

# On monogamy of four-qubit entanglement

S. Shelly Sharma<sup>1,\*</sup> and N. K. Sharma<sup>2,†</sup>

<sup>1</sup>*Departamento de Física, Universidade Estadual de Londrina, Londrina 86051-990, PR Brazil*

<sup>2</sup>*Departamento de Matematica, Universidade Estadual de Londrina, Londrina 86051-990, PR Brazil*

Our main result is a monogamy inequality satisfied by the entanglement of a focus qubit (one-tangle) in a four-qubit pure state and entanglement of subsystems. Analytical relations between three-tangles of three-qubit marginal states, two-tangles of two-qubit marginal states and unitary invariants of four-qubit pure state are used to obtain the inequality. The contribution of three-tangle to one-tangle is found to be half of that suggested by a simple extension of entanglement monogamy relation for three qubits. On the other hand, an additional contribution due to a two-qubit invariant which is a function of three-way correlations is found. We also show that four-qubit monogamy inequality conjecture of ref. [PRL 113, 110501 (2014)], in which three-tangles are raised to the power  $\frac{3}{2}$ , does not estimate the residual correlations, correctly, for certain subsets of four-qubit states. A lower bound on residual four-qubit correlations is obtained.

## I. INTRODUCTION

Entanglement is a necessary ingredient of any quantum computation and a physical resource for quantum cryptography and quantum communication [1]. It has also found applications in other areas such as quantum field theory [2], statistical physics [3], and quantum biology [4]. Multipartite entanglement that comes into play in quantum systems with more than two subsystems, is a resource for multiuser quantum information tasks. Since the mathematical structure of multipartite states is much more complex than that of bipartite states, the characterization of multipartite entanglement is a far more challenging task [5].

Monogamy is a unique feature of quantum entanglement, which determines how entanglement is distributed amongst the subsystems. Three-qubit entanglement is known to satisfy a quantitative constraint, known as CKW monogamy inequality [6]. In recent articles [7–9], it has been shown that the most natural extension of CKW inequality to four-qubit entanglement is violated by some of the four-qubit states and different ways to extend the monogamy inequality to four-qubits have been conjectured. For a subclass of four-qubit generic states, an extension of strong monogamy inequality to negativity and squared negativity [10] is satisfied, however, there exist four-qubit states for which negativity and squared negativity are not strongly monogamous. Three-qubit states show two distinct types of entanglement. As we go to four qubits, additional degrees of freedom make it possible for new entanglement types to emerge. It is signalled by the fact that corresponding to the three-qubit invariant that detects genuine three-way entanglement of a three-qubit pure state, a four-qubit pure state has five three-qubit invariants for each set of three qubits [11]. An  $m$ -qubit invariant is understood to be a function of state coefficients which remains invariant under the action of a local unitary transformation on the state of any one of the  $m$  qubits. A valid discussion of entanglement monogamy for four qubits, therefore, must include contributions from invariants that detect new entanglement types.

This article is an attempt to identify, analytically, the contributions of two-tangles (pairwise entanglement), three-tangles (genuine three-way entanglement) and four tangles to entanglement of a focus qubit with the three remaining qubits (one-tangle) in a four qubit state. To do this, we express one-tangle in terms of two-qubit invariants. Monogamy inequality constraint on four qubit entanglement is obtained by comparing the one-tangle with upper bounds on two-tangles and three-tangles [12] defined on two and three qubit marginal states. Contribution of three-tangles to one-tangle is found to be half of what is expected from a direct generalization of CKW inequality to four qubits. The difference arises due to new entanglement modes that are available to four qubits. It is verified that the "residual entanglement", obtained after subtracting the contributions of two-tangles and three-tangles from one-tangle, is greater or equal to genuine four-tangle. Genuine four-tangle [11, 13] is a degree eight function of state coefficients of the pure state. Besides that, the "residual entanglement" also contains contributions from square of degree-two four-tangle [14, 15] and degree-four invariants which quantify the entanglement of a given pair of qubits with its complement in a four-qubit pure state [14].

---

\*Electronic address: shelly@uel.br

†Electronic address: nsharma@uel.br

## II. ONE-TANGLE OF A FOCUS QUBIT IN A FOUR-QUBIT STATE

We start by expressing one-tangle of a focus qubit in a four-qubit state in terms of two-qubit invariant functions of state coefficients. Entanglement of qubit  $A_1$  with  $A_2$  in a two-qubit pure state

$$|\Psi_{12}\rangle = \sum_{i_1, i_2} a_{i_1 i_2} |i_1 i_2\rangle; \quad (i_m = 0, 1) \quad (1)$$

is quantified by two-tangle defined as

$$\tau_{1|2}(|\Psi_{12}\rangle) = 2|D^{00}|, \quad (2)$$

where  $D^{00} = a_{00}a_{11} - a_{10}a_{01}$  is a two-qubit invariant. Here  $a_{i_1 i_2}$  are the state coefficients. On a four qubit pure state, however, for each choice of a pair of qubits one identifies nine two-qubit invariants. Three-tangle [6] of a three-qubit pure state is defined in terms of modulus of a three-qubit invariant. On the most general four qubit state, on the other hand, we have five three-qubit invariants corresponding to a given set of three qubits. Four-qubit invariant that quantifies the sum of three-way and four-way correlations of a three-qubit partition in a pure state is known to be a degree-eight invariant [11], which is a function of three-qubit invariants. It is natural to expect that the monogamy inequality for four qubits takes into account the entanglement modes available exclusively to four-qubit system. To understand, how various two-tangles and three-tangles add up to generate total entanglement of a focus qubit in a pure four-qubit state, we follow the steps listed below:

- (1) Write down one-tangle of focus qubit as a sum of two-qubit invariants.
- (2) Express two-tangles, three-tangles and four-tangle or the upper bounds on the tangles in terms of two-qubit invariants.
- (3) Rewrite one-tangle in terms of tangles defined on two- and three-qubit reduced states and "residual four-qubit correlations".
- (4) Compare the "residual four-qubit correlations" with the lower bound on four-qubit correlations written in terms of four-qubit invariants.

To facilitate the identification of two-qubit and three-qubit invariants, we use the formalism of determinants of two by two matrices of state coefficients referred to as negativity fonts. For more on definition and physical meaning of determinants of negativity fonts, please refer to section (VI) of ref. [11].

For the purpose of this article, we write down and use the determinants of negativity fonts of a four-qubit state when qubit  $A_1$  is the focus qubit. On a four-qubit pure state, written as

$$|\Psi_{1234}\rangle = \sum_{i_1, i_2, i_3 i_4} a_{i_1 i_2 i_3 i_4} |i_1 i_2 i_3 i_4\rangle, \quad (i_m = 0, 1), \quad (3)$$

where state coefficients  $a_{i_1 i_2 i_3 i_4}$  are complex numbers and  $i_m$  refers to the basis state of qubit  $A_m$ , ( $m = 1, 2, 3$ ), we identify the determinants of two-way negativity fonts to be  $D_{(A_3)_{i_3}(A_4)_{i_4}}^{00} = a_{00i_3i_4}a_{11i_3i_4} - a_{10i_3i_4}a_{01i_3i_4}$ ,  $D_{(A_2)_{i_2}(A_4)_{i_4}}^{00} = a_{0i_20i_4}a_{1i_21i_4} - a_{1i_20i_4}a_{0i_21i_4}$ , and  $D_{(A_2)_{i_2}(A_3)_{i_3}}^{00} = a_{0i_2i_30}a_{1i_2i_31} - a_{1i_2i_30}a_{0i_2i_31}$ . Besides that we also have  $D_{(A_4)_{i_4}}^{00i_3} = a_{00i_3i_4}a_{11,i_3\oplus 1,i_4} - a_{10i_3i_4}a_{01,i_3\oplus 1,i_4}$  (three-way),  $D_{(A_3)_{i_3}}^{00i_4} = a_{00i_3i_4}a_{11i_3,i_4\oplus 1} - a_{10i_3i_4}a_{01i_3,i_4\oplus 1}$  (three-way),  $D_{(A_2)_{i_2}}^{00i_4} = a_{0i_20i_4}a_{1i_21i_4\oplus 1} - a_{1i_20i_4}a_{0i_21i_4\oplus 1}$  (three-way), and  $D^{00i_3i_4} = a_{00i_3i_4}a_{11,i_3\oplus 1,i_4\oplus 1} - a_{10i_3i_4}a_{01,i_3\oplus 1,i_4\oplus 1}$  (four-way), as the determinants of negativity fonts.

One-tangle given by  $\tau_{1|234}(|\Psi_{1234}\rangle) = 4 \det(\rho_1)$ , where  $\rho_1 = \text{Tr}_{A_2 A_3 A_4}(|\Psi_{1234}\rangle \langle \Psi_{1234}|)$ , quantifies the entanglement of qubit  $A_1$  with  $A_2, A_3$  and  $A_4$ . It is four times the square of negativity of partial transpose of four-qubit pure state with respect to qubit  $A_1$  [18]. Negativity, in general, does not satisfy the monogamy relation. However, it has been shown by He and Vidal [19] that negativity can satisfy monogamy relation in the setting provided by disentangling theorem. It is easily verified that

$$\begin{aligned} \tau_{1|234}(|\Psi_{1234}\rangle) = 4 & \left[ \sum_{i_3, i_4=0}^1 \left| D_{(A_3)_{i_3}(A_4)_{i_4}}^{00} \right|^2 + \sum_{i_2, i_4=0}^1 \left| D_{(A_2)_{i_2}(A_4)_{i_4}}^{00} \right|^2 + \sum_{i_2, i_3=0}^1 \left| D_{(A_2)_{i_2}(A_3)_{i_3}}^{00} \right|^2 \right. \\ & \left. + \sum_{i_3, i_4=0}^1 \left| D_{(A_4)_{i_4}}^{00i_3} \right|^2 + \sum_{i_3, i_4=0}^1 \left| D_{(A_3)_{i_3}}^{00i_4} \right|^2 + \sum_{i_2, i_4=0}^1 \left| D_{(A_2)_{i_2}}^{00i_4} \right|^2 + \sum_{i_3, i_4=0}^1 \left| D^{00i_3i_4} \right|^2 \right]. \quad (4) \end{aligned}$$

One-tangle depends on 2-way, 3-way and 4-way correlations of focus qubit  $A_1$  with the rest of the system.

### III. DEFINITIONS OF TWO-TANGLES AND THREE-TANGLE

This section contains the definitions of two-tangles and three-tangles for pure and mixed three-qubit states. Consider a three-qubit pure state

$$|\Psi_{123}\rangle = \sum_{i_1, i_2, i_3} a_{i_1 i_2 i_3} |i_1 i_2 i_3\rangle, \quad i_m = 0, 1, \quad (5)$$

Using the notation from ref. [11], we define  $D_{(A_3)_{i_3}}^{00} = a_{00i_3}a_{11i_3} - a_{10i_3}a_{01i_3}$ , ( $i_3 = 0, 1$ ) (the determinant of a two-way negativity font) and  $D^{00i_3} = a_{00i_3}a_{11i_3+1} - a_{10i_3}a_{01i_3+1}$ , ( $i_3 = 0, 1$ ) (the determinant of a three-way negativity font). Entanglement of qubit  $A_1$  with the rest of the system is quantified by one-tangle  $\tau_{1|23}(|\Psi_{123}\rangle) = 4 \det(\rho_1)$ , where  $\rho_1 = \text{Tr}_{A_2 A_3}(|\Psi_{123}\rangle\langle\Psi_{123}|)$ . One can verify that

$$\tau_{1|23}(|\Psi_{123}\rangle) = 4 \sum_{i_3=0}^1 \left| D_{(A_3)_{i_3}}^{00} \right|^2 + 4 \sum_{i_3=0}^1 \left| D^{00i_3} \right|^2 + 4 \sum_{i_2=0}^1 \left| D_{(A_2)_{i_2}}^{00} \right|^2. \quad (6)$$

For qubit pair  $A_1 A_2$  in  $|\Psi_{123}\rangle$ , we identify three two-qubit invariants that is

$$D_{(A_3)_0}^{00}, \frac{(D^{000} + D^{001})}{2}, D_{(A_3)_1}^{00} \quad (7)$$

while for the pair  $A_1 A_3$  two-qubit invariants are

$$D_{(A_2)_0}^{00}, \frac{(D^{000} - D^{001})}{2}, D_{(A_2)_1}^{00}. \quad (8)$$

These two-qubit invariants transform under a unitary  $U = \frac{1}{\sqrt{1+|x|^2}} \begin{bmatrix} 1 & -x^* \\ x & 1 \end{bmatrix}$  on the third qubit in a way analogous to the complex functions  $A(x)$ ,  $B(x)$  and  $C(x)$  of Appendix A. Then the invariants corresponding to  $I^{(1)}$ ,  $I^{(2)}$  and  $\tau^2$  are three-qubit invariants for the given choice of qubit pair. In Table I, we enlist the correspondence of two-qubit invariants for qubit pairs  $A_1 A_2$  and  $A_1 A_3$ , with complex numbers  $A$ ,  $B$  and  $C$  of Appendix A and set the notation for invariants corresponding to  $I^{(1)}$ ,  $I^{(2)}$  and  $\tau^2$ .

TABLE I: Two-tangles and three-tangle in terms of two-qubit invariants of a three-qubit pure state. Here  $i = 0, 1$ ,  $I^{(1)} = |A|^2 + \frac{1}{2}|B|^2 + |C|^2$ ,  $I^{(2)} = |B^2 - 4AC|$  and  $I = 4I^{(1)} - 2I^{(2)}$  (Appendix A).

	Two-qubit invariants			Three-qubit invariants			
Qubit Pair	$A$	$B$	$C$	$I^{(1)}$	$I^{(2)}$	$4I^{(2)}$	$\tau^2$
$A_1 A_2$	$D_{(A_3)_0}^{00}$	$D^{000} + D^{001}$	$D_{(A_3)_1}^{00}$	$N_{A_3}$	$ I_{3,4}( \Psi_{123}\rangle) $	$\tau_{1 2 3}( \Psi_{123}\rangle)$	$[\tau_{1 2}(\rho_{12})]^2$
$A_1 A_3$	$D_{(A_2)_0}^{00}$	$D^{000} - D^{001}$	$D_{(A_2)_1}^{00}$	$N_{A_2}$	$ I_{3,4}( \Psi_{123}\rangle) $	$\tau_{1 2 3}( \Psi_{123}\rangle)$	$[\tau_{1 3}(\rho_{13})]^2$

One-tangle in terms of three-qubit invariants listed in column five of Table I reads as

$$\tau_{1|23}(|\Psi_{123}\rangle) = 4N_{A_3} + 4N_{A_2}. \quad (9)$$

Three tangle [6] of pure state  $|\Psi_{123}\rangle$  is equal to the modulus of the polynomial invariant of degree four that is

$$\tau_{1|2|3}(|\Psi_{123}\rangle) = 4|I_{3,4}(|\Psi_{123}\rangle)|,$$

where

$$\begin{aligned} I_{3,4}(|\Psi_{123}\rangle) &= (D^{000} + D^{001})^2 - 4D_{(A_3)_0}^{00}D_{(A_3)_1}^{00} \\ &= (D^{000} - D^{001})^2 - 4D_{(A_2)_0}^{00}D_{(A_2)_1}^{00}. \end{aligned} \quad (10)$$

The entanglement measure  $\tau_{1|2|3}(|\Psi_{123}\rangle)$  is extended to a mixed state of three qubits via convex roof extension that is

$$[\tau_{1|2|3}(\rho_{123})]^{\frac{1}{2}} = \min_{\{p_i, |\phi_{123}^{(i)}\rangle\}} \sum_i p_i \left[ \tau_{1|2|3}(|\phi_{123}^{(i)}\rangle) \right]^{\frac{1}{2}}, \quad (11)$$

where minimization is taken over all complex decompositions  $\{p_i, |\phi_{123}^{(i)}\rangle\}$  of  $\rho_{123}$ . Here  $p_i$  is the probability of finding the normalized state  $|\phi_{123}^{(i)}\rangle$  in the mixed state  $\rho_{123}$ .

Two-tangle of the state  $\rho_{12} = \sum_i p_i |\phi_{12}^{(i)}\rangle \langle \phi_{12}^{(i)}|$  is constructed through convex roof extension as

$$\tau_{1|2}(\rho_{12}) = 2 \min_{\{p_i, \phi_{12}^{(i)}\}} \sum_i p_i |D^{00}(|\phi_{12}^{(i)}\rangle)|. \quad (12)$$

Two-tangle  $\tau_{1|2}(\rho_{12}) = C(\rho_{12})$ , where  $C(\rho_{12})$  is the concurrence [16, 17]. One can verify that the invariants  $N_{A_3}$ ,  $N_{A_2}$ , corresponding two-tangles, and three-tangle saturate the inequalities corresponding to Eq. (A7) that is

$$4N_{A_3} = [\tau_{1|2}(\rho_{12})]^2 + \frac{1}{2}\tau_{1|2|3}(|\Psi_{123}\rangle), \quad (13)$$

and

$$4N_{A_2} = [\tau_{1|3}(\rho_{13})]^2 + \frac{1}{2}\tau_{1|2|3}(|\Psi_{123}\rangle). \quad (14)$$

Since  $\tau_{1|23}(|\Psi_{123}\rangle) = 4(N_{A_2} + N_{A_3})$ , we obtain

$$\tau_{1|23}(|\Psi_{123}\rangle) \geq [\tau_{1|2}(\rho_{12})]^2 + [\tau_{1|3}(\rho_{13})]^2, \quad (15)$$

which is the well known CKW inequality. From Eqs. (13) and (14), the distribution of entanglement in a three-qubit state and its two-qubit marginals satisfies the following relation:

$$\tau_{1|23}(|\Psi_{123}\rangle) = [\tau_{1|2}(\rho_{12})]^2 + [\tau_{1|3}(\rho_{13})]^2 + \tau_{1|2|3}(|\Psi_{123}\rangle). \quad (16)$$

Moduli of two-qubit invariants, which depend only on the determinants of three-way negativity fonts, are used to define new two-tangles on the state  $\rho_{123} = \sum_i p_i |\phi_{123}^{(i)}\rangle \langle \phi_{123}^{(i)}|$  via

$$\tau_{1|p}^{(new)}(\rho_{123}) = \min_{\{p_i, \phi_{123}^{(i)}\}} \sum_i p_i \left( 2 |T_{1p}(|\phi_{123}^{(i)}\rangle)| \right), \quad (17)$$

where

$$T_{12}(|\Psi_{123}\rangle) = D^{000}(|\Psi_{123}\rangle) + D^{001}(|\Psi_{123}\rangle), \quad (18)$$

and

$$T_{13}(|\Psi_{123}\rangle) = D^{000}(|\Psi_{123}\rangle) - D^{001}(|\Psi_{123}\rangle). \quad (19)$$

### A. What does $\tau_{1|p}^{(new)}(\rho_{123})$ measure?

To understand the correlations represented by  $\tau_{1|p}^{(new)}(\rho_{123})$ , we examine a generic three-qubit state in its canonical form. A state is said to be in the canonical form when it is expressed as a superposition of minimal number of local basis product states (LBPS) [20]. The state coefficients of this form carry all the information about the non-local properties of the state, and do so minimally. Starting from a generic state in the basis  $|i_1 i_2 i_3\rangle$  (Eq. (5)), local unitary transformations allow us to write it in a form with the minimal number of LBPS. As a first step towards writing the state  $|\Psi_{123}\rangle$  in canonical form with respect to  $r^{th}$  qubit, we chose a unitary that results in a state on which one of the two-way two-qubit invariants is zero that is  $D_{(A_r)_0}^{00} = 0$  or  $D_{(A_r)_1}^{00} = 0$ . For example a unitary  $U^3 = \frac{1}{\sqrt{1+|x|^2}} \begin{bmatrix} 1 & -x^* \\ x & 1 \end{bmatrix}$ ; with  $x^* = \frac{(D^{000} + D^{001}) \pm \sqrt{I_{3,4}}}{2D_{(A_3)_1}^{00}}$ , acting on qubit  $A_3$  gives a state  $U^3 |\Psi_{123}\rangle = \sum_{i_1, i_2, i_3} b_{i_1 i_2 i_3} |i_1 i_2 i_3\rangle$ , such that  $b_{000}b_{110} - b_{100}b_{010} = 0$ . After eliminating  $b_{100}$ , the state reads as

$$U^3 |\Psi_{123}\rangle = b_{000} \left( |0\rangle_1 + \frac{b_{110}}{b_{010}} |1\rangle_1 \right) \left( |0\rangle_2 + \frac{b_{010}}{b_{000}} |1\rangle_2 \right) |0\rangle_3 + \sum_{i_1, i_2} b_{i_1 i_2 i_3} |i_1 i_2 1\rangle. \quad (20)$$

It is straight forward to write down the local unitaries  $U^1$  and  $U^2$  that lead to the canonical form,

$$|\Psi_{123}\rangle_c = c_{000}|000\rangle + c_{001}|001\rangle + c_{101}|101\rangle + c_{011}|011\rangle + c_{111}|111\rangle. \quad (21)$$

Notice that on canonical state,  $[4|T_{12}(|\Psi_{123}\rangle_c)|]^2 = [4|T_{13}(|\Psi_{123}\rangle_c)|]^2 = \tau_{1|2|3}(|\Psi_{123}\rangle)$ . Next we determine the range of values that  $[4|T_{12}(|\Psi_{123}\rangle)|]^2$  takes on a generic state  $|\Psi_{123}\rangle$ .

Combining the definition of  $N_{A_3}$  (from Table I) for the state  $|\Psi_{123}\rangle$ , with the result of Eq. (13), that is

$$\begin{aligned} 4N_{A_3} &= 4\left|D_{(A_3)_0}^{00}\right|^2 + 2\left|D^{000} + D^{001}\right|^2 + 4\left|D_{(A_3)_1}^{00}\right|^2 \\ &= [\tau_{1|2}(\rho_{12})]^2 + \frac{1}{2}\tau_{1|2|3}(|\Psi_{123}\rangle), \end{aligned} \quad (22)$$

we obtain

$$4|T_{12}(|\Psi_{123}\rangle)|^2 = \tau_{1|2|3}(|\Psi_{123}\rangle) - D_c, \quad (23)$$

where

$$D_c = 8\left|D_{(A_3)_0}^{00}\right|^2 + 8\left|D_{(A_3)_1}^{00}\right|^2 - 2[\tau_{1|2}(\rho_{12})]^2.$$

Since  $4\left(\left|D_{(A_3)_0}^{00}\right|^2 + \left|D_{(A_3)_1}^{00}\right|^2\right) \geq [\tau_{1|2}(\rho_{12})]^2$  ( $D_c \geq 0$ ), the value of  $|T_{12}(|\Psi_{123}\rangle)|$  satisfies

$$\tau_{1|2|3}(|\Psi_{123}\rangle) \geq 4|T_{12}(|\Psi_{123}\rangle)|^2 \geq 0. \quad (24)$$

In general, the difference  $D_c = \tau_{p|q|r}(|\Psi_{pqr}\rangle) - 4|T_{pq}(|\Psi_{pqr}\rangle)|^2$  measures the distance of a given three-qubit state from its canonical form with respect to  $r^{th}$  qubit.

On a pure state of three qubits,  $\tau_{1|2}^{(new)}(|\Psi_{123}\rangle) = \min |T_{12}(U^3|\Psi_{123}\rangle)| = 0$ . The state on which  $|T_{12}(U^3|\Psi_{123}\rangle)| = 0$ , is obtained by a unitary transformation  $U^3 = \frac{1}{\sqrt{1+|x|^2}} \begin{bmatrix} 1 & -x^* \\ x & 1 \end{bmatrix}$ , such that

$$x^* = \frac{\left|D_{(A_3)_0}^{00}\right|^2 - \left|D_{(A_3)_1}^{00}\right|^2 \pm \frac{1}{4}\sqrt{J_0}}{D_{(A_3)_1}^{00}(D^{000} + D^{001})^* + (D_{(A_3)_0}^{00})^*(D^{000} + D^{001})}, \quad (25)$$

where three-qubit invariant  $J_0$  reads as

$$J_0 = [\tau_{1|2}(\rho_{12})]^2 \left[ [\tau_{1|2}(\rho_{12})]^2 + \tau_{1|2|3}(|\Psi_{123}\rangle) \right]. \quad (26)$$

Next, consider the three-qubit mixed state  $\rho_{123} = \sum_{i=0,1} |\Phi_{123}^{(i)}\rangle \langle \Phi_{123}^{(i)}|$ , where  $|\Phi_{123}^{(i)}\rangle$  is an un-normalized state. Let the set of two-qubit invariants for the pair  $A_1A_2$  in the state  $|\Phi_{123}^{(i)}\rangle$  be

$$\left(D_{(A_3)_0}^{00}\right)^{(i)}, \frac{1}{2}(D^{000} + D^{001})^{(i)}, \left(D_{(A_3)_1}^{00}\right)^{(i)}. \quad (27)$$

New two-qubit invariant (Eq. (17)) on  $\rho_{123}$  is given by

$$\tau_{1|2}^{(new)}(\rho_{123}) = 2 \min_{\{|\Phi_{123}^{(i)}\rangle\}} \left[ |T_{12}(|\Phi_{123}^{(0)}\rangle)| + |T_{12}(|\Phi_{123}^{(1)}\rangle)| \right]. \quad (28)$$

where from Eq. (23),

$$4|T_{12}(|\Phi_{123}^{(i)}\rangle)|^2 = \tau_{1|2|3}(|\Phi_{123}^{(i)}\rangle) - D_c^{(i)}, \quad (29)$$

with  $D_c^{(i)}$  defined as

$$D_c^{(i)} = 8 \left| \left( D_{(A_3)_0}^{00} \right)^{(i)} \right|^2 + 8 \left| \left( D_{(A_3)_1}^{00} \right)^{(i)} \right|^2 - 2 \left[ \tau_{1|2} \left( \left| \Phi_{123}^{(i)} \right\rangle \right) \right]^2. \quad (30)$$

If  $U^3$  and  $V^3$  are the local unitaries on the third qubit such that  $T_{12} \left( \left| U^3 \Phi_{123}^{(0)} \right\rangle \right) = 0$  and  $T_{12} \left( \left| V^3 \Phi_{123}^{(1)} \right\rangle \right) = 0$ , then

$$\tau_{1|2}^{(new)}(\rho_{123}) = 2 \min \left\{ \left| T_{12} \left( \left| U^3 \Phi_{123}^{(1)} \right\rangle \right) \right|, \left| T_{12} \left( \left| V^3 \Phi_{123}^{(0)} \right\rangle \right) \right| \right\}. \quad (31)$$

Obviously, the value of  $\tau_{1|2}^{(new)}(\rho_{123})$  satisfies either the condition

$$\tau_{1|2|3} \left( \left| \Phi_{123}^{(0)} \right\rangle \right) \geq \left[ \tau_{1|2}^{(new)}(|\rho_{123}\rangle) \right]^2 \geq 0, \quad (32)$$

or the constraint

$$\tau_{1|2|3} \left( \left| \Phi_{123}^{(1)} \right\rangle \right) \geq \left[ \tau_{1|2}^{(new)}(|\rho_{123}\rangle) \right]^2 \geq 0. \quad (33)$$

#### IV. TANGLES AND THREE-QUBIT INVARIANTS OF A FOUR-QUBIT STATE

In this section, we identify relevant combinations of two-qubit invariants that remain invariant under a local unitary on the third qubit. Three-qubit invariants that we look for are the ones related to tangles of three-qubit reduced states obtained from four-qubit pure state by tracing out the degrees of freedom of the fourth qubit. For any given pair of qubits in a general four-qubit state, there are nine two-qubit invariants. Of the six degree-four three-qubit invariants constructed from the set of nine two-qubit invariants, one is defined only on the pure state. Five remaining invariants are functions of three-tangles and two-tangles. In Table II, we identify sets of two-qubit invariants of a four-qubit state which transform under a unitary,  $U = \frac{1}{\sqrt{1+|x|^2}} \begin{bmatrix} 1 & -x^* \\ x & 1 \end{bmatrix}$  on the third qubit in the same way as the functions  $A(x)$ ,  $B(x)$  and  $C(x)$  of Appendix A.

TABLE II: Two-tangles and three-tangles in terms of two-qubit invariants of a four-qubit pure state. Here  $i = 0, 1$ ,  $I^{(1)} = |A|^2 + \frac{1}{2}|B|^2 + |C|^2$ ,  $I^{(2)} = |B^2 - 4AC|$  and  $\tau^2 = 4I^{(1)} - 2I^{(2)}$  (Appendix A).

	Two-qubit invariants			Three-qubit invariants		
Qubit pair $\downarrow$	$A$	$B$	$C$	$I^{(1)}$	$I^{(2)}$	$\tau^2$
$A_1 A_2$	$D_{(A_3)_0(A_4)_i}^{00}$	$D_{(A_4)_i}^{000} + D_{(A_4)_i}^{001}$	$D_{(A_3)_1(A_4)_i}^{00}$	$N_{A_4}^{(i)}$	$ I_{3,4} \left( \left  \Phi_{123}^{(i)} \right\rangle \right) $	$\left[ \tau_{1 2} \left( \left  \Phi_{123}^{(i)} \right\rangle \right) \right]^2$
$A_1 A_2$	$D_{(A_3)_0}^{000} + D_{(A_3)_0}^{001}$	$\begin{pmatrix} D_{(A_3)_0}^{0000} + D_{(A_3)_0}^{0010} \\ + D_{(A_3)_0}^{0001} + D_{(A_3)_0}^{0011} \end{pmatrix}$	$D_{(A_3)_1}^{000} + D_{(A_3)_1}^{001}$	$M_{A_3}$	$ I_{3,4}^{(new)}( \Psi_{1234}\rangle) $	$\left[ \tau_{1 2}^{(new)}(\rho_{124}) \right]^2$
$A_1 A_3$	$D_{(A_2)_i(A_4)_0}^{00}$	$D_{(A_2)_i}^{000} + D_{(A_2)_i}^{001}$	$D_{(A_2)_i(A_4)_1}^{00}$	$N_{A_2}^{(i)}$	$ I_{3,4} \left( \left  \Phi_{134}^{(i)} \right\rangle \right) $	$\left[ \tau_{1 3} \left( \left  \Phi_{134}^{(i)} \right\rangle \right) \right]^2$
$A_1 A_3$	$D_{(A_4)_0}^{000} - D_{(A_4)_0}^{001}$	$\begin{pmatrix} D_{(A_4)_0}^{0000} + D_{(A_4)_0}^{0001} \\ - D_{(A_4)_0}^{0010} - D_{(A_4)_0}^{0011} \end{pmatrix}$	$D_{(A_4)_1}^{000} - D_{(A_4)_1}^{001}$	$M_{A_4}$	$ I_{3,4}^{(new)}( \Psi_{1234}\rangle) $	$\left[ \tau_{1 3}^{(new)}(\rho_{123}) \right]^2$
$A_1 A_4$	$D_{(A_2)_0(A_3)_i}^{00}$	$D_{(A_3)_i}^{000} - D_{(A_3)_i}^{001}$	$D_{(A_2)_1(A_3)_i}^{00}$	$N_{A_3}^{(i)}$	$ I_{3,4} \left( \left  \Phi_{124}^{(i)} \right\rangle \right) $	$\left[ \tau_{1 4} \left( \left  \Phi_{124}^{(i)} \right\rangle \right) \right]^2$
$A_1 A_4$	$D_{(A_2)_0}^{000} - D_{(A_2)_0}^{001}$	$\begin{pmatrix} D_{(A_2)_0}^{0000} + D_{(A_2)_0}^{0010} \\ - D_{(A_2)_0}^{0001} - D_{(A_2)_0}^{0011} \end{pmatrix}$	$D_{(A_2)_1}^{000} - D_{(A_2)_1}^{001}$	$M_{A_2}$	$ I_{3,4}^{(new)}( \Psi_{1234}\rangle) $	$\left[ \tau_{1 4}^{(new)}(\rho_{134}) \right]^2$

Three-qubit invariants listed in the last three columns depend on two-qubit invariants of columns two to four in the same way as  $I^{(1)}$ ,  $I^{(2)}$  and  $\tau^2$  depend on  $A$ ,  $B$  and  $C$ , for example three-qubit invariants in the third row of Table II read as

$$N_{A_4}^{(i)} = \left| D_{(A_3)_0(A_4)_i}^{00} \right|^2 + \frac{1}{2} \left| D_{(A_4)_i}^{000} + D_{(A_4)_i}^{001} \right|^2 + \left| D_{(A_3)_1(A_4)_i}^{00} \right|^2, (i = 0, 1) \quad (34)$$

and

$$|I_{3,4}(|\Phi_{123}^i\rangle)\rangle| = \left| \left( D_{(A_4)_i}^{000} + D_{(A_4)_i}^{001} \right)^2 - 4D_{(A_3)_0(A_4)_i}^{00} D_{(A_3)_1(A_4)_i}^{00} \right|, \quad (35)$$

and satisfy the inequality

$$4N_{A_4}^{(i)} \geq \left[ \tau_{1|2}(|\Phi_{123}^{(i)}\rangle) \right]^2 + \frac{1}{2} \left| 4I_{3,4}(|\Phi_{123}^{(i)}\rangle) \right|. \quad (36)$$

Here  $|\Phi_{123}^{(i)}\rangle$  is the un-normalized state defined through  $|\Psi_{1234}\rangle = \sum_{i_4=0,1} |\Phi_{123}^{(i_4)}\rangle |i_4\rangle$ .

Upper bound on two-tangle calculated by using the method of ref. [11] shows that

$$\sum_{i=0,1} \left[ \tau_{1|2}(|\Phi_{123}^{(i)}\rangle) \right]^2 \geq \left[ \tau_{1|2}^{up}(\rho_{12}) \right]^2 \geq \left[ \tau_{1|2}(\rho_{12}) \right]^2. \quad (37)$$

Similarly the upper bounds on  $\tau_{1|2|3}(\rho_{123})$  for the nine families of four-qubit states, calculated in ref. [12] satisfy the condition

$$\sum_{i=0,1} \left| 4I_{3,4}(|\Phi_{123}^{(i)}\rangle) \right| \geq \tau_{1|2|3}^{up}(\rho_{123}) \geq \tau_{1|2|3}(\rho_{123}). \quad (38)$$

Combining the conditions of Eqs. (37) and (38), with inequality of Eq. (36), the sum of two-tangle and three-tangle satisfies the inequality

$$4 \sum_{i=0,1} N_{A_4}^{(i)} \geq \left[ \tau_{1|2}(\rho_{12}) \right]^2 + \frac{1}{2} \tau_{1|2|3}(\rho_{123}). \quad (39)$$

On  $\rho_{124} = \sum_i |\Phi_{124}^i\rangle \langle \Phi_{124}^i|$ , new two-qubit invariant (Eq. (17)) is defined as

$$\tau_{1|2}^{(new)}(\rho_{124}) = 2 \min_{\{|\Phi_{124}^{(i)}\rangle\}} \sum_i |T_{12}(|\Phi_{124}^{(i)}\rangle)|.$$

where  $T_{12}(|\Phi_{124}^{(i)}\rangle) = D_{(A_3)_i}^{000}(|\Phi_{124}^{(i)}\rangle) + D_{(A_3)_i}^{001}(|\Phi_{124}^{(i)}\rangle)$ . New three-qubit tangle on a pure state is defined as  $\tau_{1|2|3}^{(new)}(|\Psi_{1234}\rangle) = 4 \left| (I_3)_{A_4}^{(new)}(|\Psi_{1234}\rangle) \right|$ , where

$$(I_3)_{A_4}^{(new)}(|\Psi_{1234}\rangle) = (D^{0000} + D^{0010} + D^{0001} + D^{0011})^2 - 4 \left( D_{(A_3)_0}^{000} + D_{(A_3)_0}^{001} \right) \left( D_{(A_3)_1}^{000} + D_{(A_3)_1}^{001} \right).$$

The invariants  $M_{A_3}$ ,  $\tau_{1|2|3}^{(new)}(|\Psi_{1234}\rangle)$  and  $\left[ \tau_{1|2}^{(new)}(\rho_{124}) \right]^2$  satisfy the inequality (analogous to Eq. (A7)),

$$4M_{A_3} - \frac{1}{2} \tau_{1|2|3}^{(new)}(|\Psi_{1234}\rangle) \geq \left[ \tau_{1|2}^{(new)}(\rho_{124}) \right]^2. \quad (40)$$

Using a similar argument, three-qubit invariants listed in lines 5 and 6 of Table II satisfy the inequalities

$$4 \sum_i N_{A_2}^{(i)} \geq \left[ \tau_{1|3}(\rho_{13}) \right]^2 + \frac{1}{2} \tau_{1|3|4}(\rho_{134}), \quad (41)$$

and

$$4M_{A_4} - \frac{1}{2} \tau_{1|3|4}^{(new)}(|\Psi_{1234}\rangle) \geq \left[ \tau_{1|3}^{(new)}(\rho_{123}) \right]^2, \quad (42)$$

where three-tangle defined on pure four-qubit state reads as  $\tau_{1|3|4}^{(new)}(|\Psi_{1234}\rangle) = 4 \left| (I_3)_{A_2}^{(new)}(|\Psi_{1234}\rangle) \right|$ , and

$$\tau_{1|3}^{(new)}(\rho_{123}) = 2 \min_{\{|\Phi_{123}^{(i)}\rangle\}} \sum_i |T_{13}(|\Phi_{123}^{(i)}\rangle)|.$$

Using invariants of local unitaries on qubits  $A_1$  and  $A_4$ , and definitions given in lines 7 and 8 of Table II, we obtain the inequalities

$$4 \sum_i N_{A_3}^{(i)} \geq [\tau_{1|4}(\rho_{14})]^2 + \frac{1}{2} \tau_{1|2|4}(\rho_{124}) \quad (43)$$

and

$$4M_{A_2} - \frac{1}{2} \tau_{1|2|4}^{(new)}(\Psi_{1234}) \geq [\tau_{1|4}^{(new)}(\rho_{134})]^2, \quad (44)$$

where new three-tangle reads as  $\tau_{1|2|4}^{(new)}(\Psi_{1234}) = 4 \left| (I_3)_{A_3}^{(new)}(|\Psi_{1234}\rangle) \right|$ , and

$$\tau_{1|4}^{(new)}(\rho_{134}) = 2 \min_{\{|\Phi_{134}^{(i)}\rangle\}} \sum_i \left| D_{(A_2)_i}^{000}(|\Phi_{134}^{(i)}\rangle) - D_{(A_2)_i}^{001}(|\Phi_{134}^{(i)}\rangle) \right|.$$

The relations between two-tangles, three-tangles and three-qubit invariants listed in column (5) in Table II (Eqs. (39-44)) are important to obtain the monogamy inequality satisfied by one-tangle.

## V. MONOGAMY OF FOUR-QUBIT ENTANGLEMENT

To obtain the relation between tangles of reduced states and one-tangle of the focus qubit, firstly, we identify the three-qubit invariant combinations of two-qubit invariants in Eq. (4). It is found that a four-qubit invariant of degree two, which is defined only on the pure state, is also needed. Genuine four-tangle  $\tau_{1|2|3|4}(|\Psi_{1234}\rangle)$  (Eq. (B8) appendix B), defined in refs. [11, 13] is a degree-eight function of state coefficients. However, the degree-two four-qubit invariant which is equal to invariant H of refs. [14, 15], is known to have the form,

$$I_{4,2}(|\Psi_{1234}\rangle) = D^{0000} - D^{0010} - D^{0001} + D^{0011}. \quad (45)$$

Four-tangle defined as  $\tau_{1|2|3|4}^{(0)}(|\Psi_{1234}\rangle) = 2 |I_{4,2}(|\Psi_{1234}\rangle)|$ , is non zero on a GHZ state and vanishes on W-like states of four qubits. However, since  $\tau_{1|2|3|4}^{(0)}(|\Psi_{1234}\rangle)$  fails to vanish on product of entangled states of two qubits, it is not a measure of genuine four-way entanglement. By direct substitution, one-tangle of Eq. (4) can be rewritten in terms of three-qubit invariants listed in column five of Table II and square of four-qubit invariant  $\tau_{1|2|3|4}^{(0)}(|\Psi_{1234}\rangle)$  that is

$$\tau_{1|234} = 4 \sum_{q=2}^4 \sum_{i=0}^1 N_{A_q}^{(i)} + 2 \sum_{q=2}^4 M_{A_q} + \frac{1}{4} \left[ \tau_{1|2|3|4}^{(0)}(|\Psi_{1234}\rangle) \right]^2. \quad (46)$$

By making use of the inequalities given by Eqs. (39-44), one-tangle of Eq. (46) is found to satisfy the constraint

$$\begin{aligned} \tau_{1|234}(|\Psi_{1234}\rangle) &\geq \sum_{p=2}^4 [\tau_{1|p}(\rho_{1p})]^2 + \frac{1}{2} \sum_{\substack{(p,q)=2 \\ q>p}}^4 \tau_{1|p|q}(\rho_{1pq}) \\ &\quad + \frac{1}{2} [\tau_{1|2}^{(new)}(\rho_{123})]^2 + \frac{1}{2} [\tau_{1|3}^{(new)}(\rho_{134})]^2 + \frac{1}{2} [\tau_{1|4}^{(new)}(\rho_{124})]^2. \end{aligned} \quad (47)$$

This is the monogamy relation that governs the distribution of quantum entanglement in subsystems of a four-qubit state, when qubit  $A_1$  is the focus qubit. Similar inequalities can be written down for other possible choices of focus qubit.

Parameter  $\Delta$ , which quantifies the residual correlations not accounted for by entanglement of reduced states, is defined as the difference between the entanglement of focus qubit  $A_1$  with the rest of the system and contributions from sum of two-way and three-way tangles of focus qubit, that is

$$\begin{aligned} \Delta &= \tau_{1|234}(|\Psi_{1234}\rangle) - \sum_{p=2}^4 [\tau_{1|p}(\rho_{1p})]^2 - \frac{1}{2} \sum_{\substack{(p,q)=2 \\ q>p}}^4 \tau_{1|p|q}(\rho_{1pq}) \\ &\quad - \frac{1}{2} [\tau_{1|2}^{(new)}(\rho_{123})]^2 - \frac{1}{2} [\tau_{1|3}^{(new)}(\rho_{134})]^2 - \frac{1}{2} [\tau_{1|4}^{(new)}(\rho_{124})]^2. \end{aligned} \quad (48)$$

All contributions to  $\Delta$  are invariant with respect to local unitaries on any one of the four qubits. A possible lower bound on  $\Delta$  which, partially, accounts for residual four-qubit correlations depends on the sum of three-qubit invariants  $\tau_{1|p|q}^{(new)}(|\Psi_{1234}\rangle)$ , four-qubit invariant  $\tau_{1|2|3|4}^{(0)}(|\Psi_{1234}\rangle)$  and genuine four-tangle  $\tau_{1|2|3|4}(|\Psi_{1234}\rangle)$ . It will be interesting to write residual four-qubit correlations as a permutationally invariant combination of four-qubit invariants defined on the state  $|\Psi_{1234}\rangle$  such that all possible modes of four-way entanglement are accounted for.

## VI. LOWER BOUND ON RESIDUAL FOUR-QUBIT CORRELATIONS

In this section, we examine the known four-qubit invariants to identify the possible candidates to represent four-way correlations and construct a lower bound on residual correlations. Transformation equation of determinant of a two-way negativity font due to the action of local unitaries on the two remaining qubits, yields a two-variable polynomial of degree four. Invariants of the polynomials corresponding to qubit pairs  $A_1A_2$ ,  $A_1A_3$ , and  $A_1A_4$  are four-qubit invariants. For instance, four-qubit invariants corresponding to  $U^3U^4D_{(A_3)_0(A_4)_0}^{00} = 0$ , are  $N_{A_4}^{(0)} + \frac{1}{2}M_{A_3} + N_{A_4}^{(1)}$  and

$$\begin{aligned} J^{A_1A_2} &= (I_3)_{A_4}^{(new)}(|\Psi_{1234}\rangle) - 4 \left( D_{(A_4)_0}^{000} + D_{(A_4)_0}^{001} \right) \left( D_{(A_4)_1}^{000} + D_{(A_4)_1}^{001} \right) \\ &\quad + 8 \left( D_{(A_3)_1(A_4)_0}^{00} D_{(A_3)_0(A_4)_1}^{00} + D_{(A_3)_0(A_4)_0}^{00} D_{(A_3)_1(A_4)_1}^{00} \right). \end{aligned} \quad (49)$$

Notice that  $|4J^{A_1A_2}|$  contains contribution from  $\tau_{1|2|3}^{(new)}(|\Psi_{1234}\rangle)$ .

Similarly, four-qubit invariant that quantifies the entanglement of  $A_1$  and  $A_3$  in  $N_{A_2}^{(0)} + \frac{1}{2}M_{A_4} + N_{A_2}^{(1)}$  is obtained from

$$\begin{aligned} J^{A_1A_3} &= (I_3)_{A_2}^{(new)}(|\Psi_{1234}\rangle) - 4 \left( D_{(A_2)_0}^{000} + D_{(A_2)_0}^{001} \right) \left( D_{(A_2)_1}^{000} + D_{(A_2)_1}^{001} \right) \\ &\quad + 8 \left( D_{(A_2)_1(A_4)_0}^{00} D_{(A_2)_0(A_4)_1}^{00} + D_{(A_2)_0(A_4)_0}^{00} D_{(A_2)_1(A_4)_1}^{00} \right) \end{aligned} \quad (50)$$

while the sum  $N_{A_3}^{(0)} + \frac{1}{2}M_{A_2} + N_{A_3}^{(1)}$  contains correlations between qubits  $A_1$  and  $A_4$  which depend on the four-qubit invariant that reads as

$$\begin{aligned} J^{A_1A_4} &= (I_3)_{A_3}^{(new)}(|\Psi_{1234}\rangle) - 4 \left( D_{(A_3)_0}^{000} - D_{(A_3)_0}^{001} \right) \left( D_{(A_3)_1}^{000} - D_{(A_3)_1}^{001} \right) \\ &\quad + 8 \left( D_{(A_2)_0(A_3)_1}^{00} D_{(A_2)_1(A_3)_0}^{00} + D_{(A_2)_0(A_3)_0}^{00} D_{(A_2)_1(A_3)_1}^{00} \right). \end{aligned} \quad (51)$$

It is easily verified [14] that  $J^{A_1A_i}$  ( $i = 2, 3, 4$ ) are not independent invariants but satisfy the constraint

$$4|J^{A_1A_2} + J^{A_1A_3} + J^{A_1A_4}| = 3 \left[ \tau_{1|2|3|4}^{(0)}(|\Psi_{1234}\rangle) \right]^2.$$

By construction,  $\beta^{A_1A_i} = \frac{4}{3}|J^{A_1A_i}|$  is the entanglement of qubit pair  $A_1A_i$  in the four-qubit state due to two-way, three-way and four-way correlations. It is easily verified that  $|J^{A_1A_2}| = |J^{A_3A_4}|$ ,  $|J^{A_1A_3}| = |J^{A_2A_4}|$ , and  $|J^{A_1A_4}| = |J^{A_2A_3}|$ , as such,  $\sum_{i \neq p} \beta^{A_pA_i}$  does not depend on the choice of focus qubit. A four-qubit state has four-way correlations if and only if at least two of the three  $\beta^{A_1A_i}$  ( $i = 2, 3, 4$ ) are non-zero. Recalling that one-tangle has a contribution from four-qubit invariant  $\tau_{1|2|3|4}^{(0)}(|\Psi_{1234}\rangle)$ , and genuine four-tangle  $\tau_{1|2|3|4}$  (Eq. (B8)), a lower bound on residual four-way correlations for the set of state, satisfying  $\sum_{i,j=2;(i < j)}^4 \beta^{A_1A_i} \beta^{A_1A_j} \neq 0$ , can be written as

$$\delta(|\Psi_{1234}\rangle) = \frac{1}{4} \sum_{i=2}^4 \beta^{A_1A_i} + \frac{1}{4} \left[ \tau_{1|2|3|4}^{(0)}(|\Psi_{1234}\rangle) \right]^2 + \frac{1}{2} \sqrt{\tau_{1|2|3|4}}. \quad (52)$$

All contributions to  $\delta(|\Psi_{1234}\rangle)$  are invariant with respect to local unitaries on any one of the four qubits as well as the choice of focus qubit.

On a four-qubit GHZ state

$$|GHZ_4\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle), \quad (53)$$

which is known to have only four-qubit correlations

$$\tau_{1|234} = \tau_{1|2|3|4} = \left[ \tau_{1|2|3|4}^{(0)}(|\Psi_{1234}\rangle) \right]^2 = 1,$$

while  $\beta^{A_1 A_i} = \frac{1}{3}$  ( $i = 2, 3, 4$ ). Therefore, as expected  $\tau_{1|234} = \delta(|\Psi_{1234}\rangle)$ . While on the state

$$|\Phi\rangle = \frac{1}{2}(|1111\rangle + |1100\rangle + |0010\rangle + |0001\rangle), \quad (54)$$

with no three-tangles and two-tangles, we have  $\beta^{A_1 A_2} = \frac{2}{3}$ ,  $\beta^{A_1 A_3} = \beta^{A_1 A_4} = \frac{1}{3}$  and

$$\tau_{1|234} = \tau_{1|2|3|4} = 1; \tau_{1|2|3|4}^{(0)}(|\Psi_{1234}\rangle) = 0,$$

therefore  $\delta(|\Phi\rangle) = \frac{5}{6}$ . In this case not all four-way correlations are accounted for by  $\delta(|\Phi\rangle)$ .

## VII. STATES VIOLATING GENERAL MONOGAMY INEQUALITY

As mentioned in ref. [7] a natural extension of CKW inequality to four-qubit states reads as

$$\tau_{1|234}(|\Psi_{1234}\rangle) \geq \sum_{p=2}^4 [\tau_{1|p}(\rho_{1p})]^2 + \sum_{\substack{(p,q)=2 \\ q>p}}^4 \tau_{1|p|q}(\rho_{1pq}). \quad (55)$$

Regula et al. [7] have analysed arbitrary pure states  $|\Psi_{1234}\rangle$  of four-qubit systems and shown that a subset of these states violates the inequality of Eq. (55). Based on numerical evidence, the authors provide a conjectured monogamy inequality. For the case of four-qubits with  $A_1$  as focus qubit, the monogamy inequality of Eq. (9) in ref. [7] reads as

$$\begin{aligned} \tau_{1|234}(|\Psi_{1234}\rangle) \geq & [\tau_{1|2}(\rho_{12})]^2 + [\tau_{1|3}(\rho_{13})]^2 + [\tau_{1|4}(\rho_{14})]^2 \\ & + [\tau_{1|2|3}(\rho_{123})]^{\frac{3}{2}} + [\tau_{1|2|4}(\rho_{124})]^{\frac{3}{2}} + [\tau_{1|3|4}(\rho_{134})]^{\frac{3}{2}}. \end{aligned} \quad (56)$$

Here three tangles are raised to the power  $\frac{3}{2}$ , so that the “residual four tangle” may not become negative. We denote the residual correlations, calculated from inequality of Eq. (55) by  $\Delta_1$  and that from inequality proposed in ref. [7] (Eq. (56)) by  $\Delta_2$ . In a more recent article [9], it has been clarified that the states leading to violations of the strong monogamy inequality belong to the degenerate subclasses (with  $a = c$  or  $b = c$ ) of  $|G_{abc}^{(2)}\rangle$  [21] defined as

$$\begin{aligned} |G_{abc}^{(2)}\rangle = & \frac{a+b}{2}(|0000\rangle + |1111\rangle) + \frac{a-b}{2}(|0011\rangle + |1100\rangle) \\ & + c(|0101\rangle + |1010\rangle) + |0110\rangle. \end{aligned} \quad (57)$$

For the choice  $b = c = ia$  with  $a \geq 0$ , we obtain the class of single parameter states  $|G_{a,ia,ia}^{(2)}\rangle$ . It is found that with qubit  $A_1$  as the focus qubit, one-tangle takes the value

$$\tau_{1|234} = \frac{8a^2 + 16a^4}{(4a^2 + 1)^2}, \quad (58)$$

while  $\tau_{1|2|3|4}^{(0)} = \frac{2a^2}{(4a^2+1)}$  and genuine four-tangle  $\tau_{1|2|3|4} = 0$ . Three tangles for the states read as  $\tau_{1|2|3} = \tau_{1|2|4} = \tau_{1|3|4} = \frac{8a^3}{(4a^2+1)^2}$ . All new two-tangles take value zero. Two-qubit states obtained from  $|G_{a,ia,ia}^{(2)}\rangle$  after tracing out a pair of qubits are  $X$  states. Two-tangles for these states were calculated numerically.

Figure I displays the residual four-qubit correlations,  $\Delta$  (Olive green line-Solid) from our monogamy relation (Eq. (48)),  $\Delta_1$  (red line-Dot) obtained by using a generalization of CKW inequality (Eq. (55)), as well as  $\Delta_2$  (orange line-Dash) from monogamy conjecture of ref. [7] (Eq. (56)), for the states  $|G_{a,ia,ia}^{(2)}\rangle$  ( $0 \leq a \leq 5$ ). As expected  $\Delta_1$  becomes negative for a certain range of values of  $a$ , while  $\Delta$  and  $\Delta_2$  remain positive for the set of states being considered. Since three-tangle varies from zero to one, and  $(\tau_{1|p|q})^{\frac{3}{2}} > \frac{1}{2}\tau_{1|p|q}$  for  $\tau_{1|p|q} > 0.25$ , it is obvious that

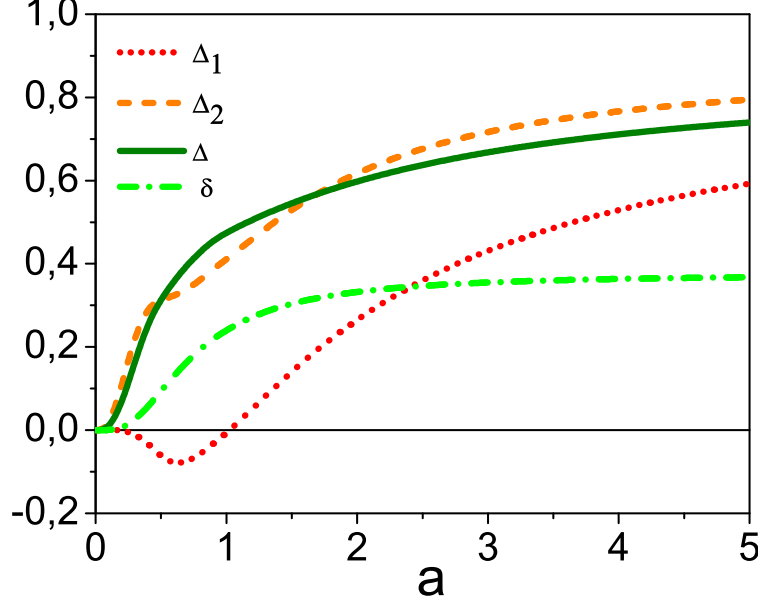


FIG. 1: Plot of  $\Delta$  (Olive green line-Solid) from our monogamy relation Eq. (48),  $\Delta_1$  (red line-Dot) from monogamy inequality of Eq. (55) and  $\Delta_2$  (orange line-Dash) using monogamy conjecture of ref. [7] Eq. (56), for the states  $|G_{a,ia,ia}^{(2)}\rangle$  ( $0 \leq a \leq 5$ ). Figure also displays a plot of our lower bound on residual four-qubit correlations  $\delta(|G_{a,ia,ia}^{(2)}\rangle)$  (green line-Dash Dot).

the conjecture of ref. [7] either overestimates or underestimates the four-way correlations in classes of four-qubit states with finite three tangles, but no contribution from new two-tangles. The lower bound on residual four-qubit correlations is found to be (Eq. (52))

$$\delta(|G_{a,ia,ia}^{(2)}\rangle) = \frac{6a^4}{(4a^2 + 1)^2}. \quad (59)$$

Figure I also shows a plot (green line-Dash Dot) of  $\delta(|G_{a,ia,ia}^{(2)}\rangle)$ .

## VIII. CONCLUSIONS

Monogamy inequality of Eq. (47), obtained analytically by expressing the tangle in terms of two-qubit and three-qubit invariants, is satisfied by all classes of four-qubit states. In particular, it is satisfied on the set of states  $|G_{a,ia,ia}^{(2)}\rangle$  that violate the entanglement monogamy relation obtained by a generalization of CKW inequality. The residual four-qubit correlations obtained by subtracting two-tangles and three-tangles from one-tangle of focus qubit represent contributions from all possible four-qubit entanglement modes. Unlike the case of three-qubits, where a single degree-four invariant quantifies three-way correlations in a pure state, a combination of four-qubit invariants is needed to quantify all possible modes of four-qubit pure state entanglement. Lower bound constructed from four-qubit invariants of degree 2, 4, and 8 is permutationally invariant and partially accounts for residual four-way correlations. To close the question, it will be interesting to find a single four-qubit invariant or write a combination of known four-qubit invariants which accounts for the residual correlations in all four-qubit pure states.

On the states  $|G_{a,ia,ia}^{(2)}\rangle$ , value of genuine four-tangle is found to be zero, while the residual entanglement is non zero. Since on  $|G_{a,ia,ia}^{(2)}\rangle$  all three tangles are finite for  $a \neq 0$ , four qubits are entangled through three-way correlations. Four-qubit states may, likewise, be entangled due to combination of three-way and two-way correlations or only pairwise correlations. Consider the subset of pure states in which four-qubit entanglement arises due to three-way

correlations, while genuine four-way correlations are absent. The monogamy inequality conjecture for four qubits in which quantifiers of three-way entanglement (three tangles) should be raised to the power  $\frac{3}{2}$  [7] does not estimate the residual correlations, correctly, for these states. A simple example is the state

$$|\Psi\rangle = \frac{1}{2} (|0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle),$$

for which  $\Delta = 0$ , while Eq. (56) predicts  $\Delta_2 = \frac{3}{8}$ . In general, the approach of ref. [7], where an  $n$ -tangle is raised to an arbitrary power to account for  $n$ -way correlations in one-tangle, is not likely to account for  $n$ -way correlations, correctly, for all the states. A simple calculation, on the same lines as for the case of four qubits, shows that as more qubits are added the contribution to one-tangle from three-tangles defined on three-qubit pure states is always multiplied by a factor of  $\frac{1}{2}$ . In addition, one-tangle may have a contribution from new-two tangles. For a given value of three-tangle, contribution from corresponding new two-tangles does not have a simple relationship with the three-tangle. In a multi-qubit state, the contribution of degree four  $n$ -tangle ( $\tau_{1|2|\dots|n}$ ) to degree four one-tangle will be multiplied by a factor of  $(\frac{1}{2})^{n-2}$ . Our approach paves the way to understanding scaling of entanglement distribution as qubits are added to obtain larger multiqubit quantum systems.

Financial support from Universidade Estadual de Londrina PR, Brazil is acknowledged.

### Appendix A: Upper bound on a function of complex numbers

In this section, we obtain mathematical relations used in section IV to define necessary invariants on four-qubit and three-qubit states. Let  $A(x)$ ,  $B(x)$  and  $C(x)$  satisfy the set of equations

$$A(x) = \frac{1}{1 + |x|^2} (A - x^* B + (x^*)^2 C), \quad (\text{A1})$$

$$B(x) = \frac{1}{1 + |x|^2} (B(1 - |x|^2) - 2x^* C + 2xA), \quad (\text{A2})$$

$$C(x) = \frac{1}{1 + |x|^2} (C + xB + x^2 A), \quad (\text{A3})$$

where  $A$ ,  $B$ , and  $C$  are complex numbers. One can verify that

$$I^{(1)} = |A|^2 + \frac{1}{2} |B|^2 + |C|^2 = |A(x)|^2 + \frac{1}{2} |B(x)|^2 + |C(x)|^2. \quad (\text{A4})$$

Consider a function of complex variable  $x$  defined as

$$\tau = 2 \min_x (|A(x)| + |C(x)|) = \min_x I(x), \quad (\text{A5})$$

Value of  $x$  for which  $A(x) = 0$  is

$$x_0^* = \frac{B}{2C} \pm \frac{1}{2C} \sqrt{B^2 - 4AC}.$$

The discriminant  $B^2 - 4AC$  is, obviously, an invariant, so we define  $I^{(2)} = |B^2 - 4AC|$ . Substituting the value of  $x_0$  in Eq. (A3), we obtain

$$I(x_0) = 2 |C(x_0)| = 2 \sqrt{I^{(1)} - \frac{1}{2} I^{(2)}}; I^{(1)} \geq \frac{1}{2} I^{(2)} \quad (\text{A6})$$

From the definition of  $\tau$ ,  $I(x_0)$  is an upper bound on  $I$  that is

$$\tau^2 \leq 4I^{(1)} - 2I^{(2)}. \quad (\text{A7})$$

## Appendix B: Genuine four-tangle

Three-qubit and four-qubit invariants relevant to quantifying three-way and genuine four-way entanglement of state (3) had been constructed in ref. [11]. Degree four invariants of interest for the triple  $A_1 A_3 A_4$  in state  $|\Psi_{1234}\rangle$  comprise a set denoted by  $\left\{(I_{3,4})_{A_2}^{4-m,m} : m = 0 \text{ to } 4\right\}$ . The elements in the set  $\left\{(I_{3,4})_{A_2}^{4-m,m} : m = 0 \text{ to } 4\right\}$  are invariant with respect to local unitaries on qubits  $A_1$ ,  $A_3$ , and  $A_4$ . The three-qubit invariants for  $A_1 A_3 A_4$  in terms of two-qubit invariants for the pair  $A_1 A_3$  read as:

$$(I_3)_{A_2}^{4,0} = \left(D_{(A_2)_0}^{000} + D_{(A_2)_0}^{001}\right)^2 - 4D_{(A_2)_0(A_4)_0}^{00} D_{(A_2)_0(A_4)_1}^{00}, \quad (\text{B1})$$

$$\begin{aligned} (I_3)_{A_2}^{3,1} &= \frac{1}{2} \left(D_{(A_2)_0}^{000} + D_{(A_2)_0}^{001}\right) (D^{0000} + D^{0001} - D^{0010} - D^{0011}) \\ &\quad - \left[D_{(A_2)_0(A_4)_1}^{00} \left(D_{(A_4)_0}^{000} - D_{(A_4)_0}^{001}\right) + D_{(A_3)_0(A_4)_0}^{00} \left(D_{(A_4)_1}^{000} - D_{(A_4)_1}^{001}\right)\right], \end{aligned} \quad (\text{B2})$$

$$\begin{aligned} (I_3)_{A_2}^{2,2} &= \frac{1}{6} (D^{0000} + D^{0001} - D^{0010} - D^{0011})^2 \\ &\quad - \frac{2}{3} \left(D_{(A_4)_0}^{000} - D_{(A_4)_0}^{001}\right) \left(D_{(A_4)_1}^{000} - D_{(A_4)_1}^{001}\right) \\ &\quad + \frac{1}{3} \left(D_{(A_2)_0}^{000} + D_{(A_2)_0}^{001}\right) \left(D_{(A_2)_1}^{000} + D_{(A_2)_1}^{001}\right) \\ &\quad - \frac{2}{3} \left(D_{(A_2)_0(A_4)_1}^{00} D_{(A_2)_1(A_4)_0}^{00} + D_{(A_2)_0(A_4)_0}^{00} D_{(A_2)_1(A_4)_1}^{00}\right) \end{aligned} \quad (\text{B3})$$

$$\begin{aligned} (I_3)_{A_2}^{1,3} &= \frac{1}{2} \left(D_{(A_2)_1}^{000} + D_{(A_2)_1}^{001}\right) (D^{0000} + D^{0001} - D^{0010} - D^{0011}) \\ &\quad - \left[D_{(A_2)_1(A_4)_1}^{00} \left(D_{(A_4)_0}^{000} - D_{(A_4)_0}^{001}\right) + D_{(A_2)_1(A_4)_0}^{00} \left(D_{(A_4)_1}^{000} - D_{(A_4)_1}^{001}\right)\right]. \end{aligned} \quad (\text{B4})$$

$$(I_3)_{A_2}^{0,4} = \left(D_{(A_2)_1}^{000} + D_{(A_2)_1}^{001}\right)^2 - 4D_{(A_2)_1(A_4)_0}^{00} D_{(A_2)_1(A_4)_1}^{00}. \quad (\text{B5})$$

Four-qubit invariant that quantifies the sum of three-way and four-way correlations of triple  $A_1 A_3 A_4$ , reads as

$$16N_{4,8}^{A_1 A_3 A_4} = 16 \left(6 \left|(I_3)_{A_2}^{2,2}\right|^2 + 4 \left|(I_3)_{A_2}^{3,1}\right|^2 + 4 \left|(I_3)_{A_2}^{1,3}\right|^2 + \left|(I_3)_{A_2}^{4,0}\right|^2 + \left|(I_3)_{A_2}^{0,4}\right|^2\right), \quad (\text{B6})$$

while degree-eight invariant that detects genuine four-body entanglement of a four-qubit state is given by

$$I_{4,8} = 3 \left((I_3)_{A_2}^{2,2}\right)^2 - 4 (I_3)_{A_2}^{3,1} (I_3)_{A_2}^{1,3} + (I_3)_{A_2}^{4,0} (I_3)_{A_2}^{0,4}. \quad (\text{B7})$$

Invariant  $I_{4,8}$  is expressed here as a function of  $A_1 A_3 A_4$  invariants. Being independent of the choice of focus qubit,  $I_{4,8}$  can also be written as a function of  $A_1 A_2 A_3$  invariants or  $A_1 A_2 A_4$  invariants. Sets  $\left\{(I_{3,4})_{A_4}^{4-m,m} : m = 0 \text{ to } 4\right\}$  for qubits  $A_1 A_2 A_3$  and  $\left\{(I_{3,4})_{A_3}^{4-m,m} : m = 0 \text{ to } 4\right\}$  for the triple  $A_1 A_2 A_4$ , can be constructed from two-qubit invariants of properly selected pair of qubits [11]. Four-tangle based on degree-eight invariant is defined [11] as

$$\tau_{1|2|3|4} = 16 |12 (I_{4,8})|. \quad (\text{B8})$$

---

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition, 10th ed.* (Cambridge University Press, New York, UK, 2011).

- [2] P. Calabrese, J. Cardy, and E. Tonni, Phys. Rev. Lett. 109, 130502 (2012).
- [3] S. Sahling, G. Remenyi, C. Paulsen, P. Monceau, V. Saligrama, C. Marin, A. Revcolevschi, L. P. Regnault, S. Raymond and J. E. Lorenzo, Nature Physics 11, 255–260 (2015).
- [4] N. Lambert, Y. N. Chen, Y. C. Chen, C. M. Li, G. Y. Chen, and F. Nori, Quantum biology, Nat. Phys. 9, 10 (2013).
- [5] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009).
- [6] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A 61, 052306 (2000).
- [7] B. Regula, S. Di Martino, S. Lee, and G. Adesso, PRL 113, 110501 (2014).
- [8] B. Regula, S. Di Martino, S. Lee, and G. Adesso, Phys. Rev. Lett. 116, 049902(E)(2016)
- [9] B. Regula, A Osterloh and G. Adesso, Phys. Rev. A 93,052338 (2016).
- [10] S. Karmakar, A. Sen, A. Bhar, and D. Sarkar, Phys. Rev. A 93, 012327 (2016).
- [11] S. S. Sharma and N. K. Sharma, Quantum Inf Process, Vol. 15, 4973 (2016).
- [12] S. S. Sharma and N. K. Sharma, arXiv:1610.03916 [quant-ph](2016).
- [13] S. S. Sharma and N. K. Sharma, AIP Conference Proceedings 1633, 35 (2014).
- [14] S. S. Sharma and N. K. Sharma, Phys. Rev. A 82, 052340 (2010).
- [15] J. G. Luque and J. Y. Thibon, Phys. Rev. A 67, 042303 (2003).
- [16] S. Hill and W. K. Wootters, Phys. Rev. Lett. 78, 5022 (1997).
- [17] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
- [18] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
- [19] H. He and G. Vidal, Phys. Rev. A 91, 012339 (2015).
- [20] A. Acin, A. Andrianov, E. Jane, R. Tarrach, J. Phys. A: Math. Gen. 34, 6725 (2001).
- [21] F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, Phys. Rev. A 65, 052112 (2002).