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Bimodal behavior of post-measured entropy and one-way quantum deficit for two-qubit X states

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Abstract A method for calculating the one-way quantum deficit is developed. It involves a careful study of post-measured entropy shapes. We discovered that in some regions of X-state space the post-measured entropy $\tilde{S}$ as a function of measurement angle $\theta \in [0, \pi/2]$ exhibits a bimodal behavior inside the open interval $(0, \pi/2)$, i.e., it has two interior extrema: one minimum and one maximum. Furthermore, cases are found when the interior minimum of such a bimodal function $\tilde{S}(\theta)$ is less than that one at the endpoint $\theta = 0$ or $\pi/2$. This leads to the formation of a boundary between the phases of one-way quantum deficit via *finite* jumps of optimal measured angle from the endpoint to the interior minimum. Phase diagram is built up for a two-parameter family of X states. The subregions with variable optimal measured angle are around 1% of the total region, with their relative linear sizes achieving 17.5%, and the fidelity between the states of those subregions can be reduced to $F = 0.968$. In addition, a correction to the one-way deficit due to the interior minimum can achieve 2.3%. Such conditions are favorable to detect the subregions with variable optimal measured angle of one-way quantum deficit in an experiment.

Keywords X density matrix · Post-measured entropy · Unimodal and bimodal functions · One-way quantum deficit

1 Introduction

Quantum correlation is a key feature of quantum mechanics and it lies at the heart of quantum information science. Besides the quantum entanglement and discord, the one-way quantum deficit is one of the most important measures of quantum correlation [1,2,3,4]. The entanglement is identical to the discord

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and one-way deficit for the pure quantum states, whereas the discord and one-way deficit coincide in considerably more general cases — they are the same for the Bell-diagonal states and even for the X states with zero Bloch vector for one qubit (i.e., with a single maximally mixed marginal) if the local measurements are performed on this qubit [5].

Definitions of quantum discord Q and one-way quantum deficit Δ involve the minimization procedure to obtain the optimal measurement performed on one part of bipartite system. This procedure for the two-qubit systems with X density matrix is reduced to the minimization problem on one variable – the polar angle $\theta \in [0, \pi/2]$ (see Refs. [6,7,8,9]). Moreover, a formula for the quantum discord is presented in a partially analytic (piecewise-analytical-numerical) form [10,11,12],

$$Q = \min\{Q_0, Q_\vartheta, Q_{\pi/2}\}. \tag{1}$$

Here, the subfunctions (branches) Q_0 and $Q_{\pi/2}$ are the analytical expressions (corresponding to the discord with optimal measurement angles equaling zero and $\pi/2$, respectively) and only the third branch Q_ϑ requires to perform numerical minimization to obtain state-dependent minimizing angle $\vartheta \in (0, \pi/2)$ if, of course, the interior minimum exists. Equations for 0- and $\pi/2$-boundaries separating respectively the Q_0 and $Q_{\pi/2}$ regions with the Q_ϑ one can be written as [10,11,12]

$$Q''(0) = 0, \qquad Q''(\pi/2) = 0. \tag{2}$$

Here $Q''(0)$ and $Q''(\pi/2)$ are the second derivatives of the measurement-dependent discord function $Q(\theta)$ with respect to θ at the endpoints $\theta = 0$ and $\pi/2$, correspondingly. The equations (2) are based on the unimodality hypothesis for the function $Q(\theta)$ which is confirmed for different classes of X states [12,13]. Notice that Eqs. (2) reflect the bifurcation mechanism of appearance of the minimum inside the interval $(0, \pi/2)$.

On the other hand, as mentioned above, there is a close connection between the one-way quantum deficit and quantum discord. Therefore it would be tempting to propose that similar properties are valid for the measurement-dependent one-way quantum deficit function $\varDelta(\theta) = \tilde{S}(\theta) - S$, where S is the pre-measurement entropy.

Recently, the authors [14] have claimed the result which is reduced to the statement that the one-way quantum deficit $\Delta = \min_\theta \varDelta(\theta)$ for the general X states is given by

$$\Delta = \begin{cases} \varDelta(\vartheta), & \varDelta''(0) < 0 \text{ and } \varDelta''(\pi/2) < 0,\ \vartheta \in (0, \pi/2); \\ \min\{\varDelta(0), \varDelta(\pi/2)\}, & \text{others.} \end{cases} \tag{3}$$

If the function $\varDelta(\theta)$ is monotonic or has single extremum inside the interval $(0, \pi/2)$ this conclusion takes place.

In the present paper we show that the post-measured entropy and consequently the measurement-dependent one-way quantum deficit can display more general behavior which refutes the relation (3). We discuss the difficulties arisen from a new type of behavior and propose, instead of Eq. (3), the method giving the correct calculation of one-way deficit for two-qubit X states.

2 Results and discussion

Let us consider a two-parameter family of X states

$$\rho_{AB} = q_1|\Psi^+\rangle\langle\Psi^+| + q_2|\Psi^-\rangle\langle\Psi^-| + (1-q_1-q_2)|00\rangle\langle 00|, \tag{4}$$

where $|\Psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$. This family generalizes the class of special X states from Ref. [14] which corresponds to the case $q_1 = 0$.

The density matrix (4) in open form is given as

$$\rho_{AB} = \begin{pmatrix} 1-q_1-q_2 & 0 & 0 & 0 \\ 0 & (q_1+q_2)/2 & (q_1-q_2)/2 & 0 \\ 0 & (q_1-q_2)/2 & (q_1+q_2)/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

Eigenvalues of this matrix equal

$$\lambda_1 = 1-q_1-q_2, \quad \lambda_2 = q_1, \quad \lambda_3 = q_2, \quad \lambda_4 = 0. \tag{6}$$

Owing to the non-negativity requirement for any density matrix, one obtains that the domain of definition for the parameters (arguments) q_1 and q_2 is restricted by conditions

$$q_1 \geq 0, \quad q_2 \geq 0, \quad q_1 + q_2 \leq 1. \tag{7}$$

Thus, the domain in plane (q_1, q_2) is the triangle $\mathcal{T}$ which is shown in Fig. 1.

One-way quantum deficit (quantum work deficit) for a bipartite state ρ_{AB} is defined as the minimal increase of entropy after a von Neumann measurement on one party (without loss of generality, say, B) [15,16,17]

$$\Delta = \min_{\{\Pi_k\}} S(\tilde\rho_{AB}) - S(\rho_{AB}), \tag{8}$$

where

$$\tilde\rho_{AB} = \sum_k (I\otimes\Pi_k)\rho_{AB}(I\otimes\Pi_k)^+ \tag{9}$$

is the weighted average of post-measured states and $S(\cdot)$ means the von Neumann entropy. In Eqs. (8) and (9), Π_k $(k = 0, 1)$ are the general orthogonal projectors

$$\Pi_k = V\pi_k V^+, \tag{10}$$

where $\pi_k = |k\rangle\langle k|$ and transformations $\{V\}$ belong to the special unitary group SU_2. Rotations V may by parametrized by two angles θ and ϕ (polar and azimuthal, respectively):

$$V = \begin{pmatrix} \cos(\theta/2) & -e^{-i\phi}\sin(\theta/2) \\ e^{i\phi}\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \tag{11}$$

with $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$.

Using Eq. (6) one gets the pre-measured entropy

$$S(q_1, q_2) \equiv S(\rho_{AB}) = -q_1 \log q_1 - q_2 \log q_2 - (1-q_1-q_2)\log(1-q_1-q_2). \tag{12}$$

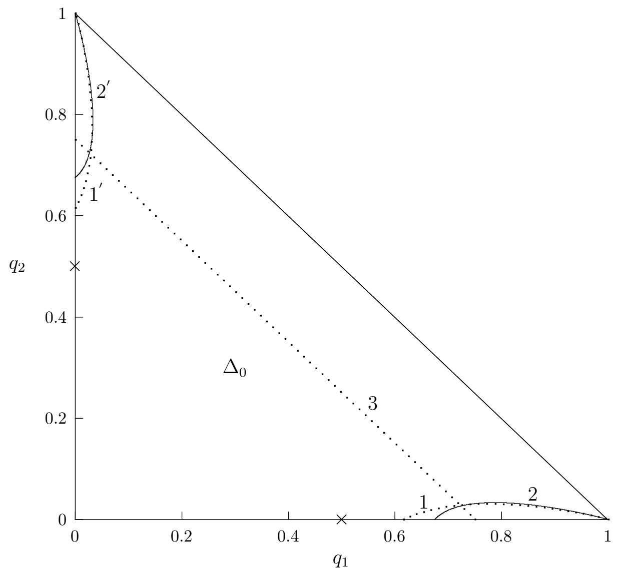

Fig. 1 Triangle $\mathcal{T}$ in the plane (q_1, q_2) with vertices $(0,0)$, $(0,1)$, and $(1,0)$ is the permitted region for the parameters q_1 and q_2. Dotted lines 1 and $1'$ are the boundaries defined by the equation $\varDelta_0 = \varDelta_{\pi/2}$. Solid lines 2 and $2'$ are the $\pi/2$-boundaries. Dotted line 3 is the path $q_1 + q_2 = 0.75$. Crosses ($\times$) at the points $(0, 0.5)$ and $(0.5, 0)$ mark the 0-boundaries

Eigenvalues of the matrix $\tilde{\rho}_{AB}$ are equal to

$$\begin{aligned} \varLambda_{1,2} &= \frac{1}{4}[\![1 + (1 - q_1 - q_2)\cos\theta \pm \{[1 - q_1 - q_2 + (1 - 2q_1 - 2q_2)\cos\theta]^2 \\ &\quad +(q_1 - q_2)^2 \sin^2\theta\}^{1/2}]\!] \\ \varLambda_{3,4} &= \frac{1}{4}[\![1 - (1 - q_1 - q_2)\cos\theta \pm \{[1 - q_1 - q_2 - (1 - 2q_1 - 2q_2)\cos\theta]^2 \\ &\quad +(q_1 - q_2)^2 \sin^2\theta\}^{1/2}]\!]. \end{aligned} \tag{13}$$

It is seen that the azimuthal angle ϕ has dropped out from the given expressions. This is due to the fact that one pair of non-diagonal entries of the density matrix (5) vanishes. Using Eqs. (13) we arrive at the post-measured entropy (entropy after measurement)

$$\tilde{S}(\theta; q_1, q_2) \equiv S(\tilde{\rho}_{AB}) = h_4(\varLambda_1, \varLambda_2, \varLambda_3, \varLambda_4), \tag{14}$$

where $h_4(x_1, x_2, x_3, x_4) = -\sum_{i=1}^{4} x_i \log x_i$ with additional condition $x_1 + x_2 + x_3 + x_4 = 1$ is the quaternary entropy function.

Notice that function $\tilde{S}$ of argument θ is invariant under the transformation $\theta \to \pi - \theta$ therefore it is enough to restrict oneself by values of $\theta \in [0, \pi/2]$. Moreover, the pre- and post-measured entropies S and $\tilde{S}$, as functions of q_1 and q_2, are symmetric under the exchange $q_1 \rightleftharpoons q_2$.

Equations (12)–(14) define the measurement-dependent one-way deficit function $\Delta(\theta) = \tilde{S}(\theta) - S$. Direct calculations show that for every choice of model parameters the function $\tilde{S}(\theta)$ and hence $\Delta(\theta)$ possess an important property, namely their first derivatives with respect to θ identically equal zero at both endpoints $\theta = 0$ and $\theta = \pi/2$:

$$\tilde{S}'(0) = \Delta'(0) \equiv 0, \qquad \tilde{S}'(\pi/2) = \Delta'(\pi/2) \equiv 0. \tag{15}$$

From Eqs. (13) and (14) we get the expressions for the post-measurement entropy at the endpoint $\theta = 0$,

$$\tilde{S}_0(q_1, q_2) = -(1 - q_1 - q_2)\log(1 - q_1 - q_2) - (q_1 + q_2)\log[(q_1 + q_2)/2], \tag{16}$$

and at the second endpoint $\theta = \pi/2$:

$$\tilde{S}_{\pi/2}(q_1, q_2) = \log 2 + h((1 + \sqrt{(1 - q_1 - q_2)^2 + (q_1 - q_2)^2})/2), \tag{17}$$

where $h(x) = -x\log x - (1 - x)\log(1 - x)$ is the Shannon binary entropy function. Together with Eq. (12) they supply us with explicit expressions for the one-way deficit at the endpoints: $\Delta_0 = \Delta(0)$ and $\Delta_{\pi/2} = \Delta(\pi/2)$. In particular, if q_1 or q_2 equals zero then $\Delta_0 = q\log 2$ $(= q, \text{bit})$, where $q = \{q_1, q_2\}$.

Solving the transcendental equation

$$\Delta_0 = \Delta_{\pi/2} \tag{18}$$

or, the same, $\tilde{S}_0 = \tilde{S}_{\pi/2}$ we find the subregions in the plane (q_1, q_2), where $\Delta_{\pi/2} < \Delta_0$ (restricted in Fig. 1 by dotted curves 1 and $1'$ and corresponding Cartesian axes Oq_1 and Oq_2) and where, v.v., $\Delta_0 < \Delta_{\pi/2}$ (marked in Fig. 1 by symbol Δ_0). The curve 1 has two endpoints on the axis Oq_1: at $q_1 = 0.61554$ and $q_1 = 1$. Analogously for the curve $1'$ (see Fig. 1).

The 0- and $\pi/2$-boundaries, i.e., where respectively the second derivatives

$$\Delta''(0) = 0 \quad \text{and} \quad \Delta''(\pi/2) = 0 \tag{19}$$

or, the same, $\tilde{S}''(0) = 0$ and $\tilde{S}''(\pi/2) = 0$, will be needed below. As calculations yield,

$$\begin{aligned}\tilde{S}''(\pi/2) \; &= \frac{(q_1 - q_2)^2}{2r^3}[r^2 - (1 - 2q_1 - 2q_2)^2]\ln\frac{1 + r}{1 - r} \\ &- \frac{(1 - q_1 - q_2)^2}{1 - r^2}[1 - 2(1 - 2q_1 - 2q_2)(1 - \frac{1 - 2q_1 - 2q_2}{2r^2})],\end{aligned} \tag{20}$$

where

$$r = \sqrt{(1 - q_1 - q_2)^2 + (q_1 - q_2)^2}. \tag{21}$$

On the other hand, calculations show that the second derivative $\tilde{S}''(\theta)$ with respect to θ is finite at $\theta = 0$ only when $q_1 q_2 = 0$:

$$\tilde{S}''(0) = \frac{1 - 3q + 2q^2}{2 - 3q}\ln\frac{2(1 - q)}{q}, \tag{22}$$

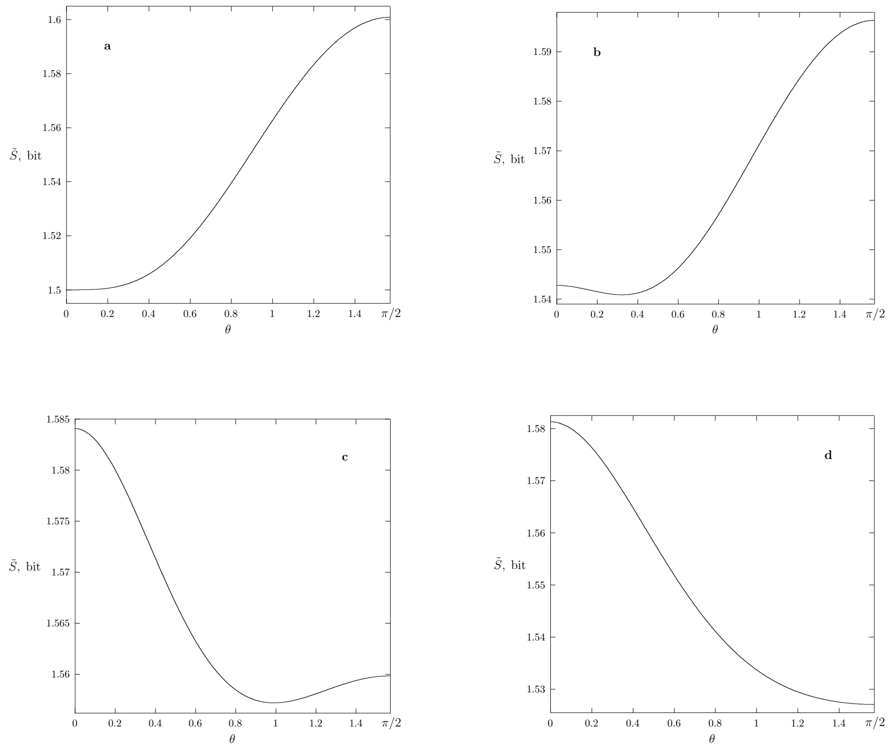

Fig. 2 Post-measurement entropy $\tilde{S}$ vs θ by $q_2 = 0$ and $q_1 = 0.5$ (a), 0.55 (b), 0.65 (c), and 0.7 (d)

where again $q = \{q_1, q_2\}$. The roots of equation $\tilde{S}''(0) = 0$ are 1/2 and 1. Thus, the bifurcation 0-boundary exists only if $q_1 = 0$ or, inversely, $q_2 = 0$ (that is, only at two points on each of the Cartesian axes Oq_1 and Oq_2). The corresponding 0-boundaries $q_1 = 1/2$, when $q_2 = 0$, and $q_2 = 1/2$, when $q_1 = 0$ are shown in Fig. 1 by the crosses.

The results of numerical solution of the equation $\tilde{S}''(\pi/2) = 0$ are presented in Fig. 1 by solid lines 2 and 2′. The endpoints for the curve 2 on the axis Oq_1 are $q_1 = 0.67515$ and $q_1 = 1$. The curves 1 and 2 intersect at the point with coordinates $q_1 = 0.739\,409$ and $q_2 = 0.029\,686$ ($q_1 + q_2 = 0.769\,095$). Analogously for the curves 1′ and 2′ with, of course, permutation of q_1 and q_2 (see again Fig. 1).

Let us consider the behavior of post-measured entropy $\tilde{S}(\theta)$ and non-minimized one-way deficit $\Delta(\theta)$ by moving along different trajectories (paths) in the triangle $\mathcal{T}$.

Start with the passing along the leg of triangle $\mathcal{T}$. Figure 2 shows the evolution of shape of the post-measured entropy $\tilde{S}(\theta; q_1, 0)$ with changing the

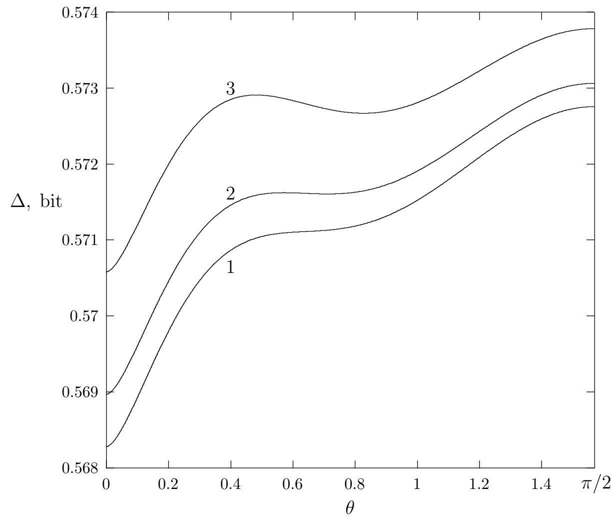

Fig. 3 Measurement-dependent one-way quantum deficit $\Delta(\theta)$ along the line $q_1+q_2 = 0.75$ by $q_1 = 0.72$ (1), 0.72015 (2), and 0.7205 (3). The bimodality appearing from an inflection point is clearly seen

parameter q_1. The curve has the monotonically increasing behavior when the argument q_1 varies from $q_1 = 0$ to $q_1 = 1/2$; see Fig. 2(a). At the point $q_1 = 1/2$ a bifurcation of the minimum at $\theta = 0$ occurs. Then, when q_1 increases from 0.5 to 0.67515, the curve $\tilde{S}(\theta)$ has, as shown in Figs. 2(b) and (c), the interior minimum, with the function $\tilde{S}(\theta)$ being here unimodal. So, the region with variable optimal angle ϑ takes up a part $0.17515 \approx 17.5\%$ on the section $[0, 1]$ of Oq_1 axis and the fidelity of states at points $(0.5, 0)$ and $(0.67515, 0)$ is equal to $F = 96.8\%$[1]. The position of such a local minimum smoothly increases from zero to $\pi/2$; see again the curves in Figs. 2(b) and (c). The values of $\tilde{S}_0$ and $\tilde{S}_{\pi/2}$ become equal at the point $q_1 = 0.61554$ ($\tilde{S}_0 = \tilde{S}_{\pi/2} = 1.57667$ bit, hence $\Delta_{\pi/2} = \Delta_0 = q_1 = 0.61554$ bit) and the depth of interior minimum is 0.01397 bit what gives a relative correction to the one-way deficit equaled $\delta\Delta = 2.3\%$. Then, at the value of $q_1 = 0.67515$, the system experiences a new sudden transition – from the branch, which is characterized by the continuously changing optimal angle ϑ in the full interval (from 0 to $\pi/2$), to the branch $\tilde{S}_{\pi/2}$ with constant optimal measurement angle equaled $\pi/2$. After this the curves of post-measured entropy exhibit monotonically decreasing behavior as illustrated in Fig. 2(d). One should emphasize here that the minimized one-way quantum deficit, $\Delta = \min_\theta \Delta(\theta)$, vs the model parameter q_1 is continuous and smooth. Nevertheless, the function $\Delta(q_1)$ has nonanalyticities at the points $q = 0.5$ and 0.67515 which manifest themselves in higher derivatives.

[1] Note for comparison that in two-photon experiments one achieves now the values of fidelity $F = 99.8(2)\%$ [18] and $F = 99.8(1)\%$ [19].

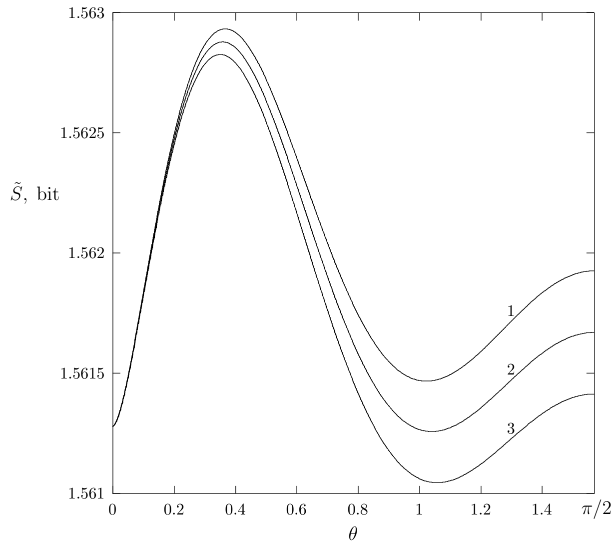

Fig. 4 Post-measured entropy $\tilde{S}$ as a function of θ by $q_2 = 0.75 - q_1$ and $q_1 = 0.7215$ (1), 0.7216 (2), and 0.7217 (3).

Consider now the behavior of post-measurement entropy and measurement-dependent one-way deficit in the bulk area of $\mathcal{T}$. We can inspect the total domain taking all possible straight-line trajectories $q_1 + q_2 = const \le 1$. The behavior of the system is, obviously, symmetric relative to the middle of such trajectories. Take, for instance, the trajectory $q_1+q_2 = 0.75$ which is shown in Fig. 1 by the straight line 3. The shape of the curve $\varDelta(\theta)$ has the monotonically increasing type in the middle of this trajectory ($q_1 = q_2 = 0.375$). However, with the increase of the value of parameter q_1, the birth of a pair of extrema from an inflection point occurs inside the interval $(0, \pi/2)$; the situation is illustrated in Fig. 3. This phenomenon happens at the value of $q_1 = 0.72015$. According to the definition (see, e.g., Ref. [20]) a function having two extrema in some interval is called bimodal on this interval.

With further increase of the q_1 value a qualitatively new effect is observed. We demonstrate it by the curves $\tilde{S}(\theta)$ shown in Fig. 4. When the parameter q_1 achieves the value of 0.72159, the position of global minimum suddenly jumps through a finite step $\varDelta\vartheta$ from zero to $\vartheta = 1.0409 \approx 60^\circ$ (see Fig. 4). As a result, the fracture is arisen on the continuous curve of minimized one-way quantum deficit $\Delta(q_1)$. The position of the fracture point is determined from the equation $\tilde{S}_0 = \tilde{S}_\vartheta$ or

$$\varDelta_0 = \varDelta_\vartheta. \tag{23}$$

After this the interior minimum lies lower than another minimum located at the endpoint $\theta = 0$. Notice that behavior of curve 3 in Fig. 4 leads to a contradiction with Eq. (3), i.e., the equation is incorrect for general X states.

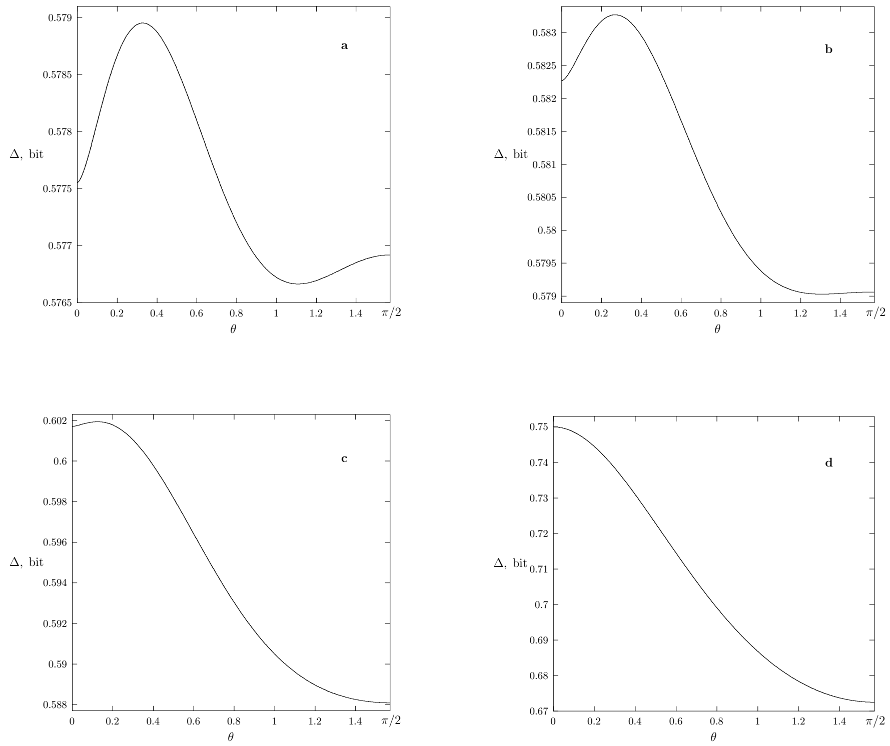

Fig. 5 Measurement-dependent one-way quantum deficit $\varDelta(\theta)$ along the line $q_1+q_2 = 0.75$ by $q_1 = 0.722$ (a), 0.723 (b), 0.727 (c), and 0.75 (d). Minimum on the curve disappears at the endpoint $\theta = \pi/2$ through the bifurcation mechanism whereas the maximum annihilates at the endpoint $\theta = 0$ via the singularity mechanism

With further increasing q_1 the interior minimum smoothly moves to the point $\theta = \pi/2$ and disappears at $q_1 = 0.72358$ when the trajectory crosses the curve 2, i.e., the $\pi/2$-boundary (see Fig. 1). The dynamics of corresponding deformations of $\varDelta(\theta)$ is depicted in Fig. 5. After crossing the $\pi/2$-boundary, the behavior of Δ undergoes to the branch $\varDelta_{\pi/2}$ up to the point of contact of trajectory with the Cartesian axis, i.e., up to $q_1 = 0.75$, where the interior maximum of $\varDelta(\theta)$ disappears at the endpoint $\theta = 0$. This happens through a new non-bifurcation (and non-inflection) mechanism. Since the second derivative $\varDelta''(\theta)$ at $\theta = 0$ diverges out of the Cartesian axes we will call this mechanism the singular one.

As a result, the one-way quantum deficit is obtained from the final equation

$$\Delta = \min\{\varDelta_0, \varDelta_\vartheta, \varDelta_{\pi/2}\}, \tag{24}$$

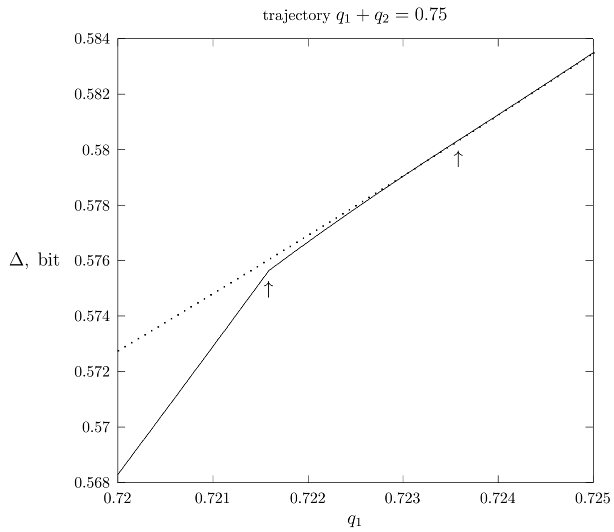

Fig. 6 One-way quantum deficit Δ vs q_1 along the path $q_1 + q_2 = 0.75$ is shown by solid line. Dotted line corresponds to the branch $\Delta_{\pi/2}$. Fraction Δ_ϑ with variable optimal measured angle lies between two arrows. The transition $\Delta_0 \leftrightarrow \Delta_\vartheta$ is displayed as a fracture on the curve $\Delta(\mathrm{q}_1)$ whereas the $\Delta_\vartheta \leftrightarrow \Delta_{\pi/2}$ one is hidden — the curve is here continuous and smooth

Table 1 Jumps of optimal measured angles, $\Delta\vartheta$, on the boundary between the phases Δ_0 and Δ_ϑ

q_1	q_2	$\Delta\vartheta$
0.5	0	$0 = 0°$
0.544 535	$0.55 - q_1$	$0.1267 \approx 7°$
0.588 104	$0.6 - q_1$	$0.2470 \approx 14°$
0.631 766	$0.65 - q_1$	$0.4020 \approx 23°$
0.676 082	$0.7 - q_1$	$0.6252 \approx 36°$
0.721 590	$0.75 - q_1$	$1.0409 \approx 60°$
0.739 409	0.029 686	$\pi/2 = 90°$

where Δ_0 and $\Delta_{\pi/2}$ are known in closed analytical forms and Δ_ϑ is found numerically. The behavior of one-way deficit along the trajectory $q_1+q_2 = 0.75$ is shown in Fig. 6.

Either totally or partially similar behavior takes place for other trajectories $q_1 + q_2 = const$ which go lower the intersection point of curves defined by equations $\Delta_0 = \Delta_{\pi/2}$ and $\Delta''(\pi/2) = 0$, i.e, when $const \leq 0.769\,095$. For example, in the case of trajectory $q_1 + q_2 = 0.65$, the bimodality appears at $q_1 \simeq 0.631$ and a jump of optimal measurement angle from zero happens at $q_1 = 0.631\,766$. Values of jump angles $\Delta\vartheta$ in different cases are collected in Table 1.

A set of points where the optimal measurement angle discontinuously changes from zero to a finite value gives the jumping (or hopping) bound-

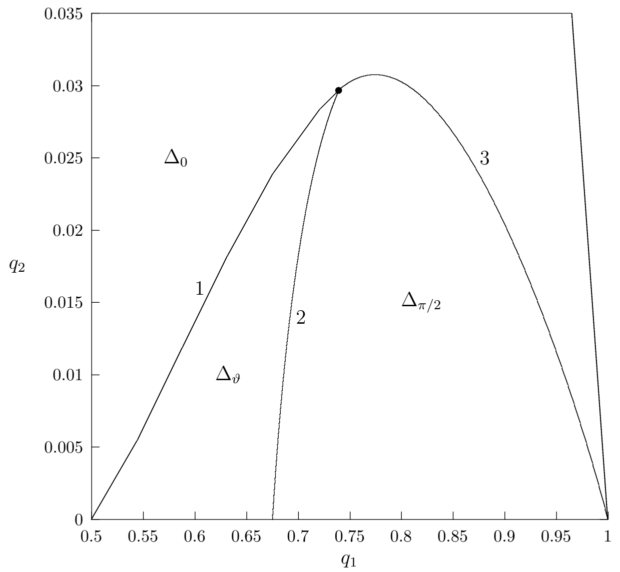

Fig. 7 A fragment of phase diagram. The boundary 1 is defined by equation $\varDelta_0 = \varDelta_\vartheta$, 2 is the $\pi/2$-boundary, and the boundary 3 is defined by equation $\varDelta_0 = \varDelta_{\pi/2}$. The black circle (•) is the intersection point of $\pi/2$-boundary with equilibrium curve of phases $\varDelta_0$ and $\varDelta_{\pi/2}$. (This figure represents a part of the domain of definition shown in Fig. 1.)

ary; it serves instead of the absent ordinary 0-boundary (see Fig. 7). Between this boundary and the $\pi/2$-one, there exists an intermediate phase (fraction) $\varDelta_\vartheta$ with state-dependent optimal measurement angle ϑ which smoothly varies from some nonzero value to $\pi/2$. The flat of two subregions with variable optimal angle, $\varDelta_\vartheta$, is near 1% of the one of total domain $\mathcal{T}$.

When *const* $> 0.769\,095$ (i.e., when the trajectories lie above the black circle shown in Fig. 7), the situation is different. With increasing q_1 from middle values to the endpoint on the axis Oq_1 the curves $\tilde{S}(\theta)$ or $\varDelta(\theta)$ are deformed from monotonically increasing shape to the shape with a single interior maximum (which is born at the point, where $\varDelta''(\pi/2) = 0$) and then a sudden transition $\varDelta_0 \to \varDelta_{\pi/2}$ occurs at the boundary defined by the relation $\varDelta_0 = \varDelta_{\pi/2}$ (line 3 in Fig. 7). Here, there is no intermediate region $\varDelta_\vartheta$ and the transition $\varDelta_0 \to \varDelta_{\pi/2}$ is characterized visually by a fracture on the curve $\Delta(q_1)$. Typical behavior of one-way deficit is shown in Fig. 8 along the trajectory $q_1 + q_2 = 0.8$.

So, the presented method to calculate the one-way quantum deficit of X states is reduced first of all to careful analyzing of the shapes of post-measured entropy or measurement-dependent one-way deficit curves. One should also solve equations for the boundaries between three possible phases (branches): Eqs. (18), (19), and (23). After this the one-way quantum deficit is obtained from the piecewise-analytical-numerical formula (24).

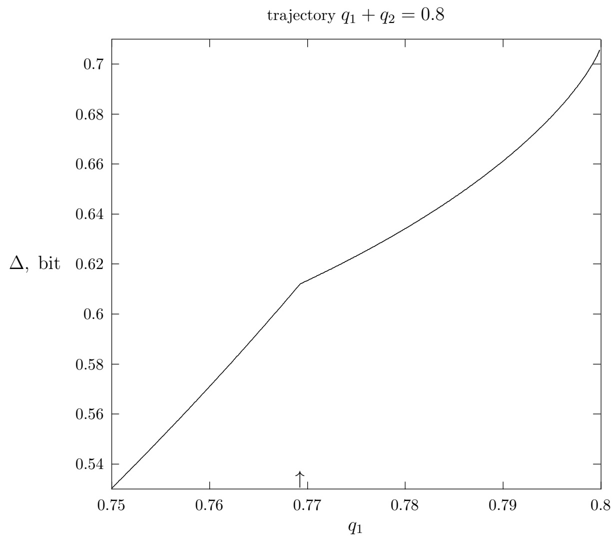

Fig. 8 Dependence of Δ vs q_1 by $q_2 = 0.8 - q_1$. Arrow marks the position of a fracture at the point $q_1 = 0.769\,269$, where the one-way deficit undergoes from the branch $\varDelta_0$ to the $\varDelta_{\pi/2}$ one

3 Summary and concluding remarks

In this paper we have found that besides the monotonic and unimodal behavior the post-measured entropy and hence the measurement-dependent one-way quantum deficit upon the measurement angle can have a new kind of behavior. Namely, these functions can exhibit the bimodal shape in the open interval $(0, \pi/2)$ for different regions in the space of X state parameters. This expands the variety of behavior for the one-way quantum deficit Δ. In particular, a new state-dependent phase (fraction) which is characterized by a *partial* interval of optimal measured angles has been found. Instead of smooth conjugation of the branches $\varDelta_0$ and $\varDelta_{\pi/2}$ this leads to a fracture on the curve of one-way deficit.

New mechanism of a boundary arising between the phases via jumping the optimal measured angle on a finite step has been discovered. Instead of bifurcation conditions (19) the boundary is now determined by a relation like (23). The study of post-measured entropy shapes is the general way to determine the correct one-way quantum deficit.

This is in contrast with the behavior of conditional entropy and, consequently, measurement-dependent quantum discord in the same regions of parameter space: their behavior is restricted by monotonic and unimodal types. In any case, this rather simple and therefore attractive picture is valid for the different specific cases and subclasses of X states [10,12,13]. In particular, such a behavior of conditional entropy is confirmed for the symmetric XXZ states

[13] those may be written in an equivalent form as

$$\rho_{AB} = q_1|\Psi^+\rangle\langle\Psi^+| + q_2|\Psi^-\rangle\langle\Psi^-| + q_3|00\rangle\langle 00| + q_4|11\rangle\langle 11| \qquad (25)$$

with $q_1 + q_2 + q_3 + q_4 = 1$.

An intriguing question remains: are there any more general shapes of curves for the post-measured entropy of X states? For instance, can this entropy have trimodal and, maybe, multimodal dependence? The answer to these and other questions should come from the future investigations of post-measurement entropy shapes in the full five-parameter X-state space.

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