy is defined as

$$R_{\alpha}(p):=\frac{1}{1-\alpha}\,\ln\Bigl(\sum\nolimits_i p_i^{\alpha}\Bigr)\,. \tag{2.4}$$

This entropy is a non-increasing function of α [38]. In the limit $\alpha\to1$, we obtain the usual Shannon entropy

$$H_1(p)=-\sum\nolimits_i p_i\,\ln p_i\,. \tag{2.5}$$

For $\alpha\in(0,1)$, the right-hand side of (2.4) is certainly concave [39]. Convexity properties of Rényi's entropies with orders $\alpha>1$ depend on dimensionality of probabilistic vectors [40, 41]. The binary Rényi entropy is concave for $0<\alpha\leq2$ [41]. We also recall that the Rényi entropy is Schur-concave.

Tsallis entropies form another important extension of the Shannon entropy. For $0<\alpha\neq1$, the Tsallis α-entropy is defined as [42]

$$H_{\alpha}(p):=\frac{1}{1-\alpha}\,\Bigl(\sum\nolimits_i p_i^{\alpha}-1\Bigr)=-\sum\nolimits_i p_i^{\alpha}\,\ln_{\alpha}(p_i). \tag{2.6}$$

The α-logarithm of positive ξ is put here as $\ln_{\alpha}(\xi)=\bigl(\xi^{1-\alpha}-1\bigr)/(1-\alpha)$. For $\alpha=1$, Tsallis' α-entropy also reduces to (2.5). The right-hand side of (2.6) is a concave function of probabilities for all $0<\alpha\neq1$. It is Schur-concave as well. Other properties of Rényi and Tsallis entropies with quantum applications are discussed in [40].

It will be convenient to use norm-like functionals. For arbitrary $\alpha>0$, we define

$$\|p\|_{\alpha}:=\Bigl(\sum\nolimits_i p_i^{\alpha}\Bigr)^{1/\alpha}\,. \tag{2.7}$$

It is a legitimate norm only for $\alpha\geq1$. For $0<\alpha\neq1$, we then have

$$R_{\alpha}(p)=\frac{\alpha}{1-\alpha}\,\ln\|p\|_{\alpha}\,. \tag{2.8}$$

By $R_{\alpha}(\mathcal{N}|\boldsymbol{\rho})$ and $H_{\alpha}(\mathcal{N}|\boldsymbol{\rho})$, we will, respectively, mean the entropies obtained by substituting the probabilities (2.3) into (2.4) and (2.6).

Let us consider two functions $g=\{g_i\}$ and $h=\{h_i\}$ of the discrete variable i that runs D points. Then the convolution of g and h is introduced as

$$(g*h)_k:=\sum_{i=0}^{D-1}g_i\,h_{k\ominus i}\,, \tag{2.9}$$

where the sign "$\ominus$" denotes the subtraction in $\mathbb{Z}/D$. The convolution scheme to build measurement operators is based on a simple but important observation.

Proposition 1 *Let $g=\{g_i\}$ and $h=\{h_i\}$ be positive-valued functions of integer variable $i\in\{0,1,\ldots,D-1\}$, and let*

$$\sum\nolimits_i h_i=1\,. \tag{2.10}$$

For $\alpha>1>\beta>0$, we then have

$$\|g*h\|_{\alpha}\leq\|g\|_{\alpha}\,, \tag{2.11}$$

$$\|g*h\|_{\beta}\geq\|g\|_{\beta}\,. \tag{2.12}$$

Proof. For $\alpha>1$, the function $\xi\mapsto\xi^{\alpha}$ has positive second derivative. Due to Jensen's inequality, for each k we obtain

$$[(g*h)_k]^{\alpha}\leq\sum\nolimits_i h_{k\ominus i}\,g_i^{\alpha}\,.$$

Summing this with respect to k gives $\|g*h\|_{\alpha}^{\alpha}\leq\|g\|_{\alpha}^{\alpha}$ due to (2.10). This completes the proof of (2.11). The function $\xi\mapsto\xi^{\beta}$ has negative second derivative for $0<\beta<1$. Rewriting the above inequalities in opposite direction, we get the claim (2.12). $\blacksquare$

The notion of majorization is posed as follows. Let us treat real-valued functions $g=\{g_i\}$ and $h=\{h_i\}$ of the index $i\in\{0,1,\ldots,D-1\}$ as D-dimensional vectors. The formula $g\prec h$ implies that, for all $0\leq k\leq D-1$,

$$\sum_{i=0}^{k}g_i^{\downarrow}\leq\sum_{i=0}^{k}h_i^{\downarrow}\,,\qquad\sum_{i=0}^{D-1}g_i=\sum_{i=0}^{D-1}h_i\,.$$

Here, the arrows down imply that the values should be put in the decreasing order. Our approach to deriving separability conditions will also use the following lemma.

Proposition 2 *Let p and q be two probability distributions supported on the same finite set; then*

$$p*q\prec p\,,\qquad p*q\prec q\,. \tag{2.13}$$

Proof. It is sufficient to prove only one of the two relations (2.13). Let us represent each probability distribution as a column with D entries. The following result is well known (see, e.g., theorem II.1.10 of [43]). The relation $p*q\prec p$ holds if and only if

$$p*q=\mathsf{T}p \tag{2.14}$$

for some doubly stochastic matrix T. A square matrix is called doubly stochastic, when its entries are positive and the sum of entries is equal to 1 in each row and in each column. In the case considered, the formula (2.14) directly follows from the definition of convolution: i-th row of T reads $q_{i\ominus j}$, where j runs from 0 to $D-1$. Hence, each row of T is obtained by the cyclic shift of the above row by one step to the right. Thus, each probability q_j appears exactly one time in any row and in any column. Actually, the matrix T is doubly stochastic. $\blacksquare$

The statement of Proposition 2 may be compared with lemma 1 of [7]. Here, we prefer to give another formulation with emphasizing the role of convolution. In addition, our scheme is rather formulated in terms of measurement operators including the case of POVMs. As lemma 1 of [7] deals with observables, it does not seem to be applicable immediately in these settings.

Let $\mathcal{E}=\bigl\{|e_i\rangle\bigr\}$ and $\mathcal{E}'=\bigl\{|e'_j\rangle\bigr\}$ be two orthonormal bases in a d-dimensional Hilbert space $\mathcal{H}$. They are said to be mutually unbiased if and only if for all i and j,

$$\bigl|\langle e_i|e'_j\rangle\bigr|=\frac{1}{\sqrt{d}}\ . \tag{2.15}$$

Several orthonormal bases form a set of mutually unbiased bases (MUBs), when them are pairwise mutually unbiased. Such bases have found use in many questions of quantum information theory (see [44] and references therein). When d is a prime power, we can certainly construct $d+1$ MUBs [44]. It is based on properties of prime powers and corresponding finite field [45, 46].

There exist measurements such that each of them uniquely determine every possible state by the measurement statistics that it alone generates. Measurements with this property are said to be informationally complete [37]. Symmetric informationally complete measurements have a symmetric structure in their elements. In d-dimensional Hilbert space, we consider a set of d^2 rank-one operators of the form

$$|f_i\rangle\langle f_i|=\frac{1}{d}\,|\phi_i\rangle\langle\phi_i|\ . \tag{2.16}$$

If the normalized vectors $|\phi_j\rangle$ all satisfy the condition

$$\bigl|\langle\phi_i|\phi_j\rangle\bigr|^2=\frac{1}{d+1}\qquad(i\neq j)\ , \tag{2.17}$$

the set of operators (2.16) is a symmetric informationally complete POVM (SIC-POVM) [47]. It was conjectured that SIC-POVMs exist in all dimensions [48]. The existence of SIC-POVMs has been shown analytically or numerically for all dimensions up to 67 [49]. Connections between MUBs and SIC-POVMs are discussed in [50, 51].

Since basic constructions of MUBs are related to prime power d, one can try to get an appropriate modification. The authors of [52] proposed the concept of mutually unbiased measurements (MUMs). Using weaker requirements, a complete set of $d+1$ MUMs exists for all d. Let us consider two POVM measurements $\mathcal{N}=\{\mathsf{N}_i\}$ and $\mathcal{N}'=\{\mathsf{N}'_j\}$. Each of them contains d elements such that

$$\mathrm{Tr}(\mathsf{N}_i)=\mathrm{Tr}(\mathsf{N}'_j)=1\ , \tag{2.18}$$

$$\mathrm{Tr}(\mathsf{N}_i\mathsf{N}'_j)=\frac{1}{d}\ . \tag{2.19}$$

The POVM elements are all of trace one, but generally not of rank one. The formula (2.19) is used instead of the squared formula (2.15). Two different elements of the same POVM $\mathcal{N}$ satisfy

$$\mathrm{Tr}(\mathsf{N}_i\mathsf{N}_j)=\delta_{ij}\,\varkappa+(1-\delta_{ij})\,\frac{1-\varkappa}{d-1}\ , \tag{2.20}$$

where $\varkappa$ is the efficiency parameter [52]. The same condition is imposed on the elements of $\mathcal{N}'$. By $\varkappa$, we characterize how close the POVM elements are to rank-one projectors [52]. In general, one satisfies [52]

$$\frac{1}{d}<\varkappa\leq 1\ .$$

For $\varkappa=1/d$ we have the trivial case, in which $\mathsf{N}_i=\mathbb{1}_d/d$ for all i. The value $\varkappa=1$, if possible, gives the standard case of mutually unbiased bases. In principle, we can only say that the maximal efficiency can be reached for prime power d. More precise bounds on $\varkappa$ depend on an explicit construction of POVM elements [52].

Similar ideas can be used to generalize SIC-POVMs. For all finite d, a common construction has been given [53]. Consider a POVM with d^2 elements N_i, which satisfy the following two conditions. First, for all $i=0,\ldots,d^2-1$ we have

$$\mathrm{Tr}(\mathsf{N}_i\mathsf{N}_i)=a\,. \tag{2.21}$$

Second, the pairwise inner products are all symmetric, namely

$$\mathrm{Tr}(\mathsf{N}_i\mathsf{N}_j)=b\qquad(i\neq j)\,. \tag{2.22}$$

Then, the operators N_i form a general SIC-POVM. Combining the conditions (2.21) and (2.22) with the completeness relation finally gives [53]

$$b=\frac{1-ad}{d(d^2-1)}\ . \tag{2.23}$$

We also get $\mathrm{Tr}(\mathsf{N}_i)=1/d$ for all i. A deviation of general SIC-POVM from the usual one is completely characterized by a. In general, this parameter is restricted as [53]

$$\frac{1}{d^3}<a\leq\frac{1}{d^2}\ .$$

The value $a=1/d^3$ gives $\mathsf{N}_i=\mathbb{1}_d/d^2$, so that the measurement is not informationally complete. The value $a=1/d^2$ is achieved, when the POVM elements are all rank-one [53]. The latter is actually the case of usual SIC-POVMs, when POVM elements appear due to (2.16). Even if usual SIC-POVMs exist in all dimensions, they are rather hard to construct. General SIC-POVMs have a similar structure that makes them appropriate in determining an informational content of a quantum state.

III. SOME FORMS OF UNCERTAINTY RELATIONS

In this section, we recall some of existing formulations of the uncertainty principle. We begin with uncertainty relations of the Maassen–Uffink type [20]. Then, majorization uncertainty relations of the papers [31, 33] will be applied. Uncertainty relations for measurements with a special structure should also be considered.

The used orthonormal bases are denoted by $\mathcal{E}=\big\{|e_i\rangle\big\}$ and $\mathcal{E}'=\big\{|e_j'\rangle\big\}$ with $i,j=0,\ldots,d-1$. If the pre-measurement state is described by normalized density matrix $\boldsymbol{\rho}\in\mathcal{L}_+(\mathcal{H})$, then the corresponding probabilities appear as $p_i=p_i(\mathcal{E}|\boldsymbol{\rho})$ and $q_j=p_j(\mathcal{E}'|\boldsymbol{\rho})$. Entropic uncertainty relations of the Maassen–Uffink type were inspired by Kraus [19] and later proved in [20]. To the orthonormal bases $\mathcal{E}=\big\{|e_i\rangle\big\}$ and $\mathcal{E}'=\big\{|e_j'\rangle\big\}$, we assign

$$\eta(\mathcal{E},\mathcal{E}'):=\max\Big\{\big|\langle e_i|e_j'\rangle\big|:\ 0\leq i,j\leq d-1\Big\}\,. \tag{3.1}$$

Due to Riesz's theorem [54], we have

$$\|p\|_\alpha\leq\eta^{2(1-\beta)/\beta}\,\|q\|_\beta\,, \tag{3.2}$$

$$\|q\|_\alpha\leq\eta^{2(1-\beta)/\beta}\,\|p\|_\beta\,, \tag{3.3}$$

where $1/\alpha+1/\beta=2$ and $\alpha>1>\beta$. Hence, various uncertainty relations in terms of generalized entropies can be derived. For some reasons, we will begin a derivation of separability conditions just with (3.2) and (3.3).

The formulas (3.2) and (3.3) are immediately generalized to POVM measurements [55]. Here, we restrict a consideration to especially important case of rank-one POVMs. Let $\mathcal{F}=\big\{|f_i\rangle\big\}$ and $\mathcal{F}'=\big\{|f_j'\rangle\big\}$ be two sets of D subnormalized vectors such that

$$\sum_{i=0}^{D-1}|f_i\rangle\langle f_i|=\mathbb{1}_d\,,\qquad\sum_{j=0}^{D-1}|f_j'\rangle\langle f_j'|=\mathbb{1}_d\,. \tag{3.4}$$

Here, we typically deal with $D>d$. The author of [56] pointed out haw the Maassen–Uffink relation is generalized to such measurements. Following [57], we are rather interested in extending just (3.2) and (3.3). Replacing (3.1) with

$$\eta(\mathcal{F},\mathcal{F}'):=\max\Big\{\big|\langle f_i|f_j'\rangle\big|:\ 0\leq i,j\leq D-1\Big\}\,, \tag{3.5}$$

the relations (3.2) and (3.3) hold under the conditions $1/\alpha+1/\beta=2$ and $\alpha>1>\beta$.

We now recall the majorization approach to uncertainty relations in finite dimensions. Applications of this approach beyond this case are discussed in [30, 34]. Let p and q denote the probabilistic vectors generated by two quantum measurements in the same prepared state. The basic idea is to majorize some binary combination of p and q by a third vector with bounding elements. To do so, the authors of [31, 33] inspected norms of submatrices of a certain unitary matrix.

To the orthonormal bases $\mathcal{E}=\big\{|e_i\rangle\big\}$ and $\mathcal{E}'=\big\{|e_j'\rangle\big\}$, one assigns the unitary $d\times d$ matrix $\mathsf{V}(\mathcal{E},\mathcal{E}')$ with entries $v_{ij}=\langle e_i|e_j'\rangle$. By $\mathcal{SUB}(\mathsf{V},k)$, we mean the set of all its submatrices of class k defined by

$$\mathcal{SUB}(\mathsf{V},k):=\big\{\mathsf{M}\in\mathbb{M}_{r\times r'}(\mathbb{C}):\ r+r'=k+1,\ \mathsf{M}\text{ is a submatrix of }\mathsf{V}\big\}\,. \tag{3.6}$$

The positive integer k runs all the values allowed by the condition $r+r'=k+1$. The majorization relations of [31, 33] are expressed in terms of quantities

$$s_k := \max\bigl\{\|\mathsf{M}\|_\infty :\ \mathsf{M}\in\mathcal{SUB}(\mathsf{V},k)\bigr\}\,. \tag{3.7}$$

It will be convenient to label these quantities by integers starting with 1. Due to completeness and orthonormality of each bases, one has $s_d=1$ and, therefore, $s_k=1$ for all $d\leq k\leq 2d-1$.

The authors of [33] proved the majorization relation

$$p\oplus q\prec\{1\}\oplus w\,, \tag{3.8}$$
$$w=(s_1,s_2-s_1,\ldots,s_d-s_{d-1})\,. \tag{3.9}$$

This majorizing vector is completed by s_d-s_{d-1}, since $s_k=1$ for $d\leq k\leq 2d-1$ and further differences are all zero. The following entropic bounds follow from (3.8). For $0<\alpha\leq 1$, it holds that

$$R_\alpha(p)+R_\alpha(q)\geq R_\alpha(w)\,. \tag{3.10}$$

For $\alpha>1$, the sum of two Rényi entropies obeys another inequality [33]

$$R_\alpha(p)+R_\alpha(q)\geq\frac{2}{1-\alpha}\,\ln\!\left(\frac{1+\|w\|_\alpha^\alpha}{2}\right). \tag{3.11}$$

Majorization relations of the tensor-product type were first considered in [31, 32]. The authors of [31] showed that

$$p\otimes q\prec w'\,, \tag{3.12}$$

where the majorizing vector is put as

$$w'=(t_1,t_2-t_1,\ldots,t_d-t_{d-1})\,,\qquad t_k=\frac{(1+s_k)^2}{4}\,. \tag{3.13}$$

Combining (3.12) with Schur-concavity of the Rényi entropy, for $\alpha>0$ we have [31, 32]

$$R_\alpha(p)+R_\alpha(q)\geq R_\alpha(w')\,. \tag{3.14}$$

For $0<\alpha\leq 1$, we will choose (3.10), since $\omega\prec\omega'$ and $R_\alpha(\omega)\geq R_\alpha(\omega')$ [33]. Nevertheless, the relation (3.14) is useful for $\alpha>1$. The sum of two Tsallis α-entropies is bounded from below similarly to (3.10). For any $\alpha>0$ we have [33]

$$H_\alpha(p)+H_\alpha(q)\geq H_\alpha(w)\,. \tag{3.15}$$

As we plan to deal with rank-one POVMs, the majorization approach should be reformulated appropriately. In principle, a general way of extension was considered in [35]. However, that paper focus on quantum operations described in terms of Kraus operators. Rank-one POVMs are so important that we prefer to give an explicit derivation. In this case, the majorization approach is based on the following statement.

Proposition 3 *Let each of sets $\mathcal{F}=\bigl\{|f_i\rangle\bigr\}$ and $\mathcal{F}'=\bigl\{|f_j'\rangle\bigr\}$ contain D subnormalized vectors that form rank-one POVM in d-dimensional space $\mathcal{H}$. Let $\mathcal{I}$ and $\mathcal{J}$ be two subsets of the set $\{0,\ldots,D-1\}$. For arbitrary density matrix $\boldsymbol{\rho}$, we have*

$$\sum\nolimits_{i\in\mathcal{I}}p_i(\mathcal{F}|\boldsymbol{\rho})+\sum\nolimits_{j\in\mathcal{J}}p_j(\mathcal{F}'|\boldsymbol{\rho})\leq 1+\|\mathsf{C}_\mathcal{I}\mathsf{C}_\mathcal{J}^\dagger\|_\infty\ . \tag{3.16}$$

Here, the $|\mathcal{I}|\times d$ matrix $\mathsf{C}_\mathcal{I}$ is formed by rows $\langle f_i|$ with $i\in\mathcal{I}$, and the $|\mathcal{J}|\times d$ matrix $\mathsf{C}_\mathcal{J}$ is formed by rows $\langle f_j'|$ with $j\in\mathcal{J}$.

Proof. For definiteness, we write $\mathcal{I}=\{i_1,\ldots,i_m\}$ and $\mathcal{J}=\{j_1,\ldots,j_n\}$, whence

$$\mathsf{C}_\mathcal{I}=\begin{pmatrix}\langle f_{i_1}|\\ \cdots\\ \langle f_{i_m}|\end{pmatrix},\qquad \mathsf{C}_\mathcal{J}=\begin{pmatrix}\langle f_{j_1}'|\\ \cdots\\ \langle f_{j_n}'|\end{pmatrix}.$$

It will be sufficient to prove the claim (3.16) for pure states. Its validity for mixed states follows by the spectral decomposition. Keeping in mind matrix relations of the form

$$\mathsf{C}_{\mathcal{I}}^{\dagger}\mathsf{C}_{\mathcal{I}}=\sum\nolimits_{i\in\mathcal{I}}|f_i\rangle\langle f_i|\,, \tag{3.17}$$

we have

$$\sum\nolimits_{i\in\mathcal{I}}p_i(\mathcal{F}|\psi)+\sum\nolimits_{j\in\mathcal{J}}p_j(\mathcal{F}'|\psi)=\langle\psi|\mathsf{G}^{\dagger}\mathsf{G}|\psi\rangle\,, \qquad \mathsf{G}=\begin{pmatrix}\mathsf{C}_{\mathcal{I}}\\ \mathsf{C}_{\mathcal{J}}\end{pmatrix}.$$

Due to properties of the spectral norm, we obtain

$$\langle\psi|\mathsf{G}^{\dagger}\mathsf{G}|\psi\rangle\leq\|\mathsf{G}^{\dagger}\mathsf{G}\|_{\infty}=\|\mathsf{G}\mathsf{G}^{\dagger}\|_{\infty}\leq\max\bigl\{\|\mathsf{C}_{\mathcal{I}}\|_{\infty}^{2},\|\mathsf{C}_{\mathcal{J}}\|_{\infty}^{2}\bigr\}+\|\mathsf{C}_{\mathcal{I}}\mathsf{C}_{\mathcal{J}}^{\dagger}\|_{\infty}\,. \tag{3.18}$$

The justification of (3.18) is very similar to the proof of proposition 2 of [35]. The definition of $\mathsf{C}_{\mathcal{I}}$ and $\mathsf{C}_{\mathcal{J}}$ is the only distinction. Let us put the complete $D\times d$ matrices such that $\mathsf{C}_{\mathcal{F}}$ is formed by all the rows $\langle f_i|$, and $\mathsf{C}_{\mathcal{F}'}$ is formed by all the rows $\langle f_j'|$. By submultiplicativity of the spectral norm, one gets

$$\|\mathsf{C}_{\mathcal{I}}\|_{\infty}\leq\|\mathsf{C}_{\mathcal{F}}\|_{\infty}\,, \qquad \|\mathsf{C}_{\mathcal{J}}\|_{\infty}\leq\|\mathsf{C}_{\mathcal{F}'}\|_{\infty}\,. \tag{3.19}$$

It follows from (3.4) that $\mathsf{C}_{\mathcal{F}}^{\dagger}\mathsf{C}_{\mathcal{F}}=\mathsf{C}_{\mathcal{F}'}^{\dagger}\mathsf{C}_{\mathcal{F}'}=\mathbb{1}_d$, whence $\|\mathsf{C}_{\mathcal{F}}\|_{\infty}=\|\mathsf{C}_{\mathcal{F}'}\|_{\infty}=1$. Combining the latter with (3.18) and (3.19) completes the proof. ∎

Using (3.16), we can now extend (3.10), (3.11), (3.14), and (3.15). Denoting $p=p(\mathcal{F}|\boldsymbol{\rho})$ and $q=p(\mathcal{F}'|\boldsymbol{\rho})$, these uncertainty relations are all valid with the following changes. The quantities (3.7) are now calculated with $1\leq k\leq 2D-1$ for the $D\times D$ matrix

$$\mathsf{V}(\mathcal{F},\mathcal{F}')=\bigl[\bigl[\langle f_i|f_j'\rangle\bigr]\bigr]\,. \tag{3.20}$$

Combining $\mathsf{C}_{\mathcal{F}'}^{\dagger}\mathsf{C}_{\mathcal{F}'}=\mathbb{1}_d$ with $\mathsf{V}=\mathsf{C}_{\mathcal{F}}\mathsf{C}_{\mathcal{F}'}^{\dagger}$ leads to $\mathsf{V}\mathsf{V}^{\dagger}=\mathsf{C}_{\mathcal{F}}\mathsf{C}_{\mathcal{F}}^{\dagger}$ and $\|\mathsf{V}\|_{\infty}=1$. Thus, we certainly have $s_k=1$ for $k=2D-1$. Let $D_{\star}$ denote the first index with the property $s_{D_{\star}}=1$. Since vectors of the sets $\mathcal{F}$ and $\mathcal{F}'$ are all subnormalized, we will have $D\leq D_{\star}\leq 2D-1$.

The vectors (3.9) and (3.13) should then include differences up to $s_{D_{\star}}-s_{D_{\star}-1}$ and $t_{D_{\star}}-t_{D_{\star}-1}$, respectively. In the following, the uncertainty relations of this section will be applied in deriving entanglement criteria. Among majorization-based relations, we will mainly use (3.10) and (3.15) with respect to the values of α, for which the corresponding entropy is certainly concave.

It is often expedient to check entanglement with specially designed measurements. For example, mutually unbiased measurements are treated to be capable for such purposes. Another interesting way is connected with symmetric informationally complete measurements. Here, we will use entropic uncertainty relations derived in [57, 58].

Let $\bigl\{\mathcal{E}^{(1)},\ldots,\mathcal{E}^{(K)}\bigr\}$ be a set of K MUBs in d-dimensional space $\mathcal{H}$. For $\alpha\in(0;2]$, the sum of Rényi's entropies obeys the state-independent bound [57]

$$\frac{1}{K}\sum_{t=1}^{K}R_{\alpha}(\mathcal{E}^{(t)}|\boldsymbol{\rho})\geq\ln\!\left(\frac{Kd}{d+K-1}\right). \tag{3.21}$$

Note that the right-hand side of (3.21) is independent of α, whereas Rényi's entropy does not increase with growth of α. To obtain more sensitive criteria, we should take largest orders providing concavity of entropies. Therefore, we will use (3.21) with $\alpha=1$ for arbitrary d and with $\alpha=2$ for $d=2$. For $\alpha\in(0;2]$ and arbitrary state $\boldsymbol{\rho}$ on $\mathcal{H}$, the sum of Tsallis' entropies satisfies the state-independent bound

$$\frac{1}{K}\sum_{t=1}^{K}H_{\alpha}(\mathcal{E}^{(t)}|\boldsymbol{\rho})\geq\ln_{\alpha}\!\left(\frac{Kd}{d+K-1}\right). \tag{3.22}$$

The results (3.21) and (3.22) are based on the inequality

$$\sum_{t=1}^{K}\sum_{i=0}^{d-1}p_i(\mathcal{E}^{(t)}|\boldsymbol{\rho})^2\leq\mathrm{Tr}(\boldsymbol{\rho}^2)+\frac{K-1}{d}\leq 1+\frac{K-1}{d}\ , \tag{3.23}$$

derived in [59]. Of course, the existence of K MUBs should be proved independently. We will study entropic formulation of entanglement criteria based on (3.23). It differs from the previous approach considered in [60]. Using (3.23), the authors of [60] put a specific correlation measure that is bounded from above for separable states.

For mutually unbiased measurements, the following extension of (3.23) takes place [58]. Let $\{\mathcal{N}^{(1)},\ldots,\mathcal{N}^{(K)}\}$ be a set of K MUMs of the efficiency $\varkappa$. For arbitrary $\boldsymbol{\rho}$, we then have [58]

$$\sum_{t=1}^{K}\sum_{i=0}^{d-1} p_i(\mathcal{N}^{(t)}|\boldsymbol{\rho})^2 \leq \frac{1-\varkappa+(\varkappa d-1)\,\mathrm{Tr}(\boldsymbol{\rho}^2)}{d-1}+\frac{K-1}{d}\leq \varkappa+\frac{K-1}{d}\ . \tag{3.24}$$

For $d+1$ MUMs, the inequality (3.24) is actually saturated [58]. For pure states, this result was shown in [52] and then applied for entanglement detection in [61]. For $\alpha\in(0;2]$, the inequality (3.24) gives

$$\frac{1}{K}\sum_{t=1}^{K} R_\alpha(\mathcal{N}^{(t)}|\boldsymbol{\rho})\geq \ln\!\left(\frac{Kd}{\varkappa d+K-1}\right), \tag{3.25}$$

$$\frac{1}{K}\sum_{t=1}^{K} H_\alpha(\mathcal{N}^{(t)}|\boldsymbol{\rho})\geq \ln_\alpha\!\left(\frac{Kd}{\varkappa d+K-1}\right). \tag{3.26}$$

The existence of K MUMs of some efficiency was proved up to $K=d+1$ [52].

For a SIC-POVM $\mathcal{F}=\{|f_i\rangle\}$, the index of coincidence can also be calculated exactly. For the pre-measurement state $\boldsymbol{\rho}$, it holds that [57]

$$\sum_{i=0}^{d^2-1} p_i(\mathcal{F}|\boldsymbol{\rho})^2=\frac{\mathrm{Tr}(\boldsymbol{\rho}^2)+1}{d(d+1)}\leq\frac{2}{d(d+1)}\ . \tag{3.27}$$

As was briefly noticed in [57], this result allows to build a SIC-POVM scheme for entanglement detection. The following uncertainty relations were derived due to (3.27). For $\alpha\in(0;2]$ and any density matrix $\boldsymbol{\rho}$ on $\mathcal{H}$, we have [57]

$$R_\alpha(\mathcal{F}|\boldsymbol{\rho})\geq \ln\!\left(\frac{d(d+1)}{2}\right), \tag{3.28}$$

$$H_\alpha(\mathcal{F}|\boldsymbol{\rho})\geq \ln_\alpha\!\left(\frac{d(d+1)}{2}\right). \tag{3.29}$$

Let general SIC-POVM $\mathcal{N}=\{\mathsf{N}_i\}$ be characterized by the parameter a in the sense of (2.21). For the given pre-measurement state $\boldsymbol{\rho}$, we have [62]

$$\sum_{i=0}^{d^2-1} p_i(\mathcal{N}|\boldsymbol{\rho})^2=\frac{(ad^3-1)\,\mathrm{Tr}(\boldsymbol{\rho}^2)+d(1-ad)}{d(d^2-1)}\leq\frac{ad^2+1}{d(d+1)}\ . \tag{3.30}$$

For a usual SIC-POVM, when $a=d^{-2}$, the result (3.30) is reduced to (3.27). For $\alpha\in(0;2]$ and arbitrary density matrix $\boldsymbol{\rho}$ on $\mathcal{H}$, one gets [62]

$$R_\alpha(\mathcal{N}|\boldsymbol{\rho})\geq \ln\!\left(\frac{d(d+1)}{ad^2+1}\right), \tag{3.31}$$

$$H_\alpha(\mathcal{N}|\boldsymbol{\rho})\geq \ln_\alpha\!\left(\frac{d(d+1)}{ad^2+1}\right). \tag{3.32}$$

Due to (3.30), general SIC-POVMs can be used for entanglement detection. The authors of [63] gave an appropriate form of correlation measures proposed in [60] and reformulated for usual SIC-POVMs in [57].

IV. FORMULATION OF SEPARABILITY CONDITIONS

This section is devoted to separability conditions that follow from uncertainty relations listed in the previous section. In particular, we will present separability conditions on the base of majorization uncertainty relations. Used uncertainty relations are local in the sense that they are posed for one of subsystems. A utility of entanglement criteria based on local uncertainty relations was justified in [10]. To formulate separability conditions, some definitions should be recalled.

We consider a bipartite system of d-level subsystems A and B. The tensor product $\mathcal{H}_{AB}=\mathcal{H}_A\otimes\mathcal{H}_B$ is the total Hilbert space. Any state of the total system is described by density matrix $\boldsymbol{\rho}_{AB}\in\mathcal{L}_{+}(\mathcal{H}_{AB})$. Product states written as $\boldsymbol{\rho}_A\otimes\boldsymbol{\rho}_B$ reveal no correlations between subsystems. A bipartite mixed state is called separable, when its density matrix can be represented as a convex combination of product states [64–66]. For more formal results about separable operators and states, see chapter 6 of [37].

Let us proceed to building measurements on a bipartite system according to the convolution scheme. Its advance is that POVM measurements are naturally treated in the context of entanglement detection. In a certain sense, this scheme is a genuine development of the approach studied in [13]. Total measurement operators are constructed as follows.

Definition 1 *Let $\mathcal{N}_A=\{\mathsf{N}_{Ai}\}$ and $\mathcal{N}_B=\{\mathsf{N}_{Bj}\}$ be D-outcome POVMs in $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. We call a POVM $\mathcal{M}(\mathcal{N}_A,\mathcal{N}_B)=\{\mathsf{\Pi}_k\}$ to be constructed according to the convolution scheme, when*

$$\mathsf{\Pi}_k:=\sum_{i=0}^{D-1}\mathsf{N}_{Ai}\otimes\mathsf{N}_{Bk\ominus i}\,. \tag{4.1}$$

Here, the sign "$\ominus$" denotes the subtraction in $\mathbb{Z}/D$ and $k\in\{0,1,\ldots,D-1\}$.

For each of two subsystems, we will use several measurements marked by the label t. For each $\mathcal{M}^{(t)}$ built according to Definition 1, one has

$$p(\mathcal{M}^{(t)}|\boldsymbol{\rho}_A\otimes\boldsymbol{\rho}_B)=p(\mathcal{N}_A^{(t)}|\boldsymbol{\rho}_A)*p(\mathcal{N}_B^{(t)}|\boldsymbol{\rho}_B)\,. \tag{4.2}$$

For product states, each resolution $\mathcal{M}^{(t)}$ generates the convolution of two distributions assigned to local measurements. Together with entropic bounds, this fact allows us to get inequalities that are satisfied by any convex combination of product states. The convolution operation is also important in deriving entropic entanglement criteria for a bipartite system with continuous-variables [67, 68]. They have recently been extended to multipartite systems [69]. Our first result is posed as follows.

Proposition 4 *Let each of sets $\mathcal{F}_A^{(1)}$ and $\mathcal{F}_A^{(2)}$ of subnormalized vectors form rank-one POVM in $\mathcal{H}_A$, and let each of sets $\mathcal{F}_B^{(1)}$ and $\mathcal{F}_B^{(2)}$ of subnormalized vectors form rank-one POVM in $\mathcal{H}_B$. Let two POVMs $\mathcal{M}^{(t)}(\mathcal{F}_A^{(t)},\mathcal{F}_B^{(t)})$ in $\mathcal{H}_A\otimes\mathcal{H}_B$ be constructed from these sets according to Definition 1. If state $\boldsymbol{\rho}_{AB}$ is separable and $1/\alpha+1/\beta=2$, then*

$$R_\alpha(\mathcal{M}^{(1)}|\boldsymbol{\rho}_{AB})+R_\beta(\mathcal{M}^{(2)}|\boldsymbol{\rho}_{AB})\geq-2\,\ln\eta_S\,, \tag{4.3}$$

$$H_\alpha(\mathcal{M}^{(1)}|\boldsymbol{\rho}_{AB})+H_\beta(\mathcal{M}^{(2)}|\boldsymbol{\rho}_{AB})\geq\ln_\mu\!\big(\eta_S^{-2}\big)\,, \tag{4.4}$$

where $S=A,B$, maximal entropic parameter $\mu=\max\{\alpha,\beta\}$, and $\eta_S=\eta(\mathcal{F}_S^{(1)},\mathcal{F}_S^{(2)})$ according to (3.5).

Proof. We will further assume that $\alpha>1>\beta$. The Shannon case $\alpha=\beta=1$ is finally reached by taking the corresponding limit. It is sufficient to prove (4.3) and (4.4) only for one of the cases $S=A,B$. For brevity, we also denote

$$Q_{AB}^{(t)}=p(\mathcal{M}^{(t)}|\boldsymbol{\rho}_{AB})\,,\qquad p_A^{(t)}=p(\mathcal{F}_A^{(t)}|\boldsymbol{\rho}_A)\,,\qquad q_B^{(t)}=p(\mathcal{F}_B^{(t)}|\boldsymbol{\rho}_B)\,, \tag{4.5}$$

where $t=1,2$, reduced densities $\boldsymbol{\rho}_A=\mathrm{Tr}_B(\boldsymbol{\rho}_{AB})$ and $\boldsymbol{\rho}_B=\mathrm{Tr}_A(\boldsymbol{\rho}_{AB})$.

We first suppose that $\boldsymbol{\rho}_{AB}$ is a product state appeared as $\boldsymbol{\rho}_A\otimes\boldsymbol{\rho}_B$. Combining $Q_{AB}^{(t)}=p_A^{(t)}*q_B^{(t)}$ with (2.11) and (2.12), for $\alpha>1>\beta$ we have the inequality

$$\big\|Q_{AB}^{(1)}\big\|_\alpha\leq\|p_A^{(1)}\|_\alpha\leq\eta_A^{2(1-\beta)/\beta}\,\|p_A^{(2)}\|_\beta\leq\eta_A^{2(1-\beta)/\beta}\,\big\|Q_{AB}^{(2)}\big\|_\beta \tag{4.6}$$

and its "twin" with swapped $Q_{AB}^{(1)}$ and $Q_{AB}^{(2)}$. In general, we cannot assume concavity of the Rényi α-entropy for $\alpha>1$. Hence, we should extend our results to separable states before obtaining final entropic inequalities.

Each separable state can be represented as a convex combination of product states,

$$\boldsymbol{\rho}_{AB}=\sum\nolimits_\lambda\lambda\,\boldsymbol{\rho}_{A\lambda}\otimes\boldsymbol{\rho}_{B\lambda}\,.$$

Here, density matrices are all normalized so that $\sum_\lambda\lambda=1$. For the above combination of product states, we obtain

$$Q_{AB}^{(t)}=\sum\nolimits_\lambda\lambda\,Q_{AB\lambda}^{(t)}\,, \tag{4.7}$$

where each $Q^{(t)}_{AB\lambda}$ corresponds to the product $\boldsymbol{\rho}_{A\lambda}\otimes\boldsymbol{\rho}_{B\lambda}$. Following [70, 71], at this step we use the Minkowski inequality. Assuming $\alpha>1>\beta>0$, this inequality gives

$$\|Q^{(1)}_{AB}\|_{\alpha}=\Big\|\sum\nolimits_{\lambda}\lambda\,Q^{(1)}_{AB\lambda}\Big\|_{\alpha}\leq\sum\nolimits_{\lambda}\lambda\,\|Q^{(1)}_{AB\lambda}\|_{\alpha}\,, \tag{4.8}$$

$$\sum\nolimits_{\lambda}\lambda\,\|Q^{(2)}_{AB\lambda}\|_{\beta}\leq\Big\|\sum\nolimits_{\lambda}\lambda\,Q^{(2)}_{AB\lambda}\Big\|_{\beta}=\|Q^{(2)}_{AB}\|_{\beta}\,. \tag{4.9}$$

For each λ, the quantities $\|Q^{(1)}_{AB\lambda}\|_{\alpha}$ and $\|Q^{(2)}_{AB\lambda}\|_{\beta}$ obey (4.6). By (4.8) and (4.9), the relation (4.6) and its "twin" written in terms of $Q^{(t)}_{AB}$ are also valid for all separable state.

To complete the proof, we shall convert (4.6) into entropic inequalities. The Rényi entropies are represented via norm-like functionals according to (2.8). To get (4.3) with $\alpha>1>\beta$, we take the logarithm of both the sides of (4.6) and use the link $(\alpha-1)/\alpha=(1-\beta)/\beta$. The inequality with swapped probability distributions is then obtained by a parallel argument. Together, these inequality are joined into (4.3) under the condition $1/\alpha+1/\beta=2$ solely. The case of Tsallis entropies is not so immediate. Following [55], we can examine a minimization problem under the restrictions imposed by (4.6) and its "twin". Calculations resulting in (4.4) are very similar to the derivation given in appendix of [55]. ■

The statement of Proposition 4 provides entropic separability conditions based on local uncertainty relations of the Maassen–Uffink type. These formulas hold under the restriction $1/\alpha+1/\beta=2$. The latter reflects the fact that the Maassen–Uffink result is derived from Riesz's theorem [54]. An alternative viewpoint is that such uncertainty relations follow from the monotonicity of the quantum relative entropy [72]. The used scheme of building total measurements also leads to separability conditions on the base of majorization uncertainty relations. Our second result is posed as follows.

Proposition 5 *Let each of sets $\mathcal{F}^{(1)}_{A}$ and $\mathcal{F}^{(2)}_{A}$ of subnormalized vectors form rank-one POVM in $\mathcal{H}_A$, and let each of sets $\mathcal{F}^{(1)}_{B}$ and $\mathcal{F}^{(2)}_{B}$ of subnormalized vectors form rank-one POVM in $\mathcal{H}_B$. Let two POVMs $\mathcal{M}^{(t)}(\mathcal{F}^{(t)}_{A},\mathcal{F}^{(t)}_{B})$ in $\mathcal{H}_A\otimes\mathcal{H}_B$ be constructed from these sets according to Definition 1. For $S=A,B$, we introduce $D\times D$ matrix $\mathsf{V}_S=\mathsf{V}(\mathcal{F}^{(1)}_{S},\mathcal{F}^{(2)}_{S})$ due to (3.20). To each of such two matrices, we assign the sequence of numbers according to (3.7) and the majorizing vector w_S, where $S=A,B$. For each separable state $\boldsymbol{\rho}_{AB}$ and $0<\alpha\leq1$, there holds*

$$R_{\alpha}(\mathcal{M}^{(1)}|\boldsymbol{\rho}_{AB})+R_{\alpha}(\mathcal{M}^{(2)}|\boldsymbol{\rho}_{AB})\geq R_{\alpha}(w_S)\,. \tag{4.10}$$

For each separable state $\boldsymbol{\rho}_{AB}$ and $\alpha>0$, there holds

$$H_{\alpha}(\mathcal{M}^{(1)}|\boldsymbol{\rho}_{AB})+H_{\alpha}(\mathcal{M}^{(2)}|\boldsymbol{\rho}_{AB})\geq H_{\alpha}(w_S)\,. \tag{4.11}$$

Proof. In view of symmetry between subsystems, we will prove (4.10) and (4.11) only for one of the cases $S=A,B$. Combining (2.13) with (4.2) and using the notation (4.5) again, for each product state we write

$$R_{\alpha}\big(Q^{(t)}_{AB}\big)\geq R_{\alpha}\big(p^{(t)}_{A}\big)\,,$$
$$H_{\alpha}\big(Q^{(t)}_{AB}\big)\geq H_{\alpha}\big(p^{(t)}_{A}\big)\,.$$

It is essential here that both the α-entropies are Schur-concave. By the majorization-based relation (3.10) and (3.15), for a product state we have

$$R_{\alpha}\big(Q^{(1)}_{AB}\big)+R_{\alpha}\big(Q^{(2)}_{AB}\big)\geq R_{\alpha}\big(p^{(1)}_{A}\big)+R_{\alpha}\big(p^{(2)}_{A}\big)\geq R_{\alpha}(w_A)\qquad(0<\alpha\leq1)\,, \tag{4.12}$$

$$H_{\alpha}\big(Q^{(1)}_{AB}\big)+H_{\alpha}\big(Q^{(2)}_{AB}\big)\geq H_{\alpha}\big(p^{(1)}_{A}\big)+H_{\alpha}\big(p^{(2)}_{A}\big)\geq H_{\alpha}(w_A)\qquad(0<\alpha<\infty)\,. \tag{4.13}$$

As the Rényi α-entropy is concave for $0<\alpha\leq1$, the formula (4.12) implies (4.10) for all separable states. The claim (4.11) follows from (4.13) due to concavity of the Tsallis α-entropy. ■

In Proposition 5, we deal with separability conditions derived from majorization uncertainty relations. Unlike (4.3), the condition (4.10) is restricted to the range $0<\alpha\leq1$, where Rényi's entropy is concave irrespectively to dimensionality of probabilistic vectors. For $\alpha>1$, concavity properties actually depend on dimensionality of probabilistic vectors. So, the binary Rényi entropy is concave for all $0<\alpha\leq2$ [41]. Since the case of qubits is very important, we give two separability conditions additional to (4.10). Here, the majorization-based relations (3.11) and (3.14) will be used.

Assuming $D=2$, we consider pairs $\{\mathcal{E}_A^{(1)},\mathcal{E}_A^{(2)}\}$ and $\{\mathcal{E}_B^{(1)},\mathcal{E}_B^{(2)}\}$ of orthonormal bases, each in two dimensions. For product states of a two-qubit system and $\alpha>1$, one has

$$R_\alpha\bigl(Q_{AB}^{(1)}\bigr)+R_\alpha\bigl(Q_{AB}^{(2)}\bigr)\geq\frac{2}{1-\alpha}\,\ln\!\left(\frac{1+\|w_S\|_\alpha^\alpha}{2}\right), \tag{4.14}$$

$$R_\alpha\bigl(Q_{AB}^{(1)}\bigr)+R_\alpha\bigl(Q_{AB}^{(2)}\bigr)\geq R_\alpha(w_S^{\prime})\,. \tag{4.15}$$

The vectors ω_S and $\omega_S^{\prime}$ are obtained for unitary 2×2 matrix $\mathsf{V}_S=\mathsf{V}(\mathcal{E}_S^{(1)},\mathcal{E}_S^{(2)})$ with $S=A,B$ in line with (3.9) and (3.13). We cannot extend (4.14) and (4.15) to separable states without entropic concavity. When $d=2$ and $1<\alpha\leq2$, for each separable state $\boldsymbol{\rho}_{AB}$ we finally get

$$R_\alpha(\mathcal{M}^{(1)}|\boldsymbol{\rho}_{AB})+R_\alpha(\mathcal{M}^{(2)}|\boldsymbol{\rho}_{AB})\geq\frac{2}{1-\alpha}\,\ln\!\left(\frac{1+\|w_S\|_\alpha^\alpha}{2}\right), \tag{4.16}$$

$$R_\alpha(\mathcal{M}^{(1)}|\boldsymbol{\rho}_{AB})+R_\alpha(\mathcal{M}^{(2)}|\boldsymbol{\rho}_{AB})\geq R_\alpha(w_S^{\prime})\,. \tag{4.17}$$

For a two-qubit system, majorization-based separability conditions in terms of Rényi's entropies are given in the range $0<\alpha\leq2$.

We shall now proceed to separability conditions related to local measurements with a special structure. In the case of MUBs, the following statement takes place.

Proposition 6 *Let $\{\mathcal{E}_A^{(1)},\ldots,\mathcal{E}_A^{(K)}\}$ be a set of K MUBs in $\mathcal{H}_A$. Let $\{\mathcal{E}_B^{(1)},\ldots,\mathcal{E}_B^{(K)}\}$ be a set of K MUBs in $\mathcal{H}_B$. Let K POVMs $\mathcal{M}^{(t)}(\mathcal{E}_A^{(t)},\mathcal{E}_B^{(t)})$ in $\mathcal{H}_A\otimes\mathcal{H}_B$ be constructed from these MUBs according to Definition 1. For each separable state $\boldsymbol{\rho}_{AB}$ and $\alpha\in(0;2]$, there holds*

$$\frac{1}{K}\sum_{t=1}^{K}H_\alpha(\mathcal{M}^{(t)}|\boldsymbol{\rho}_{AB})\geq\ln_\alpha\!\left(\frac{Kd}{d+K-1}\right), \tag{4.18}$$

where $d=\dim(\mathcal{H}_A)=\dim(\mathcal{H}_B)$.

Proof. Again, we will prove (4.18) only for one of the cases $S=A,B$. The Tsallis α-entropy is Schur-concave for all $\alpha\in(0;2]$. Combining this fact with (2.13) and (3.22), for any product we have

$$\frac{1}{K}\sum_{t=1}^{K}H_\alpha(\mathcal{M}^{(t)}|\boldsymbol{\rho}_A\otimes\boldsymbol{\rho}_B)\geq\frac{1}{K}\sum_{t=1}^{K}H_\alpha(\mathcal{E}_A^{(t)}|\boldsymbol{\rho}_A)\geq\ln_\alpha\!\left(\frac{Kd}{d+K-1}\right). \tag{4.19}$$

By concavity of Tsallis' α-entropy, we extend the latter to all separable states. ∎

In the particular case $\alpha=1$, we have

$$\frac{1}{K}\sum_{t=1}^{K}H_1(\mathcal{M}^{(t)}|\boldsymbol{\rho}_{AB})\geq\ln\!\left(\frac{Kd}{d+K-1}\right),$$

whenever $\boldsymbol{\rho}_{AB}$ is separable. This separability condition is actually those that can be derived from the Rényi-entropy bound (3.21). When d is not specified, we can use concavity of Rényi's α-entropy only for $0<\alpha\leq1$. In addition, it does not increase with growth of α. For $d=2$, however, the Rényi α-entropy is concave up to $\alpha=2$. Thus, for a two-qubit system we write the condition

$$\frac{1}{K}\sum_{t=1}^{K}R_2(\mathcal{M}^{(t)}|\boldsymbol{\rho}_{AB})\geq\ln\!\left(\frac{2K}{K+1}\right), \tag{4.20}$$

where $K=2,3$ and $\boldsymbol{\rho}_{AB}$ is separable. The result remains formally valid for $K=1$, but the bound becomes trivial here.

Detecting entanglement, we should use as many complementary measurement as possible. When the dimensionality d of subsystems is a prime power, $d+1$ mutually unbiased bases exist. For other values of d, we may apply mutually unbiased measurements, since a complete set of $d+1$ MUMs exists in all dimensions [52]. Hence, entanglement detection with MUMs is of interest. The following claim is derived from (3.26) similarly to the proof of Proposition 6.

Proposition 7 *Let* $\{\mathcal{N}_A^{(1)},\ldots,\mathcal{N}_A^{(K)}\}$ *be a set of* K *MUMs of the efficiency* $\varkappa_A$ *in* $\mathcal{H}_A$*, and let* $\{\mathcal{N}_B^{(1)},\ldots,\mathcal{N}_B^{(K)}\}$ *be a set of* K *MUMs of the efficiency* $\varkappa_B$ *in* $\mathcal{H}_B$*. Let* K *POVMs* $\mathcal{M}^{(t)}(\mathcal{N}_A^{(t)},\mathcal{N}_B^{(t)})$ *in* $\mathcal{H}_A\otimes\mathcal{H}_B$ *be constructed from these MUMs according to Definition 1. For each separable state* $\boldsymbol{\rho}_{AB}$ *and* $\alpha\in(0;2]$*, there holds*

$$\frac{1}{K}\sum_{t=1}^{K} H_\alpha(\mathcal{M}^{(t)}|\boldsymbol{\rho}_{AB})\geq\ln_\alpha\!\left(\frac{Kd}{\varkappa_S d+K-1}\right), \tag{4.21}$$

where $S=A,B$ *and* $d=\dim(\mathcal{H}_A)=\dim(\mathcal{H}_B)$*.*

Entropic uncertainty relations for symmetric informationally complete measurements also lead to separability conditions. Note that such separability conditions are formulated for a single measurement. We give formulations for a usual SIC-POVM and then for a general one.

Proposition 8 *Let* $\mathcal{F}_A$ *and* $\mathcal{F}_B$ *be two sets of subnormalized vectors that form SIC-POVMs in* $\mathcal{H}_A$ *and* $\mathcal{H}_B$*, respectively. Let POVM* $\mathcal{M}(\mathcal{F}_A,\mathcal{F}_B)$ *in* $\mathcal{H}_A\otimes\mathcal{H}_B$ *be constructed according to Definition 1. For each separable state* $\boldsymbol{\rho}_{AB}$ *and* $\alpha\in(0;2]$*, there holds*

$$H_\alpha(\mathcal{M}|\boldsymbol{\rho}_{AB})\geq\ln_\alpha\!\left(\frac{d(d+1)}{2}\right), \tag{4.22}$$

where $d=\dim(\mathcal{H}_A)=\dim(\mathcal{H}_B)$*.*

Proof. The Tsallis α-entropy is Schur-concave for all $\alpha\in(0;2]$. Combining this fact with (2.13) and (3.29), for any product we have

$$H_\alpha(\mathcal{M}|\boldsymbol{\rho}_A\otimes\boldsymbol{\rho}_B)\geq H_\alpha(\mathcal{F}_A|\boldsymbol{\rho}_A)\geq\ln_\alpha\!\left(\frac{d(d+1)}{2}\right). \tag{4.23}$$

By concavity of Tsallis' α-entropy, the latter is extended to all separable states. ■

General SIC-POVM exist in all finite dimensions [53]. Moreover, they can be obtained within a unifying framework. Even if usual SIC-POVMs exist in all dimensions, they may be difficult to implement. Thus, entanglement detection with general SIC-POVM may be more appropriate. The following claim is derived from (3.32) similarly to the proof of Proposition 8.

Proposition 9 *Let* $\mathcal{N}_A$ *and* $\mathcal{N}_B$ *be two general SIC-POVMs in* $\mathcal{H}_A$ *and* $\mathcal{H}_B$*, respectively. Let POVM* $\mathcal{M}(\mathcal{N}_A,\mathcal{N}_B)$ *in* $\mathcal{H}_A\otimes\mathcal{H}_B$ *be constructed according to Definition 1. For each separable state* $\boldsymbol{\rho}_{AB}$ *and* $\alpha\in(0;2]$*, there holds*

$$H_\alpha(\mathcal{M}|\boldsymbol{\rho}_{AB})\geq\ln_\alpha\!\left(\frac{d(d+1)}{a_S d^2+1}\right), \tag{4.24}$$

where $S=A,B$ *and* $d=\dim(\mathcal{H}_A)=\dim(\mathcal{H}_B)$*.*

We have derived a lot of separability conditions in terms of the Rényi and Tsallis entropies. The violation of any separability condition certifies that the measured state of a bipartite system is entangled. We also note that the presented conditions actually concern biseparability. The problem of detecting multipartite entanglement is generally more complicated [73, 74]. This issue was addressed in [75–78].

V. DISCUSSION

In this section, we will apply the presented conditions to some states, for which separability limits are already known. We also compare the two types of derived criteria with each other and also with previous criteria given in the literature. Bipartite separability conditions are often tested with density matrices of the form

$$(1-c)\boldsymbol{\varrho}_{sep}+c\,|\Phi\rangle\langle\Phi|\,.$$

Here, the density matrix $\boldsymbol{\varrho}_{sep}$ is separable, $|\Phi\rangle$ is a maximally entangled state, and real constant $c\in[0;1]$. Taking $\boldsymbol{\varrho}_{sep}$ to be the completely mixed state, the above form is a bipartite case of Werner states [64]. A bipartite Werner state is separable if and only if [79]

$$c\leq\frac{1}{d+1}\ , \tag{5.1}$$

where d is the dimensionality of each of two subsystems. The authors of [79] also presented necessary and sufficient conditions for multipartite Werner states.

We begin with a two-qubit system. The corresponding Pauli matrices are denoted as $\boldsymbol{\sigma}_x$, $\boldsymbol{\sigma}_y$, $\boldsymbol{\sigma}_z$. For a bipartite system of two qubits, the inequality (5.1) gives $c \leq 1/3$. The entangled pure state $|\Phi\rangle$ will be taken as

$$|\Phi\rangle = \frac{1}{\sqrt{2}} \big(|z_0 z_0\rangle + |z_1 z_1\rangle \big) \,, \tag{5.2}$$

where $\big\{|z_0\rangle, |z_1\rangle\big\}$ is the eigenbasis of $\boldsymbol{\sigma}_z$. We further consider a family of density matrices

$$\boldsymbol{\varrho}_\Phi = \frac{1-c}{4}\, \mathbb{1}_2 \otimes \mathbb{1}_2 + c\,|\Phi\rangle\langle\Phi| \,. \tag{5.3}$$

Let us take the basis $\big\{|z_0\rangle, |z_1\rangle\big\}$ and a rotated basis $\big\{|u_0\rangle, |u_1\rangle\big\}$ such that

$$|u_0\rangle := \cos\theta\,|z_0\rangle + \sin\theta\,|z_1\rangle \,, \qquad |u_1\rangle := \sin\theta\,|z_0\rangle - \cos\theta\,|z_1\rangle \,, \tag{5.4}$$

where $\theta \neq 0$. So, we wish to deal not only with two mutually unbiased bases. This point is essential for comparing majorization-based separability conditions with separability conditions of the Maassen–Uffink type. For each of two qubits, we actually take the observables $\boldsymbol{\sigma}_z$ and

$$|u_0\rangle\langle u_0| - |u_1\rangle\langle u_1| \,. \tag{5.5}$$

For $\theta = \pi/4$, the second basis $\big\{|u_0\rangle, |u_1\rangle\big\}$ gives the eigenbasis of $\boldsymbol{\sigma}_x$, whence (5.5) reads as it. The total measurements $\mathcal{M}^{(z)}$ and $\mathcal{M}^{(u)}$ respectively contain the projectors

$$\Lambda_0^{(z)} = |z_0 z_0\rangle\langle z_0 z_0| + |z_1 z_1\rangle\langle z_1 z_1| \,, \qquad \Lambda_1^{(z)} = |z_0 z_1\rangle\langle z_0 z_1| + |z_1 z_0\rangle\langle z_1 z_0| \,, \tag{5.6}$$

$$\Lambda_0^{(u)} = |u_0 u_0\rangle\langle u_0 u_0| + |u_1 u_1\rangle\langle u_1 u_1| \,, \qquad \Lambda_1^{(u)} = |u_0 u_1\rangle\langle u_0 u_1| + |u_1 u_0\rangle\langle u_1 u_0| \,. \tag{5.7}$$

For θ in the quadrant I, we have $\eta_S = \max\{\cos\theta, \sin\theta\} \equiv \eta$ and

$$w = \big(\eta, 1-\eta\big) \,, \tag{5.8}$$

$$w' = \frac{1}{4} \big(1 + 2\eta + \eta^2, 3 - 2\eta - \eta^2 \big) \,. \tag{5.9}$$

Combining these observations with (4.3), (4.4), (4.10), (4.11), (4.16), and (4.17), we obtain a lot of separability conditions for the case considered.

When $\theta = \pi/4$, we have $\eta = 1/\sqrt{2}$ and two MUBs, namely the eigenbases of $\boldsymbol{\sigma}_z$ and $\boldsymbol{\sigma}_x$. By calculations, we obtain

$$\langle\Phi|\Lambda_i^{(z)}|\Phi\rangle = \delta_{i0} \,, \qquad \langle\Phi|\Lambda_j^{(x)}|\Phi\rangle = \delta_{j0} \,,$$

where δ_{ij} is the Kronecker symbol. On the completely mixed state, each of two measurements generates the uniform distribution. For the state (5.3), we twice obtain the pair of probabilities $(1 \pm c)/2$. By inspection, the best detection among relations of the Maassen–Uffink type is provided by (4.3) for the choice $\alpha = \infty$ and $\beta = 1/2$. Here, we have the condition

$$-\ln\!\left(\frac{1+c}{2}\right) + \ln\!\Big(1 + \sqrt{1-c^2}\Big) \geq \ln 2 \,,$$

which is equivalent to $c \leq 1/\sqrt{2}$. So, the entropic separability conditions of the form (4.3) detect entanglement when $c > 1/\sqrt{2} \approx 0.7071$. The same range takes place for the criterion that considers the sum of maximal probabilities in two measurements. It is rather natural since that criterion is also based on the Maassen–Uffink approach [9]. The result quoted follows from a general formulation by substituting $d = 2$. Performing a direct optimization in the qubit case allows us to improve restrictions [9]. On the other hand, this approach becomes hardly appropriate with growth of the dimensionality.

Let us proceed to majorization-based separability conditions. The condition (4.16) gives

$$-\ln\!\left(\frac{1+c^2}{2}\right) \geq -\ln\!\left(\frac{1+\|w\|_2^2}{2}\right) , \tag{5.10}$$

where $\|w\|_2^2 = 2-\sqrt{2}$. With (5.10), we are able to detect entanglement for $c>\|w\|_2\approx 0.7654$. Calculating $\|w'\|_2$ for $\eta=1/\sqrt{2}$, the condition (4.17) reads

$$-\ln\left(\frac{1+c^2}{2}\right)\geq -\ln\|w'\|_2\,. \tag{5.11}$$

For $c>\sqrt{2\,\|w'\|_2-1}\approx 0.7450$, we can detect entanglement due to (5.11). For both the majorization-based conditions, the range of detection is slightly less than for separability conditions of the Maassen–Uffink type.

It is instructive to address bases that are not mutually unbiased. With $\theta=\pi/6$, the best result among conditions of the Maassen–Uffink type is reached by (4.3) for the choice $\alpha=\beta=1$. Separability conditions of the form (4.3) detect entanglement for $c>c_1$ with $c_1\approx 0.9347$. The best result among majorization-based conditions is provided due to (5.11). With the latter, we are able to detect entanglement for $c>c_2$ with $c_2\approx 0.8719$. Using the majorization-based separability conditions, we see more effective detection. For other θ, results were found to be similar. Separability conditions of the Maassen–Uffink type are rather preferable for MUBs. Of course, such bases are used in many schemes for entanglement detection *per se*. However, in quantum information science we may also perform measurements designed for other purposes. Statistics of such measurements may nevertheless be used additionally for entanglement detection. Here, we can apply separability conditions based on majorization uncertainty relations.

Let us compare two forms of separability conditions with using three MUBs. For a qubit, these MUBs are taken as the eigenbases of the Pauli observables. The first form deals with the so-called correlation measure introduced in [60]. Using (3.23), the authors of [60] obtained separability conditions in terms of the correlation measure. In two dimensions, the correlation measure is expressed as

$$J(\boldsymbol{\rho}_{AB})=\sum_{t=z,x,y}\sum_{i=0,1}\langle e_i^{(t)}e_i^{(t)}|\boldsymbol{\rho}_{AB}|e_i^{(t)}e_i^{(t)}\rangle\,, \tag{5.12}$$

where $|ee'\rangle\equiv|e\rangle\otimes|e'\rangle$. For separable states of a two-qubit system, the correlation measure satisfies $J\leq 2$. Simple calculations now give

$$J(\boldsymbol{\varrho}_{\Phi})=\frac{1+c}{2}+\frac{1+c}{2}+\frac{1-c}{2}=\frac{3+c}{2}\ . \tag{5.13}$$

For all $c\in[0;1]$, the right-hand side of (5.13) does not violate the separability condition $J\leq 2$. Density matrices of the form (5.3) are not separable for $c>1/3$ and all escape the entanglement detection with respect to this criterion. It is not the case for separability conditions such as (4.18) and (4.20).

For each of three measurements $\mathcal{M}^{(t)}=\bigl\{\Lambda_0^{(t)},\Lambda_1^{(t)}\bigr\}$, where $t=z,x,y$, the projectors are written according to (5.6). By direct calculations, we have

$$\langle\Phi|\Lambda_0^{(z)}|\Phi\rangle=1\,,\qquad \langle\Phi|\Lambda_0^{(x)}|\Phi\rangle=1\,,\qquad \langle\Phi|\Lambda_1^{(y)}|\Phi\rangle=1\,. \tag{5.14}$$

On the completely mixed state, each of measurements generates the uniform distribution with two outcomes. For the state (5.3), we obtain probabilities $(1\pm c)/2$ in all three cases. Substituting $d=2$, $K=3$ and $\alpha=2$, both the entropic bounds (4.18) and (4.20) lead to the condition

$$\left(\frac{1+c}{2}\right)^2+\left(\frac{1-c}{2}\right)^2=\frac{1+c^2}{2}\leq\frac{2}{3}\ ,$$

or merely $c\leq 1/\sqrt{3}$. So, the entropic separability conditions (4.18) and (4.20) detect entanglement for $c>1/\sqrt{3}\approx 0.5774$. In the considered example, the entropic approach is more effective than the method based on the correlation measure. An efficiency of separability conditions is very sensitive to the choice of local measurement bases. For conditions in terms of maximal probabilities, this fact was already mentioned in [13]. Further, the range $c>1/\sqrt{3}$ is wider than the range $c>1/\sqrt{2}$, in which separability conditions of the form (4.3) are able to detect entanglement. Our abilities to detect entanglement should increase, when the number of involved bases grows and used separability conditions are chosen properly.

Using several MUBs, we have two possible types of separability conditions, one in terms of entropies and another in terms of correlation measures. It is also instructive to compare these types with entangled states of a two-qutrit system. Let us recall the generalized Pauli operators

$$\mathsf{Z}=\begin{pmatrix}1&0&0\\0&\omega_3&0\\0&0&\omega_3^*\end{pmatrix},\qquad \mathsf{X}=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},$$

where $\omega_3 = \exp(\mathtt{i}2\pi/3)$. Four MUBs in the qutrit Hilbert space can be described as the eigenbases of the operators Z, X, ZX, and ZX^2. We shall test separability conditions on the following family of states,

$$\boldsymbol{\varrho}_\Psi = \frac{1-c}{9}\, \mathbb{1}_3 \otimes \mathbb{1}_3 + c\, |\Psi\rangle\langle\Psi| \,, \tag{5.15}$$

$$|\Psi\rangle = \frac{1}{\sqrt{3}} \big(|z_0 x_0\rangle + |z_1 x_2\rangle + |z_2 x_1\rangle \big) \,. \tag{5.16}$$

By $\{|z_i\rangle\}$, $\{|x_i\rangle\}$, $\{|y_i\rangle\}$, with $i=0,1,2$, we respectively mean the eigenbases of Z, X, and ZX. Defining the correlation measure for three MUBs similarly to (5.12), one obtains $J(\boldsymbol{\varrho}_\Psi)=1$. Separability conditions in terms of correlation measures follow from (3.23). When $d=3$ and $K=3$, for all separable states we have the condition $J\leq 5/3$. The latter is fulfilled by $J(\boldsymbol{\varrho}_\Psi)$ independently of $c\in[0;1]$. Density matrices of the form (5.15) are not separable for $c>1/4$ and all escape the entanglement detection with respect to this criterion. Note that $|\Psi\rangle$ is an eigenstate of three operators, so that

$$(\mathsf{Z}\otimes\mathsf{X})|\Psi\rangle = |\Psi\rangle \,, \qquad (\mathsf{X}\otimes\mathsf{Z})|\Psi\rangle = |\Psi\rangle \,, \qquad (\mathsf{ZX}\otimes\mathsf{ZX})|\Psi\rangle = \omega_3\, |\Psi\rangle \,. \tag{5.17}$$

Hence, we may try to rotate unitarily local measurement bases. For instance, one can take simultaneously $\{|z_i\rangle\}$ on qutrit A and $\{|x_i\rangle\}$ on qutrit B, and so on. Calculating the measure

$$\sum_{i=0,1,2} \Big(\langle z_i x_i|\boldsymbol{\varrho}_\Psi|z_i x_i\rangle + \langle x_i z_i|\boldsymbol{\varrho}_\Psi|x_i z_i\rangle + \langle y_i y_i|\boldsymbol{\varrho}_\Psi|y_i y_i\rangle \Big) = 1 \,,$$

we still see no violation.

Let us consider three measurements designed according to Definition 1. To the first pair of bases, we assign the three projectors

$$\begin{aligned}
\mathsf{\Pi}_0^{(zx)} &= |z_0x_0\rangle\langle z_0x_0| + |z_1x_2\rangle\langle z_1x_2| + |z_2x_1\rangle\langle z_2x_1| \,, \\
\mathsf{\Pi}_1^{(zx)} &= |z_0x_1\rangle\langle z_0x_1| + |z_1x_0\rangle\langle z_1x_0| + |z_2x_2\rangle\langle z_2x_2| \,, \\
\mathsf{\Pi}_2^{(zx)} &= |z_0x_2\rangle\langle z_0x_2| + |z_1x_1\rangle\langle z_1x_1| + |z_2x_0\rangle\langle z_2x_0| \,,
\end{aligned}$$

which form $\mathcal{M}^{(zx)}$. In a similar manner, we write projectors of the measurements $\mathcal{M}^{(xz)} = \{\mathsf{\Pi}_k^{(xz)}\}$ and $\mathcal{M}^{(yy)} = \{\mathsf{\Pi}_k^{(yy)}\}$. It immediately follows from (5.17) that

$$\langle\Psi|\mathsf{\Pi}_0^{(zx)}|\Psi\rangle = 1 \,, \qquad \langle\Psi|\mathsf{\Pi}_0^{(xz)}|\Psi\rangle = 1 \,, \qquad \langle\Psi|\mathsf{\Pi}_1^{(yy)}|\Psi\rangle = 1 \,.$$

On the completely mixed state, each of three measurements generates the uniform distribution with three outcomes. For the state (5.15), we have three probability distributions, each with one entry $(1+2c)/3$ and two entries $(1-c)/3$. Substituting $d=3$, $K=3$ and $\alpha=2$, the entropic bound (4.18) gives the condition

$$\left(\frac{1+2c}{3}\right)^2 + 2\left(\frac{1-c}{3}\right)^2 = \frac{1+2c^2}{3} \leq \frac{5}{9} \,,$$

or $c\leq 1/\sqrt{3}$. So, the entropic separability condition (4.18) can detect entanglement of (5.15) when $c>1/\sqrt{3}\approx 0.5774$. We again observe cases, in which the derived separability conditions are more efficient than separability conditions in terms of correlation measures. Of course, both the types essentially depend on local unitary rotations and permutations of kets in bases. However, such operations will considerably increase costs of entanglement detection. In practice, when resources are fixed, we should therefore try to use as many separability conditions as possible.

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