# Relay entanglement and clusters of correlated spins.

# S.I.Doronin and A.I. Zenchuk

<sup>2</sup>Institute of Problems of Chemical Physics, RAS, Chernogolovka, Moscow reg., 142432, Russia.

## Abstract

Considering a spin-1/2 chain, we suppose that the entanglement passes from a given pair of particles to another one, thus establishing the relay transfer of entanglement along the chain. Therefore, we introduce the relay entanglement as a sum of all pairwise entanglements in a spin chain. For more detailed studying the effects of remote pairwise entanglements, we use the partial sums collecting entanglements between the spins separated by up to a certain number of nodes. The problem of entangled cluster formation is considered, and the geometric mean entanglement is introduced as a characteristics of quantum correlations in a cluster. Generally, the life-time of a cluster decreases with an increase in its size.

PACS numbers:

#### I. INTRODUCTION

The quantum correlations are responsible for a principal advantages of quantum devises over their classical counterparts. The problems of correlations between remote systems [1, 2], formation [3–8] and transfer of quantum correlations [9–13] are getting an increasing meaning because of progress in quantum information processing which includes quantum state creation and information transfer [14–18]. Obviously, the relation between the information transfer and quantum correlations exist; however it can not be simply released. Thus, the sender-receiver correlations (entanglement) were studied in set of papers [19–24]. Nevertheless, many issues require clarification. For instance, it was shown that the sender-receiver entanglement (SR-entanglement) is vanishing in the case of perfect state transfer [22] and it also can be vanishing in the case of remote state creation [25]. It was shown in [25] that the possibility to transfer the information (i.e., the parameters of the sender's initial state) from the sender to the receiver is directly related to a certain determinant condition (associated with the informational correlation in Ref.[26]). But still this determinant condition is not straightforwardly related to the traditional measures such as entanglement [27, 28] and discord [29–31].

It follows from the above that quantum information transfer can be established without the SR-entanglement, and therefore, this entanglement can not be responsible for such a process [22, 25]. We emphasize that this fact makes principal difference between the information transfer and teleportation [32–34] where the entanglement between the sender and receiver is necessary.

Our paper is aimed at studying quantum correlations which can govern the state (information) transfer from the sender to the receiver. We assume that, similar to the simple explanation of the state transfer as an excitation transfer from one node to another, the mechanism of entanglement transfer is basically the same. Namely, if at some instant there are two large pairwise entanglements, then we can say that the entanglement passes between these pairs. Therewith these pairs can be the pairs of remote spins (not of the close neighbors only). Thus, the entanglement is transferred along the chain during some evolution period T if there are significant time-overlaps between the pairwise entanglements such that at each instant t,  $0 < t \le T$ , there are at least two large pairwise entanglements. If this is the case, then the sum of all pairwise entanglements is non-vanishing over the

whole evolution period T, although each individual pairwise entanglement is large only over the relatively small sub-interval of the period T. We propose that this propagating entanglement supplements the process of information propagation. In other words, there is an assembly of pairwise entanglements (rather then the single SR-entanglement) that governs the information transfer along the spin chain. This assembly is characterized by the above sum which we refer to as the relay entanglement. First this concept was introduced in [35]. Thus, the relay entanglement is a pairwise mechanism of entanglement propagation along the spin chain and establishes the indirect SR-entanglement. We emphasize that the indirect SR-entanglement found here does not assume arising the SR-entanglement at any instant of the evolution period T. In particular, the SR-entanglement can be absent during the whole evolution period T as was observed in [25], such behavior does not contradict our protocol. Another feature of the relay entanglement is that it arises during the evolution even if there is no initial entanglement, unlike the transfer of entangled states from the multi-qubit sender to the multi-qubit receiver [9–11] where entanglement is present initially.

Of course, the contribution to the relay entanglement from the remote pairs and from the close ones is different. For detailed study of these contributions, we introduce the so-called partial sums collecting the entanglements between the spins separated by up to a certain number of nodes. We study the dependence of the partial sums from the distance between the correlated spins and show that the result significantly depends on the parameters of the sender's and receiver's initial states. In particular, the contribution from the remote spins can be more important in certain cases.

In addition, in the course of evolution, the pairwise entanglements can form clusters, i.e., families of entanglements with large time-overlaps between each two of them. Accordingly, the set of neighboring spins, such that all possible pairwise entanglements in this set are significant at some instant, form the cluster of correlated spins at this instant [35–37]. Each individual cluster exists during some time interval, then it diffuses and another cluster appears. We study the cluster formation in dependence on the initial parameters of the sender and receiver states and estimate the life-time of a cluster. As a characteristic of the entanglement in a cluster, we introduce a geometric mean of all the pairwise entanglements inside of this cluster. Formation of entangled clusters in an evolutionary chain represents a particular way of creating the quantum subsystems with all entangled nodes. Such clusters can be candidates for quantum registers of small size.

The structure of the paper is following. The model of communication line we deal with is described in Sec.II. The relay entanglement in a symmetrical model as a characteristics of entanglement propagation is studied in Sec.III. The clusters of entangled particles arising during evolution are described in Sec.IV. The obtained results are discussed in Sec.V.

#### II. MODEL

We consider a communication line based on a homogeneous spin chain which includes the one-qubit sender (S), transmission line (TL) and one-qubit receiver (R) and evolves under the nearest-neighbor XX-Hamiltonian [38]

$$H = \sum_{i=1}^{N-1} D(I_{ix}I_{(i+1)x} + I_{iy}I_{(i+1)y}). \tag{1}$$

The initial density matrix is a tensor product state:

$$\rho_0 = \rho_0^S \otimes \rho_0^{TL} \otimes \rho_0^R, \tag{2}$$

where the transmission line TL is in the ground state initially

$$\rho^{TL} = |0\rangle_{TL} \,_{TL}\langle 0|, \tag{3}$$

and the initial states of the sender S and receiver R are represented as follows [13]:

$$\rho_0^S = U^S \Lambda^S (U^S)^+, \quad \rho_0^R = U^R \Lambda^R (U^R)^+.$$
 (4)

Here the eigenvalue and eigenvector matrices read, respectively,

$$\Lambda^{S} = \operatorname{diag}(\lambda^{S}, 1 - \lambda^{S}), \quad \Lambda^{R} = \operatorname{diag}(\lambda^{R}, 1 - \lambda^{R}), \tag{5}$$

and

$$U^{S} = \begin{pmatrix} \cos\frac{\pi\alpha_{1}}{2} & -e^{-2i\pi\alpha_{2}}\sin\frac{\pi\alpha_{1}}{2} \\ e^{2i\pi\alpha_{2}}\sin\frac{\pi\alpha_{1}}{2} & \cos\frac{\pi\alpha_{1}}{2} \end{pmatrix}, \tag{6}$$

$$U^{R} = \begin{pmatrix} \cos\frac{\pi\beta_{1}}{2} & -e^{-2i\pi\beta_{2}}\sin\frac{\pi\beta_{1}}{2} \\ e^{2i\pi\beta_{2}}\sin\frac{\pi\beta_{1}}{2} & \cos\frac{\pi\beta_{1}}{2} \end{pmatrix}.$$
 (7)

The parameters  $\lambda^S$ ,  $\lambda^R$  are referred to as the eigenvalue control parameters, while the parameters  $\alpha_i$ ,  $\beta_i$  (i = 1, 2) are called the eigenvector control parameters. The term "control" means the responsibility of these parameters for the value of entanglement. The variation intervals for these parameters are following:

$$0 \le \alpha_i \le 1, \ 0 \le \beta_i \le 1, \ i = 1, 2, 0 \le \lambda^S \le 1, \ 0 \le \lambda^R \le 1.$$
(8)

Studying the evolution of pairwise correlations, we need the density matrix of the subsystem of the ith and jth spins, which reads

$$\rho^{(ij)}(t) = \text{Tr}_{/ij} \Big( V(t) \rho_0 V^+(t) \Big), \tag{9}$$

where the trace is taken over all the spins except the *i*th and *j*th ones. In particular, if i = 1 and j = N, we have the density matrix for the SR-state.

The dynamics of communication line with the one-qubit sender and receiver and ground initial state of TL reduces to the two-excitation subspace of the whole  $2^N$ -dimensional Hilbert space related to the N-node spin-1/2 system. This significantly simplifies calculations. Nevertheless, to provide a detailed analysis given below we restrict ourselves to the spin chain of N = 10 nodes.

We consider the evolution of quantum correlations during the time interval from zero to some optimized instant. Following Ref.[25], we chose such the instant that the sum of all but one diagonal elements of the SR-density matrix  $\rho^{(1N)}$ 

$$s = \sum_{i=2}^{4} \rho_{ii}^{1N} \tag{10}$$

(we use the basis  $|00\rangle$ ,  $|01\rangle$ ,  $|10\rangle$ ,  $|11\rangle$ ) averaged over the parameters of the initial state takes the maximal value. The instant found in this way corresponds to the case when the maximal initial signal is collected at the sender and receiver, i.e., the instant suitable for the state registration at the subsystem SR. The choice of this instant is predicted by our purpose of studying those correlations which can be responsible for the information transfer between the sender and receiver.

We consider entanglement as a measure of quantum correlations. The Wootters criterion [27, 28] allows us to calculate the quantum entanglement between the nodes n and m. Traditionally the entanglement E can be represented as a monotonic function of so-called

concurrence  $C, E = -\frac{1+\sqrt{1-C^2}}{2}\log_2\frac{1+\sqrt{1-C^2}}{2} - \frac{1-\sqrt{1-C^2}}{2}\log_2\frac{1-\sqrt{1-C^2}}{2}$ , where

$$C = \max(0, 2\lambda_{max} - \sum_{i=1}^{4} \lambda_i), \quad \lambda_{max} = \max(\lambda_1, \lambda_2, \lambda_3, \lambda_4). \tag{11}$$

Here  $\lambda_i$  are the eigenvalues of the following matrix

$$\tilde{\rho}^{(nm)} = \sqrt{\rho^{nm}(\sigma_y \otimes \sigma_y)(\rho^{nm})^*(\sigma_y \otimes \sigma_y)}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{12}$$

Hereafter we consider the concurrence as a measure of entanglement.

#### III. RELAY ENTANGLEMENT IN SYMMETRICAL MODEL

#### A. Preliminary remarks and symmetrical initial state

In general, the model introduced in Sec.II possesses 6 parameters of the initial state (4-7). However, according to the results of Ref. [25], the effect of  $\alpha_2$  and  $\beta_2$  on entanglement is negligible. In addition, we consider the symmetrical model for simplicity. Thus we set

$$\lambda^R = \lambda^S = \lambda, \quad \alpha_1 = \beta_1 = \alpha, \quad \alpha_2 = \beta_2 = 0. \tag{13}$$

In this setting, averaging the sum s in (10) over the parameters of the sender's and receiver's initial state reduces to

$$\langle s \rangle = \pi \int_0^1 d\alpha \int_{\frac{1}{2}}^1 d\lambda \sin(\alpha \pi) s(\alpha, \lambda).$$
 (14)

In the case N=10, the numerical analysis shows that  $\langle s \rangle$  reaches its maximal value at t=12.238 (we use dimensionless time setting D=1).

As mentioned in Introduction, the information transfer between the sender S and receiver R is not related to the SR-entanglement [25]. There is a large region in the plane of initial eigenvalues  $(\lambda^R, \lambda^S)$  such that the information transfer occurs without the entanglement between R and S (for any values of other initial-state parameters) during the whole evolution period from zero to the certain prescribed optimal instant for state registration. For N = 10, this region of vanishing SR-entanglement, bounded by the curve B, is shown in Fig.1.

The bisectrix  $\lambda^R = \lambda^S$  in Fig.1 crosses the boundary curve B at the point  $\lambda = 0.7493$  with  $\alpha = 0.4361$ . These values have been found in a straightforward way, see Ref.[25].

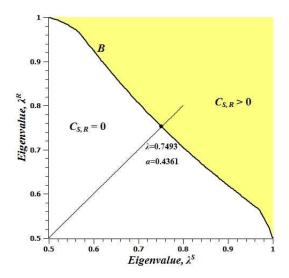


FIG. 1: The region of zero SR-entanglement  $C_{S,R}$  for the chain of N=10 nodes at the optimal instant t=12.238. The bisectrix  $\lambda^R=\lambda^S$  crosses the boundary B at the point  $\lambda=0.7493$  with  $\alpha=0.4361$ .

Namely, increasing  $\lambda$  along the bisectrix we find the critical value ( $\lambda = 0.7493$ ) such that there is no entanglement at  $\lambda$  below this value (independently on the value of  $\alpha$ ) and, at the critical value  $\lambda = 0.7493$ , the SR-entanglement is nonzero only at  $\alpha = 0.4361$ . Thus, the point ( $\lambda, \alpha$ ) =(0.7493,0.4361) on the bisectrix in Fig.1 is the point of entanglement arising.

#### B. Partial sums and relay entanglement

Our basic assumption is that the pairwise correlations are responsible for the state propagation in a quantum system so that the rest of this section is devoted to studying such combinations of pairwise entanglements which remain valuable during the whole evolution period. In this case, we can propose that the entanglement passes from a given entangled pair to another one establishing the relay propagation of entanglement along the spin chain.

Let  $C_{i,j}$  be the pairwise concurrence between the *i*th and *j*th particles. We distribute all  $C_{i,j}$  in the groups of concurrences between the equidistant nodes (|i-j| = m = const), and consider the mean concurrences  $\mathcal{C}_m$  in such groups:

$$C_m = \frac{1}{N-m} \sum_{i=1}^{N-m} C_{i,i+m}, \quad m = 1, 2, \dots, N-1,$$
(15)

as functions of the initial state parameters ( $\lambda$  and  $\alpha$ ) and time t. To illustrate the timebehavior of these functions, we depicture two of them,  $C_1$  and  $C_2$ , for the particular values  $\lambda =$ 

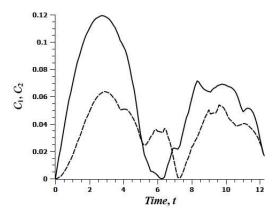


FIG. 2: The mean concurrences  $C_1$  (bold line) and  $C_2$  (dashed line) as functions of t at  $\lambda = 0.7$  and  $\alpha = 0$ . Here and below N = 10.

0.7 and  $\alpha = 0$  in Fig.2. We see that, in general,  $C_1 > C_2$ , but there is a time interval where the contribution from  $C_2$  prevails over the contribution from  $C_1$ . Thus, the entanglement between the nearest neighbors does not always dominate. This prompts us to introduce the sum of all pairwise entanglements in the chain

$$S = \sum_{i=1}^{N} C_i \tag{16}$$

as a measure of overall overlaps of pairwise concurrences. We refer to S as the relay entanglement. However, the relay entanglement does not explore the contribution from different mean concurrences  $C_i$ . To reveal this effect, we introduce the partial sums  $S_m$ 

$$S_m = \sum_{i=1}^m C_i, \quad m = 1, 2, \dots, N - 1.$$
 (17)

The relay entanglement S and the partial sums  $S_m$  are functions of t,  $\lambda$  and  $\alpha$ . There is an evident chain of inequalities:

$$S_1 \le S_2 \le \dots \le S. \tag{18}$$

To characterize the partial sums (which are functions of three variables), we consider the maximal values of these sums over the time interval  $0 \le t \le 12.238$  and their minimal values over the time interval  $1 \le t \le 12.238$ :

$$S_m^{max}(\lambda, \alpha) = \max_{0 \le t \le -12.238} S_m(\lambda, \alpha, t), \tag{19}$$

$$S_m^{min}(\lambda, \alpha) = \min_{1 \le t \le -12.238} S_m(\lambda, \alpha, t). \tag{20}$$

In the later case the shift by 1 in the time interval is conventional, we use it to remove the initial evolution interval because all pairwise entanglements are zero at t=0 due to our choice of the initial state. The contribution from the mth group is negligible if  $\frac{|S_{m-1}^{max} - S_m^{max}|}{S_{m-1}^{max}} \ll 1$ 

and 
$$\frac{|S_{m-1}^{min} - S_m^{min}|}{S_{m-1}^{min}} \ll 1.$$

To depicture the behavior of two-parametric functions  $S_m^{max}$  and  $S_m^{min}$ , we introduce the mean value and the root-mean-square deviation with respect to the parameters  $\alpha$  and  $\lambda$ . Thus, for any function S of the two parameters  $\lambda$  and  $\alpha$ , the mean value and root-meansquare deviation with respect to  $\alpha$  read:

$$\langle S \rangle_{\alpha}(\lambda) = \frac{\pi}{2} \int_{0}^{1} d\alpha \sin(\alpha \pi) S(\lambda, \alpha),$$
 (21)

$$\delta_{\alpha}S(\lambda) = \sqrt{\left\langle (S(\lambda, \alpha) - \left\langle S \right\rangle_{\alpha}(\lambda))^2 \right\rangle_{\alpha}}.$$
 (22)

Similarly, the mean value and root-mean-square deviation with respect to  $\lambda$  read:

$$\langle S \rangle_{\lambda}(\alpha) = 2 \int_{1/2}^{1} d\lambda S(\lambda, \alpha), \tag{23}$$

$$\delta_{\lambda} S(\alpha) = \sqrt{\langle (S(\lambda, \alpha) - \langle S \rangle_{\lambda}(\alpha))^{2} \rangle_{\lambda}}. \tag{24}$$

$$\delta_{\lambda}S(\alpha) = \sqrt{\left\langle (S(\lambda, \alpha) - \left\langle S \right\rangle_{\lambda}(\alpha))^{2} \right\rangle_{\lambda}}.$$
 (24)

Now we can estimate:

$$\langle S \rangle_{\alpha} - \delta_{\alpha} S \lesssim S \lesssim \langle S \rangle_{\alpha} + \delta_{\alpha} S,$$
 (25)

$$\langle S \rangle_{\lambda} - \delta_{\lambda} S \lesssim S \lesssim \langle S \rangle_{\lambda} + \delta_{\lambda} S.$$
 (26)

Therefore we can represent S graphically as the mean values  $\langle S \rangle_{\alpha}$  (or  $\langle S \rangle_{\lambda}$ ) supplemented by the root-mean-square deviations  $\delta_{\alpha}S$  (or  $\delta_{\lambda}S$ ) as the error-bars.

#### $\mathbf{C}.$ Analysis of partial sums

Accordingly, the functions  $S_m^{max}$  and  $S_m^{min}$  (m = 1, 2, 3, 9) in terms of the mean values and root-mean-square deviations with respect to  $\alpha$  are shown in Fig.3, and those functions in terms of the mean values and root-mean-square deviations with respect to  $\lambda$  are depictured in Fig.4. Both these figures show that the functions  $S_m^{max}$  and  $S_m^{min}$  are significantly different for m = 1, 2, 3, and therefore, the groups  $C_m$ , m = 1, 2, 3, are the most important in the correlation propagation. For m > 3, this difference is much less significant, except the intervals  $\lambda \gtrsim 0.8$  (Fig.3, compare graphs c,d) and  $0.2 \lesssim \alpha \lesssim 0.7$  (Fig.4, compare graphs c,d). Thus, all the concurrence groups  $\mathcal{C}_m$ ,  $m = 1, \ldots, 9$ , contribute to the entanglement propagation if the initial state parameters are inside of these intervals and only few of them contribute to the entanglement propagation otherwise.

Fig. 3a,b shows that the maxima of partial sums  $S_m$  with m=1,2 are almost independent on  $\lambda$ , but their dependence on  $\alpha$  increases with an increase in  $\lambda$ , which is indicated by the length of the error-bars. The later statement is correct for any m as shown in this figure. With an increase in m, these maxima become more  $\lambda$ -dependent especially for  $\lambda \gtrsim 0,8$ .

Fig. 4 is significantly different. Fig. 4a-c shows that the maximum of partial sums  $S_m$  increases with  $\alpha$  for small m. The dependence on  $\lambda$  (reflected by the error bars) is minimal in the middle of  $\alpha$ -interval for small m. However, with an increase in m, the  $\alpha$ -dependence becomes more complicated and  $\lambda$ -dependence becomes maximal at the middle of the  $\alpha$ -interval,  $0.2 \lesssim \alpha \lesssim 0.7$ . The largest root-mean-square deviation corresponds to  $\alpha \sim 0.4361$  (the crosspoint of the bisectrix and the boundary B in Fig.1).

#### IV. CLUSTERS OF ENTANGLED PARTICLES

As a problem directly related to the relay entanglement, we consider the formation of clusters of entangled particles, i.e., such clusters that all the pairwise entanglements inside of each of them are significant.

Before proceed to study the entanglement in clusters we notice that the symmetry of considered spin chain (the even-number chain, N = 10) causes the week concurrence in the middle pair,  $C_{56}$ . For comparison, we show  $C_{12}$  and  $C_{56}$  over the plane  $(\lambda, \alpha)$  in Fig.5. Unlike  $C_{12}$ , the concurrence  $C_{56}$  vanishes over the large part of its domain. This affects the characteristics of those clusters which include the 5th and 6th nodes.

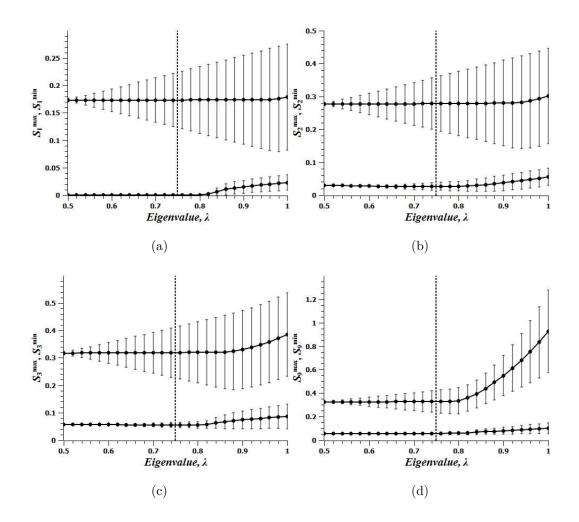


FIG. 3: The maximum  $S_m^{max} \approx \left\langle S_m^{max} \right\rangle_{\alpha} \pm \delta_{\alpha} S^{max}$  (upper curve) and minimum  $S_m^{min} \approx \left\langle S_m^{min} \right\rangle_{\alpha} \pm \delta_{\alpha} S^{min}$  (lower curve) as functions of  $\lambda$ . a) m=1, b) m=2, c) m=3, d) m=9. The vertical dotted lines correspond to the intersection point of the bisectrix and boundary curve in Fig.1 ( $\lambda=0.7493$ ).

# A. Geometric average of pairwise entanglements as measure of quantum correlations in cluster

To characterize the quantum correlations in a cluster of M particles numbered  $i, i + 1, \dots, i + M - 1$  we introduce the geometric mean of all pairwise concurrences in this cluster:

$$P_{M,i}(\lambda, \alpha, t) = \left(\prod_{n-k \le M-1} C_{i+k, i+n}\right)^{\frac{1}{\binom{M}{2}}}, \quad \binom{M}{2} = \frac{M!}{2!(M-2)!} = \frac{M(M-1)}{2}, \quad (27)$$

where M is the number of spins in the cluster (the size of the cluster), and i is the position of the first node of the cluster. Thus, the cluster involves spins from ith to (i + M - 1)th.

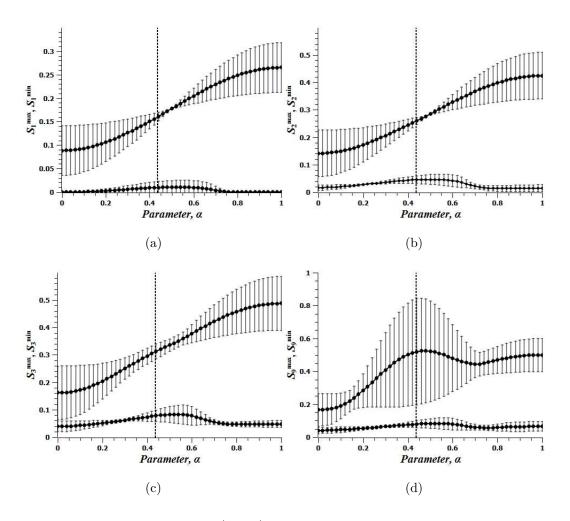


FIG. 4: The maximum  $S_m^{max} \approx \left\langle S_m^{max} \right\rangle_{\lambda} \pm \delta_{\lambda} S^{max}$  (upper curve) and the minimum  $S_m^{min} \approx \left\langle S_m^{min} \right\rangle_{\lambda} \pm \delta_{\lambda} S^{min}$  (lower curve) as functions of  $\alpha$ . a) m=1, b) m=2, c) m=3, d) m=9. The vertical dotted lines correspond to the intersection point of the bisectrix and boundary curve in Fig.1 ( $\alpha=0.4361$ )

The geometric mean  $P_{M,i}$  gets large value at some instant if all the concurrences  $C_{i,j}$ ,  $i < j \le i + M$ , are large at this instant. This function is localized in time. Generally, its non-zero t-support gets narrower with an increase in M and i; however, this rule is not strict as is shown below in Fig.9. To estimate and compare the geometric mean  $P_{M,i}$  in different clusters, we consider the maximum of  $P_{M,i}$  over the selected time interval:

$$P_{M,i}^{max}(\lambda,\alpha) = \max_{0 \le t \le 12.238} P_{M,i}(\lambda,\alpha,t). \tag{28}$$

Similar to Sec.III, we characterize the function of two variables  $P_{M,i}^{max}$  using the mean value and root-mean-square deviation with respect to the parameters  $\alpha$  and  $\lambda$  defined in formulas

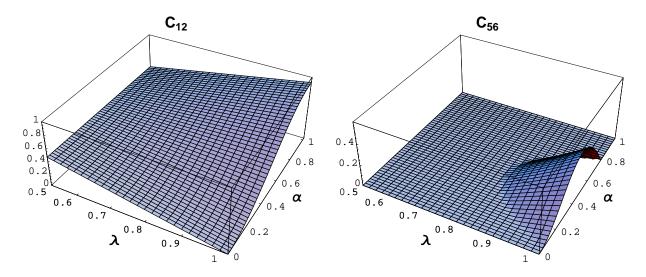


FIG. 5: The concurrences  $C_{12}$  and  $C_{56}$  over the plane  $(\lambda, \alpha)$ .

(21-24).

The functions  $P_{3,i}^{max}$ ,  $i=1,\ldots,8$ ,  $P_{4,i}^{max}$ ,  $i=1,\ldots,7$ , and  $P_{5,i}^{max}$ ,  $i=1,\ldots,6$ , in terms of the mean values and root-mean-square deviations with respect to  $\alpha$  are shown in Fig.6. The mean-values depicted in Fig.6a,b,c show that the dependence on  $\lambda$  becomes more significant for those clusters which include the 5th and 6th spins (i.e.,  $i \sim \frac{N}{2}$ ) when the concurrence  $C_{56}$  is involved into the definition (27). In addition the root-mean-square deviations (Fig.6d,e,f) show that the effect of  $\alpha$  increases with an increase in  $\lambda$  (over the interval  $\frac{1}{2} \leq \lambda \leq 1$ ),

The mean values and root-mean-square deviations of the geometric mean  $P_{M,i}^{max}$  with respect to  $\lambda$  defer from those with respect to  $\alpha$ , see Fig.7. In particular, not all the mean values are monotonic functions of  $\alpha$ . Those of them which include  $C_{56}$  in their definition have the maximum. Moreover, all the root-mean-square deviations are also non-monotonic functions. They have either maximum (those that include  $C_{56}$  in their definitions) or minimum (all others). Therefore, the effect of the eigenvalue parameter  $\lambda$  depends mainly on the position of the cluster in the chain.

In addition, all the geometric means  $P_{M,i}$  involving  $C_{56}$  in their definition (27) have the critical values  $\lambda_c$  and  $\alpha_c$  (both depending on M and i) such that the clusters exist in the intervals  $\lambda_c(M,i) \leq \lambda \leq 1$  and  $0 \leq \alpha \leq \alpha_c(M,i)$ . These critical values for M=3,4,5 are

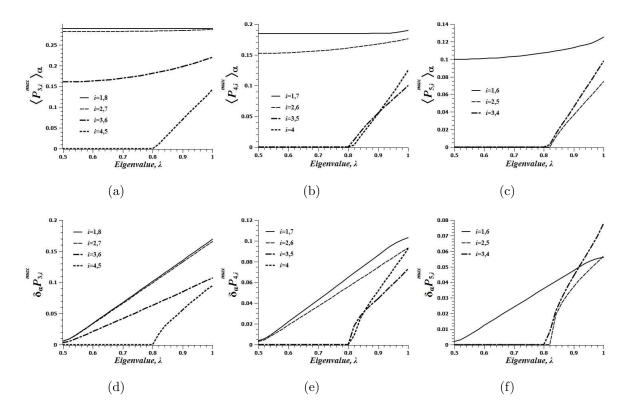


FIG. 6: The mean values  $\langle P_{M,i}^{max} \rangle_{\alpha}$  as functions of  $\lambda$ : a)  $M=3, i=1,\ldots,8$ , b)  $M=4, i=1,\ldots,7$ , c)  $M=5, i=1,\ldots,6$ . The root-mean-square deviations  $\delta_{\alpha}P_{M,i}^{max}$  as functions of  $\lambda$ : d)  $M=3, i=1,\ldots,8$ , e)  $M=4, i=1,\ldots,7$ , f)  $M=5, i=1,\ldots,6$ .

following.

$$M = 3: \quad \lambda_c(3,4) = \lambda_c(3,5) = 0.82, \quad \alpha_c(3,4) = \alpha_c(3,5) = 0.76,$$

$$M = 4: \quad \lambda_c(4,3) = \lambda_c(4,4) = \lambda_c(4,5) = 0.82, \quad \alpha_c(4,3) = \alpha_c(4,4) = \alpha_c(4,5) = 0.66,$$

$$M = 5: \quad \lambda_c(5,2) = \lambda_c(5,5) = 0.84, \quad \lambda_c(5,3) = \lambda_c(5,4) = 0.82,$$

$$\alpha_c(5,2) = \alpha_c(5,5) = 0.64, \quad \alpha_c(5,3) = \alpha_c(5,4) = 0.62.$$

$$(29)$$

The geometric mean  $P_{M,i}$  characterizes the overlap of different pairwise concurrences in the cluster of M particles. This function depends on the shape of each concurrence, their maximal values and instants of these maxima. In general, clusters  $P_{M,i}$  become less entangled with an increase in M and i, which is demonstrated in Figs.6 and 7.

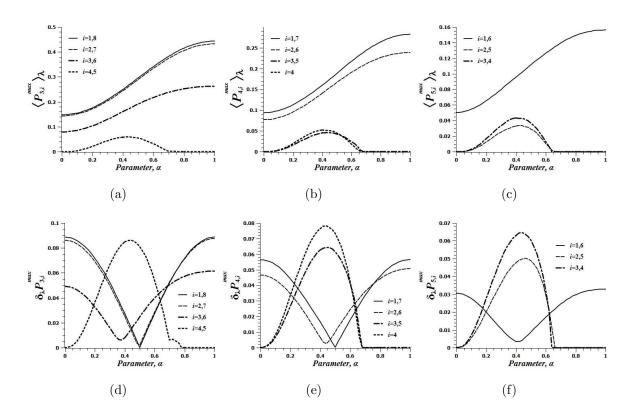


FIG. 7: The mean values  $\langle P_{M,i}^{max} \rangle_{\lambda}$  as functions of  $\alpha$ . a)  $M=3, i=1,\ldots,8$ , b)  $M=4, i=1,\ldots,7$ , c)  $M=5, i=1,\ldots,6$ . The root-mean-square deviations  $\delta_{\lambda}P_{M,i}^{max}$  as functions of  $\alpha$ . d)  $M=3, i=1,\ldots,8$ , e)  $M=4, i=1,\ldots,7$ , f)  $M=5, i=1,\ldots,6$ .

#### B. Large clusters

For completeness, we also discuss the formation of large clusters (M > 5) and depicture the geometric mean  $P_{M,i}^{max}$  for M = 6 and M = 10 in Fig.8 in terms of mean values and rootmean-square deviations. All these functions involve the concurrence  $C_{56}$  in their definition (27), so that the critical values  $\lambda_c$  and  $\alpha_c$  exist for any i. For the fixed M, both mean values of functions  $P_{M,i}^{max}$  (i.e., with respect to  $\alpha$  and  $\lambda$ ) and both root-mean-square deviations get the maximal values for i corresponding to the middle of its variation interval (i = 3) for M = 6, as shown in Fig.8.

### C. Cluster's life-time

Clearly, the clusters exist during some time intervals, which we call the cluster's life-time. We say that the cluster is destroyed if  $P_{M,i}^{max} < \varepsilon_C$ , where  $\varepsilon_C$  is some conventional value which

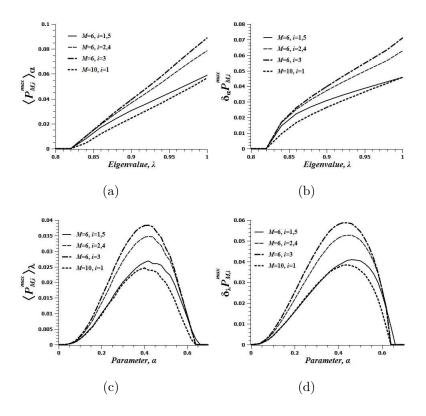


FIG. 8: The mean values and root-mean-square deviations for M=6 (solid lines) and M=10 (dashed lines). a) The mean values  $\langle P_{M,i}^{max} \rangle_{\alpha}$  as functions of  $\lambda$ ; b) The mean values  $\langle P_{M,i}^{max} \rangle_{\lambda}$  as functions of  $\alpha$ ; c) The root-mean-square deviations  $\delta_{\alpha} P_{M,i}^{max}$  as functions of  $\lambda$ ; d) The root-mean-square deviations  $\delta_{\lambda} P_{M,i}^{max}$  as functions of  $\lambda$ 

is taken to be equal 0.1 in this paper. We estimate the life-time by the formula

$$T_{M,i} = t_r - t_l, (30)$$

where  $t_r$  and  $t_l$  are such that  $\varepsilon_C < P_{M,i} \le P_{M,i}^{max}$  over the interval  $t_r \le t \le t_l$ . This period depends on both  $\lambda$  and  $\alpha$ . To illustrate this dependence, we depicture the mean values  $(\langle T_{M,i} \rangle_{\alpha}(\lambda))$  and  $\langle T_{M,i} \rangle_{\lambda}(\alpha)$  and root-mean-square deviations  $(\delta_{\alpha}T_{M,i}(\lambda))$  and  $\delta_{\lambda}T_{M,i}(\alpha)$  in Fig.9. These figures show that the clusters, whose geometric means  $P_{M,i}$  involve  $C_{56}$  in their definition (27), have restricted domain in the plane  $(\lambda,\alpha)$ . Fig.9a demonstrates that  $\langle T_{M,i} \rangle_{\alpha}$  slightly depends on  $\lambda$  for those clusters which do not involve  $C_{56}$  in the definition of  $P_{M,i}$ . The dependence of  $\langle T_{M,i} \rangle_{\lambda}$  on  $\alpha$  is significant for  $\alpha \lesssim 0.65$  which can be seen in Fig.9b. For  $\alpha \gtrsim 0.65$ ,  $\langle T_{M,i} \rangle_{\lambda}$  is almost constant. Fig.9c shows that the spread of  $T_{M,i}$  at the fixed  $\lambda$  and different  $\alpha$  becomes more significant with an increase in  $\lambda$ . Similarly, Fig.9d shows that the spread of  $T_{M,i}$  at a fixed  $\alpha$  and different  $\lambda$  is (generally) significant for  $\alpha \lesssim 0.65$ , except

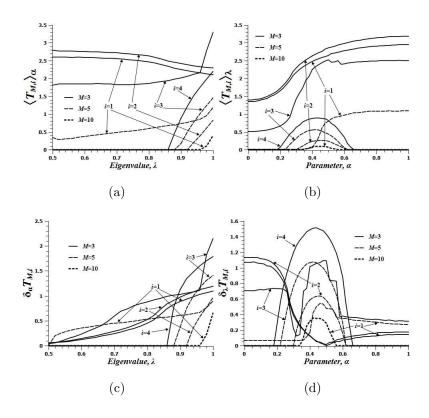


FIG. 9: The life-times of clusters. a) The mean values  $\langle T_{M,i} \rangle_{\alpha}$  as functions of  $\lambda$ ; b) The mean values  $\langle T_{M,i} \rangle_{\lambda}$  as functions of  $\alpha$ ; c) the root-mean-square deviations  $\delta_{\alpha} T_{M,i}$  as functions of  $\lambda$ ; d) the root-mean-square deviations  $\delta_{\lambda} T_{M,i}$  as functions of  $\alpha$ .

the cases  $M=3,\,i=1,2,$  when this spread is valuable for  $\alpha\lesssim 0.4.$  Generally, the life-time decreases with an increase in M.

#### V. CONCLUSIONS

Although the correlations between the sender and receiver may vanish during the whole evolution period, other pairwise correlations arise during the evolution and supplement the information propagation. While the entanglement between any two particles in an evolutionary spin chain is located in time, the sum S of all pairwise entanglements can be significant over the whole evolution period. One can say that the entanglement propagates from a given pair of entangled particles to another one; therefore we refer to this sum as the relay entanglement. Thus, the relay entanglement (the assembly of pairwise quantum entanglements) supplements information propagation. It is natural to assume that the

correlations between the nearest neighbors are most important. However, this is not always true because the entanglements between the remote nodes prevail in certain cases which is justified by the partial sums  $S_m$ , whose  $S_m^{max}$  and  $S_m^{min}$  (17-20) are described in terms of their mean values and root-mean-square deviations with respect to  $\alpha$  and  $\lambda$ , eqs.(21-24), in the model of 10-node homogeneous spin-1/2 chain with the initial state symmetrical with respect to the change  $S \leftrightarrow R$ .

During the evolution, the clusters of entangled particles arise (i.e., the clusters where all pairwise entanglements are significant), where geometric mean  $P_{M,i}$  (27) serves as the characteristics of entanglement. We study its dependence on the initial-state parameters  $\lambda$  and  $\alpha$  using the mean value and root-mean-square deviation with respect to the parameters  $\alpha$  and  $\lambda$  and graphically show that the geometric mean  $P_{M,i}$  can be almost constant over some intervals of these parameters, while it can significantly vary over other intervals. For the large clusters (M > 5 in our case), in addition, there are intervals  $\lambda < \lambda_c$  and  $\alpha > \alpha_c$  where these clusters do not exist. The life-time of the clusters decreases with an increase in the size of a cluster, however, the cluster's position in the chain also affects the lifetime. The large-size clusters are short-living in this model, while small-size clusters of such type can be candidates for quantum registers. These registers carry information encoded into the initial state of a quantum system (for instance, into the sender). Therefore, applying a proper unitary transformation to the localized entangled cluster, we can affect its state and consequently the state registered at the receiver. This phenomenon can be used for realizing certain quantum gates, where the selected cluster serves as a control element.

This work is partially supported by the program of the Presidium of RAS No. 5 "Electron resonance, spin-dependent electron effects and spin technology" and by the Russian Foundation for Basic Research, grant No.15-07-07928.

<sup>[1]</sup> L.Banchi, A. Bayat, P. Verrucchi, and S.Bose, Phys.Rev.Let. 106, 140501 (2011)

<sup>[2]</sup> B. Chen, and Zh. Song, Sci. China-Phys., Mech. Astron **53**, 1266 (2010)

<sup>[3]</sup> Datta, A., Shaji, A., Caves, C.M., Phys. Rev. Lett. 100, 050502 (2008)

<sup>[4]</sup> Lanyon, B.P., Barbieri, M., Almeida, M.P., White, A.G., Phys. Rev. Lett. 101, 200501 (2008)

<sup>[5]</sup> W.J.Nie, Yu.H.Lan, Yo.Li, and Sh.Ya.Zhu, Sci.China-Phys., Mech. Astron 57, 2276 (2014)

- [6] P. Zhang, B. You, and L.-X. Cen, Chin. Sci. Bull., 59, 3841 (2014)
- [7] J.X. Sci, W. Xu, G. Sun et al, Chin. Sci. Bull. **59**, 2547 (2014)
- [8] S. Rodriques, N. Datta, and P. J. Love, Phys. Rev. A 90, 012340 (2014)
- [9] L.Banchi, J.G.Apollaro, A.Cuccoli, R.Vaia, and P.Verrucchi, Phys. Rev. A 82, 052321 (2010)
- [10] P.Lorenz and J.Stolze, Phys.Rev.A 90, 044301 (2014)
- [11] R. Sousa and Y. Omar, New J. Phys. **16**, 123003 (2014)
- [12] T. Huan, R. Zhou, and H.Ian, Phys.Rev.A **92**, 022301 (2015)
- [13] S. I. Doronin and A. I. Zenchuk, Theor. Math. Phys. 188(2), 1259 (2016)
- [14] N.A.Peters, J.T.Barreiro, M.E.Goggin, T.-C.Wei, and P.G.Kwiat in *Quantum Communications and Quantum Imaging III*, ed. R.E.Meyers, Ya.Shih, Proc. of SPIE 5893 (SPIE, Bellingham, WA, 2005)
- [15] N.A.Peters, J.T.Barreiro, M.E.Goggin, T.-C.Wei, and P.G.Kwiat, Phys.Rev.Lett. 94, 150502 (2005)
- [16] C.H.Bennett, D.P.DiVincenzo, P.W.Shor, J.A.Smolin, B.M.Terhal, and W.K.Wootters, Phys.Rev.Lett. 87, 077902 (2001); Erratum, C.H.Bennett, D.P.DiVincenzo, P.W.Shor, J.A.Smolin, B.M.Terhal, and W.K.Wootters, Phys. Rev. Lett. 88, 099902(E) (2002)
- [17] C.H.Bennett, P.Hayden, D.W.Leung, P.W.Shor, and A.Winter, IEEE Transection on Information Theory **51**, 56 (2005)
- [18] G.L.Giorgi, Phys. Rev. A 88, 022315 (2013)
- [19] S. Bose, Phys. Rev. Lett. **91**, 207901 (2003)
- [20] L.Campos Venuti, S.M.Giampaolo, F.Illuminati, P.Zanardi, Phys. Rev. A 76, 052328 (2007)
- [21] S.I.Doronin, A.I.Zenchuk, Phys. Rev. A 81, 022321 (2010)
- [22] G.Gualdi, I.Marzoli, P.Tombesi, New J. Phys. 11, 063038 (2009)
- [23] A.Bayat and S.Bose, Phys. Rev. A P81, 012304 (2010)
- [24] S.I.Doronin, E.B.Fel'dman, and A.I.Zenchuk, Phys. Rev. A 79, 042310 (2009)
- [25] S. I. Doronin, and A. I. Zenchuk, Quantum Inf. Process. 16(3), 1 (2017)
- [26] A.I.Zenchuk, Quantum Inf. Proc. **13**(12), 2667 (2014)
- [27] W.K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998)
- [28] S.Hill and W.K.Wootters, Phys. Rev. Lett. 78, 5022 (1997)
- [29] L.Henderson and V.Vedral J.Phys.A:Math.Gen. **34**, 6899 (2001)
- [30] H.Ollivier and W.H.Zurek, Phys.Rev.Lett. 88, 017901 (2001)

- [31] W. H. Zurek, Rev. Mod. Phys. **75**, 715 (2003)
- [32] C.H.Bennett, G.Brassard, C.Crépeau, R.Jozsa, A.Peres, and W.K.Wootters, Phys. Rev. Lett. 70, 1895 (1993)
- [33] D.Bouwmeester, J.-W. Pan, K.Mattle, M.Eibl, H.Weinfurter, and A. Zeilinger, Nature 390, 575 (1997)
- [34] D. Boschi, S. Branca, F. De Martini, L. Hardy, and S. Popescu, Phys. Rev. Lett. 80, 1121 (1998)
- [35] E.B.Fel'dman and A.I. Zenchuk, Phys.Rev.A 86, 012303 (2012)
- [36] M. Tomaselli, S. Hediger, D. Suter, and R. R. Ernst, J. Chem. Phys. 105, 10672 (1996)
- [37] H. C. Krojanski and D. Suter, Phys. Rev. Lett. 93, 090501 (2004)
- [38] D. C. Mattis, The Many-Body Problem: An Encyclopedia of Exactly Solved Models in One Dimension (World Scientific, Singapore, 1993).