The dynamics of local quantum uncertainty and trace distance discord for two-qubit X states under decoherence: A comparative study

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Abstract

We employ the concepts of local quantum uncertainty and geometric quantum discord based on the trace norm to investigate the environmental effects on quantum correlations of two bipartite quantum systems. The first one concerns a two-qubit system coupled with two independent bosonic reservoirs. We show that the trace discord exhibits frozen phenomenon contrarily to local quantum uncertainty. The second scenario deals with a two level system, initially prepared in a separable state, interacting with a quantized electromagnetic radiation. Our results show that there exists an exchange of quantum correlations between the two-level system and its surrounding which is responsible of the revival phenomenon of non classical correlations.

Keywords: Local quantum uncertainty. Geometric quantum discord. Non classical correlations. Dynamics of two-qubit system. Frozen and revival quantum correlations.

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1 Introduction

Quantum information protocols exploit the quantum features of superposition and entanglement to achieve quantum tasks that are not possible using the classical laws of physics and quantum information processors offer significant advantages in the communication and processing of information [1]. Quantum teleportation [2, 3], and quantum cryptography or quantum key distribution [4] constitute the most promising applications of quantum information science. Characterizing quantum correlations in quantum systems is a challenging issue in this field of research. Several measures of quantum correlations in bipartite and multipartite quantum systems have been introduced in the literature [5, 6, 7, 8]. The concurrence and entanglement of formation are two examples of quantitative measures of entanglement [9]. Another indicator of quantum correlations in bipartite quantum systems is the quantum discord based on the von Neumann entropy. This measure was introduced Ollivier and Zurek [10] and by Henderson and Vedral [11] and goes beyond entanglement and provides the proper tool to investigate the quantum correlations in an arbitrary bipartite state even those which are separable. The computability of quantum discord based on von Neumann entropy is in general a very complex task. Indeed, it has been proven that computing entropic quantum discord is NP-complete [12] and only partial results were obtained for some special two-qubit states. To overcome these difficulties, a geometric variant of quantum discord has been proposed in several works by employing Schatten p-norms. The first geometric measure of quantum discord was formulated in [13] by using the Hilbert-Schmidt norm (p=2). Despite its ease of computability [14, 15, 16, 17], the geometric discord is not a good quantum correlations indicator. In fact, the geometric discord based on Hilbert-Schmidt distance can increase under local quantum operations on the unmeasured qubit. This drawback is due to the lack of the contractivity property that any quantum correlations quantifier should satisfy (see reference [18]). Now, it is well established that the trace norm (Bures norm with p=1) is the only Schatten p-norm that one can use to deal with the geometric measure of quantum correlations [19] (see also [20, 21]).

Recently, a new measure called local quantum uncertainty has been introduced to investigate the pairwise quantum correlations of the discord type in multipartite systems. It has all the desirable properties that every good quantum correlations quantifier should satisfy [22]. It is based on the notion of Skew information introduced by Wigner to determine the uncertainty in the measurement of an observable [23]. The local quantum uncertainty presents the advantage of being analytically computable for any qubit-qudit system [22]. It is interesting to stress that this new quantum correlation quantifier is deeply related to quantum Fisher information [24, 25, 26] which is usually used in the context of quantum metrology [27].

In this paper, to quantify the degree of quantum correlations, we will use the local quantum uncertainty and the trace quantum discord. In Section 2, we give the explicit expression of local quantum uncertainty in a two-qubit X state. We give also the expression of the geometric discord based on the trace norm. In Section 3, we will investigate the dynamics of quantum correlations in a two-qubit X state coupled to two independent reservoirs. Section 4 is devoted to the analysis of the creation of quantum correlations in a system of two 2-level atoms, interacting with a quantized radiation field, initially prepared in a separable state. Concluding remarks close this paper.

2 Quantum correlations quantifiers: local quantum uncertainty and trace distance

2.1 Local quantum uncertainty

The local quantum uncertainty is a reliable quantifier of quantum correlation in bipartite quantum systems. This is essentially due to its easiness of computability and the fact that it enjoys all necessary properties of being a good quantumness measure. It is zero for classically correlated states and invariant under local unitary operations. We notice that for a two-qubit pure state, the local quantum uncertainty reduces to the linear entropy of entanglement [22]. The local quantum uncertainty quantifies the minimal quantum uncertainty in a quantum state due to a measurement of a local observable [22]. For a bipartite quantum state ρ , the local quantum uncertainty is defined as

$$\mathcal{U}(\rho) \equiv \min_{K_1} \mathcal{I}(\rho, K_1 \otimes \mathbb{I}_2), \tag{1}$$

where K_1 is some local observable on the qubit 1, \mathbb{I}_2 is the identity operator acting on the qubit 2 and

$$\mathcal{I}(\rho, K_1 \otimes \mathbb{I}_2) = -\frac{1}{2} \text{Tr}([\sqrt{\rho}, K_1 \otimes \mathbb{I}_2]^2)$$
 (2)

is the skew information which provides an analytical tool to quantify the information content in the state ρ with respect to the observable K_1 . This quantity was introduced by Wigner and Yanase to quantify the uncertainty in mixed states [23]. The statistical idea underlying skew information is the Fisher information which plays a central role in the theory of statistical estimation and quantum metrology [24]. The local quantum uncertainty is defined through a minimization procedure over the ensemble of all Hermitian operators acting on the qubit 1 [22]. For two qubit systems ($\frac{1}{2}$ -spin particles), the expression of the local quantum uncertainty is given by [22]

$$\mathcal{U}(\rho) = 1 - \max\{\lambda_1, \lambda_2, \lambda_3\},\tag{3}$$

where λ_1, λ_2 and λ_3 are the eigenvalues of the 3×3 matrix W whose matrix elements are defined by

$$\omega_{ij} \equiv \text{Tr}\{\sqrt{\rho}(\sigma_i \otimes \mathbb{I}_2)\sqrt{\rho}(\sigma_j \otimes \mathbb{I}_2)\},\tag{4}$$

with i, j = 1, 2, 3.

The X states belong to an interesting family of two-qubit states that are used in several problems of quantum information. In the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the X states are of the form

$$\rho = \begin{pmatrix}
\rho_{11} & 0 & 0 & \rho_{14} \\
0 & \rho_{22} & \rho_{23} & 0 \\
0 & \rho_{32} & \rho_{33} & 0 \\
\rho_{41} & 0 & 0 & \rho_{44}
\end{pmatrix}.$$
(5)

In the Fano-Bloch representation, the density matrix ρ can be written as follows:

$$\rho = \frac{1}{4} \sum_{\alpha,\beta} T_{\alpha\beta} \sigma_{\alpha} \otimes \sigma_{\beta}, \tag{6}$$

where $T_{\alpha\beta} = \text{Tr}\rho (\sigma_{\alpha} \otimes \sigma_{\beta})$. For states of X type, the non vanishing matrix elements (4) are given by (see the appendix 1)

$$w_{11} = \frac{1}{4} \left[4 \left(\sqrt{\lambda_1} + \sqrt{\lambda_4} \right) \left(\sqrt{\lambda_2} + \sqrt{\lambda_3} \right) + \frac{\left(T_{11}^2 - T_{22}^2 \right) + \left(T_{12}^2 - T_{21}^2 \right) + \left(T_{03}^2 - T_{30}^2 \right)}{\left(\sqrt{\lambda_1} + \sqrt{\lambda_4} \right) \left(\sqrt{\lambda_2} + \sqrt{\lambda_3} \right)} \right], \tag{7}$$

$$w_{22} = \frac{1}{4} \left[4 \left(\sqrt{\lambda_1} + \sqrt{\lambda_4} \right) \left(\sqrt{\lambda_2} + \sqrt{\lambda_3} \right) + \frac{\left(T_{22}^2 - T_{11}^2 \right) + \left(T_{21}^2 - T_{12}^2 \right) + \left(T_{30}^2 - T_{03}^2 \right)}{\left(\sqrt{\lambda_1} + \sqrt{\lambda_4} \right) \left(\sqrt{\lambda_2} + \sqrt{\lambda_3} \right)} \right], \tag{8}$$

$$w_{33} = \frac{1}{2} \left[\left(\sqrt{\lambda_1} + \sqrt{\lambda_4} \right)^2 + \left(\sqrt{\lambda_2} + \sqrt{\lambda_3} \right)^2 \right] + \frac{1}{8} \left[\frac{\left(T_{30} + T_{03} \right)^2 - \left(T_{11} - T_{22} \right)^2 - \left(T_{12} + T_{21} \right)^2}{\left(\sqrt{\lambda_1} + \sqrt{\lambda_4} \right)^2} \right]$$
(9)

$$+\frac{1}{8}\left[\frac{(T_{03}-T_{30})^2-(T_{11}+T_{22})^2-(T_{12}-T_{21})^2}{\left(\sqrt{\lambda_2}+\sqrt{\lambda_3}\right)^2}\right],\tag{10}$$

$$w_{12} = w_{21} = \frac{1}{2} \frac{T_{11}T_{21} + T_{22}T_{12}}{(\sqrt{\lambda_1} + \sqrt{\lambda_4})(\sqrt{\lambda_2} + \sqrt{\lambda_3})},\tag{11}$$

$$w_{13} = w_{31} = w_{23} = w_{32} = 0. (12)$$

where $\lambda_i (i = 1, 2, 3, 4)$ are the eigenvalues of the density matrix ρ .

2.2 Trace measure of geometric discord

The trace norm (or 1-norm) was employed as a reliable geometric quantifier of quantum discord [19]. The expressions of trace distance quantum discord have been analytically derived for Bell-diagonal states [19, 28] and for an arbitrary two-qubit X state [21]. The trace distance quantum discord for a two-qubit state ρ is defined by

$$D_{\mathrm{T}}(\rho) = \frac{1}{2} \min_{\chi \in \Omega} ||\rho - \chi||_{1}, \tag{13}$$

where the trace distance is defined by $||\rho - \chi||_1 = \text{Tr}\sqrt{(\rho - \chi)^{\dagger}(\rho - \chi)}$. It measures the distance between the state ρ and the classical-quantum state χ belonging to the set Ω of classical-quantum states. A generic state $\chi \in \Omega$ is of the form $\chi = \sum_k p_k \Pi_{k,1} \otimes \rho_{k,2}$ where $\{p_k\}$ is a probability distribution, $\Pi_{k,1}$ are the orthogonal projector associated with the qubit 1 and $\rho_{k,2}$ is density matrix associated with the second qubit. The phase factors $\frac{\rho_{14}}{|\rho_{14}|} = e^{i\theta_{14}}$ and $\frac{\rho_{23}}{|\rho_{23}|} = e^{i\theta_{23}}$ of the off diagonal elements can be removed using the local unitary transformations

$$|0\rangle_1 \to \exp\left(\frac{-i}{2}(\theta_{14} + \theta_{23})\right)|0\rangle_1 \quad |0\rangle_2 \to \exp\left(\frac{-i}{2}(\theta_{14} - \theta_{23})\right)|0\rangle_2.$$

In this way, the anti-diagonal entries of the density matrix become positive and one gets

$$\rho \to \tilde{\rho} = \begin{pmatrix} \rho_{11} & 0 & 0 & |\rho_{14}| \\ 0 & \rho_{22} & |\rho_{23}| & 0 \\ 0 & |\rho_{23}| & \rho_{33} & 0 \\ |\rho_{14}| & 0 & 0 & \rho_{44} \end{pmatrix},$$

which rewrites in the Fano-Bloch representation as

$$\tilde{\rho} = \sum_{\alpha\beta} R_{\alpha\beta} \sigma_{\alpha} \otimes \sigma_{\beta},$$

where the non vanishing matrix elements $R_{\alpha\beta}$ are given by

$$R_{11} = 2(|\rho_{23}| + |\rho_{14}|)$$
 $R_{22} = 2(|\rho_{23}| - |\rho_{14}|)$ $R_{33} = 1 - 2(\rho_{22} + \rho_{33})$
$$R_{03} = 2(\rho_{11} + \rho_{33}) - 1$$
 $R_{30} = 2(\rho_{11} + \rho_{22}) - 1.$

The trace distance quantum discord is invariant under local transformations and we have

$$D_T(\rho) = D_T(\tilde{\rho})$$
.

The minimization in the equation (13) has been worked out for a generic two qubit-qubit X state [21]. As result, the trace distance quantum discord in the state ρ Eq. (5) takes the form

$$D_{\rm T}(\rho) = \frac{1}{2} \sqrt{\frac{R_{11}^2 R_{\rm max}^2 - R_{22}^2 R_{\rm min}^2}{R_{\rm max}^2 - R_{\rm min}^2 + R_{11}^2 - R_{22}^2}},\tag{14}$$

where

$$R_{\min}^2 = \min\{R_{11}^2, R_{33}^2\}$$
 with $R_{\max}^2 = \max(R_{33}^2, R_{22}^2 + R_{30}^2)$.

The Bell diagonal states constitute a specific instance of two-qubit X states. In the Fano-Bloch representation, they take the form

$$\rho^{\mathrm{BD}} = \frac{1}{4} [I \otimes I + \vec{c} \cdot (\vec{\sigma} \otimes \vec{\sigma})],$$

with $\vec{c} = \{c_1, c_2, c_3\}$ being a three-dimensional vector with elements satisfying $0 \le |c_i| \le 1$, and $\vec{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$ denotes the standard Pauli matrices. The trace distance discord (14) reduces to [19]

$$D_{\rm T}(\rho^{\rm BD}) = \inf\{|c_1|, |c_2|, |c_3|\},\tag{15}$$

that is the intermediate value for the absolute values of the correlation factors c_1 , c_2 , and c_3 .

3 Local quantum uncertainty and trace discord of two qubits in independent reservoirs

3.1 The density matrix

We consider a two-qubit system interacting with two independent reservoirs described by Ohmic-like spectral densities [29, 30, 31, 32]. The whole system is described by the following Hamiltonian

$$H = \sum_{j=1}^{2} \left[\frac{v_{j}}{2} \sigma_{j,3} + \sum_{k} w_{j,k} b_{j,k}^{\dagger} b_{j,k} + \sum_{k} \sigma_{j,3} \left(g_{j,k} b_{j,k}^{\dagger} + g_{j,k}^{*} b_{j,k} \right) \right].$$
 (16)

The first term of this Hamiltonian describes the two qubit (j = 1, 2) with v_j denoting the energy difference between the excited state $|1\rangle_j$ and the ground state $|0\rangle_j$ and $\sigma_{j,3}$ is the third Pauli matrix. The second term describes two independent reservoirs where $b_{j,k}^{\dagger}$ and $b_{j,k}$ are respectively the bosonic creation and the annihilation operators. They satisfy the usual bosonic commutation relations and $w_{j,k}$ denotes the frequency of the k-th mode of the reservoir coupled to the qubit j. The last term in (16) expresses the interaction part between the reservoir and the qubit. The coupling strength between the qubit j and the k-th mode is denoted by $g_{j,k}$.

To simplify our purpose, we shall assume that the two qubits are initially prepared in the following Bell-diagonal states

$$\rho_S(0) = \frac{1}{4} \left(I \otimes I + \sum_{i=1}^3 c_i \sigma_i \otimes \sigma_i \right), \tag{17}$$

where the correlation parameters satisfy the conditions $0 \le |c_i| \le 1$. Without loss of generality, we assume that $|c_1| \ge |c_2|$. The density matrix, in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle$, rewrites as

$$\rho_S(0) = \frac{1}{4} \begin{pmatrix} c_3^+ & 0 & 0 & c_- \\ 0 & c_3^- & c_+ & 0 \\ 0 & c_+ & c_3^- & 0 \\ c_- & 0 & 0 & c_3^+ \end{pmatrix}.$$

$$(18)$$

where $c_3^+ = 1 + c_3$, $c_3^- = 1 - c_3$, $c_+ = c_1 + c_2$ and $c_- = c_1 - c_2$. The states of the reservoirs in thermal equilibrium at the temperature T are given by

$$\rho_{E_j} = \frac{1}{Z_{E_j}} \exp\left(-\beta \sum_k w_{j,k} b_{j,k}^{\dagger} b_{j,k}\right),\tag{19}$$

where Z_{Ej} is the partition function of the reservoir j(j=1,2). Initially, the density matrix of the whole system reads as $\rho(0) = \rho_S(0) \otimes \rho_{E_1} \otimes \rho_{E_2}$, and the evolved state is then given by

$$\rho(t) = \exp(-iHt) \rho(0) \exp(iHt). \tag{20}$$

The density matrix of the two-qubit system is then obtained by tracing out the environment degrees of freedom.

This gives

$$\rho_S(t) = \text{Tr}_E \left[\exp\left(-iHt\right) \rho\left(0\right) \exp\left(iHt\right) \right]. \tag{21}$$

The matrix elements of $\rho_S(t)$ are given by

$$\langle ms | \rho_S(t) | nl \rangle = \text{Tr} \left[\pi_1^{mn}(t) \pi_2^{sl}(t) \rho(0) \right],$$
 (22)

with $\pi_j^{ms}(t) = \exp(iHt) \pi_j^{ms} \exp(-iHt)$ is the Heisenberg operator of qubit j and $\pi_j^{ms} = |s\rangle_j \langle m|$ where s and m take the values 0 and 1.

It is simple to see that the for s=m, the operators $\pi_j^{mm}=\frac{1}{2}(1+(-1)^m\sigma_3)$ commute with the Hamiltonian H and one has $\pi_j^{mm}(t)=\pi_j^{mm}(0)$. For different values of s and m, the operators π_j^{ms} are given by $\pi_j^{ms}(0)=|s\rangle_j\langle m|=\frac{1}{2}(\sigma_1+i(-1)^m\sigma_2)$. Therefore to determine the operators $\pi_j^{ms}(t)$ for $s\neq m$, one has to solve the following Heisenberg equations for the bosonic modes

$$i\frac{db_{j,k}(t)}{dt} = w_{j,k}b_{j,k}(t) + g_{j,k}\sigma_{j,3},$$
 (23)

and those associated with the two qubits are

$$i\frac{d\pi_{j}^{ms}(t)}{dt} = (-1)^{m+1} \left[v_{j} + 2\sum_{k} \left(g_{j,k} b_{j,k}^{\dagger}(t) + g_{j,k}^{*} b_{j,k}(t) \right) \right] \pi_{j}^{ms}(t).$$
 (24)

From the later equations, one gets

$$\pi_{j}^{01}(t) = (\pi_{j}^{10}(t))^{\dagger} = \pi_{j}^{01} \exp\left\{iv_{j}t - 2\sum_{k} \left(a_{j,k}(t)b_{j,k}^{\dagger} - a_{j,k}^{*}(t)b_{j,k}\right)\right\},\tag{25}$$

where $a_{j,k}(t) = g_{j,k} (1 - \exp(iw_{j,k}t))/w_{j,k}$.

In the Fano-Bloch representation, the matrix $\rho_S(t)$ can be expressed as follows

$$\rho_S(t) = \frac{1}{4} \left[I \otimes I + T_{11}\sigma_1 \otimes \sigma_1 + T_{12}\sigma_1 \otimes \sigma_2 + T_{21}\sigma_2 \otimes \sigma_1 + T_{22}\sigma_2 \otimes \sigma_2 + T_{33}\sigma_3 \otimes \sigma_3 \right]. \tag{26}$$

The non-vanishing correlation parameters $T_{\alpha\beta}$, occurring in (26), are given by:

$$T_{11} = [c_1 \cos(v_1 t) \cos(v_2 t) + c_2 \sin(v_1 t) \sin(v_2 t)] e^{-\gamma(t)}$$

$$T_{12} = [c_1 \cos(v_1 t) \sin(v_2 t) - c_2 \sin(v_1 t) \cos(v_2 t)] e^{-\gamma(t)}$$

$$T_{21} = [c_1 \sin(v_1 t) \cos(v_2 t) - c_2 \cos(v_1 t) \sin(v_2 t)] e^{-\gamma(t)}$$

$$T_{22} = [c_1 \sin(v_1 t) \sin(v_2 t) + c_2 \cos(v_1 t) \cos(v_2 t)] e^{-\gamma(t)}$$

$$T_{33} = c_3$$
(27)

where the time-dependent function $\gamma(t)$ is defined by:

$$\gamma(t) = \sum_{j=1}^{2} \sum_{k} 4 |g_{j,k}|^{2} w_{j,k}^{-2} \coth\left(\frac{\beta w_{j,k}}{2}\right) \left[1 - \cos\left(w_{j,k}t\right)\right]. \tag{28}$$

In the continuum limit, the term $\sum_{k} 4 |g_{j,k}|^2$ is replaced by $\int dw J_j(w) \, \delta(w_{j,k} - w)$ and to simplify our purpose we consider the situation where both reservoirs have the same spectral density $J_1(w) = J_2(w) = J(w)$. In this limiting case, $\gamma(t)$ can be expressed as:

$$\gamma(t) = 2 \int_{0}^{\infty} dw J(w) w^{-2} \coth\left(\frac{\beta w}{2}\right) [1 - \cos(wt)]. \tag{29}$$

The expression of the spectral density characterizing each reservoir is given by:

$$J(w) = \lambda \Omega^{1-s} w^s e^{-w/\Omega}, \tag{30}$$

with Ω is the cutoff frequency and λ is a dimensionless coupling constant between the system and the environment (the reservoirs). For s = 1, the reservoir is of ohmic type, for 0 < s < 1, the reservoir is sub-ohmic and for s > 1, the reservoir is called super-ohmic.

3.2 Dynamics of Local quantum uncertainty of two qubits in independent reservoirs

In the computational basis, the density matrix (26) takes the following form

$$\rho_{S}(t) = \frac{1}{4} \begin{pmatrix} c_{3}^{+} & 0 & 0 & c_{-}e^{-i(v_{2}+v_{1})t-\gamma(t)} \\ 0 & c_{3}^{-} & c_{+}e^{i(v_{2}-v_{1})t-\gamma(t)} & 0 \\ 0 & c_{+}e^{-i(v_{2}-v_{1})t-\gamma(t)} & c_{3}^{-} & 0 \\ c_{-}e^{i(v_{2}+v_{1})t-\gamma(t)} & 0 & 0 & c_{3}^{+} \end{pmatrix}.$$
(31)

which is of two-qubit X type. It follows that the Local quantum uncertainty and trace discord can be easily evaluated using the results reported hereinabove. To obtain the explicit expression of local quantum uncertainty, one computes first the elements of the matrix W. From (7), one gats

$$w_{11} = \frac{1}{2} \sqrt{\left(c_3^+ + \sqrt{(c_3^+)^2 - (c_-)^2 e^{-2\gamma(t)}}\right) \left(c_3^- + \sqrt{(c_3^-)^2 - (c_+)^2 e^{-2\gamma(t)}}\right)} + \frac{1}{2} \frac{(c_-)^2 e^{-2\gamma(t)} \cos(2v_1 t)}{\sqrt{\left(c_3^+ + \sqrt{(c_3^+)^2 - (c_-)^2 e^{-2\gamma(t)}}\right) \left(c_3^- + \sqrt{(c_3^-)^2 - (c_+)^2 e^{-2\gamma(t)}}\right)}}.$$
(32)

Using the expression (8), one obtains

$$w_{22} = \frac{1}{2} \left(\sqrt{\left(c_3^+ + \sqrt{\left(c_3^+\right)^2 - \left(c_-\right)^2 e^{-2\gamma(t)}}\right) \left(c_3^- + \sqrt{\left(c_3^-\right)^2 - \left(c_+\right)^2 e^{-2\gamma(t)}}\right)} \right)$$

$$- \frac{1}{2} \left(\frac{\left(c_-\right)^2 e^{-2\gamma(t)} \cos\left(2v_1 t\right)}{\sqrt{\left(c_3^+ + \sqrt{\left(c_3^+\right)^2 - c_-^2 e^{-2\gamma(t)}}\right) \left(c_3^- + \sqrt{\left(c_3^-\right)^2 - \left(c_+\right)^2 e^{-2\gamma(t)}}\right)}} \right)$$

$$(33)$$

The equation (9) gives

$$w_{33} = \frac{1}{4} \left(2 + \sqrt{\left(c_3^+\right)^2 - \left(c_-\right)^2 e^{-2\gamma(t)}} + \sqrt{\left(c_3^-\right)^2 - \left(c_+\right)^2 e^{-2\gamma(t)}} \right) - \frac{\left(c_-\right)^2 e^{-2\gamma(t)}}{4 \left(c_3^+ + \sqrt{\left(c_3^+\right)^2 - \left(c_-\right)^2 e^{-2\gamma(t)}}\right)} - \frac{\left(c_+\right)^2 e^{-2\gamma(t)}}{4 \left(c_3^- + \sqrt{\left(c_3^-\right)^2 - \left(c_+\right)^2 e^{-2\gamma(t)}}\right)}$$

$$(34)$$

The non vanishing off-diagonal elements (11) write

$$w_{12} = w_{21} = \frac{c_{-}c_{+}e^{-2\gamma(t)}\cos(v_{1}t)\sin(v_{1}t)}{\sqrt{\left(c_{3}^{+} + \sqrt{\left(c_{3}^{+}\right)^{2} - \left(c_{-}\right)^{2}e^{-2\gamma(t)}}\right)\left(c_{3}^{-} + \sqrt{\left(c_{3}^{-}\right)^{2} - \left(c_{+}\right)^{2}e^{-2\gamma(t)}}\right)}}.$$
(35)

The eigenvalues of the matrix W are

$$\lambda_1 = \frac{1}{2} \left[(w_{11} + w_{22}) + \sqrt{(w_{11} + w_{22})^2 - 4(w_{11}w_{22} - w_{21}w_{12})} \right], \tag{36}$$

$$\lambda_2 = \frac{1}{2} \left[(w_{11} + w_{22}) - \sqrt{(w_{11} + w_{22})^2 - 4(w_{11}w_{22} - w_{21}w_{12})} \right], \tag{37}$$

$$\lambda_3 = w_{33} \tag{38}$$

We have assumed that $|c_2| \leq |c_1|$. Thus, one has $\lambda_2 \leq \lambda_1$ and subsequently the local quantum uncertainty is given in term of

$$\lambda_{\max} = \max(\lambda_1, \lambda_3). \tag{39}$$

Therefore, one should consider two cases: $\lambda_1 \leq \lambda_3$ and $\lambda_3 \leq \lambda_1$. Hence, when $\lambda_{\text{max}} = \lambda_1$, the local quantum uncertainty is given by the following expression

$$U_{A}(\rho_{S}(t)) = \frac{1}{2} \left[2 - \sqrt{\left(c_{3}^{+} + \sqrt{\left(c_{3}^{+}\right)^{2} - \left(c_{-}\right)^{2} e^{-2\gamma(t)}}\right) \left(c_{3}^{-} + \sqrt{\left(c_{3}^{-}\right)^{2} - \left(c_{+}\right)^{2} e^{-2\gamma(t)}}\right)} \right] - \frac{c_{-}c_{+}e^{-2\gamma(t)}}{2\sqrt{\left(c_{3}^{+} + \sqrt{\left(c_{3}^{+}\right)^{2} - \left(c_{-}\right)^{2} e^{-2\gamma(t)}}\right) \left(c_{3}^{-} + \sqrt{\left(c_{3}^{-}\right)^{2} - \left(c_{+}\right)^{2} e^{-2\gamma(t)}}\right)}},$$

$$(40)$$

and when $\lambda_{max} = \lambda_3$, one gets

$$U_{A}(\rho_{S}(t)) = \frac{1}{4} \left[2 - \sqrt{\left(c_{3}^{+}\right)^{2} - \left(c_{-}\right)^{2} e^{-2\gamma(t)}} - \sqrt{\left(c_{3}^{-}\right)^{2} - \left(c_{+}\right)^{2} e^{-2\gamma(t)}} \right] + \frac{\left(c_{-}\right)^{2} e^{-2\gamma(t)}}{4 \left[c_{3}^{+} + \sqrt{\left(c_{3}^{+}\right)^{2} - \left(c_{-}\right)^{2} e^{-2\gamma(t)}}\right]} + \frac{\left(c_{+}\right)^{2} e^{-2\gamma(t)}}{4 \left[c_{3}^{-} + \sqrt{\left(c_{3}^{-}\right)^{2} - \left(c_{+}\right)^{2} e^{-2\gamma(t)}}\right]}.$$

$$(41)$$

3.3 Dynamics of trace quantum discord of two qubits in independent reservoirs

The non vanishing matrix correlations, in the Fano-Bloch representation, of the density matrix $\rho_S(t)$ (31) are gien by

$$R_{11} = c_1 e^{-\gamma(t)}$$
 $R_{22} = c_2 e^{-\gamma(t)}$ $R_{33} = c_3$

The trace quantum discord (14) takes the form

$$D_T(\rho_S(t)) = \frac{e^{-\gamma(t)}}{2} \sqrt{\frac{c_1^2 \max\{c_3^2, c_2^2 e^{-2\gamma(t)}\} - c_2^2 \min\{c_1^2 e^{-2\gamma(t)}, c_3^2\}}{\max\{c_3^2, c_2^2 e^{-2\gamma(t)}\} - \min\{c_1^2 e^{-2\gamma(t)}, c_3^2\} + (c_1^2 - c_2^2) e^{-2\gamma(t)}}}.$$
(42)

For $|c_3| \ge |c_2|e^{-\gamma(t)}$ and $|c_3| \ge |c_1|e^{-\gamma(t)}$, the trace distance discord is given by:

$$D_T(\rho_S(t)) = \frac{1}{2} |c_1| e^{-\gamma(t)}.$$
(43)

For $|c_3| \ge |c_2|e^{-\gamma(t)}$ and $|c_3| \le |c_1|e^{-\gamma(t)}$ one gets:

$$D_T(\rho_S(t)) = \frac{1}{2}|c_3|. \tag{44}$$

In this situation, it is remarkable that the quantum correlations are unaffected by the noisy environment and the geometric discord exhibits a freezing behavior. For $|c_3| \le |c_2|e^{-\gamma(t)}$ and $|c_3| \ge |c_1|e^{-\gamma(t)}$ one hase

$$D_T\left(\rho_S(t)\right) = 0,$$

reflecting the absence of quantum correlations. For $|c_3| \leq |c_2|e^{-\gamma(t)}$ and $|c_3| \leq |c_1|e^{-\gamma(t)}$, the trace distance discord is simply given by

$$D_T(\rho_S(t)) = \frac{1}{2} |c_2| e^{-\gamma(t)}.$$
 (45)

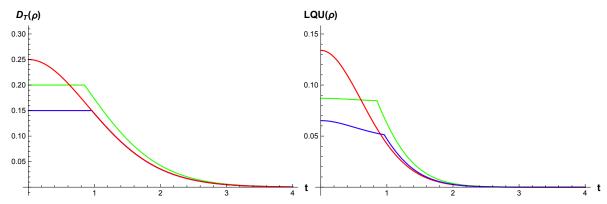


Fig 1: The local quantum uncertainty and trace distance discord for the sub-ohmic reservoirs with s=0.5, $\lambda=0.1$, $\Omega\beta=1$. $c_1=0.6$, $c_2=-0.3$, $c_3=0.4$ (green line). $c_1=-0.5$, $c_2=0$, $c_3=0.3$ (blue line). $c_1=0.5$, $c_2=-0.3$, $c_3=0.6$ (red line).

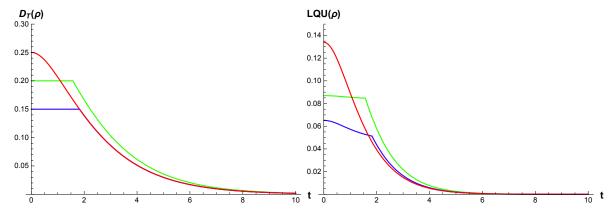


Fig 2: The local quantum uncertainty and trace distance discord for the ohmic reservoirs with s=1, $\lambda=0.1$, $\Omega\beta=1$. $c_1=0.6$, $c_2=-0.3$, $c_3=0.4$ (green line). $c_1=-0.5$, $c_2=0$, $c_3=0.3$ (blue line). $c_1=0.5$, $c_2=-0.3$, $c_3=0.6$ (red line).

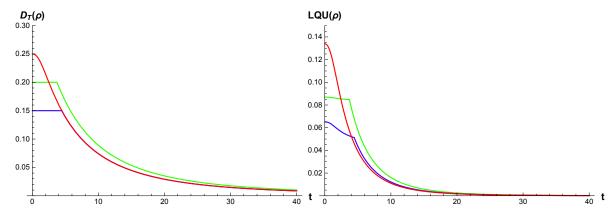


Fig 3: The local quantum uncertainty and trace distance discord for the super-ohmic with s=1.5, $\lambda=0.2$, T=0. $c_1=0.6$, $c_2=-0.3$, $c_3=0.4$ (green line). $c_1=-0.5$, $c_2=0$, $c_3=0.3$ (blue line). $c_1=0.5$, $c_2=-0.3$, $c_3=0.6$ (red line).

We begin by analyzing the behavior of local quantum uncertainty in three particular Bell states coupled to environment of ohmic, sub-ohmic and super-ohmic type. As depicted in the figures 1, 2 and 3, the local quantum uncertainty may exhibit a sudden change behavior for the states with $(c_1 = 0.6, c_2 = -0.3, c_3 = 0.4)$ and $(c_1 = -0.5, c_2 = 0, c_3 = 0.3)$. This behavior is not observable for the state with $(c_1 = 0.5, c_2 = -0.3, c_3 = 0.6)$ for which the local quantum uncertainty decays monotonically. This decrease becomes more pronounced when passing from sub-ohmic to super-ohmic regime. This is essentially due to the strong nature of the environment effect in the sub-ohmic regime. It must be noticed also that for the states with $(c_1 = 0.6, c_2 = -0.3, c_3 = 0.4)$ and $(c_1 = -0.5, c_2 = 0, c_3 = 0.3)$, the local quantum uncertainty varies almost linearly before the sudden change point. The interval of time, in which this variation is linear, depends on the nature of the system-environment coupling. It is larger in the ohmic regime in comparison with the others regimes. Let us now analyze the behavior of quantum correlations measured by trace norm. We notice first that the local quantum uncertainty and trace discord exhibit similar behavior for the considered states, except the freezing behavior which occurs before the sudden change of trace norm. We obtain a large and controllable freezing interval for states with $(c_1 = 0.6, c_2 = -0.3, c_3 = 0.4)$ evolving in Markovian environment of super-ohmic type. This interval is reduced

by the decoherence effects in the case of ohmic and sub-ohmic regimes. This freezing phenomenon exhibited by trace norm reflects that the quantum correlations in a given quantum state are not affected by the noise generated by its surrounding environment. This is surprising and questionable phenomenon does not occur when one consider local quantum uncertainty as quantifier of quantum discord. Indeed, as we have discussed above, in the interval where the trace discord is constant, the local quantum uncertainty decreases linearly until the point when the sudden change of the behavior happens.

4 Quantum correlations dynamics of two 2-level atoms interacting with an electromagnetic field

In this section, we consider two identical atoms with ground states $|g_i\rangle$ and excited states $|e_i\rangle$ (i=1,2) which are coupled with a quantized electromagnetic field [33, 34, 35, 36]. In the rotating-wave approximation, the Hamiltonian can write

$$\hat{H} = \hbar \omega_0 S^z + \sum_{\vec{k}s} \omega_k \hat{a}_{\vec{k}s}^{\dagger} \hat{a}_{\vec{k}s} - i\hbar \sum_{\vec{k}s} \left[\vec{\mu} \cdot \vec{g}_{\vec{k}s} S^+ \hat{a}_{\vec{k}s} - H.c \right], \tag{46}$$

where $\hat{a}_{\vec{k}s}$ and $\hat{a}_{\vec{k}s}^{\dagger}$ are respectively the annihilation and creation operators corresponding to the field mode $\vec{k}s$, which has wave vector \vec{k} , frequency ω_k and the index of polarization s. The coupling factor is given by

$$\vec{g}_{\vec{k}s}\left(\vec{r}_{i}\right) = \left(\frac{\omega_{k}}{2\varepsilon_{0}\hbar V}\right)^{\frac{1}{2}} \vec{e}_{\vec{k}s} e^{i\vec{k}.\vec{r}_{i}}$$

where V denotes the quantization volume and e_{ks} is the electric field polarization vector. This factor represents the mode function of the three dimensional field, evaluated at the position \vec{r}_i of the ith atom. The quantity $\vec{\mu}$ is the transition dipole moment and ω_0 is the transition frequency. The operators S^{\pm} and S^z are the collective spin operators defined by $S^{\pm} = \sum_i S_i^{\pm}$ and $S^z = \sum_i S_i^z$ with $S_i^+ = |e_i\rangle \langle g_i|$, $S_i^- = |g_i\rangle \langle e_i|$ and $S_i^z = |e_i\rangle \langle e_i| - |g_i\rangle \langle g_i|$. To study the dynamics of this system, we shall employ the following master equation [34]:

$$\frac{\partial \rho(\tau)}{\partial t} = -i\omega_0 \sum_{i=1}^{2} \left[S_i^z, \rho \right] - i\Omega_{12} \sum_{i \neq j}^{2} \left[S_i^+ S_j^-, \rho \right] - \frac{1}{2} \sum_{i,j=1}^{2} \Gamma_{ij} \left(\rho S_i^+ S_j^- + S_i^+ S_j^- \rho - 2S_j^- \rho S_i^+ \right) \tag{47}$$

where $\Gamma_{ii} \equiv \Gamma$ (i = 1, 2) is the spontaneous emission rate induced by the direct coupling of the atom with the radiation field, $\Gamma_{12} = \Gamma_{21}$ denotes the collective damping and Ω_{12} is the dipole-dipole interaction potential. They are given by [34, 35, 36]:

$$\Gamma_{ij} = \frac{3}{2} \Gamma \left\{ \left[1 - (\vec{\mu} \cdot \vec{r}_{ij})^2 \right] \frac{\sin(k_0 r_{ij})}{k_0 r_{ij}} + \left[1 - 3(\vec{\mu} \cdot \vec{r}_{ij})^2 \right] \left[\frac{\cos(k_0 r_{ij})}{(k_0 r_{ij})^2} - \frac{\sin(k_0 r_{ij})}{(k_0 r_{ij})^3} \right] \right\}, \tag{48}$$

and

$$\Omega_{ij} = \frac{3}{4}\Gamma \left\{ -\left[1 - (\vec{\mu}.\vec{r}_{ij})^2\right] \frac{\cos(k_0 r_{ij})}{k_0 r_{ij}} + \left[1 - 3(\vec{\mu}.\vec{r}_{ij})^2\right] \left[\frac{\sin(k_0 r_{ij})}{(k_0 r_{ij})^2} + \frac{\cos(k_0 r_{ij})}{(k_0 r_{ij})^3}\right] \right\}.$$
(49)

where $r_{ij} = |r_j - r_i|$ is the distance between the atoms, and $k_0 = \frac{\omega_0}{c}$. We consider the situation where the two qubits are initially prepared in their excited states $|e_1, e_2\rangle$. Using the master equation (47), the evolved

two-qubit state writes in the computational basis as

$$\rho(\tau) = \begin{pmatrix} a(\tau) & 0 & 0 & 0\\ 0 & b(\tau) & c(\tau) & 0\\ 0 & c(\tau) & b(\tau) & 0\\ 0 & 0 & 0 & 1 - a(\tau) - 2b(\tau) \end{pmatrix}.$$
 (50)

where

$$a\left(\tau\right) = e^{-2\tau},\tag{51}$$

$$b(\tau) = \frac{e^{-\tau}}{(1-\gamma)} \left[\left(1 + \gamma^2 \right) \left(\cosh\left(\gamma\tau\right) - e^{-\tau} \right) - 2\gamma \sinh\left(\gamma\tau\right) \right], \tag{52}$$

$$c(\tau) = \frac{e^{-\tau}}{(1-\gamma)} \left[2\gamma \left(\cosh(\gamma \tau) - e^{-\tau} \right) - \left(1 + \gamma^2 \right) \sinh(\gamma \tau) \right], \tag{53}$$

with $\tau = \Gamma t$ and $\gamma = \Gamma_{12}/\Gamma$.

The density matrix $\rho(\tau)$ rewrites, in Fano-Bloch representation, as

$$\rho\left(\tau\right) = \frac{1}{4} \sum_{\alpha\beta} T_{\alpha\beta} \sigma_{\alpha} \otimes \sigma_{\beta} \tag{54}$$

where the nonvanishing correlation matrix elements are given by

$$T_{11} = T_{22} = 2c(\tau)$$
 $T_{33} = 1 - 4b(\tau)$ $T_{03} = T_{30} = 2(a(\tau) + b(\tau)) - 1,$ (55)

in term of the time dependent functions $a(\tau)$, $b(\tau)$ and $c(\tau)$ given respectively by (51), (52) and (53).

4.1 Dynamics of Local quantum uncertainty

To determine the local quantum uncertainty for the density matrix $\rho(\tau)$, one employs the results obtained in section 2. Thus, the elements of the matrix W (4) are given by

$$w_{11} = w_{22} = \left(\sqrt{b(\tau) + c(\tau)} + \sqrt{b(\tau) - c(\tau)}\right) \left(\sqrt{a(\tau)} + \sqrt{1 - a(\tau) - 2b(\tau)}\right), \tag{56}$$

$$w_{33} = 1 - 2b(\tau) + 2\sqrt{b(\tau)^2 - c(\tau)^2},$$
(57)

which can be rewritten also as

$$w_{11} = w_{22} = \frac{e^{-\tau}}{(1 - \gamma^2)} \left[\sqrt{(1 - \gamma^2) e^{-\tau}} + \sqrt{4\gamma \sinh(\gamma \tau) + 2(1 - \gamma^2) \sinh(\tau) - 2(1 + \gamma^2) (\cosh(\gamma \tau) - e^{-\tau})} \right] \left[(1 + \gamma) \sqrt{e^{-\gamma \tau} - e^{-\tau}} + (1 - \gamma) \sqrt{e^{\gamma \tau} - e^{-\tau}} \right],$$

$$w_{33} = 1 + \frac{2e^{-\tau}}{(1 - \gamma^2)} \left[2\gamma \sinh{(\gamma \tau)} - \left(1 + \gamma^2 \right) \left(\cosh{(\gamma \tau)} - e^{-\tau} \right) + \sqrt{1 + e^{-2\tau} - 2e^{-\tau} \cosh{(\gamma \tau)}} \right].$$

In the situation where the wo atoms are very close ($r_{12} \to 0$), we have $\Gamma_{12} \to \Gamma$. This means that $\gamma \to 1$ and $b(\tau) = c(\tau) = e^{-2\tau}\tau$. In this case, the elements of the matrix W are simply given by

$$w_{11} = w_{22} = \sqrt{2b(\tau) a(\tau)} + \sqrt{2b(\tau) (1 - a(\tau) - 2b(\tau))}$$
(58)

$$w_{33} = 1 - 2b(\tau) \tag{59}$$

which rewrite as

$$w_{11} = w_{22} = \sqrt{2e^{-2\tau}\tau} \left[\sqrt{e^{-2\tau}} + \sqrt{1 - e^{-2\tau}(1 + 2\tau)} \right]$$
 and $w_{33} = 1 - 2e^{-2\tau}\tau$.

4.2 Dynamics of geometric quantum discord

To determine the analytic expression of the trace distance discord of this system from the equation (14), it is necessary to calculate the expressions of T_{max}^2 and T_{min}^2 by comparing T_{33}^2 with $T_{22}^2 + T_{30}^2$ and T_{11}^2 with T_{33}^2 . It is simple to check that the difference $T_{33}^2 - T_{22}^2 - T_{30}^2$ is negative and $T_{max}^2 = T_{22}^2 + T_{30}^2$. Therefore, to get the trace discord, one has to treat separately the cases $|T_{33}| \ge |T_{11}|$ and $|T_{11}| \ge |T_{33}|$. Using the equation (14) and noticing that $T_{11} = T_{22}$ (see Eq.(55)), one verifies that the trace discord, in both cases, writes

$$D_T(\rho(\tau)) = \frac{1}{2}|T_{11}| = |c(\tau)|, \tag{60}$$

where $c(\tau)$ is given by (53).

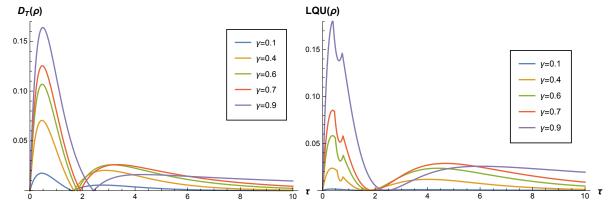


Fig 4: The trace distance discord $D_T(\rho)$ and local quantum uncertainty versus the parameter τ for the different values of γ .

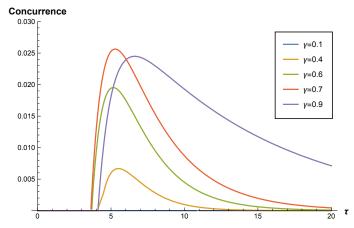


Fig 5: The variation of the Concurrence for the different values of γ .

4.3 Dynamics of Concurrence

The two atoms are initially prepared in the separable state $|e_1, e_2\rangle$. Besides the local quantum uncertainty and trace quantum discord, we shall also consider the dynamics of the concurrence. This measure, introduced by Wootters [23, 24] is given by

$$C = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\},\tag{61}$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ are the square roots of the eigenvalues of the matrix $\rho \tilde{\rho}$. The matrix $\tilde{\rho}$ that is obtained by the "spin-flipped" operation from the density matrix ρ as $\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$. For the density matrix (50), the corresponding eigenvalues are

$$\{\lambda_{1} = \lambda_{2}, \lambda_{3}, \lambda_{4}\} = \left\{\sqrt{a\left(\tau\right)\left(1 - a\left(\tau\right) - 2b\left(\tau\right)\right)}, \left|b\left(\tau\right) - c\left(\tau\right)\right|, \left|b\left(\tau\right) + c\left(\tau\right)\right|\right\},\tag{62}$$

and the concurrence (61) writes

$$C(\rho) = \begin{cases} \max\{0, -2b(\tau)\} & \text{if} \quad \lambda_{1} = \sqrt{a(\tau)(1 - a(\tau) - 2b(\tau))}, \\ \max\{0, -2(c(\tau) + \sqrt{a(\tau)(1 - a(\tau) - 2b(\tau))})\} & \text{if} \quad \lambda_{1} = |b(\tau) - c(\tau)|, \\ \max\{0, 2(c(\tau) - \sqrt{a(\tau)(1 - a(\tau) - 2b(\tau))})\} & \text{if} \quad \lambda_{1} = |b(\tau) + c(\tau)|, \end{cases}$$
(63)

The dynamics of local quantum uncertainty and trace quantum discord are plotted in the Figure 4. As depicted in Fig. 4, the behavior of both the local quantum uncertainty and trace discord show a sudden birth of quantum correlations. Recall that the initial two qubit state is separable and the dipole-dipole interaction between the two atoms leads to the generation of non classical correlations. The generated quantum correlations survive over a certain interval of time. After, the amount of quantum correlations decreases showing a degradation of the generated non classical correlations caused by the environmental effects. It must be noticed that the amount of quantum correlations is more important when the two atoms get close each other. The interaction between the two atoms enhances the quantum correlations between the components of the system. Another important aspect reported in Fig. 4 is the revival of quantum correlations. Indeed, as it can be seen from Fig. 4, local quantum uncertainty and trace discord increase, reach a maximum and decrease to vanish after finite time interval. This is followed by a revival of quantum correlations in the system. This revival can be explained by the transfer of correlations from the total system including the environment to the two qubit system. Comparing the results of Fig. 4 and Fig. 5, one notices that the sudden birth of concurrence is delayed in comparison with local quantum uncertainty and trace quantum discord. This corroborates the fact that the concurrence as quantum quantifier can not capture the total amount of quantum correlations existing in a mixed two-qubit system. It is also important to stress that the concurrence does not exhibit the revival phenomena and can not capture the quantum correlation revival.

We also considered the comparison between local quantum uncertainty, trace discord and concurrence for different values of γ related to the distance separating the two atoms. The results reported in the figures 6, 7, 8, 9, 10 and 11 show that local quantum uncertainty and trace discord behave almost identically. These figures show also that the concurrence is initially zero for a certain period of time. This means that the evolved states stay separable in this interval but contain quantum correlations that are not captured by the concurrence. In particular, we noticed that for $\gamma \to 1$, the concurrence is zero. The system of the two atoms stays separable but contains quantum correlations that are not captured by the concurrence.

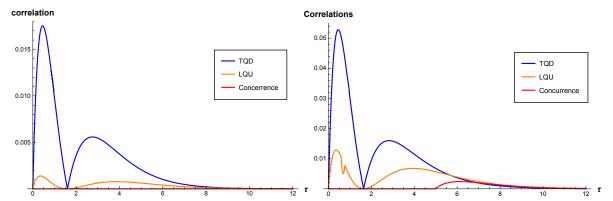


Fig 6: Local Quantum Uncertainty, trace Distance Fig 7: Local Quantum Uncertainty, trace Distance Discord and concurrence for $\gamma = 0.1$. Discord and concurrence for $\gamma = 0.3$.

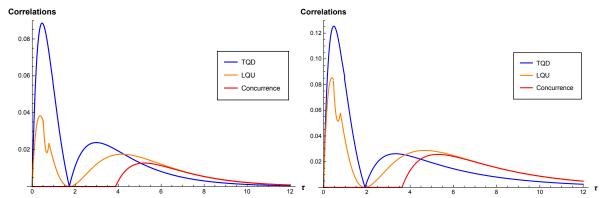


Fig 8: Local Quantum Uncertainty, trace Distance Fig 9: Local Quantum Uncertainty, trace Distance Discord and concurrence for $\gamma = 0.5$. Discord and concurrence for $\gamma = 0.7$.

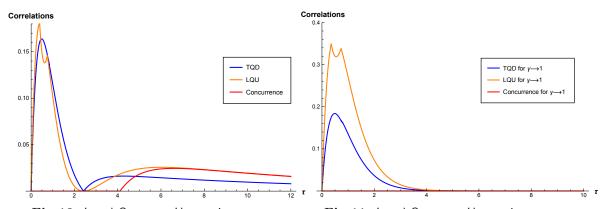


Fig 10: Local Quantum Uncertainty, trace Distance Discord and concurrence for $\gamma=0.9$.

Fig 11: Local Quantum Uncertainty, trace Distance Discord and concurrence for $\gamma \longrightarrow 1$.

5 Concluding Remarks

In this work, we considered the dynamics of quantum correlations in two specific bipartite quantum systems. The first is a two-qubit system coupled to two independent bosonic reservoirs. The second concerns a two 2-level atoms interacting with the modes of a quantized radiation field. We used the local quantum uncertainty, trace quantum discord and the concurrence to investigate the main features of the dynamics of the quantum

correlations contained in the system submitted to the environmental effects. In this sense, we have derived the analytical expression for local quantum uncertainty quantifying quantum correlations in two-qubit X states coupled to two independent bosonic reservoirs. In analyzing the dynamics of this quantifier, we considered three types of reservoirs :sub-Ohmic, Ohmic, and super-Ohmic. We also compared the local quantum uncertainty and trace quantum discord. In particular, it has been shown that the local quantum uncertainty does not exhibit the freezing phenomenon which is observed when one uses the trace norm as quantifier.

For a two 2-level atoms interacting with the bosonic modes of an electromagnetic field, we analyzed the dynamics of quantum correlations for different configurations by varying the inter-atomic distance. In this case we have utilized local quantum uncertainty, trace quantum discord and concurrence to analyze the main characteristics of quantum correlations in this system initially prepared in a separable state. We have noticed that the evolved system stays for a certain period of time separable (the concurrence is zero) contrarily to local quantum uncertainty and trace quantum discord which are non vanishing. This corroborates the fact that local quantum uncertainty and trace discord go beyond the Wootters concurrence and indicates the sudden birth of quantum correlations. This is essentially due to the dipole-dipole interaction between the two atoms which creates quantum correlations between the components of the system. A second remarkable feature is the revival phenomenon which reflects the transfer of quantum correlations from the environment to the two qubit system.

Appendices

Appendix 1:

The eigenvalues corresponding to this matrix ρ (5) are given by

$$\lambda_1 = \frac{1}{2}t_1 + \frac{1}{2}\sqrt{t_1^2 - 4d_1}, \qquad \lambda_2 = \frac{1}{2}t_2 + \frac{1}{2}\sqrt{t_2^2 - 4d_2}$$
 (64)

$$\lambda_3 = \frac{1}{2}t_2 - \frac{1}{2}\sqrt{t_2^2 - 4d_2}, \qquad \lambda_4 = \frac{1}{2}t_1 - \frac{1}{2}\sqrt{t_1^2 - 4d_1}$$
 (65)

where

$$t_1 = \rho_{11} + \rho_{44}, \quad d_1 = \rho_{11}\rho_{44} - \rho_{14}\rho_{41}, \quad t_2 = \rho_{22} + \rho_{33}, \quad d_2 = \rho_{22}\rho_{33} - \rho_{32}\rho_{23}.$$
 (66)

The square root of the density matrix ρ can be written in terms of the matrix elements ρ , in the computational basis, as follows:

$$\sqrt{\rho} = \begin{pmatrix}
\frac{\rho_{11} + \sqrt{d_1}}{\sqrt{t_1 + 2\sqrt{d_1}}} & 0 & 0 & \frac{\rho_{14}}{\sqrt{t_1 + 2\sqrt{d_1}}} \\
0 & \frac{\rho_{22} + \sqrt{d_2}}{\sqrt{t_2 + 2\sqrt{d_2}}} & \frac{\rho_{23}}{\sqrt{t_2 + 2\sqrt{d_2}}} & 0 \\
0 & \frac{\rho_{32}}{\sqrt{t_2 + 2\sqrt{d_2}}} & \frac{\rho_{33} + \sqrt{d_2}}{\sqrt{t_2 + 2\sqrt{d_2}}} & 0 \\
\frac{\rho_{41}}{\sqrt{t_1 + 2\sqrt{d_1}}} & 0 & 0 & \frac{\rho_{44} + \sqrt{d_1}}{\sqrt{t_1 + 2\sqrt{d_1}}}
\end{pmatrix}.$$
(67)

The eigenvalues $\sqrt{\lambda_1}$, $\sqrt{\lambda_2}$, $\sqrt{\lambda_3}$ and $\sqrt{\lambda_4}$ of the matrix $\sqrt{\rho}$ can be rewritten as:

$$\sqrt{\lambda_1} = \frac{1}{2}\sqrt{t_1 + 2\sqrt{d_1}} + \frac{1}{2}\sqrt{t_1 - 2\sqrt{d_1}}$$
(68)

$$\sqrt{\lambda_2} = \frac{1}{2}\sqrt{t_2 + 2\sqrt{d_2}} + \frac{1}{2}\sqrt{t_2 - 2\sqrt{d_2}} \tag{69}$$

$$\sqrt{\lambda_3} = \frac{1}{2}\sqrt{t_2 + 2\sqrt{d_2}} - \frac{1}{2}\sqrt{t_2 - 2\sqrt{d_2}} \tag{70}$$

$$\sqrt{\lambda_4} = \frac{1}{2}\sqrt{t_1 + 2\sqrt{d_1}} - \frac{1}{2}\sqrt{t_1 - 2\sqrt{d_1}}.$$
(71)

The matrix $\sqrt{\rho}$ is written in the Fano-Bloch representation as:

$$\sqrt{\rho} = \frac{1}{4} \sum_{\chi, \delta} \mathcal{R}_{\chi \delta} \sigma_{\chi} \otimes \sigma_{\delta}, \tag{72}$$

where $\chi, \delta = 0, 1, 2, 3$ and the Fano-Bloch parameters are defined by: $\mathcal{R}_{\chi\delta} = \text{Tr}\left(\sqrt{\rho}\sigma_{\chi}\otimes\sigma_{\delta}\right)$. The vanishing correlation parameters $\mathcal{R}_{\chi\delta}$ are given by:

$$\mathcal{R}_{00} = \sqrt{t_1 + 2\sqrt{d_1}} + \sqrt{t_2 + 2\sqrt{d_2}} \tag{73}$$

$$\mathcal{R}_{03} = \frac{1}{2} \frac{T_{30} + T_{03}}{\sqrt{t_1 + 2\sqrt{d_1}}} - \frac{1}{2} \frac{T_{30} - T_{03}}{\sqrt{t_2 + 2\sqrt{d_2}}}$$

$$\tag{74}$$

$$\mathcal{R}_{30} = \frac{1}{2} \frac{T_{30} + T_{03}}{\sqrt{t_1 + 2\sqrt{d_1}}} + \frac{1}{2} \frac{T_{30} - T_{03}}{\sqrt{t_2 + 2\sqrt{d_2}}}$$

$$\tag{75}$$

$$\mathcal{R}_{11} = \frac{1}{2} \frac{T_{11} + T_{22}}{\sqrt{t_2 + 2\sqrt{d_2}}} + \frac{1}{2} \frac{T_{11} - T_{22}}{\sqrt{t_1 + 2\sqrt{d_1}}}$$
(76)

$$\mathcal{R}_{12} = \frac{1}{2} \frac{T_{12} - T_{21}}{\sqrt{t_2 + 2\sqrt{d_2}}} + \frac{1}{2} \frac{T_{12} + T_{21}}{\sqrt{t_1 + 2\sqrt{d_1}}}$$
(77)

$$\mathcal{R}_{12} = \frac{1}{2} \frac{T_{12} - T_{21}}{\sqrt{t_2 + 2\sqrt{d_2}}} + \frac{1}{2} \frac{T_{12} + T_{21}}{\sqrt{t_1 + 2\sqrt{d_1}}}$$
(78)

$$\mathcal{R}_{21} = \frac{1}{2} \frac{T_{12} + T_{21}}{\sqrt{t_1 + 2\sqrt{d_1}}} - \frac{1}{2} \frac{T_{12} - T_{21}}{\sqrt{t_2 + 2\sqrt{d_2}}}$$
(79)

$$\mathcal{R}_{22} = \frac{1}{2} \frac{T_{11} + T_{22}}{\sqrt{t_2 + 2\sqrt{d_2}}} - \frac{1}{2} \frac{T_{11} - T_{22}}{\sqrt{t_1 + 2\sqrt{d_1}}}$$
(80)

$$\mathcal{R}_{33} = \sqrt{t_1 + 2\sqrt{d_1}} - \sqrt{t_2 + 2\sqrt{d_2}}. (81)$$

Appendix 2:

In this appendix, we give the expressions of the $\gamma(t)$ function for sub-ohmic, ohmic and super-ohmic reservoirs [29].

(i) Sub-Ohmic reservoirs:

The sub-Ohmic regime corresponds to the situation where 0 < s < 1. In this case, the function $\gamma(t)$ reads as [29]

$$\gamma(t) = \frac{2\lambda\Gamma(s)}{s-1} \left\{ 1 - \left(1 - \Omega^2 t^2\right)^{(1-s)/2} \cos\left[(s-1)\arctan\left(\Omega t\right)\right] \right\}$$

$$+ \frac{4\lambda\Gamma(s)}{s-1} \sum_{m=1}^{\infty} (1 + m\Omega\beta)^{1-s} \times \left\{ 1 - \left[1 + \left(\frac{\Omega t}{1 + m\Omega\beta}\right)^2\right]^{(1-s)/2} \cos\left[(s-1)\arctan\left(\frac{\Omega t}{1 + m\Omega\beta}\right)\right] \right\},$$

where $\Gamma(s)$ denotes the Gamma function. To study the evolution of the local quantum uncertainty, one should calculate the time derivative of the function $\gamma(t)$. It is given by

$$\frac{d\gamma\left(t\right)}{dt} = 2\lambda\Gamma\left(s\right)\Omega\left\{\left(1 + \Omega^{2}t^{2}\right)^{-s/2}\sin\left[s\arctan\left(\Omega t\right)\right] + 2\sum_{m=1}^{\infty}\left[\left(1 + m\Omega\beta\right)^{2} + \Omega^{2}t^{2}\right]^{-s/2}\sin\left[s\arctan\left(\frac{\Omega t}{1 + m\Omega\beta}\right)\right]\right\}$$
(82)

(ii) Ohmic reservoirs:

For reservoirs of Ohmic type (s = 1), the low-frequency behavior of the function $J(\omega)$ is linear in term of ω . The equation (29) becomes [29]

$$\gamma(t) = \lambda \left[\ln\left(1 + \Omega^2 t^2\right) + 4\ln\Gamma\left(1 + \frac{1}{\Omega\beta}\right) - 2\ln\left|\Gamma\left(1 + \frac{1}{\Omega\beta} + i\frac{t}{\beta}\right)\right|^2 \right]$$
(83)

The time derivative of the function $\gamma(t)$ is given by

$$\frac{d\gamma(t)}{dt} = 2\lambda\Omega^2 t \left[\frac{1}{1 + \Omega^2 t^2} + \sum_{m=1}^{\infty} \frac{2}{\left(1 + m\Omega\beta\right)^2 + \Omega^2 t^2} \right]. \tag{84}$$

(iii) Super-Ohmic reservoirs:

For super-Ohmic reservoirs (s > 1), the function $\gamma(t)$ reads as [29]

$$\gamma\left(t\right) = 2\lambda\Gamma\left(s-1\right)\left\{1 - \left(1 + \Omega^{2}t^{2}\right)^{1-s/2}\cos\left[\left(s-1\right)\arctan\left(\Omega t\right)\right]\right\} \\ + 4\lambda\Gamma\left(s-1\right)\sum_{m=1}^{\infty}\left(1 + m\Omega\beta\right)^{1-s}\left\{1 - \left[1 + \left(\frac{\Omega t}{1 + m\Omega\beta}\right)^{2}\right]^{(1-s)/2}\cos\left[\left(s-1\right)\arctan\left(\frac{\Omega t}{1 + m\Omega\beta}\right)\right]\right\},$$

and the derivative of $\gamma(t)$ with respect to time t is given by

$$\frac{d\gamma\left(t\right)}{dt} = 2\lambda\Omega\Gamma\left(s\right)\left(1 + \Omega^{2}t^{2}\right)^{-s/2}\sin\left[s\arctan\left(\Omega t\right)\right] + 4\lambda\Omega\Gamma\left(s\right)\sum_{m=1}^{\infty}\left[\left(1 + m\Omega\beta\right)^{2} + \Omega^{2}t^{2}\right]^{-s/2}\sin\left[s\arctan\left(\frac{\Omega t}{1 + m\Omega\beta}\right)\right].$$

Due to the complicated expression of $\gamma(t)$ and to simplify our numerical calculations, we have chosen the situation where $1 < s \le 2$ and the temperature equals zero. In this case case, the function $\gamma(t)$ becomes

$$\gamma(t) = 2\lambda\Gamma(s-1)\left\{1 - \left(1 + \Omega^2 t^2\right)^{(1-s)/2}\cos\left[(s-1)\arctan\left(\Omega t\right)\right]\right\}. \tag{85}$$

References

- M. A. Nielsen and I.L. Chuang, Quantum Computation and Quantum Information (Cambridge Univ. Press, Cambridge, 2000).
- [2] S. L. Braunstein and H. J. Kimble, Phys. Rev. Lett., 80 (1998) 869.
- [3] D. Bouwmeester, J. W.Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger, Nature, 390 (1997) 575.
- [4] C.H. Bennett and G. Brassard in *The proceeding of IEEE international conference on Computers, Systems and Signal Processing, Bangalore, India* (IEEE, New York, 1984), p. 175.
- [5] V. Vedral, Rev. Mod. Phys. **74** (2002) 197.
- [6] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, Rev. Mod. Phys. 81 (2009) 865.
- [7] O. Gühne and G. Tóth, Phys. Rep. 474 (2009) 1.

- [8] K. Modi, A. Brodutch, H. Cable, T. Paterek and V. Vedral, Rev. Mod. Phys. 84 (2012) 1655.
- [9] W. K. Wootters, Phys. Rev. Lett. **80** (1998) 2245.
- [10] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. 88 (2001) 017901.
- [11] L. Henderson and V. Vedral, J. Phys. A 34 (2001) 6899.
- [12] Y. Huang, New Journal of Physics 16 (2014) 033027.
- [13] B. Dakić, V. Vedral, and Č. Brukner, Phys. Rev. Lett. **105** (2010) 190502
- [14] B. Bellomo, R. Lo Franco and G. Compagno, Phys. Rev. A 86 (2012) 012312.
- [15] B. Bellomo, G.L. Giorgi, F. Galve, R. Lo Franco, G. Compagno and R. Zambrini, Phys. Rev. A 85 (2012) 032104.
- [16] M. Daoud and R. Ahl Laamara, Phys. Lett. A 376 (2012) 2361.
- [17] M. Daoud and R. Ahl Laamara, Int. J. Quantum Inf. 10 (2012) 1250060.
- [18] M. Piani, Phys. Rev. A 86 (2012) 034101.
- [19] F. M. Paula, T. R. de Oliveira, and M. S. Sarandy, Phys. Rev. A 87 (2013) 064101; F.M. Paula, J.D. Montealegre, A. Saguia, T.R. de Oliveira and M.S. Sarandy, EPL, 103 (2013) 50008.
- [20] T.R. Bromley, M. Cianciaruso, R. Lo Franco, G. Adesso, J. Phys. A: Math. Theor. 47 (2014) 405302
- [21] F. Ciccarello, T. Tufarelli and V. Giovannetti, New J. Phys. 16 (2014) 013038.
- [22] D. Girolami, T. Tufarelli, and G. Adesso, Phys. Rev. Lett. 110 (2013) 240402.
- [23] E. P. Wigner and M. M. Yanasse, Proc. Nat. Acad. Sci. USA 49 (1963) 910.
- [24] S. Luo, Phys. Rev. Lett. **91** (2003) 180403.
- [25] D. Petz and C. Ghinea in Quantum Probability and Related Topics, edited by R. Rebolledo and M. Orszag, World Scientific, 2011.
- [26] S. Luo, Proc. Amer. Math. Soc. 132 (2003) 885.
- [27] D. Girolami, T. Tufarelli and G. Adesso, Phys. Rev. Lett. 110 (2013) 240402.
- [28] T. Nakano, M.Piani and G. Adesso, Phys. Rev. A 88 (2013) 012117.
- [29] L. Liu, J. Zhang, S. Cheng and D. Tong, J. Phys. Soc. Japan 82 (2013) 0640002.
- [30] X. N. Hao, J. C. Hou and J. Q. Li, Quant. Inf. Proc. 15 (2016) 2819.
- [31] E. Ferraro, M. Scala, R. Migliore and A. Napoli, Non-Markovian dissipative dynamics of two coupled qubits in independent reservoirs: Comparison between exact solutions and master-equation approaches. Phys. Rev. A 80 (2009) 042112.

- [32] M. Scala, R. Migliore and A. Messina, J. Phys. A: Mathematical and Theoretical, 41 (2008) 435304.
- [33] G. Agarwal, Quantum Statistical Theories of Spontaneous Emission and their Relation to Other Approaches, vol. 70 of Springer Tracts in Modern Physics (Springer-Verlag, Berlin, 1974).
- [34] R. Tanaś and Z. Ficek, J. Opt. B: Quantum and Semiclassical Optics, 6 (2004) S90.
- [35] Z. Ficek, R. Tanaś and S. Kielich, Physica A 146 (1987) 452.
- [36] A. Auyuanet and L. Davidovich, Phys. Rev. A 82 (2010) 032112.