# Stochastic local operations and classical communication (SLOCC) and local unitary operations (LU) classifications of $n$ qubits via ranks and singular values of the spin-flipping matrices 

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#### Abstract

We construct $\ell$-spin-flipping matrices from the coefficient matrices of pure states of $n$ qubits and show that the $\ell$-spin-flipping matrices are congruent and unitary congruent whenever two pure states of $n$ qubits are SLOCC and LU equivalent, respectively. The congruence implies the invariance of ranks of the $\ell$-spin-flipping matrices under SLOCC and then permits a reduction of SLOCC classification of $n$ qubits to calculation of ranks of the $\ell$-spin-flipping matrices. The unitary congruence implies the invariance of singular values of the $\ell$-spin-flipping matrices under LU and then permits a reduction of LU classification of $n$ qubits to calculation of singular values of the $\ell$-spin-flipping matrices. Furthermore, we show that the invariance of singular values of the $\ell$-spin-flipping matrices $\Omega_{1}^{(n)}$ implies the invariance of the concurrence for even $n$ qubits and the invariance of the n-tangle for odd $n$ qubits. Thus, the concurrence and the n -tangle can be used for LU classification and computing the concurrence and the n-tangle only performs additions and multiplications of coefficients of states.


## INTRODUCTION

Entanglement is considered as a uniquely resource in quantum teleportation, quantum cryptography and quantum information and computation 1]. There are different types of entanglement for multipartite systems. The classification of entanglement plays an important rule in quantum information theory [2]-29]. The following three types of classifications of entanglement have widely been studied: LU [24]-[29], LOCC (local operations and classical communication) 30], and SLOCC [2], 3]22].

Considerable efforts have contributed to the SLOCC entanglement classification. For example, the complete SLOCC classifications of two and three qubits have been obtained. There are two (resp. six) SLOCC equivalence classes for two (resp. three) qubits. For four qubits, there are infinite SLOCC equivalence classes [3] and the infinite SLOCC classes are partitioned into nine inequivalent families [4-7]. It is known that a SLOCC classification for $n$ qubits remains unsolved because the difficulty increases rapidly as $n$ does [8 13, 22].

Recently, SLOCC invariant polynomials have been proposed for classification of entanglement [14]. Very recently, it has been shown that ranks of the coefficient matrices of states are invariant un-
der SLOCC and the invariance of the ranks can be applied to SLOCC classification of entanglement [10, 11, 22, 23].

It is known that two LU equivalent states have the same amount of entanglement and are equally used for any kind of application [28, 29]. The polynomial invariants under LU were studied [24, 25]. The necessary and sufficient conditions for $L U$ equivalence of states of n qubits were presented [29] and used for LU classifications of two to five qubits 28].

In this paper, we construct the $\ell$-spin-flipping matrices from the coefficient matrices of pure states of $n$ qubits and prove that the $\ell$-spin-flipping matrices are congruent and unitary congruent under SLOCC and LU, respectively. The congruence confirms the invariance of ranks and the unitary congruence asserts the invariance of singular values. The invariance of ranks and singular values can be used for SLOCC and LU classifications of $n$ qubits.

This paper is organized as follows. In the section 2, from the coefficient matrices of states of $n$ qubits we construct the $\ell$-spin-flipping matrices and we show that ranks of the $\ell$-spin-flipping matrices are preserved under SLOCC. In the section 3, we demonstrate how the invariant ranks are used for SLOCC classification of entanglement. In the section 4 , we argue the invariance of singular values of the $\ell$-spin-flipping matrices under LU. In the section

5, we investigate the LU classification.

## THE INVARIANCE OF RANKS OF THE $\ell$-SPIN-FLIPPING MATRICES

Let $|\psi\rangle=\sum_{i=0}^{2^{n}-1} a_{i}|i\rangle$ be any pure state of $n$ qubits, where $a_{i}$ are coefficients. It is well known that two $n$-qubit pure states $|\psi\rangle$ and $\left|\psi^{\prime}\right\rangle$ are SLOCC (resp. LU) equivalent if and only if the two states satisfy the following equation

$$
\begin{equation*}
\left|\psi^{\prime}\right\rangle=\mathcal{A}_{1} \otimes \mathcal{A}_{2} \otimes \cdots \otimes \mathcal{A}_{n}|\psi\rangle, \tag{1}
\end{equation*}
$$

where $\mathcal{A}_{i}$ are invertible (resp. unitary) 3,29$]$.
Let $C_{q_{1} \cdots q_{\ell}}(|\psi\rangle)$ be the $2^{\ell} \times 2^{n-\ell}$ coefficient matrix of the state $|\psi\rangle$ of $n$ qubits, where $q_{1}, \cdots, q_{\ell}$ are the row bits and $q_{\ell+1}, \cdots, q_{n}$ are the column bits. For example, for three qubits,

$$
C_{1}(|\psi\rangle)=\left(\begin{array}{cccc}
a_{0} & a_{1} & a_{2} & a_{3}  \tag{2}\\
a_{4} & a_{5} & a_{6} & a_{7}
\end{array}\right)
$$

Let

$$
\begin{align*}
& \Omega_{q_{1}, q_{2}, \ldots, q_{i}}^{(n)}(|\psi\rangle) \\
= & C_{q_{1}, q_{2}, \ldots, q_{i}}^{(n)}(|\psi\rangle) v^{\otimes(n-i)}\left(C_{q_{1}, q_{2}, \ldots, q_{i}}^{(n)}(|\psi\rangle)\right)^{T}, \tag{3}
\end{align*}
$$

where $v=\sqrt{-1} \sigma_{y}$ and $\sigma_{y}$ is the Pauli operator.
It is clear that $\Omega_{q_{1}, q_{2}, \ldots, q_{i}}^{(n)}(|\psi\rangle)$ is a square matrix of order $2^{i}$ for $n$ qubits. Here, we call $\Omega_{q_{1}, \ldots, q_{i}}^{(n)}(|\psi\rangle)$ a spin-flipping matrix. Clearly, when $n-i$ is even, then $\Omega_{q_{1}, \ldots, q_{i}}^{(n)}(|\psi\rangle)$ is symmetric. Otherwise, it is skewsymmetric.

Next, we define the 1-spin-flipping $\Omega_{q_{1}, \ldots, q_{i}}^{(n)}(|\psi\rangle)^{\odot} 1$ matrix as the spin-flipping $\Omega_{q_{1}, \ldots, q_{i}}^{(n)}(|\psi\rangle)$ and the $\ell$ -spin-flipping matrix $\Omega_{q_{1}, \ldots, q_{i}}^{(n)}(|\psi\rangle)^{\odot \ell}$ for $\ell>1$ as $\Omega_{q_{1}, \ldots, q_{i}}^{(n)}(|\psi\rangle)^{\odot \ell}=\Omega_{q_{1}, \ldots, q_{i}}^{(n)}(|\psi\rangle)^{\odot(\ell-1)} v^{\otimes i} \Omega_{q_{1}, \ldots, q_{i}}^{(n)}(|\psi\rangle)$.

Let $\alpha=\Pi_{k=1}^{i} \operatorname{det} \mathcal{A}_{q_{k}}$ and $\beta=\Pi_{k=i+1}^{n} \operatorname{det} \mathcal{A}_{q_{k}}$. Then, invoking the fact that $\mathcal{A}_{k}^{T} v \mathcal{A}_{k}=\left(\operatorname{det} \mathcal{A}_{k}\right) v$ and from [22], for two SLOCC equivalent states $\left|\psi^{\prime}\right\rangle$ and $|\psi\rangle$ of $n$ qubits a complicated calculation yields

$$
\begin{gather*}
\Omega_{q_{1}, \ldots, q_{i}}^{(n)}\left(\left|\psi^{\prime}\right\rangle\right)^{\odot \ell} \\
=\alpha^{\ell-1} \beta^{\ell}\left(\mathcal{A}_{q_{1}} \otimes \cdots \otimes \mathcal{A}_{q_{i}}\right) \Omega_{q_{1}, \ldots, q_{i}}^{(n)}(|\psi\rangle)^{\odot \ell} \times \\
\left(\mathcal{A}_{q_{1}} \otimes \cdots \otimes \mathcal{A}_{q_{i}}\right)^{T} . \tag{5}
\end{gather*}
$$

Eq. (5) leads to the following theorem.

Theorem 1. If two pure states $\left|\psi^{\prime}\right\rangle$ and $|\psi\rangle$ of $n$ qubits are SLOCC equivalent, then for any $\ell$ the $\ell$-spin-flipping matrices $\Omega_{q_{1}, \ldots, q_{i}}^{(n)}\left(\left|\psi^{\prime}\right\rangle\right)^{\odot \ell}$ and $\Omega_{q_{1}, \ldots, q_{i}}^{(n)}(|\psi\rangle)^{\odot \ell}$ are congruent and then have the same rank.

The contraposition of Theorem 1 says that if the $\ell$-spin-flipping matrices $\Omega_{q_{1}, \ldots, q_{i}}^{(n)}\left(\left|\psi^{\prime}\right\rangle\right)^{\odot \ell}$ and $\Omega_{q_{1}, \ldots, q_{i}}^{(n)}(|\psi\rangle)^{\odot \ell}$ have different ranks for some $\ell$, then the two states $\left|\psi^{\prime}\right\rangle$ and $|\psi\rangle$ belong to different SLOCC classes.

## SLOCC CLASSIFICATION VIA RANKS OF THE $\ell$-SPIN-FLIPPING MATRICES

Let $r(A)$ be the rank of the matrix $A$. Then, it is well known that $r(A B) \leq \min \{r(A), r(B)\}$. Here, $r\left(\Omega_{q_{1}, \cdots, q_{i}}^{(n)}(|\psi\rangle)^{\odot k}\right)$ is denoted as $r_{q_{1}, \cdots, q_{i}}^{(k)}$. Then $r_{q_{1}, \cdots, q_{i}}^{(1)} \geq \cdots \geq r_{q_{1}, \cdots, q_{i}}^{(k)} \geq \cdots$. If $\Omega_{q_{1}, \ldots, q_{i}}^{(n)}(|\psi\rangle)$ is not full, then the rank $r_{q_{1}, \cdots, q_{i}}^{(k)}$ may decrease as $k$ increases. Therefore, it is possible that the $\ell$-spinflipping matrices have different ranks for some $\ell$ for two SLOCC inequivalent $n$-qubit states. It means that the theorem 1 may be used for SLOCC classification.

For example, for the states GHZ and W of three qubits a simple calculation shows that $\Omega_{1,2}^{(3)}(|\mathrm{GHZ}\rangle)$ and $\Omega_{1,2}^{(3)}(|\mathrm{W}\rangle)$ have the same rank 2. But for GHZ, $r_{1,2}^{(2)}=2$ while for $\mathrm{W}, r_{1,2}^{(2)}=1$. So, in light of Theorem 1 GHZ and W are SLOCC inequivalent.

Next we show how Theorem 1 is used for SLOCC classification.
(1). For any state $|\psi\rangle$ of two qubits, $\Omega_{1}^{(2)}(|\psi\rangle)=$ $\left(a_{0} a_{3}-a_{1} a_{2}\right) v$. It is trivial to see that the rank of $\Omega_{1}^{(2)}(|\psi\rangle)$ is 2 or 0 . Thus, we obtain a complete classification under SLOCC for two qubits.
(2). By using $r_{1,2}^{(1)}, r_{1,2}^{(2)}$, and $r_{1,2}^{(3)}$, a tedious calculation yields a SLOCC classification of three qubits in Table I.

TABLE I. The SLOCC classification of three qubits

| states | $r_{1,2}^{(1)} r_{1,2}^{(2)} r_{1,2}^{(3)}$ | states | $r_{1,2}^{(1)} r_{1,2}^{(2)} r_{1,2}^{(3)}$ |
| :---: | :---: | :---: | :---: |
| GHZ | 222 | W | 210 |
| A-BC,B-AC | 200 | C-AB, $\|000\rangle$ | 000 |

By Table I, we can determine that to which SLOCC class a state of three qubits belongs. For example, let $|\xi\rangle=\frac{1}{2 \sqrt{2}}\left(\sum_{i=0}^{6}|i\rangle-|7\rangle\right)$. Then
$r\left(\left(\Omega_{1,2}^{(3)}(|\xi\rangle)^{\odot \ell}\right)=2, \ell \geq 1\right.$. So, $|\xi\rangle$ belongs to GHZ class.
(3). By using $r_{1,2}^{(1)}, r_{1,2}^{(2)}$, and $r_{1,2}^{(3)}$, we obtain a complete SLOCC classification of the states of Acin et al.'s canonical form in Table II.

It is well known that any state of three qubits can be written in the following Acin et al.'s canonical form:

$$
\begin{align*}
|A\rangle= & \lambda_{0}|000\rangle+\lambda_{1} e^{i \varphi}|100\rangle \\
& +\lambda_{2}|101\rangle+\lambda_{3}|110\rangle+\lambda_{4}|111\rangle \tag{6}
\end{align*}
$$

where $\lambda_{i} \geq 0, i=0,1,2,4,0 \leq \varphi \leq \pi$, and $\sum_{i=0}^{4} \lambda_{i}^{2}=1$ [26, 27]. Via $r_{1,2}^{(1)}, r_{1,2}^{(2)}, r_{1,2}^{(3)}$, and Table I, a tedious and straightforward calculation yields the complete SLOCC classification of the states of the Acin's canonical form in Table II. Thus, we recover Dür et al.'s six SLOCC classes obtained via local ranks [3]. Here, each one of the six classes is parametrized by parameters $\lambda_{1}, \lambda_{2}, \lambda_{3}$, and $\lambda_{4}$. For example, the GHZ class is described by $\lambda_{0} \lambda_{4} \neq 0$. For another example, we name as the SLOCC W class the SLOCC class in Table II characterized by the parameters $\lambda_{0} \neq 0, \lambda_{4}=0$, and $\lambda_{2} \lambda_{3} \neq 0$. We can show that the W state belongs to the SLOCC W class below. Let $|\vartheta\rangle=\frac{1}{\sqrt{3}}(|000\rangle+|101\rangle+|110\rangle)$. One can test that $\sigma_{x} \otimes I \otimes I|\vartheta\rangle=|\mathrm{W}\rangle$ and $|\vartheta\rangle$ belongs to the SLOCC W class.

TABLE II. Complete SLOCC classification of states of Acin et al.'s canonical form

|  |  | $r_{1,2}^{(1)} r_{1,2}^{(2)} r_{1,2}^{(3)}$ | classes |
| :---: | :---: | :---: | :---: |
| $\lambda_{0}=0, \lambda_{4} \neq 0$ | $\lambda_{2} \lambda_{3}=\lambda_{1} \lambda_{4} e^{i \varphi}$ | 000 | A-B-C |
|  | $\lambda_{2} \lambda_{3} \neq \lambda_{1} \lambda_{4} e^{i \varphi}$ | 200 | A-BC |
| $\lambda_{0} \neq 0, \lambda_{4}=0$ | $\lambda_{2}=\lambda_{3}=0$ | 000 | A-B-C |
|  | $\lambda_{2}=0, \lambda_{3} \neq 0$ | 000 | C-AB |
|  | $\lambda_{2} \neq 0, \lambda_{3}=0$ | 200 | B-AC |
| $\lambda_{0}=\lambda_{4}=0$ | $\lambda_{2} \lambda_{3} \neq 0$ | 210 | W |
|  | $\lambda_{2} \lambda_{3}=0$ | 000 | A-B-C |
| $\lambda_{0} \lambda_{4} \neq 0$ | $\lambda_{2} \lambda_{3} \neq 0$ | 200 | A-BC |

## THE INVARIANCE OF SINGULAR VALUES OF THE $\ell$-SPIN-FLIPPING MATRICES UNDER LU

Here, two matrices $A$ and $B$ are called unitary congruent if there is a unitary matrix $P$ such that $B=P A P^{T}$. When two states are LU equivalent, $\mathcal{A}_{i}$ in Eq. (11) are unitary. Thus, Eq. (5) leads to the following theorem.

Theorem 2. If two pure states $\left|\psi^{\prime}\right\rangle$ and $|\psi\rangle$ of $n$ qubits are LU equivalent, then the $\ell$-spin-flipping matrices $\Omega_{q_{1}, \ldots, q_{i}}^{(n)}\left(\left|\psi^{\prime}\right\rangle\right)^{\odot \ell}$ and $\Omega_{q_{1}, \ldots, q_{i}}^{(n)}(|\psi\rangle)^{\odot \ell}$ are unitary congruent, and then
(1) have the same ranks,
(2) have the same singular values, and
(3) $\Omega_{q_{1}, \ldots, q_{i}}^{(n)}\left(\left|\psi^{\prime}\right\rangle\right)$ and $\Omega_{q_{1}, \ldots, q_{i}}^{(n)}(|\psi\rangle)$ have the same absolute values of determinants.

First we demonstrate how to partition Verstraete et al.'s nine families under LU invoking absolute values of determinants. For example, for the family $L_{a_{4}}$ [4], $\mid \operatorname{det} \Omega_{1,2}^{(4)}\left(\left|L_{a_{4}}\right\rangle\right)\left|=\left|a^{4}\right|^{2}\right.$. Thus, two states of the family $L_{a_{4}}$ with different values of $|a|$ are different under LU. Similarly, under LU we can partition the families $G_{a b c d}, L_{a b c_{2}}, L_{a_{2} b_{2}}$, and $L_{a b_{3}}$ in [4].

Next we use the following example to explain that the invariance of singular values is more powerful than the invariance of the ranks for LU classification. Let us consider two states $\left|W_{1}\right\rangle$ and $\left|W_{2}\right\rangle$ of the SLOCC W class of three qubits in Table II, where $\left.\left|W_{1}\right\rangle=\frac{1}{2}(|000\rangle+|100\rangle+|101\rangle+\mid 110)\right)$ and $\left|W_{2}\right\rangle=$ $\lambda_{0}|000\rangle+\lambda_{2}|101\rangle+\lambda_{3}|110\rangle$, where $\lambda_{0}^{2}=\frac{1}{4}\left(\frac{1}{2}-\frac{\sqrt{2}}{4}\right)$, $\lambda_{2}^{2}=\frac{1}{2}+\frac{\sqrt{2}}{4}$, and $\lambda_{3}^{2}=\frac{3}{4}\left(\frac{1}{2}-\frac{\sqrt{2}}{4}\right)$. A calculation makes Table III. From Table III, clearly we cannot distinguish $\left|W_{1}\right\rangle$ and $\left|W_{2}\right\rangle$ under LU via the ranks $r_{1,2}^{(1)}, r_{1,2}^{(2)}$, and $r_{1,2}^{(3)}$ or the singular values $\sigma_{1}, \sigma_{2}, \sigma_{3}$, and $\sigma_{4}$ of $\Omega_{1,2}^{(3)}\left(\left|W_{i}\right\rangle\right)$. Whereas, we can distinguish them under LU via the singular values $\eta_{1}, \eta_{2}, \eta_{3}$, and $\eta_{4}$ of $\Omega_{1,2}^{(3)}\left(\left|W_{i}\right\rangle\right)^{\odot 2}$.

TABLE III. Two LU inequivalent states of three qubits

| state | $r_{1,2}^{(1)} r_{1,2}^{(2)} r_{1,2}^{(3)}$ | $\sigma_{1}^{2} \sigma_{2}^{2} \sigma_{3}^{2} \sigma_{4}^{2}$ | $\eta_{1}^{2} \eta_{2}^{2} \eta_{3}^{2} \eta_{4}^{2}$ |
| :--- | :---: | :---: | :---: |
| $\left\|W_{1}\right\rangle$ | 210 | $\frac{1}{8} \frac{1}{8} 00$ | $\frac{1}{4096} \frac{1}{4096} 00$ |
| $\left\|W_{2}\right\rangle$ | 210 | $\frac{1}{8} \frac{1}{8} 00$ | $\frac{3}{16384} \frac{3}{16384} 00$ |

## LU CLASSIFICATION OF $n$ QUBITS VIA SINGULAR VALUES OF THE $\ell$-SPIN-FLIPPING MATRICES

## LU classification of even $n$ qubits invoking singular values of $\Omega_{1}^{(n)}$

Singular values of $\Omega_{1}^{(n)}$ are just the concurrence for even $n$ qubits

For any state $|\psi\rangle$ of even $n$ qubits, let $t_{1}$ and $t_{2}$ be the singular values of the skew-symmetric
matrix $\Omega_{1}^{(n)}(|\psi\rangle)$. Then, a calculation yields that

$$
\begin{equation*}
t_{1}=t_{2}=\left|\sum_{i=0}^{2^{n-1}-1}(-1)^{N(i)} a_{i} a_{2^{n}-i-1}\right| \tag{7}
\end{equation*}
$$

Here $N(i)$ is the number of 1 s in the $n$-bit binary representation $i_{n-1} \ldots i_{1} i_{0}$ of $i$. That is, $N(i)$ is the parity of $i$. It is known that the singular values of $\Omega_{1}^{(n)}(|\psi\rangle)$ in Eq. (7) are just the concurrence of even $n$ qubits for the state $|\psi\rangle[20]$. For two qubits, $t_{1}=$ $t_{2}=\left|a_{0} a_{3}-a_{1} a_{2}\right|$.

LU classification of even $n$ qubits invoking the invariance of the concurrence for even $n$ qubits

Theorem 2 and Eq. (7) lead to the following theorem.

Theorem 3. If two pure states of even $n$ qubits are LU equivalent then the two states have the same concurrence for even $n$ qubits.

In comparison, if two pure states of even $n$ qubits are SLOCC equivalent then either their concurrences for even $n$ qubits both vanish or neither vanishes 20].

From Theorem 3, if two states of even $n$ qubits have different concurrences, then they belong to different LU equivalence classes. For example, for two qubits the Bell states $\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$ have the maximal concurrence $\frac{1}{2}$. let $|\zeta\rangle=\frac{1}{\sqrt{3}}|00\rangle+\frac{\sqrt{2}}{\sqrt{3}}|11\rangle$. The concurrence of $|\zeta\rangle$ is $\frac{\sqrt{2}}{3}$. In light of Theorem $3,|\zeta\rangle$ is LU inequivalent to the Bell state though $|\zeta\rangle$ is SLOCC equivalent to the Bell state. For two qubits, the Schmidt coefficients are used for LU classification [28].

Let $c$ denote the concurrence. Clearly, $0 \leq c \leq \frac{1}{2}$. And let the family, denoted as $F_{c}$, consist of all the states with the same concurrence $c$. It means that if two states are LU equivalent then they belong to the same family $F_{c}$. For example, GHZ and $|0 \cdots 0\rangle$ of any even $n$ qubits belong to $F_{1 / 2}$ and $F_{0}$, respectively. Clearly, for any even $n$ qubits there is a one to one correspondence between the set $\left\{F_{c} \mid c \in[0,1 / 2]\right\}$ of the families $F_{c}$ and the interval $[0,1 / 2]$.

## LU classification of odd $n$ qubits invoking singular values of $\Omega_{1}^{(n)}$

A product of singular values of $\Omega_{1}^{(n)}(|\psi\rangle)$ is just the n-tangle for odd $n$ qubits

For any state $|\psi\rangle$ of odd $n$ qubits, the symmetric matrix $\Omega_{1}^{(n)}(|\psi\rangle)$ can be written as $\left(\begin{array}{ll}e_{11} & e_{12} \\ e_{12} & e_{22}\end{array}\right)$. We calculate $e_{11}, e_{12}$, and $e_{22}$ in Appendix A. Let $t_{1}$ and $t_{2}$ be the singular values of $\Omega_{1}^{(n)}(|\psi\rangle)$. Then, a calculation yields that $t_{1}^{2}=\frac{\Delta+\sqrt{\Delta^{2}-4 D}}{2}$ and $t_{2}^{2}=$ $\frac{\Delta-\sqrt{\Delta^{2}-4 D}}{2}$, where $\Delta=2\left|e_{12}\right|^{2}+\left|e_{11}\right|^{2}+\left|e_{22}\right|^{2}$ and $D=\left|e_{11} e_{22}-e_{12}^{2}\right|^{2}$.

## LU classification of odd n qubits invoking the invariance of the $n$-tangle for odd $n$ qubits

Note that $t_{1}^{2}+t_{2}^{2}=\Delta$ and $t_{1} t_{2}=\left|e_{11} e_{22}-e_{12}^{2}\right|$. Recall that $\left|e_{11} e_{22}-e_{12}^{2}\right|$ is just the n-tangle of odd $n$ qubits [19, 22]. Thus, the invariance of singular values of $\Omega_{1}^{(n)}(|\psi\rangle)$ implies that the n-tangle and $\Delta$ are invariant under LU. For example, the above states $\left|W_{1}\right\rangle$ and $\left|W_{2}\right\rangle$ are different in $\Delta$, so they are LU inequivalent.

We can conclude the following theorem from Theorem 2 and the above discussion.

Theorem 4. If two pure states of odd $n$ qubits are LU equivalent then they have the same n -tangle for odd $n$ qubits.

In comparison, if two pure states of odd $n$ qubits are SLOCC equivalent then either their n-tangles for odd $n$ qubits both vanish or neither vanishes [19].

From Theorem 4, one can see that two pure states of odd $n$ qubits are LU inequivalent provided that the two states are different in n-tangle. For $|\mathrm{GHZ}\rangle$, the $n$-tangle is $1 / 4$, so any state whose n-tangle is not $1 / 4$ is LU inequivalent to $|\mathrm{GHZ}\rangle$. It is known that the n-tangle of odd $n$ qubits is between 0 and $1 / 4$. We can define the family $F_{g}$ to be the set of all the states whose $n$-tangles are $g$. Thus, if two pure states of odd $n$ qubits are LU equivalent then they belong to the same family $F_{g}$. It is plain to see that for any odd $n$ qubits, there is a one to one correspondence between the set $\left\{F_{g} \mid g \in[0,1 / 4]\right\}$ of the families $F_{g}$ and the interval $[0,1 / 4]$.

## LU classification of three qubits invoking singular values of $\Omega_{1,2}^{(3)}$

$$
\text { Singular values of } \Omega_{1,2}^{(3)}
$$

A straightforward calculation yields that the singular values of $\Omega_{1,2}^{(3)}(|\psi\rangle)$ are $S, S, 0,0$, where $S$ is put in Appendix A. For any state $|A\rangle$ of Acin et al.'s canonical form in Eq. (6), $S^{2}$ reduces to

$$
\begin{equation*}
S^{2}=\lambda_{0}^{2} \lambda_{2}^{2}+\lambda_{0}^{2} \lambda_{4}^{2}+\left|\lambda_{1} \lambda_{4} e^{i \varphi}-\lambda_{2} \lambda_{3}\right|^{2} . \tag{8}
\end{equation*}
$$

Specially, for the SLOCC GHZ class in Table II, we have $S$ in Eq. (8); for the SLOCC W class in Table II, $S^{2}=\lambda_{2}^{2}\left(\lambda_{0}^{2}+\lambda_{3}^{2}\right)$; for the SLOCC A-BC class, $S=\left|\lambda_{1} \lambda_{4} e^{i \varphi}-\lambda_{2} \lambda_{3}\right|$; for the SLOCC B-AC class, $S=\lambda_{0} \lambda_{2}$; for the SLOCC C-AB class and the SLOCC $|000\rangle$ class, $S=0$.

## LU classification of Acin et al.'s canonical form for three qubits invoking singular values of $\Omega_{1,2}^{(3)}$

Recall that for three qubits, the space of the normalized states in Eq. (6) is partitioned into nine families under LU [26, 27]. It is well known that pure states of three qubits were partitioned into six SLOCC classes: GHZ, W, A-BC, B-AC, C-AB, and A-B-C [3]. In terms of the singular values, we partition each one of the SLOCC classes GHZ, W, A-BC, $\mathrm{B}-\mathrm{AC}$, and $\mathrm{C}-\mathrm{AB}$ in Table IV under LU as follows.

In light of Theorem 4, one can know that two states are LU inequivalent if the two states are different in $S$. Next, let us demonstrate how to partition the SLOCC class W under LU. Let the family $F_{S}$ consist of all the states of the SLOCC W class with the same value of $S$. Thus, we obtain a one to one correspondence between the set $\left\{F_{s} \mid S \in(0, \sqrt{2} / 3]\right\}$ of the LU families $F_{s}$ and the interval $(0, \sqrt{2} / 3]$. Similarly, we can partition GHZ, B-AC, and A-BC under LU. See Table IV.

For the SLOCC C-AB class, $S=0$. Note that Acin et al.'s canonical form for SLOCC C-AB class in Table II reduces to $|\varsigma\rangle_{A B}|0\rangle_{C}$, where $|\varsigma\rangle_{A B}=$ $\left(\lambda_{0}|00\rangle+\lambda_{1} e^{i \varphi}|10\rangle+\lambda_{3}|11\rangle\right)_{A B}$, where $\lambda_{0} \lambda_{3} \neq 0$. Let $c$ be the concurrence of $|\varsigma\rangle_{A B}$ and the family $F_{c}$ consist of all the states $|\varsigma\rangle_{A B}|0\rangle_{C}$ with the same value in $c=\lambda_{0} \lambda_{3}$. Thus, there is a one to one correspondence between the set $\left\{F_{c} \mid c \in(0,1 / 2]\right\}$ of the families $F_{c}$ and the interval $(0,1 / 2]$.

TABLE IV. LU classification of the states of Acin et al.'s canonical form

| SLOCC | the set of LU families |
| :---: | :---: |
| GHZ class | $\left\{F_{s} \mid S \in(0,1 / 2]\right\}$ |
| W class | $\left\{F_{s} \mid S \in(0, \sqrt{2} / 3]\right\}$ |
| B-AC class | $\left\{F_{s} \mid S \in(0,1 / 2]\right\}$ |
| A-BC class | $\left\{F_{s} \mid S \in(0,1 / 2]\right\}$ |
| C-AB class | $\left\{F_{c} \mid c \in(0,1 / 2]\right\}$ |
| $\|000\rangle$ class | single family |

## SUMMARY

In this paper, from the coefficient matrices of states of $n$ qubits we construct the $\ell$-spin-flipping matrices and show that the $\ell$-spin-flipping matrices are congruent and unitary congruent under SLOCC and LU , respectively. Thus, the ranks and the singular values of the $\ell$-spin-flipping matrices are invariant under SLOCC and LU, respectively.

The invariance of ranks of the spin-flipping matrices provides a simple way of classifying $n$-qubit states under SLOCC. For example, we obtain complete SLOCC classifications of two and three qubits. The invariance of singular values of the spin-flipping matrices $\Omega_{1}^{(n)}$ implies the invariance of the concurrence for even $n$ qubits and the invariance of the n-tangle for odd $n$ qubits. The invariance of the concurrence and the invariance of the n-tangle permit LU classifications for even $n$ qubits and odd $n$ qubits, respectively. See Table V. It only performs additions and multiplications of coefficients of states to compute the concurrence for even $n$ qubits and the $n$-tangle for odd $n$ qubits in comparison to the methods 28, 29].

TABLE V. SLOCC and LU classification invoking the concurrence and n-tangle

| qubits | $\psi^{\prime}$ and $\psi$ are SLOCC inequivalent |
| :--- | :---: |
| even $n$ | if only one of their concurrences is 0 <br> if only one of their n-tangles is 0 |
| qubits $n$ | $\psi^{\prime}$ and $\psi$ are LU inequivalent |
| even $n$ | if their concurrences are different |
| odd $n$ | if their n-tangles are different |

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## APPENDIX A. SOME EXPRESSIONS

$$
\begin{gathered}
e_{12}=\sum_{i=0}^{2^{n-1}-1}(-1)^{N(i)} a_{i} a_{2^{n}-1-i} \\
e_{11}=2 \sum_{i=0}^{2^{n-2}-1}(-1)^{N(i)} a_{i} a_{2^{n-1}-1-i} \\
e_{22}=2 \sum_{i=0}^{2^{n-2}-1}(-1)^{N(i)} a_{2^{n-1}+i} a_{2^{n}-1-i} \\
S^{2}=\left|a_{0} a_{3}-a_{1} a_{2}\right|^{2}+\left|a_{0} a_{5}-a_{1} a_{4}\right|^{2} \\
+\left|a_{0} a_{7}-a_{1} a_{6}\right|^{2}+\left|a_{2} a_{5}-a_{3} a_{4}\right|^{2} \\
+\left|a_{2} a_{7}-a_{3} a_{6}\right|^{2}+\left|a_{4} a_{7}-a_{5} a_{6}\right|^{2}
\end{gathered}
$$

[1] Nielsen,M.A., Chuang, I.L.: Quantum Computation and Quantum Information (Cambridge Univ. Press, Cambridge, 2000).
[2] Bennett, C. H., Brassard, G., Crèpeau, C., Jozsa, R., Peres, A., Wootters, W. K.: Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. Phys. Rev. Lett., 70, 1895 (1993)
[3] Dür, W., Vidal, G., Cirac, J.I.: Three qubits can be entangled in two inequivalent ways. Phys. Rev. A 62, 062314 (2000)
[4] Verstraete, F., Dehaene, J., DeMoor, B., Verschelde, H.: Four qubits can be entangled in nine different ways. Phys. Rev. A 65, 052112 (2002)
[5] Miyake, A.: Classification of multipartite entangled states by multidimensional determinants. Phys. Rev. A 67, 012108 (2003)
[6] Borsten, L., Dahanayake, D., Duff, M.J., Marrani, A., Rubens, W.: Four-qubit entanglement classification from string theory. Phys. Rev. Lett. 105, 100507 (2010)
[7] Viehmann, O., Eltschka, C., Siewert, J.: Polynomial invariants for discrimination and classification of four-qubit entanglement. Phys. Rev. A 83, 052330 (2011)
[8] Bastin, T., Krins, S., Mathonet, P., Godefroid, M., Lamata, L., Solano, E.: Operational Families of Entanglement Classes for Symmetric N-Qubit States. Phys. Rev. Lett. 103, 070503 (2009)
[9] Ribeiro, P., Mosseri, R.: Entanglement in the Symmetric Sector of $n$ Qubits. Phys. Rev. Lett. 106, 180502 (2011).
[10] Li, X., Li, D.: Classification of general n-qubit states under stochastic local operations and classical communication in terms of the rank of coefficient matrix. Phys. Rev. Lett. 108, 180502 (2012)
[11] Li, X., Li, D.: Method for classifying multiqubit states via the rank of the coefficient matrix and its application to four-qubit states. Phys. Rev. A 86, 042332 (2012)
[12] Sharma, S.S., Sharma, N.K.: Classification of multipartite entanglement via negativity fonts. Phys. Rev. A 85, 042315 (2012)
[13] Gour, G.,Wallach, N.R.: Classification of multipartite entanglement of all finite dimensionality. Phys. Rev. Lett. 111, 060502 (2013)
[14] Luque, J.-G., Thibon, J.-Y.: Polynomial invariants of four qubits. Phys. Rev. A 67, 042303 (2003)
[15] Leifer, M.S., Linden, N., Winter, A.: Measuring polynomial invariants of multiparty quantum states. Phys. Rev. A 69, 052304 (2004)
[16] Levay, P.: On the geometry of a class of N-qubit entanglement monotones. J. Phys. A: Math. Gen. 38, 9075, (2005)
[17] Osterloh, A., Siewert, J.: Constructing N-qubit entanglement monotones from antilinear operators. Phys. Rev. A 72, 012337 (2005)
[18] Li, D., Li, X., Huang, H., Li, X.: Stochastic local operations and classical communication invariant and the residual entanglement for $n$ qubits. Phys. Rev. A 76, 032304 (2007)
[19] Li, D.: The n-tangle of odd n qubits. Quantum Inf. Process 11, 481 (2012).
[20] Li, X., Li, D.: Polynomial invariants of degree 4 for even-n qubits and their applications in entanglement classification. Phys. Rev. A 88, 022306 (2013)
[21] Coffman, V., Kundu, J., Wootters, W.K.: Distributed entanglement. Phys. Rev. A 61, 052306 (2000)
[22] Li, X., Li, D.: Entanglement classification and invariant-based entanglement measures. Phys. Rev. A 91, 012302 (2015)
[23] Wang, Shuhao, Lu, Yao, Gao, Ming, Cui, Jianlian, Li, Junlin: Classification of arbitrary-dimensional multipartite pure states under stochastic local operations and classical communication using the rank of coefficient matrix. J. Phys. A: Math. Theor. 46, 105303 (2013)
[24] Grassl, M., Rötteler, M., Beth, T.: Computing local invariants of quantum-bit systems. Phys. Rev. A 58, 1833 (1998)
[25] Rains, E.M.: Polynomial invariants of quantum codes. IEEE trans. inf. theory 46, 54 (2000)
[26] Acin, A., Andrianov, A., Costa, L., Jane, E., Latorre, J.L., Tarrach, R.: Generalized Schmidt decomposition and classification of three-quantum-bit states. Phys. Rev. Lett. 85, 1560 (2000)
[27] Acin, A., Andrianov, A., Jane, E., Tarrach, R.: Three-qubit pure-state canonical forms. J. Phys. A: Math. Theor. 34, 6725 (2001)
[28] Kraus, B: Local unitary equivalence and entangle-
ment of multipartite pure states. Phys. Rev. A 82, 032121 (2010)
[29] Kraus, B.: Local Unitary Equivalence of Multipartite Pure States. Phys. Rev. Lett. 104, 020504 (2010)
[30] Sárosi, G., Lévay, P.: Entanglement in fermionic Fock space. J. Phys. A: Math. Theor. 47, 115304 (2014)

