# Weak limit theorem for a nonlinear quantum walk 

Masaya Maeda, Hironobu Sasaki, Etsuo Segawa, Akito Suzuki, Kanako Suzuki

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#### Abstract

This paper continues the study of large time behavior of a nonlinear quantum walk begun in [13]. In this paper, we provide a weak limit theorem for the distribution of the nonlinear quantum walk. The proof is based on the scattering theory of the nonlinear quantum walk and the limit distribution is obtained in terms of its asymptotic state.


## 1 Introduction

This paper continues the study of a one-dimensional nonlinear quantum walk (NLQW) begun in 13], where we developed a scattering theory for NLQW. The model treated there covers a nonlinear optical Galton board [17, [15], a quantum walk with a feed-forward quantum coin [20], nonlinear discrete dynamics [11, and a model exhibiting topological phenomena [4]. For more details on earlier works, we refer to the previous paper [13]. In a forthcoming companion paper [14], we numerically study a solitonic behavior of NLQW. In this paper, we study a weak limit theorem (WLT) for NLQW. The WLT for the one-dimensional (linear) quantum walk (QW) was first found by Konno [9], proved in [10, and then generalized by several authors [1, 2, 3, 5, 6, 7, 8, 12, 19, 21. The WLT states that

$$
\frac{X_{t}}{t} \text { converges in law to a random variable } V \text { as } t \rightarrow \infty
$$

where $X_{t}$ is a random variable denoting the position of a quantum walker at time $t=0,1,2, \ldots$.. Because $X_{t} / t$ is the average velocity of the walker, $V$ is interpreted as the asymptotic velocity of the walker and hence WLT well describes the asymptotic behavior of the walker. Here, the probability distribution of $X_{t}$ is naturally defined according to Born's rule as

$$
P\left(X_{t}=x\right)=\left\|\Psi_{t}(x)\right\|_{\mathbb{C}^{2}}^{2}, \quad x \in \mathbb{Z}
$$

where $\Psi_{t}$ is the state of the walker at time $t$, which is in the state space $\mathcal{H}:=l^{2}\left(\mathbb{Z} ; \mathbb{C}^{2}\right)$. The state evolution is governed by

$$
\Psi_{t+1}(x)=P(x+1) \Psi_{t}(x+1)+Q(x-1) \Psi_{t}(x-1), \quad x \in \mathbb{Z}, t=0,1,2, \ldots
$$

where $P(x)$ and $Q(x) \in M(2 ; \mathbb{C})$ satisfy $P(x)+Q(x)=: C(x) \in U(2)$. More precisely, the state at time $t$ is given by $\Psi_{t}=U_{\mathrm{L}}^{t} \Psi_{0}$, where $\Psi_{0}$ is the initial state, which is a normalized vector in $\mathcal{H}$, and $U_{\mathrm{L}}$ is the evolution operator defined as follows. Let $\hat{C}$ be the coin operator defined as the multiplication by $C(x)$ and $S$ be the shift operator, i.e., $(\hat{C} \Psi)(x):=C(x) \Psi(x)$ and $(S \Psi)(x):=$ ${ }^{\mathrm{t}}\left(\Psi_{1}(x+1), \Psi_{2}(x-1)\right)(x \in \mathbb{Z})$ for $\Psi={ }^{\mathrm{t}}\left(\Psi_{1}, \Psi_{2}\right) \in \mathcal{H}$. The evolution operator $U_{\mathrm{L}}$ is a unitary operator defined as $U_{\mathrm{L}}=S \hat{C}$. As shown in [21], if $C(x)=C_{0}+O\left(|x|^{-1-\epsilon}\right)$ with some $C_{0} \in U(2)$ and
$\epsilon>0$ independent of $x$, then WLT is proved and the limit distribution $\mu_{V}$ is expressed in terms of the wave operator $W_{+}:=\mathrm{s}-\lim _{t \rightarrow \infty} U_{\mathrm{L}}^{-t} U_{0}^{t} \Pi_{\mathrm{ac}}\left(U_{0}\right)$, where $U_{0}=S \hat{C}_{0}$ and $\Pi_{\mathrm{ac}}\left(U_{0}\right)$ is the projection onto the subspace of absolute continuity. See also [1, 18, 19 for anisotropic cases.

In the case of NLQW, the dynamics is governed by

$$
\begin{equation*}
u(t+1, x)=(\hat{P} u(t))(x+1)+(\hat{Q} u(t))(x-1), \quad x \in \mathbb{Z}, t=0,1,2, \ldots, \tag{1.1}
\end{equation*}
$$

where $t \mapsto u(t):=u(t, \cdot) \in \mathcal{H}$ is in $l^{\infty}(\mathbb{N} \cup\{0\} ; \mathcal{H})$. $\hat{P}$ and $\hat{Q}$ are nonlinear maps on $\mathcal{H}$ and give a norm preserving nonlinear map $\hat{C}: \mathcal{H} \ni u \mapsto \hat{C} u:=\hat{P} u+\hat{Q} u$. Although the dynamics is similar to the linear quantum walk, it does not define a quantum system. However,

$$
\begin{equation*}
p_{t}(x):=\|u(t, x)\|_{\mathbb{C}^{2}}^{2}, \quad x \in \mathbb{Z} . \tag{1.2}
\end{equation*}
$$

defines a probability distribution. Indeed, similarly to the linear quantum walk, the dynamics (1.1) is expressed as $u(t+1, \cdot)=U u(t, \cdot), t=0,1,2, \ldots$, where $U:=S \hat{C}$ is a nonlinear map on $\mathcal{H}$. Because $U$ preserves the norm, (1.2) defines the probability distribution provided that the initial state $u(0, \cdot)=u_{0} \in \mathcal{H}$ is a normalized vector. We use $X_{t}$ to denote the random variable that follows (1.2), i.e., $P\left(X_{t}=x\right)=p_{t}(x)$. Of course $X_{t}$ never describes the position of a walker that occupies any single position in $\mathbb{Z}$, but we dare to call $X_{t}$ the position of a nonlinear quantum walker in analogy with the linear quantum walk. It is mathematically more convenient to study the limit behavior of $X_{t}$ than the distribution $p_{t}(x)$ itself.

In this paper, we consider a nonlinear coin given by

$$
(\hat{C} u)(x)=C_{\mathrm{N}}\left(g\left|u_{1}(x)\right|^{2}, g\left|u_{2}(x)\right|^{2}\right) u(x), \quad x \in \mathbb{Z} \quad \text { for } u={ }^{\mathrm{t}}\left(u_{1}, u_{2}\right) \in \mathcal{H},
$$

where $g>0$ controls the strength of the nonlinearity and $C_{\mathrm{N}}:[0, \infty) \times[0, \infty) \ni\left(s_{1}, s_{2}\right) \mapsto$ $C_{\mathrm{N}}\left(s_{1}, s_{2}\right) \in U(2)$ with $C_{\mathrm{N}}(0,0)=: C_{0} \in U(2)$. As was shown in [13], in the weak nonlinear regime, $U(t) u_{0}$ scatters, i.e., $\lim _{t \rightarrow \infty}\left\|u(t, \cdot)-U_{0}^{t} u_{+}\right\|_{\mathcal{H}}=0$ with some asymptotic state $u_{+} \in \mathcal{H}$. The aim of this paper is to establish WLT for $X_{t}$ that follows (1.2) and prove that the limit distribution is given by

$$
\mu_{V}(d v)=w(v) f_{\mathrm{K}}(v ;|a|) d v,
$$

where $w(v)$ is a function expressed in terms of $u_{+}$and $f_{\mathrm{K}}(v ; r)$ is the Konno function $(r>0)$.
The rest of this paper is organized as follows. Sec. 2 is devoted to reviewing the results of [13]. We state our main results in Sec. 3 and give proofs in Sec. 4.

## 2 Preliminaries

In this section, we review the definition of NLQW and results obtained in [13]. Throughout this paper, we set $\mathcal{H}=l^{2}\left(\mathbb{Z} ; \mathbb{C}^{2}\right)$ and drop the subscript $\mathcal{H}$ in the norm and inner product when there is no ambiguity. Let

$$
C_{\mathrm{N}}:[0, \infty) \times[0, \infty) \rightarrow U(2)
$$

satisfy

$$
C_{0}:=C_{\mathrm{N}}(0,0)=\left(\begin{array}{cc}
a & b \\
-\bar{b} & \bar{a}
\end{array}\right) \quad \text { with }|a|^{2}+|b|^{2}=1 \text { and } 0<|a|<1 .
$$

We define a nonlinear coin operator $\hat{C}$ as

$$
\begin{equation*}
(\hat{C} u)(x)=C_{\mathrm{N}}\left(g\left|u_{1}(x)\right|^{2}, g\left|u_{2}(x)\right|^{2}\right) u(x), \quad x \in \mathbb{Z} \quad \text { for } u={ }^{\mathrm{t}}\left(u_{1}, u_{2}\right) \in \mathcal{H}, \tag{2.1}
\end{equation*}
$$

where $g>0$ is a constant that controls the strength of the nonlinearity. Let $u_{0} \in \mathcal{H}$ be the initial state of a walker with $\left\|u_{0}\right\|=1$. The state $u(t)$ of the walker at time $t=1,2, \ldots$ is defined by induction as follows.

$$
u(0)=u_{0}, \quad u(t+1)=U u(t), \quad t=0,1,2, \ldots,
$$

where $U=S \hat{C}$. We then define a nonlinear evolution operator $U(t)$ as

$$
U(t) u_{0}=u(t)
$$

Similarly, we define a linear coin operator $\hat{C}_{0}$ as $\left(\hat{C}_{0} u\right)(x)=C_{0} u(x)$ and set $U_{0}=S \hat{C}_{0}$. By scattering, we mean the following:

Definition 2.1. We say $U(t) u_{0}$ scatters if there exists $u_{+} \in \mathcal{H}$ such that

$$
\lim _{t \rightarrow \infty}\left\|U(t) u_{0}-U_{0}^{t} u_{+}\right\|=0
$$

We use $U_{g=1}(t)$ to denote the evolution operator $U(t)$ that has the nonlinear coin $\hat{C}$ defined in (2.1) with $g=1$. As mentioned in the previous paper [13], the smallness of $\left\|u_{0}\right\|_{\mathcal{H}}$ and $\left\|u_{0}\right\|_{l^{1}}$ corresponds to the smallness of $g$, because

$$
U(t) u_{0}=\frac{1}{\sqrt{g}} U_{g=1}(t) v_{0} \quad \text { with } v_{0}:=\sqrt{g} u_{0}
$$

Thus, the result in 13 is reformulated as follows. We use $\|A\|_{\mathbb{C}^{2} \rightarrow \mathbb{C}^{2}}$ tot denote the operator norm of the matrix $A$, i.e., $\|A\|_{\mathbb{C}^{2} \rightarrow \mathbb{C}^{2}}:=\sup _{v \in \mathbb{C}^{2},\|v\|_{\mathbb{C}^{2}}=1}\|A v\|_{\mathbb{C}^{2}}$.

Theorem $2.2\left([13)\right.$. Assume that $C_{\mathrm{N}} \in C^{1}(\Omega ; U(2))$ with some domain $\Omega$ including $[0, \infty) \times$ $[0, \infty)$ and there exists $c_{0}>0$ and $m \geq 2$ such that $\left\|C_{\mathrm{N}}\left(s_{1}, s_{2}\right)-C_{0}\right\|_{\mathbb{C}^{2} \rightarrow \mathbb{C}^{2}} \leq c_{0}\left(s_{1}+s_{2}\right)^{m}$ and $\left\|\partial_{s_{j}} C_{\mathrm{N}}\left(s_{1}, s_{2}\right)\right\|_{\mathbb{C}^{2} \rightarrow \mathbb{C}^{2}} \leq c_{0}\left(s_{1}+s_{2}\right)^{m-1}$ for $j=1$ or 2 . Let $u_{0} \in \mathcal{H}$ be a normalized vector. Suppose in addition that either of the following conditions holds: (1) $m \geq 3$; (2) $m=2$ and $u_{0} \in l^{1}\left(\mathbb{Z}, \mathbb{C}^{2}\right)$. Then $U(t) u_{0}$ scatters if $g$ is sufficiently small.

## 3 Weak limit theorem

Our aim is to establish the weak limit theorem for the position $X_{t}$ of a walker at time $t$ that follows the probability distribution

$$
\begin{equation*}
P\left(X_{t}=x\right)=\|u(t, x)\|_{\mathbb{C}^{2}}^{2}, \quad x \in \mathbb{Z}, \tag{3.1}
\end{equation*}
$$

where $u(t, \cdot):=U(t) u_{0}$ with $\left\|u_{0}\right\|_{\mathcal{H}}=1$. By Theorem 2.2, if $g$ is sufficiently small, then $U(t) u_{0}$ scatters, i.e., there exists $u_{+} \in \mathcal{H}$ such that

$$
\lim _{t \rightarrow \infty}\left\|U(t) u_{0}-U_{0}^{t} u_{+}\right\|=0
$$

Let $\hat{v}_{0}$ be the asymptotic velocity operator for $U_{0}=S \hat{C}_{0}$, which is a unique self-adjoint operator such that

$$
\begin{equation*}
e^{i \xi \hat{v}_{0}}=\mathrm{s}-\lim _{t \rightarrow+\infty} e^{i \xi \hat{x}_{0}(t) / t} \tag{3.2}
\end{equation*}
$$

Here $\hat{x}_{0}(t)=U_{0}^{-t} \hat{x} U_{0}$ is the Heisenberg operator of the position operator $\hat{x}$. See [5, 21] for more details. We give the precise definition of $\hat{v}_{0}$ in (4.2). We use $E_{A}(\cdot)$ to denote the spectral projection of a self-adjoint operator $A$.

Theorem 3.1 (weak limit theorem). Let $X_{t}, \hat{v}_{0}$, and $u_{+}$be as above. Then there exists a random variable $V$ such that $X_{t} / t$ converges in law to $V$, whose distribution $\mu_{V}$ is given by

$$
\mu_{V}(d v)=d\left\|E_{\hat{v}_{0}}(v) u_{+}\right\|^{2}
$$

In what follows, we provide an explicit formula for the density function of $\mu_{V}$ obtained in Theorem 3.1. To this end, we proceed along the lines of [19]. Let $f_{\mathrm{K}}$ be the Konno function defined for all $r>0$ as

$$
f_{\mathrm{K}}(v ; r)= \begin{cases}\frac{\sqrt{1-r^{2}}}{\pi\left(1-v^{2}\right) \sqrt{r^{2}-v^{2}},} & |v|<r \\ 0, & |v| \geq 0\end{cases}
$$

Similarly to [19], we introduce operators

$$
K_{j, m}: \mathcal{H} \rightarrow \mathcal{G}:=L^{2}\left([-|a|,|a|], f_{\mathrm{K}}(v ;|a|) d v / 2\right), \quad j=1,2, m=0,1
$$

as follows. Let $\mathbb{T}:=\mathbb{R} / 2 \pi \mathbb{Z}$. Because $U_{0}$ is translation invariant, it can be decomposed by the Fourier transformation $F: \mathcal{H} \rightarrow L^{2}\left(\mathbb{T} ; \mathbb{C}^{2} ; d k / 2 \pi\right)$ and the Fourier transform $F U_{0} F^{-1}$ is the multiplication operator by

$$
\hat{U}_{0}(k)=\left(\begin{array}{cc}
e^{i k} a & e^{i k} b \\
-e^{-i k} \bar{b} & e^{-i k} \bar{a}
\end{array}\right) \in U(2), \quad k \in \mathbb{T}
$$

We use $\varphi_{j}(k)$ to denote the normalized eigenvectors of $\hat{U}_{0}(k)$ corresponding to the eigenvalues

$$
\lambda_{j}(k)=|a| \cos \left(k+\theta_{a}\right)+i(-1)^{j-1} \sqrt{|b|^{2}+|a|^{2} \sin \left(k+\theta_{a}\right)}, \quad j=1,2 .
$$

Let $k_{k, m}:[-|a|,|a|] \rightarrow I_{m}:=\left[\pi(m-1 / 2)-\theta_{a}, \pi(m+1 / 2)-\theta_{a}\right]$ be a function defined as

$$
k_{j, m}(v)=-\theta_{a}+m \pi+\arcsin \left(\frac{(-1)^{j+m}|b| v}{|a| \sqrt{1-v^{2}}}\right), \quad j=1,2, m=0,1
$$

where $\theta_{a} \in[0,2 \pi)$ is the argument of $a$. By direct calculation, $k_{j, m}$ is differentiable in $(-|a|,|a|)$ and

$$
\frac{d}{d v} k_{j, m}=(-1)^{j+m} \pi f_{\mathrm{K}}(v,|a|)
$$

We now define the operators $K_{j, m}$ as

$$
\left(K_{j, m} u\right)(v)=\left\langle\varphi_{j}\left(k_{j, m}(v)\right), \hat{u}\left(k_{j, m}(v)\right)\right\rangle_{\mathbb{C}^{2}}, \quad v \in[-|a|,|a|]
$$

where $\hat{u}$ is the Fourier transform of $u \in \mathcal{H}$.
Theorem 3.2. Let $u_{+}$and $V$ be as Theorem 3.1, Then

$$
\mu_{V}(d v)=w(v) f_{\mathrm{K}}(v,|a|) d v
$$

where

$$
w(v)=\frac{1}{2} \sum_{j=1,2} \sum_{m=0,1}\left|\left(K_{j, m} u_{+}\right)(v)\right|^{2}, \quad v \in[-|a|,|a|] .
$$

## 4 Proofs of Theorems

The proofs of Theorem 3.1 and 3.2 proceed along the same lines as that of 21] [Corollary 2.4]. We suppose that $\left\|u_{0}\right\|_{\mathcal{H}}=1$. Let $\hat{x}$ be the position operator. By (3.1), the characteristic function of $X_{t} / t$ is given by

$$
\begin{equation*}
\mathbb{E}\left(e^{i \xi X_{t} / t}\right)=\left\langle U(t) u_{0}, e^{i \xi \hat{x} / t} U(t) u_{0}\right\rangle, \quad \xi \in \mathbb{R} \tag{4.1}
\end{equation*}
$$

where $\mathbb{E}(X)$ denotes the expectation value of a random variable $X$. The asymptotic velocity operator $\hat{v}_{0}$ in (3.2) is defined via the Fourier transform: $F v_{0} F^{-1}$ is the multiplication operator by

$$
\begin{equation*}
\hat{v}_{0}(k)=\sum_{j=1,2} v_{j}(k)\left|\varphi_{j}(k)\right\rangle\left\langle\varphi_{j}(k)\right|, \quad k \in \mathbb{T} \tag{4.2}
\end{equation*}
$$

where

$$
v_{j}(k):=\frac{i}{\lambda_{j}(k)} \frac{d}{d k} \lambda_{j}(k)=\frac{(-1)^{j}|a| \sin \left(k+\theta_{a}\right)}{\sqrt{|b|^{2}+\sin ^{2}\left(k+\theta_{a}\right)}}
$$

As was shown in [19], $v_{j}: I_{m} \rightarrow[-|a|,|a|]$ is the inverse function of $k_{j, m}$.
Lemma 4.1.

$$
\lim _{t \rightarrow \infty}\left\langle U(t) u_{0}, e^{i \xi \hat{x} / t} U(t) u_{0}\right\rangle=\left\langle u_{+}, e^{i \xi \hat{v}_{0}} u_{+}\right\rangle
$$

Proof. A direct calculation yields

$$
\begin{aligned}
& \left|\left\langle U(t) u_{0}, e^{i \xi \hat{x} / t} U(t) u_{0}\right\rangle-\left\langle u_{+}, e^{i \xi \hat{v}_{0}} u_{+}\right\rangle\right| \\
& \quad \leq\left|\left\langle U(t) u_{0}-U_{0}^{t} u_{0}, e^{i \xi \hat{x} / t} U(t) u_{0}\right\rangle\right|+\left|\left\langle U_{0}^{t} u_{0}, e^{i \xi \hat{x} / t}\left(U(t) u_{0}-U_{0}^{t} u_{0}\right)\right\rangle\right| \\
& \quad+\left|\left\langle U_{0}^{t} u_{0}, e^{i \xi \hat{x} / t} U_{0}^{t} u_{0}\right\rangle-\left\langle u_{+}, e^{i \xi \hat{v}_{0}} u_{+}\right\rangle\right\rangle \mid \\
& = \\
& : I_{1}(t)+I_{2}(t)+I_{3}(t)
\end{aligned}
$$

Because $e^{i \xi \hat{x} / t}$ and $U(t)$ preserve the norm and $U(t) u_{0}$ scatters, $\lim _{t \rightarrow \infty} I_{1}(t)=\lim _{t \rightarrow \infty} I_{2}(t)=0$. By (3.2), $\lim _{t \rightarrow \infty} I_{3}(t)=0$. Hence the proof is completed.

Proof of Theorem 3.1. By (4.1) and Lemma 4.1

$$
\begin{equation*}
\lim _{t \rightarrow \infty} E\left(e^{i \xi X_{t} / t}\right)=\left\langle u_{+}, e^{i \xi \hat{v}_{0}} u_{+}\right\rangle=\int_{[-|a|,|a|]} e^{i \xi v} d\left\|E_{\hat{v}_{0}}(v) u_{+}\right\|^{2} \tag{4.3}
\end{equation*}
$$

where we have used the spectral theorem. The right-hand side in the above equation is equal to the characteristic function of a random variable $V$ following the probability distribution $\mu_{V}=$ $\left\|E_{\hat{v}_{0}}(\cdot) u_{+}\right\|^{2}$. This completes the proof of Theorem 3.1.

In what follows, we prove Theorem 3.2. The following lemma is proved similarly to [19].
Lemma 4.2. We use $\hat{G}$ to denote the multiplication operator on $\mathcal{G}$ by a Borel function $G$ : $[-|a|,|a|] \rightarrow \mathbb{C}$. Then

$$
G\left(\hat{v}_{0}\right)=\sum_{j=1,2} \sum_{m=0,1} K_{j, m}^{*} \hat{G} K_{j, m}
$$

Proof of Theorem 3.2. It suffices to prove

$$
\begin{equation*}
\left\langle u_{+}, e^{i \xi \hat{v}_{0}} u_{+}\right\rangle=\int_{[-|a|,|a|]} e^{i \xi v} w(v) f_{K}(v ;|a|) d v, \quad \xi \in \mathbb{R} \tag{4.4}
\end{equation*}
$$

Let $G(v)=e^{i \xi v}$. By Lemma 4.2, the left-hand side of (4.4) is

$$
\left\langle u_{+}, e^{i \xi \hat{v}_{0}} u_{+}\right\rangle=\sum_{j=1,2} \sum_{m=0,1}\left\langle K_{j, m} u_{+}, \hat{G} K_{j, m} u_{+}\right\rangle_{\mathcal{G}} .
$$

Because

$$
\left\langle K_{j, m} u_{+}, \hat{G} K_{j, m} u_{+}\right\rangle_{\mathcal{G}}=\int_{[-|a|,|a|]} e^{i \xi v}\left|\left(K_{j, m} u_{+}\right)(v)\right|^{2} f_{\mathrm{K}}(v ;|a|) d v / 2
$$

the proof of theorem is complete.

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Masaya Maeda, Hironobu Sasaki
Department of Mathematics and Informatics, Faculty of Science, Chiba University, Chiba 2638522, Japan

E-mail Address: maeda@math.s.chiba-u.ac.jp, sasaki@math.s.chiba-u.ac.jp
Etsuo Segawa
Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan
E-mail Address: e-segawa@m.tohoku.ac.jp
Akito Suzuki
Division of Mathematics and Physics, Faculty of Engineering, Shinshu University, Nagano 3808553, Japan

E-mail Address: akito@shinshu-u.ac.jp
Kanako Suzuki
College of Science, Ibaraki University, 2-1-1 Bunkyo, Mito 310-8512, Japan
E-mail Address: kanako.suzuki.sci2@vc.ibaraki.ac.jp

