# A Review on Quantum Search Algorithms 

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The use of superposition of states in quantum computation, known as quantum parallelism, has significant advantage in terms of speed over the classical computation. It can be understood from the early invented quantum algorithms such as Deutsch's algorithm, Deutsch-Jozsa algorithm and its variation as Bernstein-Vazirani algorithm, Simon algorithm, Shor's algorithms etc. Quantum parallelism also significantly speeds up the database search algorithm, which is important in computer science because it comes as a subroutine in many important algorithms. Quantum database search of Grover achieves the task of finding the target element in an unsorted database in a time quadratically faster than the classical computer. We review the Grover quantum search algorithms for a singe and multiple target elements in a database. The partial search algorithm of Grover and Radhakrishnan and its optimization by Korepin, called GRK algorithm are also discussed.

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## I. INTRODUCTION

Quantum computation has the advantage of speed [1, 2] over its classical counterpart which makes the quantum computation more favorable. Although building a full-fledged quantum computer [3] is still far from reality, some of the research works such as Shor's algorithm and Grover algorithm have attracted much attention in the theoretical side. In the experimental side some success with a small number of quantum bits have already been achieved.

Peter Shor showed [4, 5] that it is possible for a quantum algorithm to compute factorization in polynomial-time. L. K. Grover, on the other hand, showed [6] 8] that it is possible to search for a single target item in an unsorted database, i.e., the elements of the database are not arranged in any specific order, in a time which is quadratically faster than what a classical computer needs to complete the same task. Here time is measured in terms of the number of queries to the oracle one needs to complete a task. Grover algorithm needs $\mathcal{O}(\sqrt{N})$ queries to the oracle. Although Grover algorithm can not perform a task exponentially faster than classical computer still it is quite popular because of its wide rage of applications such as a subroutine of some large algorithms in computer science. It can be shown that the quantum algorithm of Grover is the fastest algorithm, i.e., optimal [9-11] to search in an unsorted database.

Instead of looking for the target element in the whole database at once it is sometimes natural to divide the database into several blocks and then look for the particular block which contains the target element. This is called quantum partial search algorithm, first studied by Grover and Radhakrishnan [12], which can be optimized [13 16] and further generalized to hierarchical quantum partial search algorithm [2, 17, 18].

The purpose of this article is to review the basic concepts of quantum search algorithms. In our daily life we encounter databases which contain many elements. The database may be arranged in a particular order, i.e. sorted or may not have any order at all, i.e. unsorted. For example, consider the telephone directory which has a large number of contact details of individuals. This example is particularly interesting because it serves both as a sorted and an unsorted database. When we look for the names, which are arranged in lexicographical order, then the the telephone directory is an example of sorted database. However when we look for a telephone number then the telephone directory becomes an example of an unsorted
database. The job of a quantum search algorithm is to find a specific element, usually called the target item or the solution from the vast number of elements in a database. Typically classical computer takes a time proportional to the size of the database. Quantum search algorithms, which are based on the principle of quantum mechanics, promise to significantly reduce the computation time for the same database search.

This review article is arranged in the following fashion. In section we provide an introduction to the quantum search algorithms. To understand how quantum mechanics can be exploited in our favor a set of historically important quantum algorithms are discussed in [II which distinguish between balanced and constant functions. In section [II] we give an elaborate account of the famous Grover search algorithm and in section IV we discuss the quantum partial search algorithm and its optimized version known as GRK algorithm [2]. Finally in section $\square$ we conclude.

## II. FIRST QUANTUM ALGORITHMS

Here we discuss how quantum mechanics and its principle of superposition can have profound impact on computations. Algorithms such as Deutsch's algorithm, Deutsch-Jozsa algorithm, Bernstein-Vazirani algorithm, Simon algorithm, Shor's algorithm are the first algorithms which made use of quantum superposition to perform a certain task sufficiently faster than classical computer [19]. Therefore, before we move to quantum search algorithms we in this section discuss some of these algorithms.

## A. Deutsch's algorithm

Consider Boolean functions $f$ which act on qubits as

$$
\begin{equation*}
f:\{0,1\} \rightarrow\{0,1\} \tag{1}
\end{equation*}
$$

The four functions in eq. (1) are the following $f(0)=0, f(0)=1, f(1)=0$ and $f(1)=1$. Alternatively we can say there are two constant functions $f(0)=f(1)=0, f(0)=f(1)=1$ and two balanced functions $f(0)=0 \neq f(1)=1, f(0)=1 \neq f(1)=0$. If we use a classical computer to know what the functions $f$ do then we have to run the classical computer twice. First we have to find $f(0)$ which could be either 0 or 1 and then $f(1)$ which could be again either 0 or 1 .

However in quantum computing each input corresponds to a quantum state vector. So there are two state vectors $|0\rangle$ and $|1\rangle$. Instead of feeding single basis state we can prepare a superposition of these two states to extract global information regarding the function $f$. There is a quantum black box, called oracle, which does a unitary transformation on the input vectors. The unitary operator $U_{f}$ corresponding to the function $f$ acts on a two-qubit state as the following

$$
\begin{equation*}
U_{f}|x\rangle \otimes|y\rangle \rightarrow|x\rangle \otimes|f(x) \oplus y\rangle, \tag{2}
\end{equation*}
$$

where $x, y \in\{0,1\}, \oplus$ is the addition modulo 2 and $\otimes$ is the tensor product. Note that if we use $|0\rangle$ or $|1\rangle$ as the input state then still in quantum computer we have to query the oracle twice. It can be easily understood from the fact that the qubit $|y\rangle$ flips if the input of the first qubit is mapped to $f(x)=1$. For $f(x)=0|y\rangle$ remains in the same state. Therefore the function is constant if for both inputs, which we have to provide twice, we see that $|y\rangle$ either flip or remains unchanged. For balanced function $|y\rangle$ will flip for one input and remains unchanged for other input.

To speedup the process we can instead prepare a superposition of basis inputs which is done using Hadamard transform $H$ to the qubits as

$$
\begin{equation*}
H|x\rangle=\sqrt{\frac{1}{2}} \sum_{y=0}^{1}(-1)^{x y}|y\rangle, \quad x \in\{0,1\} \tag{3}
\end{equation*}
$$

In $|0\rangle$ and $|1\rangle$ basis the matrix representation of the Hadamard transform is

$$
H=\sqrt{\frac{1}{2}}\left(\begin{array}{cc}
1 & 1  \tag{4}\\
1 & -1
\end{array}\right)
$$

The state of the two qubits after the Hadamard transform becomes

$$
\begin{equation*}
H|0\rangle=\sqrt{\frac{1}{2}}(|0\rangle+|1\rangle), \quad H|1\rangle=\sqrt{\frac{1}{2}}(|0\rangle-|1\rangle) . \tag{5}
\end{equation*}
$$

The unitary operator $U_{f}$ acts on the state $|x\rangle \otimes H|1\rangle$ as the eigenvalue equation

$$
\begin{align*}
U_{f}|x\rangle \otimes H|1\rangle & =|x\rangle \otimes \sqrt{\frac{1}{2}}(|0+f(x)\rangle-|1+f(x)\rangle) \\
& =(-1)^{f(x)}|x\rangle \otimes \sqrt{\frac{1}{2}}(|0\rangle-|1\rangle) \\
& =(-1)^{f(x)}|x\rangle \otimes H|1\rangle \tag{6}
\end{align*}
$$

Since the oracle state $H|1\rangle$ is fixed we can discard it from eq. (6) and simply write

$$
\begin{equation*}
U_{f}|x\rangle=(-1)^{f(x)}|x\rangle \tag{7}
\end{equation*}
$$

Here we remark that eq. (77) can be regarded as the reflection about a plane perpendicular to the target element. We have considered $x$ to be a single qubit here, however eq. (77) is also valid when $x$ is a $n$-qubit.

In Deutsch's algorithm Hadamard transform is applied on the state of two qubits $|0\rangle \otimes|1\rangle$

$$
\begin{equation*}
H|0\rangle \otimes H|1\rangle=\sqrt{\frac{1}{2}}(|0\rangle+|1\rangle) \otimes \sqrt{\frac{1}{2}}(|0\rangle-|1\rangle) . \tag{8}
\end{equation*}
$$

Then using eq. (77) the oracle's unitary transformation $U_{f}$ on $H|0\rangle \otimes H|1\rangle$ can be written as

$$
\begin{equation*}
U_{f} H|0\rangle \otimes H|1\rangle=\sqrt{\frac{1}{2}}\left((-1)^{f(0)}|0\rangle+(-1)^{f(1)}|1\rangle\right) \otimes \sqrt{\frac{1}{2}}(|0\rangle-|1\rangle) . \tag{9}
\end{equation*}
$$

Hadamard transform on the first qubit of eq. (9) gives

$$
\begin{align*}
H U_{f} H|0\rangle \otimes H|1\rangle & =\frac{1}{2}\left[\left((-1)^{f(0)}+(-1)^{f(1)}\right)|0\rangle+\left((-1)^{f(0)}-(-1)^{f(1)}\right)|1\rangle\right] \\
& \otimes \sqrt{\frac{1}{2}}(|0\rangle-|1\rangle) \tag{10}
\end{align*}
$$

Measurement on the first qubit in eq. (10) shows that when the function $f$ is constant, i.e. $f(0)=f(1)$, we obtain the outcome $|0\rangle$. On the other hand, when the function $f$ is balanced, i.e. $f(0) \neq f(1)$, we obtain the outcome $|1\rangle$.

The superposition of $|0\rangle$ and $|1\rangle$ does the job of finding whether the function $f$ is constant or balanced in just one query to the quantum oracle. This is called quantum parallelism.

## B. Deutsch-Jozsa algorithm

In Deutsch's algorithm we had a single qubit input to the quantum oracle, also known as quantum black box. However what happens if the input is a n-qubit, an element of a $N=2^{n}$ dimensional Hilbert space. Will the time to find out whether the function $f$ is constant or balanced increase? Here the function is said to be constant if $f(x)=0$ or $f(x)=1$ for all $N=2^{n}$ n-qubit inputs. The function $f$ is said to be balanced if $f(x)=0$ for exactly half of the input n-qubits and $f(x)=1$ for the other half of the inputs. With a classical computer we need a huge amount of time, i.e. $2^{n-1}+1$ numbers of queries in the worst case to find
out if the function is constant or balanced. However using Deutsch-Jozsa algorithm we can find the answer in just one oracle query.

Since we have now $n$-qubit state $|0\rangle^{n}$ we have to apply $n$ Hadamard transforms

$$
\begin{equation*}
H^{(n)}=H \otimes_{1} H \otimes_{2} \cdots \otimes_{n} H, \tag{11}
\end{equation*}
$$

where $\otimes_{i}$ is the $i$-th direct product. The action of $H^{(n)}$ on a general $n$-qubit state $|x\rangle$ is given by

$$
\begin{align*}
H^{(n)}|x\rangle & =\prod_{i=1}^{n} \sqrt{\frac{1}{2}} \sum_{y_{i}=0}^{1}(-1)^{x_{i} y_{i}}\left|y_{i}\right\rangle, x_{i} \in\{0,1\} \\
& =\sqrt{\frac{1}{2^{n}}} \sum_{y=0}^{2^{n}-1}(-1)^{x . y}|y\rangle \tag{12}
\end{align*}
$$

where $x . y=\oplus_{i=1}^{n} x_{i} . y_{i}$ is the scalar product modulo 2 .
In Deutsch-Jozsa algorithm Hadamard transform is applied on the state $|0\rangle^{n} \otimes|1\rangle$

$$
\begin{equation*}
H^{(n)}|0\rangle^{n} \otimes H|1\rangle=\left(\sqrt{\frac{1}{2^{n}}} \sum_{x=0}^{2^{n}-1}|x\rangle\right) \otimes \sqrt{\frac{1}{2}}(|0\rangle-|1\rangle) . \tag{13}
\end{equation*}
$$

The unitary transformation $U_{f}$ on $H^{(n)}|0\rangle \otimes H|1\rangle$ can be written as

$$
\begin{equation*}
U_{f} H^{(n)}|0\rangle^{n} \otimes H|1\rangle=\left(\sqrt{\frac{1}{2^{n}}} \sum_{x=0}^{2^{n}-1}(-1)^{f(x)}|x\rangle\right) \otimes \sqrt{\frac{1}{2}}(|0\rangle-|1\rangle) . \tag{14}
\end{equation*}
$$

Now applying the Hadamard transform on the $n$-qubit in eq. (14) we obtain

$$
\begin{equation*}
H U_{f} H^{(n)}|0\rangle^{n} \otimes H|1\rangle=\left(\frac{1}{2^{n}} \sum_{x=0}^{2^{n}-1} \sum_{y=0}^{2^{n}-1}(-1)^{f(x)}(-1)^{x . y}|y\rangle\right) \otimes \sqrt{\frac{1}{2}}(|0\rangle-|1\rangle) . \tag{15}
\end{equation*}
$$

If the function $f(x)$ is constant then

$$
\begin{equation*}
\frac{1}{2^{n}} \sum_{x=0}^{2^{n}-1}(-1)^{f(x)}(-1)^{x \cdot y}=(-1)^{f(x)} \frac{1}{2^{n}} \sum_{x=0}^{2^{n}-1}(-1)^{x \cdot y}=(-1)^{f(x)} \delta_{y, 0} \tag{16}
\end{equation*}
$$

Using eq. (16) in eq. (15) we obtain

$$
\begin{align*}
H U_{f} H^{(n)}|0\rangle^{n} \otimes H|1\rangle & =\sum_{y=0}^{2^{n}-1}(-1)^{f(x)} \delta_{y, 0}|y\rangle \otimes \sqrt{\frac{1}{2}}(|0\rangle-|1\rangle) \\
& =(-1)^{f(x)}|0\rangle^{n} \otimes \sqrt{\frac{1}{2}}(|0\rangle-|1\rangle) . \tag{17}
\end{align*}
$$

So for constant function we obtain $|0\rangle^{n}$ output state with unit probability

$$
\begin{equation*}
\left.\left.\right|^{n}\langle 0| H U_{f} H^{(n)}|0\rangle^{n}\right|^{2}=\left|(-1)^{f(x)}\right|^{2}=1 \tag{18}
\end{equation*}
$$

We have dropped the second qubit $H|1\rangle$ while evaluating the probability in eq. (18), because it remains fixed. On the other hand if the function $f(x)$ is balanced then $f(x)=0$ for half, i.e. $2^{n-1}$ values of $x$ and $f(x)=1$ for another half, i.e. $2^{n-1}$ values of $x$, which amounts to vanishing probability of obtaining $|0\rangle^{n}$

$$
\begin{equation*}
\left.\left.\right|^{n}\langle 0| H U_{f} H^{(n)}|0\rangle^{n}\right|^{2}=\left|\frac{1}{2^{n}} \sum_{x=0}^{2^{n}-1}(-1)^{f(x)}\right|^{2}=0 \tag{19}
\end{equation*}
$$

It is clear from the measurement in eq. (18) and eq. (19) that constant and balanced function can be distinguished by running the quantum black-box once.

## C. Bernstein-Vazirani algorithm

This algorithm is just a variation of the above discussed Deutsch-Jozsa algorithm, where instead of the $|0\rangle^{n}$ output we get a constant n-bit output $a$. The problem is the following: We have a function

$$
\begin{equation*}
f_{a}(x)=a . x \tag{20}
\end{equation*}
$$

where we have to find out the n-bit constant $a$ with the help of an algorithm. Replacing $f(x)=a \cdot x$ in eq. (15) we obtain

$$
\begin{equation*}
H U_{f} H^{(n)}|0\rangle^{n} \otimes H|1\rangle=\left(\frac{1}{2^{n}} \sum_{x=0}^{2^{n}-1} \sum_{y=0}^{2^{n}-1}(-1)^{a . x}(-1)^{x \cdot y}|y\rangle\right) \otimes \sqrt{\frac{1}{2}}(|0\rangle-|1\rangle) \tag{21}
\end{equation*}
$$

However we note that

$$
\begin{equation*}
\frac{1}{2^{n}} \sum_{x=0}^{2^{n}-1}(-1)^{a . x}(-1)^{x . y}=\delta_{y, a} \tag{22}
\end{equation*}
$$

Using eq. (221) in eq. (21) we obtain

$$
\begin{equation*}
H U_{f} H^{(n)}|0\rangle^{n} \otimes H|1\rangle=\sum_{y=0}^{2^{n}-1} \delta_{y, a}|y\rangle \otimes \sqrt{\frac{1}{2}}(|0\rangle-|1\rangle)=|a\rangle \otimes \sqrt{\frac{1}{2}}(|0\rangle-|1\rangle) . \tag{23}
\end{equation*}
$$

When we measure the first n-bit we obtain the value of $a$ in just a single quantum query. In classical computer we get just a single bit output each time. So classical computer would requires $n$ queries to find the value of $a$.

## III. FULL DATABASE SEARCH

Let us consider a set $\mathcal{D}=\left\{a_{0}, a_{1}, \cdots, a_{N-1}\right\}$ containing $N$ number of elements. Assume that one of the $N$ elements is a marked one, which we have to find out. One of the legitimate questions in computing is how fast one can find out the marked element or the solution. If the elements in the set are completely unsorted then the classical computer can find the marked element in $\mathcal{O}(N)$ queries/time. Grover investigated the same problem quantum mechanically and found that it is possible to devise a quantum algorithm, now known as the Grover algorithm, which can find the marked element in $\mathcal{O}(\sqrt{N})$ queries. This is a quadratic speed up in time over the classical algorithm. Bellow we discuss the famous Grover algorithm which has been extensively investigated in the literature.

## A. Grover algorithm

A database which we encounter in practice may have a single target item/element or sometimes it may have multiple target elements. Grover search can efficiently search both types of database, however the database with multiple target elements are faster to search than with single target element as can be understood from the following two sub-subsections.

## 1. Single target Grover algorithm

We associate the $N$ elements of the set $\mathcal{D}$ with the basis vectors of a $N$-dimensional Hilbert space $\mathcal{H}$ spanned by orthonormal basis vectors $\left\{\left|a_{i}\right\rangle \mid\left\langle a_{i} \mid a_{j}\right\rangle=\delta_{i j}, i=0,1, \cdots, N-1\right\}$. Now consider an initial unit vector $|\Theta\rangle$, which can be written in terms of the basis vectors as

$$
\begin{equation*}
|\Theta\rangle=\sum_{i=0}^{N-1} \cos \alpha_{i}\left|a_{i}\right\rangle \tag{24}
\end{equation*}
$$

where the direction cosines $\cos \left(\alpha_{i}\right) s$ satisfy $\sum_{i=0}^{N-1} \cos ^{2} \alpha_{i}=1$. To start with an equal probability for all the elements we assume the direction cosines to be same in all directions, i.e., $\alpha_{i}=\pi / 2-\theta$, which simplifies the initial unit vector (24) as

$$
\begin{equation*}
|\Theta\rangle=\sum_{i=0}^{N-1} \sin \theta\left|a_{i}\right\rangle=\sum_{i=0}^{N-1} \sqrt{\frac{1}{N}}\left|a_{i}\right\rangle \tag{25}
\end{equation*}
$$

One of the basis vectors let $\left|a_{T}\right\rangle$ be assigned to the target element, which has a probability

$$
\begin{equation*}
\mathcal{P}_{T}=\left|\left\langle a_{T} \mid \Theta\right\rangle\right|^{2}=\sin ^{2} \theta=\frac{1}{N} \tag{26}
\end{equation*}
$$

of obtaining it if measured in the state $|\Theta\rangle$. In order to increase the probability of getting the marked state $\left|a_{T}\right\rangle$ Grover exploited an unitary transformation $\mathcal{G}$, which we call Grover iteration:

$$
\begin{equation*}
\mathcal{G}=-\mathcal{I}_{\Theta} \mathcal{I}_{T} \tag{27}
\end{equation*}
$$

where the two reflection operators $\mathcal{I}_{T}$ and $\mathcal{I}_{\Theta}$ are given as

$$
\begin{align*}
& \mathcal{I}_{T}=\mathbb{I}-2\left|a_{T}\right\rangle\left\langle a_{T}\right|  \tag{28}\\
& \mathcal{I}_{\Theta}=\mathbb{I}-2|\Theta\rangle\langle\Theta| \tag{29}
\end{align*}
$$

To understand the action of both the reflection operators let us consider a general vector

$$
\begin{equation*}
|\psi\rangle=\sum_{i=0}^{N-1} c_{i}\left|a_{i}\right\rangle \tag{30}
\end{equation*}
$$

where $c_{i} \mathrm{~s}$ are the constant coefficients. $\mathcal{I}_{T}$ only reflects the $\left|a_{T}\right\rangle$ component and keeps the other components unchanged as can be seen from the expression

$$
\begin{equation*}
\mathcal{I}_{T}|\psi\rangle=-c_{T}\left|a_{T}\right\rangle+\sum_{i=0, i \neq T}^{N-1} c_{i}\left|a_{i}\right\rangle \tag{31}
\end{equation*}
$$

For the particular case of the state associated with the marked element $\left|a_{T}\right\rangle$ it simply becomes $\mathcal{I}_{T}\left|a_{T}\right\rangle=-\left|a_{T}\right\rangle$. On the other hand $-\mathcal{I}_{\Theta}$ inverts the coefficients $c_{i}$ of the vector $|\psi\rangle$ about the double average of their coefficients as

$$
\begin{equation*}
-\mathcal{I}_{\Theta}|\psi\rangle=\sum_{i=0}^{N-1}\left(2 \bar{c}-c_{i}\right)\left|a_{i}\right\rangle \tag{32}
\end{equation*}
$$

where $\bar{c}$ is the average of all the coefficients given by $\bar{c}=\frac{1}{N} \sum_{i=0}^{N-1} c_{i}$. One Grover iteration $\mathcal{G}$ acts on a general vector $|\psi\rangle$ as

$$
\begin{equation*}
\mathcal{G}|\psi\rangle=-\mathcal{I}_{\Theta} \mathcal{I}_{T}|\psi\rangle=\left(2 \tilde{c}+c_{T}\right)\left|a_{T}\right\rangle+\sum_{i=0, i \neq T}^{N-1}\left(2 \tilde{c}-c_{i}\right)\left|a_{i}\right\rangle \tag{33}
\end{equation*}
$$

where now the average being $\tilde{c}=\frac{1}{N}\left(-c_{T}+\sum_{i=0, i \neq T}^{N-1} c_{i}\right)$. For our purpose it is helpful to consider the action of the Grover iteration $\mathcal{G}$ on the initial state $|\Theta\rangle$ in eq. (25), which simply gives

$$
\begin{equation*}
\mathcal{G}|\Theta\rangle=\sin (2+1) \theta\left|a_{T}\right\rangle+\sum_{i=0, i \neq T}^{N-1} \cos (2+1) \theta \tan \theta\left|a_{i}\right\rangle \tag{34}
\end{equation*}
$$

Applying the same Grover iteration $j$ times on the initial state we obtain

$$
\begin{equation*}
\mathcal{G}^{j}|\Theta\rangle=\sin (2 j+1) \theta\left|a_{T}\right\rangle+\sum_{i=0, i \neq T}^{N-1} \cos (2 j+1) \theta \tan \theta\left|a_{i}\right\rangle \tag{35}
\end{equation*}
$$

Assuming that now the initial state is aligned with the target vector, i.e. $\mathcal{G}^{j}|\Theta\rangle=\left|a_{T}\right\rangle$ after $j$ successive applications of the Grover iteration we obtain the optimal number of quantum query to the oracle necessary for large database

$$
\begin{equation*}
j=\lim _{N \rightarrow \infty}\left(\frac{\pi}{4} \sqrt{N}-\frac{1}{2}\right)=\frac{\pi}{4} \sqrt{N} \tag{36}
\end{equation*}
$$

This is clearly a quadratic speed up over the classical algorithm to search for a marked element on a set of $N$ unsorted elements. Of course $j$ estimated under the above assumption may make it a non-integer in general. In that case, we have to take the integer closest to the number $\frac{\pi}{4} \sqrt{N}$.

To easily understand the action of $\mathcal{G}^{j}$ on the initial state vector $|\Theta\rangle$ let us consider the eigenvalue problem

$$
\begin{equation*}
\mathcal{G}^{j}|\phi\rangle=E^{j}|\phi\rangle . \tag{37}
\end{equation*}
$$

On the plane defined by the vectors $\left|a_{T}\right\rangle$ and $|\Theta\rangle$ eq. (37) has the following two eigenvectors

$$
\begin{equation*}
|\phi\rangle_{ \pm}=\frac{1}{\sqrt{2}}\left|a_{T}\right\rangle \pm \frac{i}{\sqrt{2}} \sum_{i=0, i \neq T}^{N-1} \tan \theta\left|a_{i}\right\rangle \tag{38}
\end{equation*}
$$

with their corresponding eigenvalues $E_{ \pm}^{j}=e^{ \pm i 2 \theta j}$. In terms of these eigenvectors the initial state vector can be expressed as

$$
\begin{equation*}
|\Theta\rangle=-\sqrt{2} i\left(e^{i \theta}|\phi\rangle_{+}-e^{-i \theta}|\phi\rangle_{-}\right) . \tag{39}
\end{equation*}
$$

Acting $\mathcal{G}^{j}$ on the expression of eq. (39) we immediately obtain

$$
\begin{equation*}
\mathcal{G}^{j}|\Theta\rangle=-\sqrt{2} i\left(e^{i(2 j+1) \theta}|\phi\rangle_{+}-e^{-i(2 j+1) \theta}|\phi\rangle_{-}\right), \tag{40}
\end{equation*}
$$

which once written in terms of the original basis $\left|a_{i}\right\rangle$ reduces to the expression of eq. (35).
a. Example with single target: Let us consider an example, where there are $N=4$ elements and one of the element is marked. We need to find out the marked element among the four elements. Naively we may think that classically we can find the marked element in one search, two searches, three searches or in the worst case in four searches. On average we need $\frac{1+2+3+4}{4}=2 \frac{1}{2}$ searches to find the target element. However, since we know there is a marked element it is not necessary to perform a forth search. Therefore, on average we only need to perform $\frac{1+2+3+3}{4}=2 \frac{1}{4}$ number of classical searches to find the target element. However quantum mechanically, using Grover algorithm, we can find the marked element in just a single query. In this case $\sin \theta=\sqrt{\frac{1}{N}}=\frac{1}{2}$. So, the angle between the initial state and the state perpendicular to the target state is $\theta=30^{\circ}$. One query to the black box will further rotate the initial state $2 \theta=60^{\circ}$ towards the target element. Now the total angle between the initial state and the sate perpendicular to the target state is $2 \theta+\theta=90^{\circ}$, which means the initial state is now completely aligned with the target state.

We can also exploit eq. (31) and eq. (32) to understand the the above example in a alternative manner. Note that $\mathcal{I}_{T}$ just inverts the sign of the amplitude of the target element and $\mathcal{I}_{\Theta}$ inverts the amplitudes of the basis vectors about the double average. For the database of $N=4$ elements each basis element in the initial state $|\Theta\rangle$ has an amplitude $c_{i}=\frac{1}{\sqrt{N}}=\frac{1}{2}$. After the action of $\mathcal{I}_{T}$ the amplitude of only the target element changes from $c_{T}=\frac{1}{2}$ to $-c_{T}=-\frac{1}{2}$. The average of the four amplitudes then reduces from $\bar{c}=\frac{1}{2}$ to $\tilde{c}=\frac{1}{4}$. Then $\mathcal{I}_{\Theta}$ inverts the amplitude about the double average, which can be seen from state in eq. (33). The amplitude of the target element after one Grover iteration is thus amplified to $2 \tilde{c}+c_{T}=1$ and the amplitudes of all the other basis elements vanish $2 \tilde{c}-c_{i}=0$.

## 2. Multiple targets Grover algorithm

In the above analysis there is just a single marked element in the set. We now consider the case when there are $M$ number of marked elements in the set $\mathcal{D}$ of $N$ number of elements. We discuss this algorithm with the help of a generalized method known as the amplitude amplification, which was studied by Brassard et al [3]. Let us first divide the Hilbert space $\mathcal{H}$ into two mutually orthogonal sub-spaces $\mathcal{H}_{T}$ and $\mathcal{H}_{n T}$. $\mathcal{H}_{T}$ is the target space of dimensions $M$, where the basis elements are associated with $M$ target elements and $\mathcal{H}_{n T}$ is the Hilbert space of non-target elements of dimensions $N-M$, where the basis vectors are associated
with all the $N-M$ non-target elements. An unit vector in the target space can be written in terms of the basis elements of the target space as

$$
\begin{equation*}
\left|A_{T}\right\rangle=\sum_{i=1}^{M} \tilde{a}_{i}\left|a_{i}\right\rangle, \quad \sum_{i=1}^{M}\left|\tilde{a}_{i}\right|^{2}=1 \tag{41}
\end{equation*}
$$

where we have rearranged the the basis vectors such that first $M$ basis vectors correspond to the target space and rest belongs to the non-target space. Similarly, an unit vector in the non-target space can be written as

$$
\begin{equation*}
\left|A_{n T}\right\rangle=\sum_{i=M+1}^{N} \bar{a}_{i}\left|a_{i}\right\rangle, \quad \sum_{i=M+1}^{N}\left|\bar{a}_{i}\right|^{2}=1 . \tag{42}
\end{equation*}
$$

We again start with the same initial vector (25) but in terms of the unit basis vectors (41) with $\tilde{a}_{i}=\sqrt{\frac{1}{M}}$ and (42) with $\bar{a}_{i}=\sqrt{\frac{1}{N-M}}$

$$
\begin{equation*}
|\tilde{\Theta}\rangle=\sqrt{\frac{M}{N}}\left|A_{T}\right\rangle+\sqrt{\frac{N-M}{N}}\left|A_{n T}\right\rangle \tag{43}
\end{equation*}
$$

The probability of obtaining a target element if measured in the initial state (43) would be equal to the probability obtaining the basis state (41) in the initial state (43) as

$$
\begin{equation*}
\tilde{\mathcal{P}}_{T}=\left|\left\langle A_{T} \mid \tilde{\Theta}\right\rangle\right|^{2}=\sin ^{2} \tilde{\theta}=\frac{M}{N} \tag{44}
\end{equation*}
$$

Here we remark that we chose specific coefficients in the basis vectors (41) and (42) so that the initial state becomes a state with same direction cosines in all directions. However we could have kept the coefficients arbitary.

The probability of getting the marked state $\left|A_{T}\right\rangle$ can be increased by the application Grover iteration $\tilde{\mathcal{G}}$, which is defines as

$$
\begin{equation*}
\tilde{\mathcal{G}}=-\mathcal{I}_{\tilde{\Theta}} \mathcal{I}_{A_{T}} \tag{45}
\end{equation*}
$$

where the two reflection operators $\mathcal{I}_{A_{T}}$ and $\mathcal{I}_{\tilde{\Theta}}$ are given as

$$
\begin{align*}
\mathcal{I}_{A_{T}} & =\mathbb{I}-2\left|A_{T}\right\rangle\left\langle A_{T}\right|  \tag{46}\\
\mathcal{I}_{\tilde{\Theta}} & =\mathbb{I}-2|\tilde{\Theta}\rangle\langle\tilde{\Theta}| \tag{47}
\end{align*}
$$

To understand the action of $\tilde{\mathcal{G}}^{j}$ on the initial state vector $|\tilde{\Theta}\rangle$ let us consider the eigenvalue problem

$$
\begin{equation*}
\tilde{\mathcal{G}}^{j}|\phi\rangle=\tilde{E}^{j}|\phi\rangle . \tag{48}
\end{equation*}
$$

In terms of the unit vectors (41) and (42) the eigenvalue equation (48) has the following two eigenvectors

$$
\begin{equation*}
|\tilde{\phi}\rangle_{ \pm}=\frac{1}{\sqrt{2}}\left|A_{T}\right\rangle \pm \frac{i}{\sqrt{2}}\left|A_{n T}\right\rangle \tag{49}
\end{equation*}
$$

with their corresponding eigenvalues $\tilde{E}_{ \pm}^{j}=e^{ \pm i 2 \tilde{\theta} j}$. In terms of these eigenvectors the initial state vector can be expressed as

$$
\begin{equation*}
|\tilde{\Theta}\rangle=-\sqrt{2} i\left(e^{i \tilde{\theta}}|\tilde{\phi}\rangle_{+}-e^{-i \tilde{\theta}}|\tilde{\phi}\rangle_{-}\right) . \tag{50}
\end{equation*}
$$

Acting $\tilde{\mathcal{G}}^{j}$ on the expression of eq. (50) we obtain

$$
\begin{equation*}
\tilde{\mathcal{G}}^{j}|\Theta\rangle=-\sqrt{2} i\left(e^{i(2 j+1) \tilde{\theta}}|\tilde{\phi}\rangle_{+}-e^{-i(2 j+1) \tilde{\theta}}|\tilde{\phi}\rangle_{-}\right) \tag{51}
\end{equation*}
$$

which can be rewritten in terms of the basis vectors $\left|A_{T}\right\rangle$ and $\left|A_{n T}\right\rangle$ as

$$
\begin{equation*}
\tilde{\mathcal{G}}^{j}|\Theta\rangle=\sin (2 j+1) \tilde{\theta}\left|A_{T}\right\rangle+\cos (2 j+1) \tilde{\theta}\left|A_{n T}\right\rangle . \tag{52}
\end{equation*}
$$

After $j$ successive application of the Grover iteration the initial state is aligned with the target unit vector, i.e. $\tilde{\mathcal{G}}^{j}|\tilde{\Theta}\rangle=\left|A_{T}\right\rangle$. For a large database of $N$ elements with $M$ target items the optimal number of quantum queries necessary to find a target item becomes

$$
\begin{equation*}
j=\lim _{N \rightarrow \infty}\left(\frac{\pi}{4} \sqrt{\frac{N}{M}}-\frac{1}{2}\right)=\frac{\pi}{4} \sqrt{\frac{N}{M}} \tag{53}
\end{equation*}
$$

a. Example with multiple targets: Let us consider an example which is similar to the example of four elements in a database discussed in IIA 1a, however this time there are multiple target elements instead of just one. For our purpose only the ratio of the number of elements $N$ in the database with the number of target elements $M$ matters. We consider the ratio to be $\frac{N}{M}=4$. The angle between the orthogonal to unit vector $\left|A_{T}\right\rangle$ in the target state and the initial state $|\Theta\rangle$ can be obtained from eq. (44) as $\tilde{\theta}=30^{\circ}$. One Grover iteration rotates the initial state $|\Theta\rangle$ towards the target state $\left|A_{T}\right\rangle$ by an amount $2 \tilde{\theta}=60^{\circ}$. After one Grover search the angle between the orthogonal to the target state and the initial state is $2 \tilde{\theta}+\tilde{\theta}=90^{\circ}$, which means the initial state is now completely aligned with the unit target state.

## 3. Generic unitary transformation for Grover search

In the discussion of Grover search algorithm in subsection III A 1 we have implicitly exploited the Walsh-Hadamard(WH) transformation $H^{(n)}$ as an unitary transformation. Note that the initial state in eq. (25), which is an equal weighted superposition of all basis states can be obtained from the state $|0\rangle^{n}$ by the application of WH transformation

$$
\begin{equation*}
\left|\Theta_{H^{(n)}}\right\rangle=|\Theta\rangle=H^{(n)}|0\rangle^{n}=\sqrt{\frac{1}{N}} \sum_{i=0}^{N-1}\left|a_{i}\right\rangle . \tag{54}
\end{equation*}
$$

Then the reflection operator $\mathcal{I}_{\Theta}$ in eq. (29) can be obtained as

$$
\begin{equation*}
\mathcal{I}_{\Theta}=H^{n}\left(\mathbb{I}-2|0\rangle^{n n}\langle 0|\right)\left(H^{n}\right)^{-1}=\mathbb{I}-2|\Theta\rangle\langle\Theta| . \tag{55}
\end{equation*}
$$

Instead of using $H^{(n)}$ we can also choose any generic unitary operator $U$ [20] which can act on the Hilbert space $\mathcal{H}$ of $N$ basis states describing $N=2^{n}$ elements of the Grover search. The initial state we now consider for our purpose is given by

$$
\begin{equation*}
\left|\Theta_{U}\right\rangle=U|0\rangle^{n} . \tag{56}
\end{equation*}
$$

Then the reflection operator corresponding to the state in eq. (56) can be written as

$$
\begin{equation*}
\mathcal{I}_{\Theta_{U}}=U\left(\mathbb{I}-2|0\rangle^{n n}\langle 0|\right) U^{-1}=\mathbb{I}-2\left|\Theta_{U}\right\rangle\left\langle\Theta_{U}\right| \tag{57}
\end{equation*}
$$

As usual $\left|a_{T}\right\rangle$ is the target element which we have to find out from the $N$ elements and $\mathcal{I}_{T}$ is the corresponding reflection operator. The amplitude of the target element $\left|a_{T}\right\rangle$ in the initial state $\left|\Theta_{U}\right\rangle$ is

$$
\begin{equation*}
\mathcal{A}_{T \Theta_{U}}=\sin \theta_{U}=\left\langle a_{T} \mid \Theta_{U}\right\rangle=\left\langle a_{T}\right| U|0\rangle^{n} \tag{58}
\end{equation*}
$$

When the probability of getting the target element in the initial state is low then eq. (58) can be approximated as

$$
\begin{equation*}
\mathcal{A}_{T \Theta_{U}}=\lim _{\theta_{U} \rightarrow 0} \sin \theta_{U}=\theta_{U} \tag{59}
\end{equation*}
$$

We can now construct the Grover iteration as

$$
\begin{equation*}
G_{U}=-\mathcal{I}_{\Theta_{U}} \mathcal{I}_{T} \tag{60}
\end{equation*}
$$

One Grover iteration moves the initial state by angle $2 \theta_{U}$ towards the target element. Assuming that after $j_{U}$ number of iterations the initial state will align with the target element then we obtain

$$
\begin{equation*}
j_{U}=\lim _{\mathcal{A}_{T \Theta_{U} \rightarrow 0}}\left(\frac{\pi}{4} \frac{1}{\mathcal{A}_{T \Theta_{U}}}-\frac{1}{2}\right)=\frac{\pi}{4} \frac{1}{\mathcal{A}_{T \Theta_{U}}} \tag{61}
\end{equation*}
$$

When the unitary operator $U=H^{(n)}$ the amplitude of the target element in the initial state becomes $\mathcal{A}_{T \Theta_{U}}=\sqrt{\frac{1}{N}}$, then eq. (61) reduces to the standard result in eq. (36).

Here we remark that when there is no apparent knowledge of the whereabouts of the target element in a database then the WH transformation is the most suitable unitary transformation because it produces an initial state which is an equal superposition of all the basis states. For many target elements the average amplitude of the target elements in the initial state is largest and the amplitude of the target elements are known.

However there can have some problems where we may have more knowledge about the target element/elements or there are some order/structure in the database. The generic unitary transformation then becomes important, because one can choose the unitary operator $U$ accordingly so as to get faster search. The Grover search is then a search of a structured database as opposed to the unstructured search discussed in sections III A 1 and III A 2.
a. Example of a structured Grover search: Here we consider an example of a structured Grover search which is discussed in refs. [21, 22]. Let us consider a function $F\left(a_{i}, b_{i}\right)$ which takes two $n$-bits $\left(a_{i}, b_{i}\right), i=1,2, \cdots, N$ as inputs and the output is zero for all $\left(a_{i}, b_{i}\right) \mathrm{s}$ except at $\left(a_{T}, b_{T}\right)$, where $F\left(a_{T}, b_{T}\right)=1$. This is an example of a database of $N^{2}$ elements and one of then $\left(a_{T}, b_{T}\right)$ is the target element. Classical computer needs $\mathcal{O}\left(N^{2}\right)$ time in the worst case to find the target element. However Grover algorithm needs $\mathcal{O}(N)$ oracle calls to find out the target element with close to one probability.

The number of oracle calls can further be reduced if we know there is some structure which can help to minimize the time of search. Let us assume that there is another function $G\left(a_{i}\right)$ which takes one $n$-bits $a_{i}, i=1,2, \cdots, N$ as input and the output is zero for all $a_{i} \mathrm{~S}$ except for $M \leq N a_{i}$, where $G\left(a_{i}\right)=1$ and $a_{T}$ also belongs to those $M a_{i}$, i.e. $G\left(a_{T}\right)=1$.

The case $M=N$ is not interesting because $G\left(a_{i}\right)=1$ for all the inputs and therefore does not reduce the search time for the target element $\left(a_{T}, b_{T}\right)$. For the case $M=1$ we may first use $G\left(a_{i}\right)$ to find $a_{T}$ in $\frac{\pi}{4} \sqrt{N}$ number of Grover iterations. Then we can use $F\left(a_{T}, b_{i}\right)$ to find $a_{T}, b_{T}$ in $\frac{\pi}{4} \sqrt{N}$ number of Grover iterations, in total $\frac{\pi}{2} \sqrt{N}$ iterations are needed.

Let us now consider the case $1<M<N$, and assume that $M$ is known. The result is also valid for $M=1$ and $M=N$ cases. Now the classical computer can find the target element in $\mathcal{O}(M N)$ repetitions. The quantum algorithm can find the target element in $\mathcal{O}(\sqrt{M N})$ oracle calls which is a quadratic speed up in time.

The function $F\left(a_{i}, b_{i}\right)$ acts on a tensor product space $\mathcal{H}_{12}=\mathcal{H}_{1} \otimes \mathcal{H}_{2}$ of dimensions $N^{2}$ and basis elements are $\left|a_{i}\right\rangle \otimes\left|b_{i}\right\rangle$, where $\left|a_{i}\right\rangle$ are the basis elements of $\mathcal{H}_{1}$ and $\left|b_{i}\right\rangle$ are the basis elements of $\mathcal{H}_{2}$. Both of the Hilbert spaces $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$ have dimensions $N$. The initial state we consider is given by

$$
\begin{equation*}
\left|\Theta_{12}\right\rangle=\left|\Theta_{1}\right\rangle \otimes\left|\Theta_{2}\right\rangle \tag{62}
\end{equation*}
$$

where the initial state on both the Hilbert spaces are given by

$$
\begin{align*}
\left|\Theta_{1}\right\rangle & =\left(\sqrt{\frac{1}{N}} \sum_{i=1}^{N}\left|a_{i}\right\rangle\right) \otimes \mathbb{I}  \tag{63}\\
\left|\Theta_{2}\right\rangle & =\mathbb{I} \otimes\left(\sqrt{\frac{1}{N}} \sum_{i=1}^{N}\left|b_{i}\right\rangle\right) \tag{64}
\end{align*}
$$

With all the basis states corresponding to $G\left(a_{i}\right)=1$ we prepare another state by equal superposition

$$
\begin{equation*}
\left|\Theta_{0}\right\rangle=\left(\sqrt{\frac{1}{M}} \sum_{G\left(a_{i}\right)=1}\left|a_{i}\right\rangle\right) \otimes \mathbb{I} \tag{65}
\end{equation*}
$$

We can now construct the reflection operators corresponding to $\left|\Theta_{1}\right\rangle,\left|\Theta_{2}\right\rangle$ and $|\Theta\rangle$ as

$$
\begin{align*}
& \mathcal{I}_{\Theta_{1}}=\left(\mathbb{I}-2\left|\Theta_{1}\right\rangle\left\langle\Theta_{1}\right|\right) \otimes \mathbb{I}  \tag{66}\\
& \mathcal{I}_{\Theta_{2}}=\mathbb{I} \otimes\left(\mathbb{I}-2\left|\Theta_{2}\right\rangle\left\langle\Theta_{2}\right|\right)  \tag{67}\\
& \mathcal{I}_{\Theta_{0}}=\left(\mathbb{I}-2\left|\Theta_{0}\right\rangle\left\langle\Theta_{0}\right|\right) \otimes \mathbb{I} \tag{68}
\end{align*}
$$

The other two reflection operators we need are

$$
\begin{align*}
& \mathcal{I}_{T_{1}}=\left(\mathbb{I}-2 \sum_{G\left(a_{i}\right)=1}\left|a_{i}\right\rangle\left\langle a_{i}\right|\right) \otimes \mathbb{I},  \tag{69}\\
& \mathcal{I}_{T_{12}}=\mathbb{I} \otimes \mathbb{I}-2\left|a_{T}\right\rangle\left\langle a_{T}\right| \otimes\left|b_{T}\right\rangle\left\langle b_{T}\right| . \tag{70}
\end{align*}
$$

Firstly, the Grover iteration

$$
\begin{equation*}
\mathcal{G}_{1}=-\mathcal{I}_{\Theta_{1}} \mathcal{I}_{T_{1}} \tag{71}
\end{equation*}
$$

is performed $j_{1}=\frac{\pi}{4} \sqrt{\frac{N}{M}}$ times on the initial state $\left|\Theta_{12}\right\rangle$, which only transforms the initial state vector $\left|\Theta_{1}\right\rangle$ to the state $\left|\Theta_{0}\right\rangle$

$$
\begin{equation*}
\mathcal{G}_{1}{ }^{j_{1}}\left|\Theta_{12}\right\rangle=\mathcal{G}_{1}^{j_{1}}\left|\Theta_{1}\right\rangle \otimes\left|\Theta_{2}\right\rangle \cong\left|\Theta_{0}\right\rangle \otimes\left|\Theta_{2}\right\rangle, \tag{72}
\end{equation*}
$$

We now define a reflection operator $\mathcal{I}_{T_{0}}$ as

$$
\begin{equation*}
\mathcal{I}_{T_{0}}=\mathcal{G}_{12}^{j_{12} \dagger} \mathcal{I}_{T_{12}} \mathcal{G}_{12}^{j_{12}}, \tag{73}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{G}_{12}=-\mathcal{I}_{\Theta_{2}} \mathcal{I}_{T_{12}} \tag{74}
\end{equation*}
$$

Note that after $j_{12}=\frac{\pi}{4} \sqrt{N}$ iterations by $\mathcal{G}_{12}$ we can obtain the target state in the following way

$$
\begin{align*}
\mathcal{G}_{12}{ }^{j_{12}}\left|a_{i}\right\rangle \otimes\left|\Theta_{2}\right\rangle & =\left|a_{i}\right\rangle \otimes\left|\Theta_{2}\right\rangle, \text { for } a_{i} \neq a_{K},  \tag{75}\\
& =\left|a_{T}\right\rangle \otimes\left|b_{T}\right\rangle, \text { for } a_{i}=a_{K} . \tag{76}
\end{align*}
$$

The reflection operator $\mathcal{I}_{T_{0}}$ defined in eq. (73) will act on the $M$ dimensional Hilbert space with basis elements $a_{i}$ for which $G\left(a_{i}\right)=1$. It reflects the target element $a_{T}$ about a plane perpendicular to $\left|a_{T}\right\rangle$. In particular its action is given by

$$
\begin{align*}
\mathcal{I}_{T_{0}}\left|a_{i}\right\rangle \otimes\left|\Theta_{2}\right\rangle & =\left|a_{i}\right\rangle \otimes\left|\Theta_{2}\right\rangle, \text { for } a_{i} \neq a_{K}  \tag{77}\\
& =-\left|a_{T}\right\rangle \otimes\left|\Theta_{2}\right\rangle, \text { for } a_{i}=a_{K} \tag{78}
\end{align*}
$$

We can now define a Grover iteration

$$
\begin{equation*}
\mathcal{G}_{0}=-\mathcal{I}_{\Theta_{0}} \mathcal{I}_{T_{0}}, \tag{79}
\end{equation*}
$$

which will find a target element $\left|a_{T}\right\rangle$ from the database of $M$ elements for which $G\left(a_{i}\right)=1$. Applying $\mathcal{G}_{0}$ on the state of eq. (72) $j=\frac{\pi}{4} \sqrt{M}$ times we obtain

$$
\begin{equation*}
\mathcal{G}_{0}^{j} \mathcal{G}_{1}^{j_{1}}\left|\Theta_{12}\right\rangle \cong \mathcal{G}_{0}^{j}\left|\Theta_{0}\right\rangle \otimes\left|\Theta_{2}\right\rangle \cong\left|a_{T}\right\rangle \otimes\left|\Theta_{2}\right\rangle, \tag{80}
\end{equation*}
$$

Finally, iterating the state in eq. (80) $j_{12}$ times by $\mathcal{G}_{12}$ we obtain

$$
\begin{equation*}
\mathcal{G}_{12}^{j_{12}} \mathcal{G}_{0}^{j_{1}} \mathcal{G}^{j_{1}}\left|\Theta_{12}\right\rangle \cong \mathcal{G}_{12}^{j_{12}}\left|a_{T}\right\rangle \otimes\left|\Theta_{2}\right\rangle \cong\left|a_{T}\right\rangle \otimes\left|b_{T}\right\rangle \tag{81}
\end{equation*}
$$

From the expansion

$$
\begin{equation*}
\mathcal{G}_{12}^{j_{12}} \mathcal{G}_{0}^{j} \mathcal{G}_{1}^{j_{1}}=\mathcal{G}_{12}^{j_{12}}\left(-\mathcal{I}_{\Theta_{0}} \mathcal{G}_{12}^{j_{12} \dagger} \mathcal{I}_{T_{12}} \mathcal{G}_{12}^{j_{12}}\right)^{j} \mathcal{G}_{1}^{j_{1}} \tag{82}
\end{equation*}
$$

we obtain the total oracle queries $j_{T}$ in large database $N$ and large $M$ limit

$$
\begin{equation*}
j_{T}=\lim _{N, M \rightarrow \infty}\left(j_{12}+2 j_{12} j+j_{1}\right)=\frac{\pi^{2}}{8} \sqrt{N M} \tag{83}
\end{equation*}
$$

This is quadratically faster than the classical time of $\mathcal{O}(N M)$ and even faster than the quantum unstructured Grover search for $M<N$ which takes time of $\mathcal{O}(N)$.

## 4. Proof of optimization of Grover algorithm

Grover search is the fastest algorithm for the problem of finding the target element from an unstructured database. No other algorithm can search for the target element shorter than $\mathcal{O}(\sqrt{N})$ oracle queries.

Consider an initial state $\left|\psi_{0}\right\rangle$ which evolves to a state $\left|\psi_{J}^{a_{i}}\right\rangle=U_{a_{i}}\left|\psi_{0}\right\rangle$ after $J$ oracle queries. We assume that after $J$ number of queries the evolved state is very very close to the target state $\left|a_{i}\right\rangle$

$$
\begin{equation*}
\left\langle\psi_{J}^{a_{i}} \mid a_{i}\right\rangle \approx 1, \quad \text { for } \quad i=1,2, \cdots, N \tag{84}
\end{equation*}
$$

The same initial state $\left|\psi_{0}\right\rangle$ evolves to a state $\left|\psi_{J}\right\rangle=U\left|\psi_{0}\right\rangle$ after $J$ empty oracle queries. Question is how far the state $\left|\psi_{J}^{a_{i}}\right\rangle$ has has drifted from $\left|\psi_{J}\right\rangle$ can be qualified in terms of the lower bound as

$$
\begin{equation*}
\left.\sum_{i=1}^{N}| | \psi_{J}^{a_{i}}\right\rangle-\left.\left|\psi_{J}\right\rangle\right|^{2} \geq 2 N-2 \sqrt{N} \tag{85}
\end{equation*}
$$

In Grover's algorithm $\left|\psi_{0}\right\rangle=|\Theta\rangle$ is the state with equal superposition of all the basis elements. The unitary operator $U_{a_{i}}$ is the Grover iteration applied $J$ times

$$
\begin{equation*}
U_{a_{i}}=\left(-\mathcal{I}_{\Theta} \mathcal{I}_{a_{i}}\right)^{J}=\left[-(\mathbb{I}-2|\Theta\rangle\langle\Theta|)\left(\mathbb{I}-2\left|a_{i}\right\rangle\left\langle a_{i}\right|\right)\right]^{J} \tag{86}
\end{equation*}
$$

Then

$$
\begin{equation*}
\left|\psi_{J}^{a_{i}}\right\rangle=U_{a_{i}}\left|\psi_{0}\right\rangle=\left(-\mathcal{I}_{\Theta} \mathcal{I}_{a_{i}}\right)^{J}|\Theta\rangle \approx\left|a_{i}\right\rangle . \tag{87}
\end{equation*}
$$

The empty oracle operator $U$ is given by

$$
\begin{equation*}
U=\left(-\mathcal{I}_{\Theta} \mathbb{I}\right)^{J}=[-(\mathbb{I}-2|\Theta\rangle\langle\Theta|) \mathbb{I}]^{J} \tag{88}
\end{equation*}
$$

where the oracle operator is just the identity operator. $U$ does not change the initial state at all

$$
\begin{equation*}
\left|\psi_{J}\right\rangle=U\left|\psi_{0}\right\rangle=U|\Theta\rangle=|\Theta\rangle . \tag{89}
\end{equation*}
$$

Substituting the the results from eq. (87) and eq. (89) in the left hand side of eq. (85) we obtain $\left.\sum_{i=1}^{N}| | \psi_{J}^{a_{i}}\right\rangle-\left.\left|\psi_{J}\right\rangle\right|^{2}=2 N-2 \sqrt{N}$, which saturates the inequality.

Given the inequality in eq. (85) in terms of the the number of elements in a database $N$ we now need another inequality which will provide a bound in terms of the number of iterations $J$. This inequality is given in terms of the lower bound as

$$
\begin{equation*}
\left.\sum_{i=1}^{N}| | \psi_{J}^{a_{i}}\right\rangle-\left.\left|\psi_{J}\right\rangle\right|^{2} \leq 4 J^{2} \tag{90}
\end{equation*}
$$

From eq. (85) and eq. (90) we obtain in large $N$ limit

$$
\begin{equation*}
J \geq \sqrt{\frac{N}{2}}=\mathcal{O}(\sqrt{N}) \tag{91}
\end{equation*}
$$

In this proof we have assumed the probability of obtaining a target state to be unity. In general by considering probability close to unity one can refine the lower bound on the number of searches $J$ in eq. (91). However upto some small factor the query time is $\mathcal{O}(\sqrt{N})$, which can not be reduced by any algorithm.

## B. Adiabatic evolution for database search

In recent years there have been several attempts to realize Grover search algorithm by adiabatic evolution [23 25] of a suitably chosen Hamiltonian. In this subsection we state one such work which shows that adiabatic approximation can be utilized to find a target item in $\mathcal{O}(\sqrt{N})$ time which is equivalent to what Grover algorithm needs.

According to the adiabatic theorem if a Hamiltonian changes slowly with time then the system initially in a ground state will always remain in the instantaneous ground state of the system. We can exploit it by starting from a Hamiltonian whose states are known and
then adiabatically evolving the Hamiltonian to a Hamiltonian whose ground state would be the desired state we are looking for, i.e. the target state.

Let us start with the Schrödinger equation of a time dependent system with Hamiltonian $H(t)$

$$
\begin{equation*}
i \hbar \frac{\partial}{\partial t} \psi_{A}(t)=H(t) \psi_{A}(t) \tag{92}
\end{equation*}
$$

where $\psi_{A}(t)$ is a state of the system. The eigenvalue equation for this system is given by

$$
\begin{equation*}
H(t) \psi_{n}(t)=E_{n}(t) \psi_{n}(t) \tag{93}
\end{equation*}
$$

where $E_{n}(t), n=1,2, \cdots$ are the time dependent eigenvalues corresponding to the time dependent eigenstates $\psi_{n}(t)$. Note that if the Hamiltonian is time independent then the eigenvalues are also time independent and the eigenstates only acquire phase factor when it evolves. After a long time of evolution the system initially in $\psi_{1}(t)$ state will be found in $\psi_{2}(t)$ state with amplitude $\epsilon$

$$
\begin{equation*}
\epsilon \sim\left|\frac{\left\langle\psi_{2}(t)\right| \frac{d H(t)}{d t}\left|\psi_{1}(t)\right\rangle}{\left(E_{2}(t)-E_{1}(t)\right)^{2}}\right| \ll 1 \tag{94}
\end{equation*}
$$

It is useful to consider even more strict condition to ensure that the system remains in its instantaneous ground state. Is is assumed that the maximum of the numerator and the minimum of the denominator in the interval $T$ in eq. (94) satisfy

$$
\begin{equation*}
\frac{\left.\max _{0 \leq t \leq T}\left|\left\langle\psi_{2}(t)\right| \frac{d H(t)}{d t}\right| \psi_{1}(t)\right\rangle \mid}{\min _{0 \leq t \leq T}\left(E_{2}(t)-E_{1}(t)\right)^{2}} \leq \epsilon . \tag{95}
\end{equation*}
$$

One can exploit the condition (95) to obtain a lower bound on time $T$ to evolve the state from $\psi_{1}(0)$ to $\psi_{1}(T)$.

As an explicit example consider the Hamiltonian

$$
\begin{equation*}
H_{\Theta}=\mathbb{I}-|\Theta\rangle\langle\Theta|, \tag{96}
\end{equation*}
$$

whose ground state $|\Theta\rangle$ is the uniform superposition of all the basis elements in the Hilbert space of dimension $N$ defined in eq. (25). It is assumed that the system is initially in the this ground state. Then to evolve the state $|\Theta\rangle$ to the target state $\left|a_{T}\right\rangle$ we have to consider a Hamiltonian of the form

$$
\begin{equation*}
H_{T}=\mathbb{I}-\left|a_{T}\right\rangle\left\langle a_{T}\right|, \tag{97}
\end{equation*}
$$

whose ground state is the target state $\left|a_{T}\right\rangle$. The Hamiltonian which will evolve the state $|\Theta\rangle$ to the target state $\left|a_{T}\right\rangle$ is given by

$$
\begin{equation*}
H(t)=(1-s(t)) H_{\Theta}+s(t) H_{T} \tag{98}
\end{equation*}
$$

where the parameter $s(t)$ depends on time. Consider a simple liner form $s(t)=\frac{t}{T}$, where $T$ is the time over which the system evolves. The difference between the lowest two eigenvalues $E_{1}(t), E_{2}(t)$ is given by

$$
\begin{equation*}
E_{2}(t)-E_{1}(t)=\frac{1}{\sqrt{N}} \sqrt{N-4(N-1) s(1-s)} \tag{99}
\end{equation*}
$$

The difference in eigenvalues is minimum i.e, $\min _{0 \leq t \leq T}\left(E_{2}(t)-E_{1}(t)\right)^{2}=1 / N$ at $s=1 / 2$. The matrix element in the numerator in eq. (95) can be simplified as

$$
\begin{equation*}
\left\langle\psi_{2}(t)\right| \frac{d H(t)}{d t}\left|\psi_{1}(t)\right\rangle=\frac{d s}{d t}\left\langle\psi_{2}(t)\right| \frac{d H(t)}{d s}\left|\psi_{1}(t)\right\rangle=\frac{1}{T}\left\langle\psi_{2}(t)\right| \frac{d H(t)}{d s}\left|\psi_{1}(t)\right\rangle \sim \frac{1}{T} . \tag{100}
\end{equation*}
$$

Here we have assumed that the matrix element $\left\langle\psi_{2}(t)\right| \frac{d H(t)}{d s}\left|\psi_{1}(t)\right\rangle \sim 1$. Putting the result of eq. (100) and the minimum eigenvalue difference in eq. (95) we obtain the time required

$$
\begin{equation*}
T \geq \frac{N}{\epsilon} \tag{101}
\end{equation*}
$$

which is equivalent to what a classical computer would take to find the target element. Since $s=t / T$ does not solve the purpose, we assume that the dependence of $s$ on time $t$ is governed by the the adiabatic approximation eq. (94), which can be rewritten as

$$
\begin{equation*}
\frac{d s}{d t} \simeq \epsilon\left(E_{2}(t)-E_{1}(t)\right)^{2}=\epsilon \frac{1}{N}(N-4(N-1) s(1-s)) \tag{102}
\end{equation*}
$$

where again we have assumed $\left\langle\psi_{2}(t)\right| \frac{d H(t)}{d s}\left|\psi_{1}(t)\right\rangle \sim 1$. Integrating eq. (102) we obtain

$$
\begin{equation*}
t=\frac{1}{2 \epsilon} \frac{N}{\sqrt{N-1}}(\arctan \sqrt{N-1}(2 s-1)+\arctan \sqrt{N-1}) . \tag{103}
\end{equation*}
$$

The evolution time $T$ can be obtained by setting $s=1$ in eq. (103)

$$
\begin{equation*}
t=\frac{1}{\epsilon} \frac{N}{\sqrt{N-1}} \arctan \sqrt{N-1} \tag{104}
\end{equation*}
$$

When the number of elements in a database is large $N \gg 1$ we get the time required to find the target element from eq. (104) as

$$
\begin{equation*}
T=\frac{\pi}{2 \epsilon} \sqrt{N} \tag{105}
\end{equation*}
$$

The is a quadratic speed up apart from a factor of inverse of error probability.
This algorithm by adiabatic evolution can be extended to the cases when there are many target elements. This time we consider a Hamiltonian of the form

$$
\begin{equation*}
\tilde{H}_{T}=\mathbb{I}-\sum_{\text {target elements }}\left|a_{i}\right\rangle\left\langle a_{i}\right| . \tag{106}
\end{equation*}
$$

Then the time dependent Hamiltonian under which the initial state $|\Theta\rangle$ will be evolved is given by

$$
\begin{equation*}
\tilde{H}(t)=(1-s(t)) H_{\Theta}+s(t) \tilde{H}_{T} \tag{107}
\end{equation*}
$$

The difference in energy between the ground state and the first excited state is now given by

$$
\begin{equation*}
E_{2}(t)-E_{1}(t)=\frac{1}{\sqrt{N}} \sqrt{N-4(N-M) s(1-s)} \tag{108}
\end{equation*}
$$

If we consider $s=t / T$ then we obtain

$$
\begin{equation*}
T \geq \frac{N}{M \epsilon} \tag{109}
\end{equation*}
$$

However if the adiabatic change is considered to be local in the parameter $s$, then the required evolution time becomes

$$
\begin{equation*}
T=\frac{\pi}{2 \epsilon} \sqrt{\frac{N}{M}} \tag{110}
\end{equation*}
$$

which is in agreement with the Grover algorithm with multiple targets.

## IV. PARTIAL DATABASE SEARCH

In reality sometimes we do not need a full search of a database rather only a partial search is enough. For example, suppose we want to look for details of contacts of a specific surname in a telephone directory. If there are eight different surnames in the telephone directory then it can be divided into eight blocks each associated with a surname. In terms of binary the state of an element of the telephone directory with $N=2^{n}$ entries can be written as $\left|a_{1}, a_{2}, a_{3}, \cdots, a_{n}\right\rangle$. Since there are only eight blocks we can assign first three binaries $a_{1}, a_{2}, a_{3}$ to the surnames. Since all the entries in a block share the same surname the first three binaries of the states in a block will be same.
a. Some attempts to partial search: The purpose of a partial search instead of a full Grover search is to achieve a grater speed than the Grover search. However not all partial searches are always advantageous. Let us consider a naive partial search in which first the database of $N$ elements is divided into $K$ blocks. Just randomly choose a block and make a full Grover search which requires $\frac{\pi}{4} \sqrt{\frac{N}{K}}$ queries. To obtain the target item and the target block one has to perform full Grover search in $K-1$ blocks separately in the worst case, which requires $(K-1) \frac{\pi}{4} \sqrt{\frac{N}{K}}$ queries. One can see that this is $\frac{K-1}{\sqrt{K}}$ times the full Grover search. Only for $K=2$ the factor $\frac{K-1}{\sqrt{K}}$ is less than one. For more than two blocks therefore this naive partial search is not faster than the full Grover search.

Another example which is also inefficient for database search with more than two blocks is the binary search. In this search the number of blocks should be of the form $K=2^{k}$ for some positive number $k$. First divide the whole database in two blocks and perform a standard Grover search in any one of the blocks which requires $\frac{\pi}{4} \sqrt{\frac{N}{2}}$ iterations. If the target item is not found then take the remaining block and divide that into two sub-blocks and repeat the previous procedure. We keep on repeating this procedure until we are left with the last block. The total number of queries is obtained by taking the sum of all the searches as $\frac{\pi}{4} \sqrt{N}\left(\sum_{i=1}^{k} \sqrt{\frac{1}{2^{i}}}\right)$. Again the factor $\sum_{i=1}^{k} \sqrt{\frac{1}{2^{i}}}$ is greater than one for $K \geq 4$, making the binary search inefficient compared to the Grover search for more than two blocks.
b. Grover and Radhakrishnan's simple partial search: The fact that the partial search can be advantageous over the full Grover search can be understood from a simple algorithm discussed by Grover and Radhakrishnan. Let us divide the database into $K$ blocks and perform a full Grover search on elements of $K-1$ randomly chosen blocks which requires $\frac{\pi}{4} \sqrt{N}\left(\sqrt{\frac{K-1}{K}}\right)$ queries. Note that the factor $\sqrt{\frac{K-1}{K}}$ is always less than one which suggests that this partial search algorithm is always more efficient than the Grover search algorithm.

## A. Single target GRK partial search algorithm

Partial search algorithm is a combination of both global search and simultaneous local search in each block. Grover and Radhakrishnan first devised a scheme for a partial database search which was latter optimized by Korepin. The database of $N$ elements which are divided into $K$ blocks are first subjected to a global Grover search $\mathcal{G}$. After $j_{1}$ Grover iterations the
initial state $|\Theta\rangle$ defined in eq. (25) becomes

$$
\begin{equation*}
\mathcal{G}^{j_{1}}|\Theta\rangle=\sin \left(2 j_{1}+1\right) \theta\left|a_{T}\right\rangle+\sum_{i=0, i \neq T}^{N-1} \cos \left(2 j_{1}+1\right) \theta \tan \theta\left|a_{i}\right\rangle \tag{111}
\end{equation*}
$$

Then to perform the local iterations let us consider the initial state of $\alpha$ block as

$$
\begin{equation*}
\left|\Theta_{\alpha}\right\rangle=\sum_{\alpha \text { block }}^{N / K} \sqrt{\frac{K}{N}}\left|a_{i}\right\rangle, \quad \alpha=1,2, \cdots, K \tag{112}
\end{equation*}
$$

which is obtained by equal superposition of all the elements in the block. The target element $\left|a_{T}\right\rangle$ should belong to one block which we call target block. If we measure the probability of obtaining the target element in the initial state of a block then for all initial states of individual blocks the probability will vanish except for the initial state $\left|\Theta_{T}\right\rangle$ of the target block the finite probability is given by

$$
\begin{equation*}
\mathcal{P}_{T}=\left|\left\langle a_{T} \mid \Theta_{T}\right\rangle\right|^{2}=\sin ^{2} \theta_{1}=\frac{K}{N} \tag{113}
\end{equation*}
$$

The local iteration in each block $\mathcal{G}_{\alpha}$ can be written as

$$
\begin{equation*}
\mathcal{G}_{\alpha}=-\mathcal{I}_{\Theta_{\alpha}} \mathcal{I}_{T}, \quad \alpha=1,2, \cdots, K \tag{114}
\end{equation*}
$$

where the local reflections $\mathcal{I}_{\Theta_{\alpha}}$ are given by

$$
\begin{equation*}
\mathcal{I}_{\Theta_{\alpha}}=\mathbb{I}-2\left|\Theta_{\alpha}\right\rangle\left\langle\Theta_{\alpha}\right|, \quad \alpha=1,2, \cdots, K \tag{115}
\end{equation*}
$$

Taking a direct sum of all the local iterations we obtain the local Grover iteration $\mathcal{G}^{L}$

$$
\begin{equation*}
\mathcal{G}^{L}=\oplus_{\alpha=1}^{K} \mathcal{G}_{\alpha}=-\left(\oplus_{\alpha=1}^{K} \mathcal{I}_{\Theta_{\alpha}}\right) \mathcal{I}_{T} \tag{116}
\end{equation*}
$$

Note that except from $\mathcal{G}_{T}$, which act on the target block component, all the other local iterations $\mathcal{G}_{\alpha}$ act trivially on $\mathcal{G}^{j_{1}}|\Theta\rangle$. The action of $\mathcal{G}_{\alpha}$ on the respective initial states are given by

$$
\begin{equation*}
\mathcal{G}_{\alpha}\left|\Theta_{\alpha}\right\rangle=-\mathcal{I}_{\Theta_{\alpha}} \mathcal{I}_{T}\left|\Theta_{\alpha}\right\rangle=-\mathcal{I}_{\Theta_{\alpha}}\left|\Theta_{\alpha}\right\rangle=\left|\Theta_{\alpha}\right\rangle, \quad \alpha \neq T, \alpha=1,2, \cdots, K \tag{117}
\end{equation*}
$$

To know how $\mathcal{G}_{T}$ acts on the target block state let us consider the eigenvalue equation

$$
\begin{equation*}
\mathcal{G}_{T}\left|\phi_{1}\right\rangle=E\left|\phi_{1}\right\rangle \tag{118}
\end{equation*}
$$

which has the following two eigenvalues

$$
\begin{equation*}
\left|\phi_{1}\right\rangle_{ \pm}=\frac{1}{\sqrt{2}}\left|a_{T}\right\rangle \pm \frac{i}{\sqrt{2}} \sum_{i \neq T}^{\text {target block }} \tan \theta_{1}\left|a_{i}\right\rangle \tag{119}
\end{equation*}
$$

with their corresponding eigenvalues $E_{ \pm}=e^{ \pm i 2 \theta_{1}}$. Let us now write the state $\mathcal{G}^{j_{1}}|\Theta\rangle$ in eq. (111) in terms of the eigenvectors $\left|\phi_{1}\right\rangle_{ \pm}$and the initial states of the non-target blocks $\left|\Theta_{\alpha}\right\rangle, \alpha \neq T$ as

$$
\begin{align*}
\mathcal{G}^{j_{1}}|\Theta\rangle= & \frac{1}{\sqrt{2}}\left(\sin \left(2 j_{1}+1\right) \theta-i \frac{\cos \left(2 j_{1}+1\right) \theta_{1} \tan \theta}{\tan \theta_{1}}\right)\left|\phi_{1}\right\rangle_{+} \\
& +\frac{1}{\sqrt{2}}\left(\sin \left(2 j_{1}+1\right) \theta+i \frac{\cos \left(2 j_{1}+1\right) \theta_{1} \tan \theta}{\tan \theta_{1}}\right)\left|\phi_{1}\right\rangle_{-} \\
& +\sum_{\alpha=1, \alpha \neq T}^{K} \frac{\cos \left(2 j_{1}+1\right) \theta \tan \theta}{\sin \theta_{1}}\left|\Theta_{\alpha}\right\rangle . \tag{120}
\end{align*}
$$

After $j_{2}$ operations with the local Grover operator $\mathcal{G}^{L}$ on the expression of eq. (120) we immediately obtain

$$
\begin{align*}
\mathcal{G}^{L j_{2}} \mathcal{G}^{j_{1}}|\Theta\rangle= & \frac{e^{i 2 j_{2} \theta_{1}}}{\sqrt{2}}\left(\sin \left(2 j_{1}+1\right) \theta-i \frac{\cos \left(2 j_{1}+1\right) \theta_{1} \tan \theta}{\tan \theta_{1}}\right)\left|\phi_{1}\right\rangle_{+} \\
& +\frac{e^{-i 2 j_{2} \theta_{1}}}{\sqrt{2}}\left(\sin \left(2 j_{1}+1\right) \theta+i \frac{\cos \left(2 j_{1}+1\right) \theta_{1} \tan \theta}{\tan \theta_{1}}\right)\left|\phi_{1}\right\rangle_{-} \\
& +\sum_{\alpha=1, \alpha \neq T}^{K} \frac{\cos \left(2 j_{1}+1\right) \theta \tan \theta}{\sin \theta_{1}}\left|\Theta_{\alpha}\right\rangle . \tag{121}
\end{align*}
$$

It is useful to write the above state $\mathcal{G}^{L j_{2}} \mathcal{G}^{j_{1}}|\Theta\rangle$ in terms of the basis vectors $\left|a_{i}\right\rangle$ as

$$
\begin{align*}
\mathcal{G}^{L^{j_{2}}} \mathcal{G}^{j_{1}}|\Theta\rangle= & \mathcal{C}_{T}\left|a_{T}\right\rangle+\mathcal{C}_{T B} \sum_{i \neq T}^{\text {target block }} \tan \theta_{1}\left|a_{i}\right\rangle \\
& +\mathcal{C}_{N T B} \sum_{\text {non-target blocks }}\left|a_{i}\right\rangle,
\end{align*}
$$

where the constant coefficients are given by

$$
\begin{align*}
\mathcal{C}_{T} & =\sin \left(2 j_{1}+1\right) \theta \cos 2 j_{2} \theta_{1}+\frac{\cos \left(2 j_{1}+1\right) \theta \tan \theta}{\tan \theta_{1}} \sin 2 j_{2} \theta_{1}  \tag{123}\\
\mathcal{C}_{T B} & =-\sin \left(2 j_{1}+1\right) \theta \sin 2 j_{2} \theta_{1}+\frac{\cos \left(2 j_{1}+1\right) \theta \tan \theta}{\tan \theta_{1}} \cos 2 j_{2} \theta_{1}  \tag{124}\\
\mathcal{C}_{N T B} & =\cos \left(2 j_{1}+1\right) \theta \tan \theta \tag{125}
\end{align*}
$$

To eliminate the components associated with the non-target blocks we make a final global Grover iteration to the vector $\mathcal{G}^{L^{j_{2}}} \mathcal{G}^{j_{1}}|\Theta\rangle$ in eq. (122). For convenience we perform a transformation with $-\mathcal{I}_{T} \mathcal{I}_{\Theta}$ instead of the Grover iteration $\mathcal{G}$ however in large blocks limit both the results are equivalent. There is also other operator such as $\mathcal{I}_{\Theta}$ which has been used by Grover and Radhakrishnan to perform the final operation. However in this case the amplitude of the target element becomes negative. The state after final transformation becomes

$$
\begin{align*}
|\mathcal{F}\rangle=\left(-\mathcal{I}_{T} \mathcal{I}_{\Theta}\right) \mathcal{G}^{L^{j_{2}}} \mathcal{G}^{j_{1}}|\Theta\rangle= & \left(\mathcal{C}_{T}-2 \overline{\mathcal{C}}\right)\left|a_{T}\right\rangle+\left(2 \overline{\mathcal{C}}-\mathcal{C}_{T B} \tan \theta_{1}\right) \sum_{i \neq T}^{\text {target block }}\left|a_{i}\right\rangle \\
& +\left(2 \overline{\mathcal{C}}-\mathcal{C}_{N T B}\right) \sum_{\text {non-target blocks }}\left|a_{i}\right\rangle
\end{align*}
$$

where the average amplitude is given by

$$
\begin{equation*}
\overline{\mathcal{C}}=\frac{1}{N}\left(\mathcal{C}_{T}+\mathcal{C}_{T B} \cot \theta_{1}+(K-1) \frac{\mathcal{C}_{N T B}}{\sin ^{2} \theta_{1}}\right) \tag{127}
\end{equation*}
$$

To evaluate eq. (126) we have used the formula of eq. (32) for the action of $-\mathcal{I}_{\Theta}$ on a generic state. Since the projection of the state $\left(-\mathcal{I}_{T} \mathcal{I}_{\Theta}\right) \mathcal{G}^{L^{j_{2}}} \mathcal{G}^{j_{1}}|\Theta\rangle$ on non-target blocks should vanish we obtain from eq. (126)

$$
\begin{equation*}
\mathcal{C}_{N T B}=\frac{2}{N}\left(\mathcal{C}_{T}+\mathcal{C}_{T B} \cot \theta_{1}+(K-1) \frac{\mathcal{C}_{N T B}}{\sin ^{2} \theta_{1}}\right) \tag{128}
\end{equation*}
$$

Substituting the values of $\mathcal{C}_{T}, \mathcal{C}_{T B}$ and $\mathcal{C}_{N T B}$ in eq. (128) and simplifying we obtain a condition

$$
\begin{align*}
& -\frac{1}{\sin \theta \cos \theta}\left(\frac{1}{2}-\frac{\sin ^{2} \theta}{\sin ^{2} \theta_{1}}\right) \cos \left(2 j_{1}+1\right) \theta \\
& =\sin \left(2 j_{1}+1\right) \theta \cos 2 j_{2} \theta_{1}+\frac{\tan \theta}{\tan \theta_{1}} \cos \left(2 j_{1}+1\right) \theta \sin 2 j_{2} \theta_{1} \\
& -\cot \theta_{1} \sin \left(2 j_{1}+1\right) \theta \sin 2 j_{2} \theta_{1}+\frac{\tan \theta}{\tan ^{2} \theta_{1}} \cos \left(2 j_{1}+1\right) \theta \cos 2 j_{2} \theta_{1} \tag{129}
\end{align*}
$$

which ensures that the non-target elements vanish from the final state. Thus we obtain the final state $\left|\mathcal{F}_{T}\right\rangle$, which is aligned with the target block

$$
\begin{align*}
\left|\mathcal{F}_{T}\right\rangle & =\sin \omega\left|a_{T}\right\rangle+\cos \omega \sum_{i \neq T}^{\text {target block }} \tan \theta_{1}\left|a_{i}\right\rangle \\
& =\left(\mathcal{C}_{T}-\mathcal{C}_{N T B}\right)\left|a_{T}\right\rangle+\left(\mathcal{C}_{N T B} \cot \theta_{1}-\mathcal{C}_{T B}\right) \sum_{i \neq T}^{\text {target block }} \tan \theta_{1}\left|a_{i}\right\rangle .
\end{align*}
$$

The block angle $\omega$ which only depends on the number of blocks $K$ of a database is given by

$$
\begin{equation*}
\tan \omega=\frac{\sin \left(2 j_{1}+1\right) \theta \cos 2 j_{2} \theta_{1}+\cos \left(2 j_{1}+1\right) \theta \tan \theta\left(\frac{\sin 2 j_{2} \theta_{1}}{\tan \theta_{1}}-1\right)}{\sin \left(2 j_{1}+1\right) \theta \sin 2 j_{2} \theta_{1}+\frac{\cos \left(2 j_{1}+1\right) \theta \tan \theta}{\tan \theta_{1}}\left(1-\cos 2 j_{2} \theta_{1}\right)} . \tag{131}
\end{equation*}
$$

## 1. Large database limit

Let us now consider the large database limit $N \rightarrow \infty$. We also consider the blocks of the database to be very large $\frac{N}{K} \rightarrow \infty$ so that the number of blocks $K$ in a database remains finite. In these limits the two rotation angles in eq. (26) and eq. (113) respectively reduce to

$$
\begin{equation*}
\lim _{\theta \rightarrow 0} \sin \theta \rightarrow \theta \rightarrow \sqrt{\frac{1}{N}}, \quad \lim _{\theta_{1} \rightarrow 0} \sin \theta_{1} \rightarrow \theta_{1} \rightarrow \sqrt{\frac{K}{N}} \tag{132}
\end{equation*}
$$

Following ref. [2] we write the number of iterations $j_{1}$ and $j_{2}$ in terms of two new parameters $\eta$ and $\beta$ as

$$
\begin{equation*}
j_{1}=\left(\frac{\pi}{4}-\frac{\eta}{\sqrt{K}}\right) \sqrt{N}, \quad j_{2}=\frac{\beta}{\sqrt{K}} \sqrt{N} \tag{133}
\end{equation*}
$$

Putting the expression for $j_{1}$ and $j_{2}$ of eq. (133) in the condition for cancellation of amplitudes eq. (129) of non-target blocks and taking the large database limit we obtain

$$
\begin{align*}
& -\sqrt{N}\left(\frac{1}{2}-\frac{1}{K}\right) \sin \frac{2 \eta}{\sqrt{K}} \\
& =\cos \frac{2 \eta}{\sqrt{K}} \cos 2 \beta+\frac{1}{\sqrt{K}} \sin \frac{2 \eta}{\sqrt{K}} \sin 2 \beta \\
& -\sqrt{\frac{N}{K}} \cos \frac{2 \eta}{\sqrt{K}} \sin 2 \beta+\frac{\sqrt{N}}{K} \sin \frac{2 \eta}{\sqrt{K}} \cos 2 \beta \tag{134}
\end{align*}
$$

Notice that the left hand side of the above equation is proportional to $\sqrt{N}$, which is a large number in our case. On the right hand side the last two terms are proportional to $\sqrt{N}$, however the first two terms are very small compared to the last two terms. Neglecting these small two terms a simple form for the cancellation of the amplitude corresponding to non-target blocks is obtained as

$$
\begin{equation*}
\tan \frac{2 \eta}{\sqrt{K}}=\frac{2 \sqrt{K} \sin 2 \beta}{K-4 \sin ^{2} \beta} \tag{135}
\end{equation*}
$$

The block angle in eq. (131) can be simplified using eq. (135) as

$$
\begin{equation*}
\lim _{N \rightarrow \infty} \tan \omega=\frac{1}{2} \cot \beta+\left(\frac{2}{K}-\frac{1}{2}\right) \tan \beta . \tag{136}
\end{equation*}
$$

Exploiting the physical constraints we can calculate the bounds of the two parameters $\eta$ and $\beta$. Since the number of queries for the global iteration as well as the number of queries for the local iteration given in eq. (133) should be non-negative $j_{1}, j_{2} \geq 0$ we obtain

$$
\begin{equation*}
\eta \leq \frac{\pi}{4} \sqrt{K}, \quad \beta \geq 0 \tag{137}
\end{equation*}
$$

The partial search algorithm have to have less number of total iterations $j_{1}+j_{2}+1$ compared to the Grover's full search algorithm

$$
\begin{equation*}
j_{1}+j_{2}+1=\left(\frac{\pi}{4}+\frac{\beta-\eta}{\sqrt{K}}\right) \sqrt{N} \leq \frac{\pi}{4} \sqrt{N} \tag{138}
\end{equation*}
$$

which implies

$$
\begin{equation*}
\beta \leq \eta \tag{139}
\end{equation*}
$$

From eq. (137) and eq. (139) we obtain

$$
\begin{equation*}
0 \leq \beta \leq \eta \leq \frac{\pi}{4} \sqrt{K} \tag{140}
\end{equation*}
$$

The expression for the parameter $\eta$ for the global iteration can be readily obtained from eq. (135) as

$$
\begin{equation*}
\eta=\frac{\sqrt{K}}{2} \arctan \left[\frac{2 \sqrt{K} \sin 2 \beta}{K-4 \sin ^{2} \beta}\right] \tag{141}
\end{equation*}
$$

where the $\arctan (x)$ is restricted to the principal branch only because of the constraint in eq. (137). The bound for the parameter $\beta$ then becomes

$$
\begin{equation*}
0 \leq \beta \leq \frac{\sqrt{K}}{2} \arctan \left[\frac{2 \sqrt{K} \sin 2 \beta}{K-4 \sin ^{2} \beta}\right] \leq \frac{\pi}{4} \sqrt{K} \tag{142}
\end{equation*}
$$

## 2. Optimization of partial search

As mentioned in the introduction the partial search of Grover and Radhakrishnan has been optimized by Korepin and the optimized version of the partial search is known as the

GRK partial search. In large database limit $N \rightarrow \infty$ the total number of queries to the quantum oracle by a partial search algorithm is given by

$$
\begin{equation*}
J(K)=\lim _{N \rightarrow \infty}\left(j_{1}+j_{2}+1\right)=\left(\frac{\pi}{4}+\frac{\beta-\eta}{\sqrt{K}}\right) \sqrt{N} \tag{143}
\end{equation*}
$$

To obtain least number of queries $J(K)$ we have to minimize

$$
\begin{equation*}
\Lambda(\beta)=\beta-\eta(\beta) \tag{144}
\end{equation*}
$$

Note that the partial search will be more efficient than the full global search if the parameter $\Lambda(\beta)$ defined above is negative. Let us assume that the function $\Lambda(\beta)$ has a minima at some point and the first derivative with respect to $\beta$ vanishes

$$
\begin{equation*}
\frac{d}{d \beta} \Lambda(\beta)=\frac{16(K-1) \sin ^{4} \beta-4 K^{2} \sin ^{2} \beta+K^{2}}{16(K-1) \sin ^{4} \beta-8 K \sin ^{2} \beta-K^{2}}=0 \tag{145}
\end{equation*}
$$

The two solutions of eq. (145) are given by

$$
\sin ^{2} \beta=\left\{\begin{array}{l}
\frac{K}{4(K-1)},  \tag{146}\\
\frac{K}{4}, \text { for } \quad K \leq 4
\end{array}\right.
$$

The second derivative of $\Lambda(\beta)$ is given by

$$
\begin{align*}
\frac{d^{2}}{d \beta^{2}} \Lambda(\beta) & =\frac{16 K \sin 2 \beta(K-1)(K-2) \cos ^{2} 2 \beta}{\left(16(K-1) \sin ^{4} \beta-8 K \sin ^{2} \beta-K^{2}\right)^{2}} \\
& +\frac{4 K \sin 2 \beta\left[16(K-1) \cos 2 \beta+(K-2)^{2}(K+2)\right]}{\left(16(K-1) \sin ^{4} \beta-8 K \sin ^{2} \beta-K^{2}\right)^{2}} \tag{147}
\end{align*}
$$

Note that for the number of blocks $K=2,3$ and 4 we have to consider the two solutions in eq. (146), where as for $K \geq 5$ only one solution $\sin ^{2} \beta=\frac{K}{4(K-1)}$ is valid.

For $K=2$ we notice from eq. (146) that the two solutions coincide. In this case $\sin ^{2} \beta=\frac{K}{4(K-1)}=\frac{K}{4}=\frac{1}{2} \Longrightarrow \beta=\frac{\pi}{4}$ and $\eta=\frac{\pi}{2 \sqrt{2}}$, which correspond to $j_{1}=0$ and $j_{2}=\frac{\pi}{4 \sqrt{2}} \sqrt{N}$. For $K=3$ and 4 the global minimum is at $\sin ^{2} \beta=\frac{K}{4(K-1)}$. Therefore for $K \geq 2$ the global minimum is achieved for

$$
\begin{align*}
\beta & =\arcsin \left(\sqrt{\frac{K}{4(K-1)}}\right)  \tag{148}\\
\eta & =\frac{\sqrt{K}}{2} \arctan \left(\frac{\sqrt{3 K-4}}{K-2}\right) \tag{149}
\end{align*}
$$

## B. Multiple targets GRK partial search algorithm

In the previous section we considered only one target element and therefore partial search was to find out the single target blocks. However there may have several target elements and several target blocks. Here we provide a generalization of the partial search to find one of the target blocks. Let us assume that we have a database of $N$ elements with $K$ blocks. Blocks with target elements are called target blocks and rest of the blocks without target elements are called non-target blocks. There are $B=\frac{N}{M}$ numbers of elements in each block. There are $K_{T}$ target blocks and each target block has $B_{T}$ number of target elements. So in total there are $M=K_{T} B_{T}$ target elements.

The initial state we consider here is $|\tilde{\Theta}\rangle$ of eq. (43) with equal superposition of all the basis states. Iterating $j_{1}$ times with the global Grover operator $\tilde{\mathcal{G}}$ we obtain from eq. (52)

$$
\begin{equation*}
\tilde{\mathcal{G}}^{j_{1}}|\tilde{\Theta}\rangle=\sin \left(2 j_{1}+1\right) \tilde{\theta}\left|A_{T}\right\rangle+\cos \left(2 j_{1}+1\right) \tilde{\theta}\left|A_{n T}\right\rangle \tag{150}
\end{equation*}
$$

where angle between the initial state $|\tilde{\Theta}\rangle$ and the normal to the unite target state $\left|A_{T}\right\rangle$ is given by

$$
\begin{equation*}
\sin ^{2} \tilde{\theta}=\frac{M}{N}=\frac{K_{T} B_{T}}{N} \tag{151}
\end{equation*}
$$

Now we have to consider the local Grover iteration in each block for which we define the local iteration in each block $\mathcal{G}_{\alpha}$ as

$$
\begin{equation*}
\tilde{\mathcal{G}}_{\alpha}=-\mathcal{I}_{\tilde{\Theta}_{\alpha}} \mathcal{I}_{T \alpha}, \quad \alpha=1,2, \cdots, K \tag{152}
\end{equation*}
$$

The local reflections $\mathcal{I}_{\tilde{\Theta}_{\alpha}}$ and $\mathcal{I}_{T \alpha}$ are given by

$$
\begin{align*}
& \mathcal{I}_{\tilde{\Theta}_{\alpha}}=\mathbb{I}-2\left|\tilde{\Theta}_{\alpha}\right\rangle\left\langle\tilde{\Theta}_{\alpha}\right|,  \tag{153}\\
& \mathcal{I}_{T_{\alpha}}=\mathbb{I}-2\left|A_{T_{\alpha}}\right\rangle\left\langle A_{T \alpha}\right|, \tag{154}
\end{align*}
$$

where

$$
\begin{align*}
& \left|\tilde{\Theta}_{\alpha}\right\rangle=\sin \tilde{\theta}_{1}\left|A_{T \alpha}\right\rangle+\cos \tilde{\theta}_{1}\left|A_{n T \alpha}\right\rangle, \alpha=1,2, \cdots, K,  \tag{155}\\
& \left|A_{T \alpha}\right\rangle=\sqrt{\frac{1}{B_{T}}} \sum_{\alpha \text { block }}^{\text {target elements }}\left|a_{i}\right\rangle, \alpha=1,2, \cdots, K . \tag{156}
\end{align*}
$$

We also define

$$
\begin{equation*}
\left|A_{n T \alpha}\right\rangle=\sqrt{\frac{1}{B-B_{T}}} \sum_{\alpha \text { block }}^{\text {non-target elements }}\left|a_{i}\right\rangle, \alpha=1,2, \cdots, K \tag{157}
\end{equation*}
$$

Note that for blocks which do not have target elements $\mathcal{I}_{T \alpha}$ simply becomes the identity operator. The angle $\tilde{\theta}_{1}$ which measures the probability of obtaining the target unit state within a target block is given by

$$
\begin{equation*}
\sin \tilde{\theta}_{1}=\sqrt{\frac{B_{T}}{B}} \tag{158}
\end{equation*}
$$

Taking a direct sum of all the local iterations we obtain the local Grover iteration $\tilde{\mathcal{G}}^{L}$

$$
\begin{equation*}
\tilde{\mathcal{G}}^{L}=\oplus_{\alpha=1}^{K} \tilde{\mathcal{G}}_{\alpha} . \tag{159}
\end{equation*}
$$

Note that except from those $\tilde{\mathcal{G}}_{\alpha} \mathrm{s}$, which act on the target blocks, all the other local iterations $\tilde{\mathcal{G}}_{\alpha}$ act trivially on $\tilde{\mathcal{G}}^{j_{1}}|\tilde{\Theta}\rangle$.

Without loss of generality we assume that first $K_{T}$ blocks are target blocks and the rest $K-K_{T}$ are non-target blocks. Then the action of $\tilde{\mathcal{G}}_{\alpha}$ on the respective initial states are given by

$$
\begin{equation*}
\tilde{\mathcal{G}}_{\alpha}\left|\tilde{\Theta}_{\alpha}\right\rangle=-\mathcal{I}_{\tilde{\Theta}_{\alpha}} \mathcal{I}_{T \alpha}\left|\tilde{\Theta}_{\alpha}\right\rangle=-\mathcal{I}_{\tilde{\Theta}_{\alpha}}\left|\tilde{\Theta}_{\alpha}\right\rangle=\left|\tilde{\Theta}_{\alpha}\right\rangle, \quad \alpha=K_{T}+1, K_{T}+2, \cdots, K \tag{160}
\end{equation*}
$$

To know how $\tilde{\mathcal{G}}_{\alpha}, \alpha=1,2, \cdots, K_{T}$, act on the target blocks let us consider the eigenvalue equations

$$
\begin{equation*}
\tilde{\mathcal{G}}_{\alpha}\left|\phi_{1_{\alpha}}\right\rangle=\tilde{E}_{\alpha}\left|\phi_{1_{\alpha}}\right\rangle, \tag{161}
\end{equation*}
$$

which have the following two eigenvectors

$$
\begin{equation*}
\left|\phi_{1_{\alpha}}\right\rangle_{ \pm}=\frac{1}{\sqrt{2}}\left|A_{T \alpha}\right\rangle \pm \frac{i}{\sqrt{2}}\left|A_{n T \alpha}\right\rangle \tag{162}
\end{equation*}
$$

with their corresponding eigenvalues $E_{ \pm}=e^{ \pm i 2 \tilde{\theta}_{1}}$.
Let us now write the state $\tilde{\mathcal{G}}^{j_{1}}|\tilde{\Theta}\rangle$ in eq. (150) in terms of the eigenvectors $\left|\phi_{1_{\alpha}}\right\rangle_{ \pm}$and the initial states of the non-target blocks

$$
\begin{align*}
\tilde{\mathcal{G}}^{j_{1}}|\tilde{\Theta}\rangle= & \sum_{\alpha=1}^{K_{T}}\left[\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{K_{T}}} \sin \left(2 j_{1}+1\right) \tilde{\theta}-i \sqrt{\frac{B-B_{T}}{N-M}} \cos \left(2 j_{1}+1\right) \tilde{\theta}_{1}\right)\left|\phi_{1_{\alpha}}\right\rangle_{+}\right. \\
& \left.+\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{K_{T}}} \sin \left(2 j_{1}+1\right) \tilde{\theta}+i \sqrt{\frac{B-B_{T}}{N-M}} \cos \left(2 j_{1}+1\right) \tilde{\theta}_{1}\right)\left|\phi_{1_{\alpha}}\right\rangle_{-}\right] \\
& +\sum_{\alpha=K_{T}+1}^{K} \sqrt{\frac{B}{N-M}} \cos \left(2 j_{1}+1\right) \tilde{\theta}\left|\tilde{\Theta}_{\alpha}\right\rangle . \tag{163}
\end{align*}
$$

After $j_{2}$ operations with the local Grover operator $\tilde{\mathcal{G}}^{L}$ on the expression of eq. (163) we immediately obtain

$$
\begin{align*}
\left(\tilde{\mathcal{G}}^{L}\right)^{j_{2}} \tilde{\mathcal{G}}^{j_{1}}|\tilde{\Theta}\rangle= & \sum_{\alpha=1}^{K_{T}}\left[\frac{e^{i 2 \tilde{\theta}_{1} j_{2}}}{\sqrt{2}}\left(\frac{1}{\sqrt{K_{T}}} \sin \left(2 j_{1}+1\right) \tilde{\theta}-i \sqrt{\frac{B-B_{T}}{N-M}} \cos \left(2 j_{1}+1\right) \tilde{\theta}\right)\left|\phi_{1_{\alpha}}\right\rangle_{+}\right. \\
& \left.+\frac{e^{-i 2 \tilde{\theta}_{1} j_{2}}}{\sqrt{2}}\left(\frac{1}{\sqrt{K_{T}}} \sin \left(2 j_{1}+1\right) \tilde{\theta}+i \sqrt{\frac{B-B_{T}}{N-M}} \cos \left(2 j_{1}+1\right) \tilde{\theta}\right)\left|\phi_{1_{\alpha}}\right\rangle_{-}\right] \\
& +\sum_{\alpha=K_{T}+1}^{K} \sqrt{\frac{B}{N-M}} \cos \left(2 j_{1}+1\right) \tilde{\theta}\left|\tilde{\Theta}_{\alpha}\right\rangle . \tag{164}
\end{align*}
$$

It is useful to write the above state $\left(\tilde{\mathcal{G}}^{L}\right)^{j_{2}} \mathcal{G}^{j_{1}}|\tilde{\Theta}\rangle$ in terms of the basis vectors $\left|a_{i}\right\rangle$ as

$$
\begin{align*}
\left(\tilde{\mathcal{G}}^{L}\right)^{j_{2}} \mathcal{G}^{j_{1}}|\tilde{\Theta}\rangle= & \tilde{\mathcal{C}}_{T} \sum_{\text {target blocks }}^{\text {target elements }}\left|a_{i}\right\rangle+\tilde{\mathcal{C}}_{T B} \sum_{\text {target blocks }}^{\text {non-target elements }}\left|a_{i}\right\rangle \\
& +\tilde{\mathcal{C}}_{N T B} \sum_{\text {non-target blocks }}^{\text {all elements }}\left|a_{i}\right\rangle
\end{align*}
$$

where the constant coefficients are given by

$$
\begin{align*}
\tilde{\mathcal{C}}_{T} & =\sqrt{\frac{1}{M}} \sin \left(2 j_{1}+1\right) \tilde{\theta} \cos 2 j_{2} \tilde{\theta}_{1} \\
& +\sqrt{\frac{B-B_{T}}{B_{T}(N-M)}} \cos \left(2 j_{1}+1\right) \tilde{\theta} \sin 2 j_{2} \tilde{\theta}_{1},  \tag{166}\\
\tilde{\mathcal{C}}_{T B} & =-\sqrt{\frac{1}{K_{T}\left(B-B_{T}\right)}} \sin \left(2 j_{1}+1\right) \tilde{\theta} \sin 2 j_{2} \tilde{\theta}_{1} \\
& +\sqrt{\frac{1}{N-M}} \cos \left(2 j_{1}+1\right) \tilde{\theta} \cos 2 j_{2} \tilde{\theta}_{1},  \tag{167}\\
\tilde{\mathcal{C}}_{N T B} & =\sqrt{\frac{1}{N-M}} \cos \left(2 j_{1}+1\right) \tilde{\theta} . \tag{168}
\end{align*}
$$

To eliminate the components associated with the non-target blocks we make a final global Grover iteration to the vector $\left(\tilde{\mathcal{G}}^{L}\right)^{j_{2}} \tilde{\mathcal{G}}^{j_{1}}|\tilde{\Theta}\rangle$ in eq. (165). For convenience we perform a transformation with $-\mathcal{I}_{T} \mathcal{I}_{\tilde{\Theta}}$ instead of the Grover iteration $\tilde{\mathcal{G}}$ however in large blocks limit both the results are equivalent. There is also other operator such as $\mathcal{I}_{\tilde{\Theta}}$ which has been used by Grover and Radhakrishnan to perform the final operation as mentioned before. However
in this case the amplitude of the target element becomes negative. Thus the state becomes

$$
\begin{align*}
|\tilde{\mathcal{F}}\rangle=\left(-\mathcal{I}_{T} \mathcal{I}_{\tilde{\Theta}}\right)\left(\tilde{\mathcal{G}}^{L}\right)^{j_{2}} \tilde{\mathcal{G}}^{j_{1}}|\tilde{\Theta}\rangle= & \left(\tilde{\mathcal{C}}_{T}-2 \tilde{\tilde{\mathcal{C}}}\right) \sum_{\text {target blocks }}^{\text {target elements }}\left|a_{i}\right\rangle \\
& +\left(2 \overline{\tilde{\mathcal{C}}}-\tilde{\mathcal{C}}_{T B}\right) \sum_{\text {target blocks }}^{\text {non-target elements }}\left|a_{i}\right\rangle \\
& +\left(2 \overline{\tilde{\mathcal{C}}}-\tilde{\mathcal{C}}_{N T B}\right) \sum_{\text {non-target blocks }}^{\text {all elements }}\left|a_{i}\right\rangle \tag{169}
\end{align*}
$$

where the average amplitude is given by

$$
\begin{equation*}
\overline{\tilde{\mathcal{C}}}=\frac{1}{N}\left(M \tilde{\mathcal{C}}_{T}+K_{T}\left(B-B_{T}\right) \tilde{\mathcal{C}}_{T B}+\left(K-K_{T}\right) B \tilde{\mathcal{C}}_{N T B}\right) . \tag{170}
\end{equation*}
$$

To evaluate eq. (169) again we have used the formula of eq. (32) for the action of $-\mathcal{I}_{\tilde{\Theta}}$ on a generic state. Since the projection of the state $\left(-\mathcal{I}_{T} \mathcal{I}_{\tilde{\Theta}}\right)\left(\tilde{\mathcal{G}}^{L}\right)^{j_{2}} \tilde{\mathcal{G}}^{j_{1}}|\tilde{\Theta}\rangle$ on non-target blocks should vanish we obtain from eq. (169)

$$
\begin{equation*}
\tilde{\mathcal{C}}_{N T B}=\frac{2}{N}\left(M \tilde{\mathcal{C}}_{T}+K_{T}\left(B-B_{T}\right) \tilde{\mathcal{C}}_{T B}+\left(K-K_{T}\right) B \tilde{\mathcal{C}}_{N T B}\right) . \tag{171}
\end{equation*}
$$

Substituting the values of $\tilde{\mathcal{C}}_{T}, \tilde{\mathcal{C}}_{T B}$ and $\tilde{\mathcal{C}}_{N T B}$ in eq. (171) and simplifying we obtain a condition

$$
\begin{align*}
& -\frac{1}{\sin \tilde{\theta} \cos \tilde{\theta}}\left(\frac{1}{2}-\frac{\sin ^{2} \tilde{\theta}}{\sin ^{2} \tilde{\theta}_{1}}\right) \cos \left(2 j_{1}+1\right) \tilde{\theta} \\
& =\sin \left(2 j_{1}+1\right) \tilde{\theta} \cos 2 j_{2} \tilde{\theta}_{1}+\frac{\tan \tilde{\theta}}{\tan \tilde{\theta}_{1}} \cos \left(2 j_{1}+1\right) \tilde{\theta} \sin 2 j_{2} \tilde{\theta}_{1} \\
& -\cot \tilde{\theta}_{1} \sin \left(2 j_{1}+1\right) \tilde{\theta} \sin 2 j_{2} \tilde{\theta}_{1}+\frac{\tan \tilde{\theta}}{\tan ^{2} \tilde{\theta}_{1}} \cos \left(2 j_{1}+1\right) \tilde{\theta} \cos 2 j_{2} \tilde{\theta}_{1} \tag{172}
\end{align*}
$$

which ensures that the non-target elements vanish from the final state. Thus we obtain the final state $\left|\tilde{\mathcal{F}}_{T}\right\rangle$, which is aligned with the target block, as

$$
\begin{align*}
\left|\tilde{\mathcal{F}}_{T}\right\rangle & =\sin \tilde{\omega}\left|A_{T}\right\rangle+\operatorname{cõs} \omega\left|A_{n T T}\right\rangle \\
& =\sqrt{M}\left(\mathcal{C}_{T}-\mathcal{C}_{N T B}\right)\left|A_{T}\right\rangle+\sqrt{K_{T}\left(B-B_{T}\right)}\left(\mathcal{C}_{N T B}-\mathcal{C}_{T B}\right)\left|A_{n T T}\right\rangle . \tag{173}
\end{align*}
$$

where

$$
\begin{equation*}
\left|A_{n T T}\right\rangle=\sqrt{\frac{1}{K_{T}\left(B-B_{T}\right)}} \sum_{\text {target blocks }}^{\text {non-target elements }}\left|a_{i}\right\rangle \tag{174}
\end{equation*}
$$

The block angle $\tilde{\omega}$ is given by

$$
\begin{equation*}
\tan \tilde{\omega}=\frac{\sin \left(2 j_{1}+1\right) \tilde{\theta} \cos 2 j_{2} \tilde{\theta}_{1}+\cos \left(2 j_{1}+1\right) \tilde{\theta} \cot \tilde{\theta}}{\sin \left(2 j_{1}+1\right) \tilde{\theta} \sin 2 j_{2} \tilde{\theta}_{1}+\cos \left(2 j_{1}+1\right) \tilde{\theta} \tan \tilde{\theta} \cot \tilde{\theta}_{1}\left(1-\cos 2 j_{2} \tilde{\theta}_{1}\right)} . \tag{175}
\end{equation*}
$$

## 1. Large database limit

Let us now consider the large database limit $N \rightarrow \infty$. We also consider the blocks of the database to be very large $B=\frac{N}{K} \rightarrow \infty$ so that the number of blocks $K$ in a database remains finite. In these limits the two rotation angles in eq. (151) and eq. (158) respectively reduces to

$$
\begin{equation*}
\lim _{\tilde{\theta} \rightarrow 0} \sin \tilde{\theta} \rightarrow \tilde{\theta} \rightarrow \sqrt{\frac{M}{N}}, \quad \lim _{\tilde{\theta}_{1} \rightarrow 0} \sin \tilde{\theta}_{1} \rightarrow \tilde{\theta}_{1} \rightarrow \sqrt{\frac{B_{K}}{B}} \tag{176}
\end{equation*}
$$

Following ref. [2] we write the number of iterations $j_{1}$ and $j_{2}$ in terms of two new parameters $\tilde{\eta}$ and $\tilde{\beta}$ as

$$
\begin{equation*}
j_{1}=\left(\frac{\pi}{4}-\frac{\tilde{\eta} \sqrt{M}}{\sqrt{K}}\right) \sqrt{\frac{N}{M}}, \quad j_{2}=\frac{\tilde{\beta} \sqrt{M}}{\sqrt{K}} \sqrt{\frac{N}{M}} \tag{177}
\end{equation*}
$$

Putting the expression for $j_{1}$ and $j_{2}$ of eq. (177) in the condition for cancellation of amplitudes eq. (172) of non-target blocks and taking the large database limit we obtain

$$
\begin{align*}
& -\sqrt{N}\left(\frac{1}{2}-\frac{K_{T}}{K}\right) \sin \frac{2 \tilde{\eta} \sqrt{M}}{\sqrt{K}} \\
& =\sqrt{M} \cos \frac{2 \tilde{\eta} \sqrt{M}}{\sqrt{K}} \cos 2 \tilde{\beta} \sqrt{B_{T}}+\frac{\sqrt{K_{T}}}{\sqrt{K}} \sin \frac{2 \tilde{\eta} \sqrt{M}}{\sqrt{K}} \sin 2 \beta \sqrt{B_{T}} \\
& -\sqrt{\frac{N}{K}} \sqrt{K_{T}} \cos \frac{2 \tilde{\eta} \sqrt{M}}{\sqrt{K}} \sin 2 \beta \sqrt{B_{T}}+\frac{\sqrt{N}}{K} K_{T} \sin \frac{2 \tilde{\eta} \sqrt{M}}{\sqrt{K}} \cos 2 \beta \sqrt{B_{T}} . \tag{178}
\end{align*}
$$

Notice that the left hand side of the above equation is proportional to $\sqrt{N}$, which is a large number in general. The last two terms on right hand side are proportional to $\sqrt{N}$ but the first two terms are small compared to the last two terms. Neglecting these small two terms and re-scaling by $\bar{K}=\frac{K}{K_{T}}, \bar{\eta}=\tilde{\eta} \sqrt{B_{T}}, \bar{\beta}=\tilde{\beta} \sqrt{B_{T}}$ a simple form for the cancellation of the amplitude correcting to non-target blocks is obtained as

$$
\begin{equation*}
\tan \frac{2 \bar{\eta}}{\sqrt{\bar{K}}}=\frac{2 \sqrt{\bar{K}} \sin 2 \bar{\beta}}{\bar{K}-4 \sin ^{2} \bar{\beta}} \tag{179}
\end{equation*}
$$

## 2. Optimization of partial search

Similar to the previous subsection, exploiting the physical constraints, we can calculate the bounds of the two parameters $\bar{\eta}$ and $\bar{\beta}$. Since the number of queries for the global iteration as well as the number of queries for the local iteration given in eq. (177) should be non-negative $j_{1}, j_{2} \geq 0$ we obtain

$$
\begin{equation*}
\bar{\eta} \leq \frac{\pi}{4} \sqrt{\bar{K}}, \quad \bar{\beta} \geq 0 . \tag{180}
\end{equation*}
$$

The partial search algorithm have to have less number of total iterations $j_{1}+j_{2}+1$ compared to the Grover's full search algorithm, i.e.

$$
\begin{equation*}
j_{1}+j_{2}+1=\left(\frac{\pi}{4}+\frac{\bar{\beta}-\bar{\eta}}{\sqrt{\bar{K}}}\right) \sqrt{\frac{N}{M}} \leq \frac{\pi}{4} \sqrt{\frac{N}{M}} \tag{181}
\end{equation*}
$$

which implies

$$
\begin{equation*}
\bar{\beta} \leq \bar{\eta} . \tag{182}
\end{equation*}
$$

From eq. (180) and eq. (182) we obtain

$$
\begin{equation*}
0 \leq \bar{\beta} \leq \bar{\eta} \leq \frac{\pi}{4} \sqrt{\bar{K}} \tag{183}
\end{equation*}
$$

The expression for the parameter $\bar{\eta}$ for the global iteration can be readily obtained from eq. (179) as

$$
\begin{equation*}
\bar{\eta}=\frac{\sqrt{\bar{K}}}{2} \arctan \left[\frac{2 \sqrt{\bar{K}} \sin 2 \bar{\beta}}{\bar{K}-4 \sin ^{2} \bar{\beta}}\right] \tag{184}
\end{equation*}
$$

where the $\arctan (x)$ is restricted to the principal branch only because of the constraint in eq. (180). The bound for the parameter $\bar{\beta}$ then becomes

$$
\begin{equation*}
0 \leq \bar{\beta} \leq \frac{\sqrt{\bar{K}}}{2} \arctan \left[\frac{2 \sqrt{\bar{K}} \sin 2 \bar{\beta}}{K-4 \sin ^{2} \bar{\beta}}\right] \leq \frac{\pi}{4} \sqrt{\bar{K}} \tag{185}
\end{equation*}
$$

In large database limit $N \rightarrow \infty$ the total number of queries to the quantum oracle by a partial search algorithm is given by

$$
\begin{equation*}
\tilde{J}(\bar{K})=\lim _{N / M \rightarrow \infty}\left(j_{1}+j_{2}+1\right)=\left(\frac{\pi}{4}+\frac{\bar{\beta}-\bar{\eta}}{\sqrt{\bar{K}}}\right) \sqrt{\frac{N}{M}} \tag{186}
\end{equation*}
$$

To obtain least number of queries $\tilde{J}(\bar{K})$ we have to minimize

$$
\begin{equation*}
\tilde{\Lambda}(\bar{\beta})=\bar{\beta}-\bar{\eta}(\bar{\beta}) \tag{187}
\end{equation*}
$$

The global minimum is achieved for $\bar{K} \geq 2$ at

$$
\begin{align*}
& \bar{\beta}=\arcsin \left(\sqrt{\frac{\bar{K}}{4(\bar{K}-1)}}\right)  \tag{188}\\
& \bar{\eta}=\frac{\sqrt{\bar{K}}}{2} \arctan \left(\frac{\sqrt{3 \bar{K}-4}}{\bar{K}-2}\right) . \tag{189}
\end{align*}
$$

## C. Success probability in partial search

In partial search and even in full Grover search usually the number of queries are not integers. In practical purpose what we do is just take the integral value nearest to the number of queries obtained from full or partial search. This introduces some error in the final state obtained after the iterations are done. This problem can be fixed to obtain the target state or the target block with cent percent success probability. In the case of partial search we will discuss here how to obtain the target block with unit success probability. Since we need the group formulation for this purpose let us first briefly discuss the group aspect of the search algorithm.

## 1. Group formulation of search algorithm

The whole discussion of full Grover search discussed in III A 1 and III A 2 can be understood by $O(2)$ transformation on the initial state. Let us write the initial state $|\Theta\rangle$ in terms of the unit basis vectors $\left|A_{T}\right\rangle$ and $\left|A_{n T}\right\rangle$ of eqs. (41) and (42) respectively as

$$
\begin{equation*}
|\tilde{\Theta}\rangle=\binom{\sin \tilde{\theta}}{\cos \tilde{\theta}} \tag{190}
\end{equation*}
$$

In the same basis the Grover iteration $\tilde{\mathcal{G}}$ can be represented as a rotation matrix in two dimensions

$$
\tilde{\mathcal{G}}=\left(\begin{array}{cc}
\cos 2 \tilde{\theta} & \sin 2 \tilde{\theta}  \tag{191}\\
-\sin 2 \tilde{\theta} & \cos 2 \tilde{\theta}
\end{array}\right)
$$

Action of the Grover iteration $j$ times successively on the initial state becomes

$$
\tilde{\mathcal{G}}^{j}|\tilde{\Theta}\rangle=\left(\begin{array}{cc}
\cos 2 j \tilde{\theta} & \sin 2 j \tilde{\theta}  \tag{192}\\
-\sin 2 j \tilde{\theta} & \cos 2 j \tilde{\theta}
\end{array}\right)\binom{\sin \tilde{\theta}}{\cos \tilde{\theta}}=\binom{\sin (2 j+1) \tilde{\theta}}{\cos (2 j+1) \tilde{\theta}} .
$$

By assuming that the initial state has evolved to the target state, i.e.,

$$
\begin{equation*}
\binom{\sin (2 j+1) \tilde{\theta}}{\cos (2 j+1) \tilde{\theta}}=\binom{1}{0} \tag{193}
\end{equation*}
$$

we can arrive at the same result in eq. (53) and when there is only one target element then we arrive at eq. (36). This formalism can be extended to partial database search problem which has $O(3)$ group representation. Again we will discuss the multiple targets and multiple target blocks case but the discussion is equally valid for single target partial search also. In partial search there are three mutually orthogonal basis vectors. The unit vector $A_{T}$ with equal superposition of all the target elements, the unit vector $A_{n T T}$ with equal superposition of all the non target elements in the target blocks and the unit vector $A_{N}$ with equal superposition of all the elements in non-target blocks. First two unit vectors $A_{T}$ and $A_{n T T}$ have already been defined in eqs. (41) and (174) respectively. We now define the unit vector $A_{N}$ as

$$
\begin{equation*}
\left|A_{N}\right\rangle=\sqrt{\frac{1}{B\left(K-K_{T}\right)}} \sum_{\text {non-target blocks }}^{\text {all elements }}\left|a_{i}\right\rangle \tag{194}
\end{equation*}
$$

These three vectors form a three dimensional vector space on which the initial state $|\tilde{\Theta}\rangle$ can be expressed as

$$
|\tilde{\Theta}\rangle=\left(\begin{array}{c}
\sin \gamma \sin \tilde{\theta}  \tag{195}\\
\sin \gamma \cos \tilde{\theta} \\
\cos \gamma
\end{array}\right)
$$

where $\sin \gamma=\sqrt{K_{T} / K}, \sin \tilde{\theta}=\sqrt{M / N}$. The global Grover iteration $\tilde{\mathcal{G}}^{j_{1}}$ can be represented as

$$
\begin{equation*}
\tilde{\mathcal{G}}^{j_{1}}=T M_{j_{1}} T \tag{196}
\end{equation*}
$$

where $T$ and $M_{j_{1}}$ are given by

$$
T=\left(\begin{array}{ccc}
1 & 0 & 0  \tag{197}\\
0 & \cos \tilde{\theta}_{1} \sin \gamma / \cos \tilde{\theta} & \cos \gamma / \cos \tilde{\theta} \\
0 & \cos \gamma / \cos \tilde{\theta} & -\cos \tilde{\theta}_{1} \sin \gamma / \cos \tilde{\theta}
\end{array}\right)
$$

and

$$
M_{j_{1}}=\left(\begin{array}{ccc}
\cos 2 j_{1} \tilde{\theta} & \sin 2 j_{1} \tilde{\theta} & 0  \tag{198}\\
-\sin 2 j_{1} \tilde{\theta} & \cos 2 j_{1} \tilde{\theta} & 0 \\
0 & 0 & (-1)^{j_{1}}
\end{array}\right)
$$

The global Grover iteration $\tilde{\mathcal{G}}^{j_{1}}$ reads as

$$
\tilde{\mathcal{G}}^{j_{1}}=\left(\begin{array}{lll}
a_{11} & a_{12} & a_{13}  \tag{199}\\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\right),
$$

where $a_{11}=\cos 2 j_{1} \tilde{\theta}, a_{12}=\sin 2 j_{1} \tilde{\theta} \sin \gamma, a_{13}=\sin 2 j_{1} \tilde{\theta} \cos \gamma, a_{21}=-a_{12}, a_{22}=$ $(-1)^{j_{1}} \cos ^{2} \gamma+\cos 2 j_{1} \tilde{\theta} \sin ^{2} \gamma, a_{23}=\sin \gamma \cos \gamma\left[(-1)^{j_{1}+1}+\cos 2 j_{1} \tilde{\theta}\right], a_{31}=-a_{13}, a_{32}=a_{23}$ and $a_{33}=(-1)^{j_{1}} \sin ^{2} \gamma+\cos 2 j_{1} \tilde{\theta} \cos ^{2} \gamma$. Representation (199) is valid for large $N$ and large $B$ limit. The local Grover iteration $\left(\tilde{\mathcal{G}}^{L}\right)^{j_{2}}$ is represented as

$$
\left(\tilde{\mathcal{G}}^{L}\right)^{j_{2}}=\left(\begin{array}{ccc}
\cos 2 j_{2} \tilde{\theta}_{1} & \sin 2 j_{2} \tilde{\theta}_{1} & 0  \tag{200}\\
-\sin 2 j_{2} \tilde{\theta}_{1} & \cos 2 j_{2} \tilde{\theta}_{1} & 0 \\
0 & 0 & 1
\end{array}\right) .
$$

The full partial search operation can also be represented in a compact form

$$
\tilde{\mathcal{G}}\left(\tilde{\mathcal{G}}^{L}\right)^{j_{2}} \tilde{\mathcal{G}}^{j_{1}}=\left(\begin{array}{ccc}
0 & \xi_{1} & \xi_{2}  \tag{201}\\
0 & \xi_{2} & -\xi_{1} \\
-1 & 0 & 0
\end{array}\right)
$$

where $\xi_{1}=\frac{1}{2 \sqrt{K-1}}-\frac{1}{2} \sqrt{\frac{3 K-4}{K}}$ and $\xi_{2}=\frac{1}{2}+\frac{1}{2} \sqrt{\frac{3 K-4}{K(K-1)}}$ satisfying $\xi_{1}^{2}+\xi_{2}^{2}=1$.

## 2. Sure success partial search

It has been shown in ref. [26] that the partial search of Grover-Radhakrishnan-Korepin can be performed in such a way that the probability of success is unity. In multiple targets partial search we here discuss the method of obtaining the target block with certainty. In this case the process of partial search is followed as it is except in the final Grover iteration $\mathcal{I}_{T}$ and $\mathcal{I}_{\tilde{\Theta}}$ are modified by phase factors, which are suitably adjusted to obtain the target block.

After the first global Grover iteration the initial state $|\tilde{\Theta}\rangle$ becomes

$$
\tilde{\mathcal{G}}^{j_{1}}|\tilde{\Theta}\rangle=\frac{1}{\cos ^{2} \tilde{\theta}}\left(\begin{array}{c}
k_{g} \cos \tilde{\theta}  \tag{202}\\
l_{g} \cos \tilde{\theta}_{1} \sin \gamma \\
l_{g} \cos \gamma
\end{array}\right)
$$

where $k_{g}=\sin 2 j_{1} \tilde{\theta}\left(\cos ^{2} \tilde{\theta}_{1} \sin ^{2} \gamma+\cos ^{2} \gamma\right)+\cos 2 j_{1} \tilde{\theta} \cos \tilde{\theta} \sin \tilde{\theta} \quad$ and $\quad l_{g} \quad=$ $\cos 2 j_{1} \tilde{\theta}\left(\cos ^{2} \tilde{\theta}_{1} \sin ^{2} \gamma+\cos ^{2} \gamma\right)-\sin 2 j_{1} \tilde{\theta} \cos \tilde{\theta} \sin \tilde{\theta}$.

Then $j_{2}$ local Grover iterations on $\tilde{\mathcal{G}}^{j_{1}}|\tilde{\Theta}\rangle$ gives us [13]

$$
\left(\tilde{\mathcal{G}}^{L}\right)^{j_{2}} \tilde{\mathcal{G}}^{j_{1}}|\tilde{\Theta}\rangle=\frac{1}{\cos ^{2} \tilde{\theta}}\left(\begin{array}{c}
k_{g} \cos \tilde{\theta} \cos 2 j_{2} \tilde{\theta}_{1}+l_{g} \sin \gamma \cos \tilde{\theta}_{1} \sin 2 j_{2} \tilde{\theta}_{1}  \tag{203}\\
-k_{g} \cos \tilde{\theta} \sin 2 j_{2} \tilde{\theta}_{1}+l_{g} \sin \gamma \cos \tilde{\theta}_{1} \cos 2 j_{2} \tilde{\theta}_{1} \\
l_{g} \cos \gamma
\end{array}\right)=\left(\begin{array}{c}
c_{11} \\
c_{21} \\
c_{31}
\end{array}\right) .
$$

Two reflection operators in the final Grover iteration are modified as

$$
\begin{align*}
& \mathcal{I}_{T}^{p h}=\mathbb{I}-\left(\mathbb{I}-e^{2 i \phi_{1}}\right)\left|A_{T}\right\rangle\left\langle A_{T}\right|  \tag{204}\\
& \mathcal{I}_{\tilde{\Theta}}^{p h}=\mathbb{I}-\left(\mathbb{I}-e^{i\left(\phi_{1}-\phi_{2}\right)}\right)|\tilde{\Theta}\rangle\langle\tilde{\Theta}| . \tag{205}
\end{align*}
$$

Now as stated above, the final modified global Grover iteration is given by

$$
\tilde{\mathcal{G}}^{\text {final }}=-\mathcal{I}_{\tilde{\Theta}}^{p h} \mathcal{I}_{T}^{p h}=\left(\begin{array}{lll}
b_{11} & b_{12} & b_{13}  \tag{206}\\
b_{21} & b_{22} & b_{23} \\
b_{31} & b_{32} & b_{33}
\end{array}\right)
$$

where $b_{11}=-e^{i\left(\phi_{1}-\phi_{2}\right)}\left[1-\left(1-e^{2 i \phi_{1}}\right) \sin ^{2} \gamma \sin ^{2} \tilde{\theta}_{1}\right], b_{12}=\left(1-e^{2 i \phi_{1}}\right) \sin ^{2} \gamma \sin \tilde{\theta}_{1} \cos \tilde{\theta}_{1}$, $b_{13}=\left(1-e^{2 i \phi_{1}}\right) \sin \gamma \cos \gamma \sin \tilde{\theta}_{1}, \quad b_{21}=e^{i\left(\phi_{1}-\phi_{2}\right)}\left(1-e^{2 i \phi_{1}}\right) \sin ^{2} \gamma \sin \tilde{\theta}_{1} \cos \tilde{\theta}_{1}, \quad b_{22}=$ $\left(1-e^{2 i \phi_{1}}\right) \sin ^{2} \gamma \cos ^{2} \tilde{\theta}_{1}-1, \quad b_{23}=\left(1-e^{2 i \phi_{1}}\right) \sin \gamma \cos \gamma \cos \tilde{\theta}_{1}, \quad b_{31}=e^{i\left(\phi_{1}-\phi_{2}\right)}(1-$ $\left.e^{2 i \phi_{1}}\right) \sin \gamma \cos \gamma \sin \tilde{\theta}_{1}, b_{32}=b_{23}$ and $b_{33}=\left(1-e^{2 i \phi_{1}}\right) \cos ^{2} \gamma-1$.

The projection of the final state $\tilde{\mathcal{G}}^{\text {final }}\left(\tilde{\mathcal{G}}^{L}\right)^{j_{2}} \tilde{\mathcal{G}}^{j_{1}}|\tilde{\Theta}\rangle$ in the direction of unit vector $\left|A_{N}\right\rangle$ of non-target blocks should vanish

$$
\begin{equation*}
\left.\left|\left\langle A_{N}\right| \tilde{\mathcal{G}}^{\text {final }}\left(\tilde{\mathcal{G}}^{L}\right)^{j_{2}} \tilde{\mathcal{G}}^{\tilde{j}_{1}}\right| \tilde{\Theta}\right\rangle \mid=0 . \tag{207}
\end{equation*}
$$

We obtain from eq. (207) the following condition on the phases

$$
\begin{align*}
& c_{11} e^{i\left(\phi_{1}-\phi_{2}\right)}\left(1-e^{2 i \phi_{1}}\right) \sin \gamma \cos \gamma \sin \tilde{\theta} \\
+ & c_{21}\left(1-e^{2 i \phi_{1}}\right) \sin \gamma \cos \gamma \cos \tilde{\theta} \\
+ & c_{31}\left[\left(1-e^{2 i \phi_{1}}\right) \cos ^{2} \gamma-1\right]=0 \tag{208}
\end{align*}
$$

where $c_{11}, c_{21}, c_{31}$ are the three components of the state in eq. (203). For simplicity we rewrite the condition in eq. (208) in the following fashion

$$
\begin{equation*}
e^{i\left(\phi_{1}-\phi_{2}\right)}\left(1-e^{2 i \phi_{1}}\right) x+\left(1-e^{2 i \phi_{1}}\right) y+2 z=0, \tag{209}
\end{equation*}
$$

where $x=c_{11} \sin \gamma \cos \gamma \sin \tilde{\theta}, y=c_{21} \sin \gamma \cos \gamma \cos \tilde{\theta}+c_{31} \cos ^{2} \gamma$ and $z=-c / 2$. Separating the real and imaginary part from eq. (209) we obtain

$$
\begin{align*}
\sin \phi_{2} & =-\frac{y}{x} \sin \phi_{1}-\frac{z}{x \sin \phi_{1}}, \\
\cos \phi_{2} & =-\frac{y}{x} \cos \phi_{1} . \tag{210}
\end{align*}
$$

Eliminating $\phi_{2}$ from eq. (210) we get a condition on phase $\phi_{1}$ as

$$
\begin{equation*}
\cos ^{2} \phi_{1}=\frac{x^{2}-(y+z)^{2}}{x^{2}-y^{2}-2 y z} . \tag{211}
\end{equation*}
$$

Note that in order to have a solution for $\phi_{1}$ from eq. (211) the following inequality have to be satisfied

$$
\begin{equation*}
x^{2} \geq(y+z)^{2} . \tag{212}
\end{equation*}
$$

The solution for $\phi_{2}$ then can be obtained from eq. (210). Numerical study for sure success partial search has been performed in [26]. It has been shown that it is always possible to find the phases $\phi_{1}, \phi_{2}$ if the number of global and local iterations are chosen as

$$
\begin{align*}
& \tilde{j}_{1}=\left\lfloor j_{1}\right\rfloor  \tag{213}\\
& \tilde{j_{2}}=\left\lfloor j_{2}\right\rfloor+\{0,1,2\}, \tag{214}
\end{align*}
$$

where $\lfloor x\rfloor$ is the integer nearest to $x$. For the local Grover iteration it may require to perform one or two extra steps as given in eq. (214). Numerically it works well for $N \leq 10^{6}$ except for $K=2, B=2$ case.

## V. CONCLUSION

We have provided a detailed discussion on database search algorithms in this review. To understand how quantum mechanics can be exploited to expedite the process of computing we started our discussion with the Deutsch's algorithm and Deutsch-Jozsa algorithm which
can find whether a function is constant or balanced in just one oracle call compared to $\mathcal{O}(N)$ oracle calls by a classical computer. Bernstein-Vazirani algorithm, which is one variation of the Deutsch-Jozsa algorithm, is also discussed.

We then discussed Grover algorithm for database search. The database of $N$ elements can have a single or multiple target elements in it. The elements in a database can have some order(sorted database) or no order(unsorted database) at all. The unsorted database with single target element can be searched with Grover algorithm in $\mathcal{O}(\sqrt{N})$ steps compared to $\mathcal{O}(N)$ steps by a classical computer. This is an example of quadratic speed up in computation time. Similarly in the unsorted database with $M$ target elements one of the target elements can be obtained in $\mathcal{O}\left(\sqrt{\frac{N}{M}}\right)$ steps by Grover algorithm. If there is any structure/order in the database then by exploiting the structure the target element can be searched even in less time by Grover algorithm. It is not possible to devise an algorithm which can search in less time than what Grover algorithm needs, i.e $\mathcal{O}(\sqrt{N})$ oracle calls.

Instead of searching the whole database for the target element sometimes it is reasonable to divide the whole database in several blocks and then look for the block which contains the target element. Grover and Radhakrishnan found an algorithm for this type of partial search, which takes $j=\left(\frac{\pi}{4}+\frac{\beta(K)-\eta(K)}{\sqrt{K}}\right) \sqrt{N}$ steps. Korepin latter improved the partial search algorithm by optimizing the coefficient $\beta(K)-\eta(K)$. This can further be generalized to include several target elements and the final global iteration can be modified by including phase factors so that the target block is obtained with unit success probability.

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[1] M. A. Nielsen and I.L. Chuang, "Quantum Computation and Quantum Information", (Cambridge: Cambridge University Press) (2000).
[2] V. E. Korepin and Y. Xu, Int. J. Mod. Phys. B 31 5727-5758 (2009).
[3] G. Brassard, P. Høyer, M. Mosca and A. Tapp, Quantum Amplitude Amplification and Estimation, Quantum computation and Quantum information, 305 53-74 (2000).
[4] P. W. Shor, Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer, Proc: 35th Annual Symposium on Foundations of Computer Science, Santa Fe, NM, 20-22, Nov. (1994).
[5] P. W. Shor, SIAM J. Sci. Stat. Comput. 261484 (1997).
[6] L. K. Grover, A fast quantum mechanical algorithm for database search, Pro. 28th Annual ACM Symp. Theor. Comput. (STOC) 212 (1996).
[7] L. K. Grover, Phys. Rev. Lett. 79325 (1997).
[8] R. Cleve, A. Ekert, L. Henderson, C. Macchiavello and M. Mosca, Complexity 433 (1998).
[9] C. H. Bennett, E. Bernstein, G. Brassard and U. Vazirani, SIAM J. Comput. 26(5) 1510 (1997).
[10] M. Boyer, G. Brassard, P. Høyer and A. Tapp, Frotsch. Phys. 46493 (1998).
[11] C. Zalka, Phys. Rev. A 602746 (1999).
[12] L. K. Grover and J. Radhakrishnan, ACM Symp. Parallel Algorithms and Architectures, Las Vegas, Nevada, USA 186 (2005).
[13] B. S. Choi and V. E. Korepin, Quantum Inf. Process. 637 (2007).
[14] V. E. Korepin, J. Phys. A 38 L731 (2005).
[15] V. E. Korepin and J. Liao, Quantum Inf. Process. 51573 (2006).
[16] V. E. Korepin and B. C. Vallilo, Prog. Theor. Phys. 116783 (2006).
[17] V. E. Korepin and Y. Xu, Proc. SPIE (Quantum Information and Computation V) 6573 65730C (2007).
[18] V. E. Korepin and Y. Xu, Int. J. Mod. Phys. B 215187 (2007).
[19] J. Preskill, Lecture notes on quantum information and computation, http://www.theory.caltech.edu/people/preskill/ph229/index.html.
[20] L. K. Grover, Phys. Rev. Lett. 80 4329-4332 (1998).
[21] E. Farhi and S. Gutmann, arXiv: quant-ph/9711035.
[22] L. K. Grover, Chaos, Solitons and Fractals 1016951705 (1999).
[23] E. Farhi and S. Gutmann, arXiv: quant-ph/9612026.
[24] E. Farhi, J. Goldstone, S. Gutmann and M. Sipser, arXiv: quant-ph/0001106.
[25] J. Roland and N. J. Cerf, Phys. Rev. A 65042308 (2002).
[26] B.-S. Choi, T. A Walker and S. L. Braunstein, Quantum Information Processing, 6 1-8 (2007).


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