State transfer with quantum side information

Yonghae Lee and Soojoon Lee

Department of Mathematics and Research Institute for Basic Sciences,

Kyung Hee University, Seoul 02447, Korea

(Dated: October 9, 2018)

Abstract

We first consider quantum communication protocols between a sender Alice and a receiver Bob, which transfer Alice's quantum information to Bob by means of non-local resources, such as classical communication, quantum communication, and entanglement. In these protocols, we assume that Alice and Bob may have quantum side information, not transferred. In this work, these protocols are called the state transfer with quantum side information. We determine the optimal costs for non-local resources in the protocols, and study what the effects of the use of quantum side information are. Our results can give new operational meanings to the quantum mutual information and the quantum conditional mutual information, which directly provide us with an operational interpretation of the chain rule for the quantum mutual information.

I.   INTRODUCTION

There are quantum communication protocols, such as the quantum teleportation [1] and the Schumacher compression [2], which transfer quantum information from Alice to Bob. In quantum information theory, these protocols have been regarded as the leading research topics, since they can provide new operational meanings to quantum quantities, such as the von Neumann entropies [3] and the smooth entropies [4]. New operational meanings have made the quantum information theory richer through intuitive understandings of quantum phenomena.

We here consider protocols in which Alice's information can be asymptotically transferred to Bob by means of quantum/classical communication and entanglement as non-local resources. In the protocols, Alice and Bob are able to apply local operations on their states, and employ their quantum side information (QSI) in order to transfer Alice's information. We call the protocols the *state transfer with QSI*, and divide the state transfer protocols with QSI into two types: the *state redistribution with QSI* and the *state merging with QSI*. In the former Alice and Bob use quantum channels for communication from Alice to Bob, and in the latter they use classical channels.

Although there have been some protocols [5–13] which deal with Alice's or Bob's QSI, the results have not explicitly explained how the use of QSI has the effects on the optimal resource costs. In addition, when Alice and Bob can use more (or less) QSI, it has not been mentioned in literature. On this account, one can raise the following two questions: (i) How does the use of Alice's and Bob's QSI affect the optimal resource costs in the state transfer with QSI? (ii) Assume that Alice or Bob uses more (or less) QSI in the state transfer with QSI. How does the use of more (or less) QSI affect the optimal resource costs?

In order to answer the two questions, we describe a mathematical definition of the state transfer with QSI, and calculate its optimal costs for non-local resources. Then we study the effects of QSI on the optimal resource costs of the state transfer with QSI. From these results, we present new operational meanings of the quantum mutual information (QMI), quantum conditional mutual information (QCMI), and a new operational interpretation of the chain rule for the QMI [3].

This paper is organized as follows. In Sec. II we define the state transfer with QSI, and calculate its optimal costs for non-local resources. In Sec. III we study what the effects of

the use of QSI are in the state transfer with QSI. Then we give new operational meanings to the QMI and the QCMI in Sec. IV. We also present well-known examples which are special cases of the state transfer with QSI in Sec. V. Finally, in Sec. VI we summarize and discuss our results.

II. STATE TRANSFER WITH QSI

We formally define the state transfer with QSI as follows.

Definition 1 (State transfer with QSI). Let $|\psi\rangle \equiv |\psi\rangle_{A_1 \cdots A_m C_A B_1 \cdots B_n R}$ be a pure initial state, where Alice and Bob hold $A_1 \cdots A_m C_A$ and $B_1 \cdots B_n$, respectively, and R is the reference. Assume that Alice and Bob have additional systems E_A^{in}, E_A^{out} and E_B^{in}, E_B^{out} for entanglement resources, respectively. For $0 \le i \le m$ and $0 \le j \le n$, a joint operation

$$\mathcal{T}_{ij} \; : \; A_1 \cdots A_i C_A E_A^{\text{in}} \otimes B_1 \cdots B_j E_B^{\text{in}}$$
$$\longrightarrow A_1 \cdots A_i E_A^{\text{out}} \otimes C_B B_1 \cdots B_j E_B^{\text{out}}$$

is called the *state transfer with QSI of* $|\psi\rangle$ (or $\text{tr}_R(|\psi\rangle\langle\psi|)$ *with error* ε, if it consists of local operations and either qubit channels or bit channels from Alice to Bob, and satisfies

$$F\Big((\mathcal{T}_{ij} \otimes \mathbb{1}_{A_{i+1}\cdots A_m B_{j+1} \cdots B_n R}) \big(|\psi\rangle \otimes |\Phi\rangle_{E_A^{\text{in}} E_B^{\text{in}}} \big),$$
$$|\psi'\rangle \otimes |\Phi\rangle_{E_A^{\text{out}} E_B^{\text{out}}} \Big) \ge 1 - \varepsilon,$$

where C_B is Bob's system with $\dim C_B = \dim C_A$, $F(\cdot, \cdot)$ is the quantum fidelity, $|\psi'\rangle$ is a final state defined as $(\mathbb{1}_{A_1 \cdots A_m B_1 \cdots B_n R} \otimes \mathbb{1}_{C_A \to C_B}) |\psi\rangle$, and $|\Phi\rangle_{E_A^{\text{in}} E_B^{\text{in}}}$ and $|\Phi\rangle_{E_A^{\text{out}} E_B^{\text{out}}}$ are maximally entangled states with Schmidt-rank $e^{\text{in}}(\mathcal{T}_{ij})$ and $e^{\text{out}}(\mathcal{T}_{ij})$, respectively.

In addition, we call the operation $\mathcal{T}_{ij}$ the *state redistribution with QSI*, if it consists of local operations and $q(\mathcal{T}_{ij})$ qubit channels without any classical channels, and $\mathcal{T}_{ij}$ is called the *state merging with QSI*, if it consists of local operations and $c(\mathcal{T}_{ij})$ bit channels without any quantum channels.

In Definition 1, the indices i and j of $\mathcal{T}_{ij}$ mean that Alice and Bob apply local operations on their QSI $A_1 \cdots A_i$ and $B_1 \cdots B_j$ in order to transfer Alice's C_A to Bob as depicted in Fig. 1, and in this situation we say that Alice and Bob *use* their QSI $A_1 \cdots A_i$ and $B_1 \cdots B_j$.
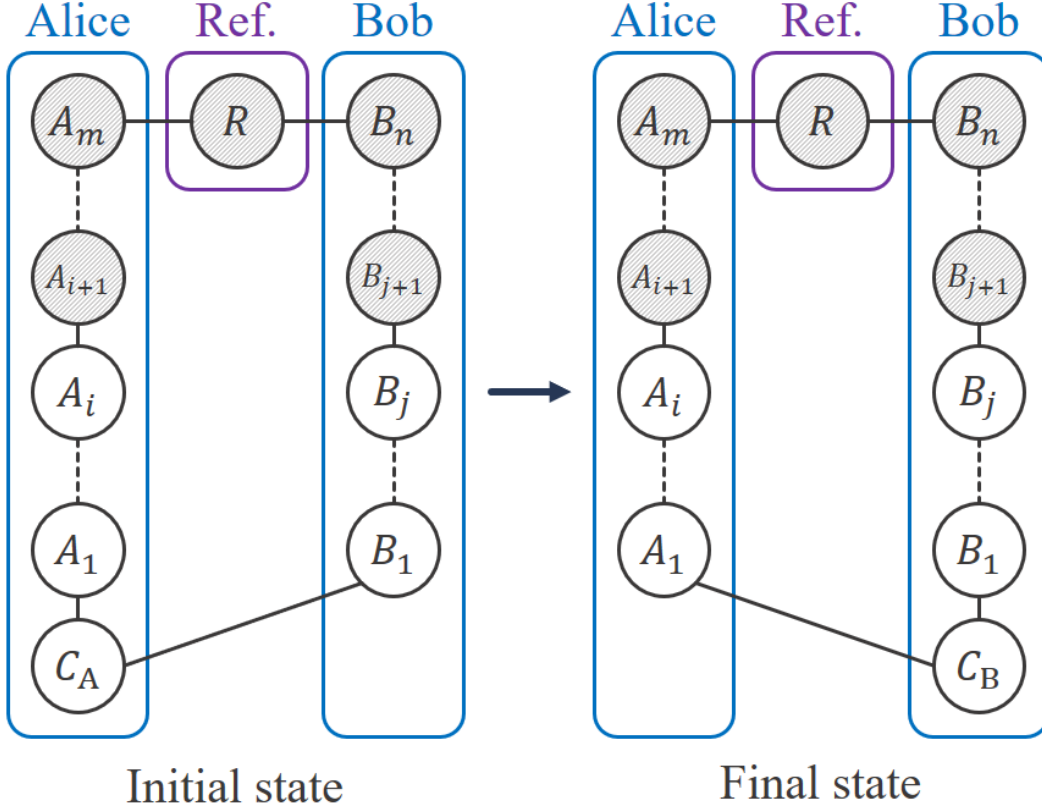
FIG. 1: The initial and final states for the state transfer with QSI of $|\psi\rangle$: C_A is a transferred part, $A_1 \cdots A_m$ and $B_1 \cdots B_n$ are Alice's and Bob's QSI, and R is the reference. In the state transfer with QSI, they use QSI $A_1 \cdots A_i$ and $B_1 \cdots B_j$, respectively, while the rests $A_{i+1} \cdots A_m$ and $B_{j+1} \cdots B_n$ are left unused.

For instance, Alice and Bob do not use any QSI if and only if $i = 0$ and $j = 0$, respectively, and they make use of the whole QSI if and only if $i = m$ and $j = n$, respectively.

We also define the optimal resource costs of the state transfer with QSI of $|\psi\rangle$ for fixed i and j.

Definition 2. For n independent and identically distributed copies of $|\psi\rangle \equiv |\psi\rangle_{A_1 \cdots A_m C_A B_1 \cdots B_n R}$, say $|\psi\rangle^{\otimes n}$, let $\mathcal{T}_{ij}^n$ be a state redistribution (or a state merging) with QSI of $|\psi\rangle^{\otimes n}$ with error ε_n, then the resource rates $(\log e^{\mathrm{in}}(\mathcal{T}_{ij}^n) - \log e^{\mathrm{out}}(\mathcal{T}_{ij}^n))/n$ and $q(\mathcal{T}_{ij}^n)/n$ (or $c(\mathcal{T}_{ij}^n)/n$) are called the *entanglement rate* and *quantum communication rate* (or *classical communication rate*) of the protocol, respectively.

For each resource rate, we call a real number r an *achievable rate* if there is a sequence

$\{\mathcal{T}_{ij}^n\}_{n\in\mathbb{N}}$ such that the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ converges to zero, and the sequence for the resource rate converges r as n tends to infinity. The smallest achievable rates for entanglement and quantum communication (or classical communication) are called the *optimal entanglement cost* and *optimal quantum communication cost* (or *optimal classical communication cost*), respectively.

We investigate the optimal resource costs for the state redistribution with QSI of $|\psi\rangle \equiv |\psi\rangle_{A_1\cdots A_m C_A B_1\cdots B_n R}$. Let $Q_{i,j}$ and $E_{i,j}$ be its optimal quantum communication and entanglement costs, respectively, when Alice and Bob use QSI $A_1\cdots A_i$ and $B_1\cdots B_j$. Let $\tilde{A} = A_1\cdots A_i$, $\tilde{B} = B_1\cdots B_j$, and $\tilde{R} = A_{i+1}\cdots A_m B_{j+1}\cdots B_n R$. Then the given state $|\psi\rangle$ becomes a four-partite state $|\psi\rangle_{\tilde{A}C_A\tilde{B}\tilde{R}}$. Since $A_{i+1}\cdots A_m$ and $B_{j+1}\cdots B_n$ are not used, and $\tilde{R}$ can be considered as the reference system of a purification $|\psi\rangle$ of a quantum state $\rho_{\tilde{A}C_A\tilde{B}}$, our state redistribution with QSI is identical to the state redistribution for $|\psi\rangle_{\tilde{A}C_A\tilde{B}\tilde{R}}$ [7, 8]. Thus, we can obtain that

$$Q_{i,j} = \frac{1}{2}I(C_A; \tilde{R}|\tilde{B})$$
$$= H(C_A) - \frac{1}{2}I(C_A; \tilde{A}) - \frac{1}{2}I(C_A; \tilde{B}),$$
$$E_{i,j} = \frac{1}{2}I(C_A; \tilde{A}) - \frac{1}{2}I(C_A; \tilde{B}),$$

where $I(\cdot;\cdot|\cdot)$ is the QCMI, $H(\cdot)$ is the von Neumann entropy and $I(\cdot;\cdot)$ is the QMI. This implies the following lemma.

Lemma 3. *For a state $\rho_{A_1\cdots A_m C_A B_1\cdots B_n}$ shared by Alice and Bob, the optimal quantum communication cost $Q_{i,j}$ and the optimal entanglement cost $E_{i,j}$ for the state redistribution with QSI can be expressed as the von Neumann entropy $H(C_A)$ and the QMI $I(C_A; A_1\cdots A_i)$ and $I(C_A; B_1\cdots B_j)$ as follows:*

$$Q_{i,j} = H(C_A) - \frac{1}{2}I(C_A; A_1\cdots A_i) - \frac{1}{2}I(C_A; B_1\cdots B_j),$$
$$E_{i,j} = \frac{1}{2}I(C_A; A_1\cdots A_i) - \frac{1}{2}I(C_A; B_1\cdots B_j). \tag{1}$$

By replacing qubit channels with bit channels, we can consider the state merging with QSI of the state $|\psi\rangle$. For each $0 \le i \le m$ and $0 \le j \le n$, let $c_{i,j}$ and $e_{i,j}$ be the optimal classical communication and entanglement costs of the state merging with QSI, respectively, when Alice and Bob employ QSI $A_1\cdots A_i$ and $B_1\cdots B_j$. Then we obtain the following lemma.

Lemma 4. *For each $0 \leq i \leq m$ and $0 \leq j \leq n$, the optimal classical communication cost $c_{i,j}$ and the optimal entanglement cost $e_{i,j}$ for the state merging with QSI can be expressed in terms of the optimal costs $Q_{i,j}$ and $E_{i,j}$ for the state redistribution with QSI as follows:*

$$c_{i,j} = 2Q_{i,j},$$
$$e_{i,j} = Q_{i,j} + E_{i,j}.$$

Proof. We first note that $Q_{i,j}$ qubit channels can be perfectly simulated with $2Q_{i,j}$ bit channels and $Q_{i,j}$ ebits by the quantum teleportation [1]. Thus by Lemma 3 Alice and Bob can perform the state merging with QSI by consuming $2Q_{i,j}$ bit channels and $Q_{i,j} + E_{i,j}$ ebits.

Now, we show that the costs of $2Q_{i,j}$ bit channels and $Q_{i,j} + E_{i,j}$ ebits are optimal for the state merging with QSI.

Suppose that the cost of $2Q_{i,j}$ bit channels is not optimal, that is, there exists $c'_{i,j}$ such that $c'_{i,j} < 2Q_{i,j}$ and the state merging with QSI can be performed with $c'_{i,j}$ bit channels. Then as in the proof of the optimality for the classical communication cost in the state merging [6], $c'_{i,j}$ bit channels can be replaced by $c'_{i,j}/2$ qubit channels and $-c'_{i,j}/2$ ebits through the coherent bit channel [14, 15]. Thus, the state redistribution with QSI can be performed with $c'_{i,j}/2$ qubit channels, which contradicts the optimality of the quantum communication cost for the state redistribution with QSI in Lemma 3. Therefore, the optimal classical communication cost is $2Q_{i,j}$.

Finally, suppose that there exists $e'_{i,j}$ such that $e'_{i,j} < Q_{i,j} + E_{i,j}$ and the state merging with QSI can be performed with $e'_{i,j}$ ebits and $2Q_{i,j}$ bit channels. Since $2Q_{i,j}$ bit channels can be replaced by $Q_{i,j}$ qubit channels and $-Q_{i,j}$ ebits, it is possible to perform the state redistribution with QSI with $e'_{i,j} - Q_{i,j}$ ebits. This contradicts the optimality of the entanglement cost for the state redistribution with QSI in Lemma 3. Therefore, the optimal entanglement cost is $Q_{i,j} + E_{i,j}$. $\qquad\square$

By Lemma 3 and Lemma 4, we can obtain that the optimal costs $c_{i,j}$ and $e_{i,j}$ of the state merging with QSI become

$$c_{i,j} = 2H(C_{\mathrm{A}}) - I(C_{\mathrm{A}}; A_1 \cdots A_i) - I(C_{\mathrm{A}}; B_1 \cdots B_j),$$
$$e_{i,j} = H(C_{\mathrm{A}}) - I(C_{\mathrm{A}}; B_1 \cdots B_j) \tag{2}$$

for $0 \leq i \leq m$ and $0 \leq j \leq n$.

III. EFFECTS OF QSI ON OPTIMAL RESOURCE COSTS IN STATE TRANS-FER WITH QSI

In this section, we investigate how the use of (additional) QSI affects the optimal resource costs in the state transfer with QSI. For this, we consider the state transfer with QSI of $\rho \equiv \rho_{A_1 \cdots A_m C_A B_1 \cdots B_n}$ which is shared by Alice and Bob as in Definition 1. In this state transfer with QSI of ρ, O denotes a type of non-local resources. For instance, O can present one of non-local resources Q, E, c, or e. Here, Q and c are qubit channels and bit channels consumed in the state transfer with QSI, respectively. E (e) is ebits consumed/generated in the state redistribution with QSI (in the state merging with QSI). For $0 \leq i \leq m$ and $0 \leq j \leq n$, if Alice and Bob use QSI $A_1 \cdots A_i$ and $B_1 \cdots B_j$ in the state transfer with QSI of ρ then the following definition enables us to quantify the effects of their QSI on an optimal resource cost R in the state transfer with QSI of ρ.

Definition 5. Let $\mathbf{E}[O]_{i,j} = O_{0,0} - O_{i,j}$. Then $\mathbf{E}[O]_{i,j}$ is called the *effect on the optimal resource cost of type O with respect to QSI $A_1 \cdots A_i$ and $B_1 \cdots B_j$ in the state transfer with QSI of ρ.*

The effect $\mathbf{E}[O]_{i,j}$ in Definition 5 appropriately measures the effect of QSI $A_1 \cdots A_i$ and $B_1 \cdots B_j$ in the state transfer with QSI of ρ, since the only difference between the optimal resource costs $R_{0,0}$ and $R_{i,j}$ is the use of QSI $A_1 \cdots A_i$ and $B_1 \cdots B_j$.

From the formulas for the optimal costs in Eqs. (1) and (2), the effect $\mathbf{E}[O]_{i,j}$ on the optimal resource cost of type O is readily calculated. Specifically, for the state redistribution with QSI of ρ, the effects $\mathbf{E}[Q]_{i,j}$ and $\mathbf{E}[E]_{i,j}$ on the optimal quantum communication cost and the optimal entanglement cost are given by

$$
\begin{aligned}
\mathbf{E}[Q]_{i,j} &= \frac{1}{2}I(C_A; A_1 \cdots A_i) + \frac{1}{2}I(C_A; B_1 \cdots B_j), \\
\mathbf{E}[E]_{i,j} &= -\frac{1}{2}I(C_A; A_1 \cdots A_i) + \frac{1}{2}I(C_A; B_1 \cdots B_j).
\end{aligned}
\tag{3}
$$

For the state merging with QSI of ρ, the effects $\mathbf{E}[c]_{i,j}$ and $\mathbf{E}[e]_{i,j}$ on the optimal classical communication cost and the optimal entanglement cost are

$$
\begin{aligned}
\mathbf{E}[c]_{i,j} &= I(C_A; A_1 \cdots A_i) + I(C_A; B_1 \cdots B_j), \\
\mathbf{E}[e]_{i,j} &= I(C_A; B_1 \cdots B_j).
\end{aligned}
\tag{4}
$$

From Eqs. (3) and (4), it is observed that the effects of QSI $A_1 \cdots A_i$ and $B_1 \cdots B_j$ can be decomposed according to Alice's QSI $A_1 \cdots A_i$ and Bob's QSI $B_1 \cdots B_j$. This means that the use of Alice's QSI $A_1 \cdots A_i$ and the use of Bob's QSI $B_1 \cdots B_j$ independently affect the optimal resource costs in the state transfer with QSI of ρ. The second observation is that all effects of QSI stem from the correlation between the part C_A and QSI $A_1 \cdots A_i$ (or $B_1 \cdots B_j$).

From these observations, it follows that the effect $\mathbf{E}[O]_{i,j}$ on the optimal resource cost of type O can be decomposed as

$$\mathbf{E}[O]_{i,j} = \mathbf{A}[O]_i + \mathbf{B}[O]_j,$$

where $\mathbf{A}[O]_i = \mathbf{E}[O]_{i,0}$ and $\mathbf{B}[O]_j = \mathbf{E}[O]_{0,j}$. Here, $\mathbf{A}[O]_i$ ($\mathbf{B}[O]_j$) indicates the effect of Alice's QSI $A_1 \cdots A_i$ (Bob's QSI $B_1 \cdots B_j$) on the optimal resource cost of type O for the state transfer with QSI of ρ. This leads us to the following theorem which provides answers about the first question.

Theorem 6. *In the state transfer with QSI of ρ, the effects of Alice's QSI $A_1 \cdots A_i$ are simply expressed as $\boldsymbol{A}[e]_i = 0$ and*

$$\boldsymbol{A}[c]_i = 2\boldsymbol{A}[Q]_i = -2\boldsymbol{A}[E]_i = I(C_A; A_1 \cdots A_i).$$

For the case of Bob's QSI $B_1 \cdots B_j$, the effects are

$$\boldsymbol{B}[c]_j = \boldsymbol{B}[e]_j = 2\boldsymbol{B}[Q]_j = 2\boldsymbol{B}[E]_j = I(C_A; B_1 \cdots B_j).$$

It is worth mentioning that since the QMI is always non-negative, the use of Bob's QSI $B_1 \cdots B_j$ can reduce all optimal resource costs of the state transfer with QSI compared to the case that Bob uses no QSI. On the other hand, the effects of Alice's QSI are somewhat different. If Alice uses her QSI $A_1 \cdots A_i$, then the optimal quantum/classical communication costs can be reduced, since the effects $\mathbf{A}[Q]_i$ and $\mathbf{A}[c]_i$ are non-negative. However, from the fact that $\mathbf{A}[e]_i = 0$ and $\mathbf{A}[E]_i$ is non-positive, the optimal entanglement cost for the state merging with QSI is unchanged and that for the state redistribution with QSI can increase. This means that even if Alice's QSI is sufficiently large, the use of the QSI cannot reduce the optimal entanglement cost of the state transfer with QSI, and can even increase that of the state merging with QSI.

In order to answer the second question about additional QSI, we need to consider the state transfer with QSI of ρ which is shared by Alice and Bob as before. Let $0 \leq i_1 \leq i_2 \leq m$ and $0 \leq j_1 \leq j_2 \leq n$. In this state transfer with QSI of ρ, Alice and Bob first use QSI $A_1 \cdots A_{i_1}$ and $B_1 \cdots B_{j_1}$. Then they use more QSI $A_1 \cdots A_{i_2}$ and $B_1 \cdots B_{j_2}$, so that QSI $A_{i_1+1} \cdots A_{i_2}$ and $B_{j_1+1} \cdots B_{j_2}$ are additionally used in this situation.

We define the effects of the use of the additional QSI $A_{i_1+1} \cdots A_{i_2}$ and $B_{j_1+1} \cdots B_{j_2}$ on the optimal resource cost of type O in the state transfer with QSI of ρ as follows.

Definition 7. Let $\mathbf{E}[O]_{i_1,j_1}^{i_2,j_2}$ be defined as

$$\mathbf{E}[O]_{i_1,j_1}^{i_2,j_2} = \mathbf{E}[O]_{i_2,j_2} - \mathbf{E}[O]_{i_1,j_1},$$

where $\mathbf{E}[O]_{i,j}$ is the effect of QSI $A_1 \cdots A_i$ and $B_1 \cdots B_j$ on the optimal resource cost of type O in the state transfer with QSI of ρ as in Definition 5. Then we call $\mathbf{E}[O]_{i_1,j_1}^{i_2,j_2}$ the *additional effect on the optimal resource cost of type O with respect to QSI $A_{i_1+1} \cdots A_{i_2}$ and $B_{j_1+1} \cdots B_{j_2}$ in the state transfer with QSI of ρ.*

Since Alice's QSI and Bob's QSI independently affect the optimal resource costs as shown in Theorem 6, the additional effect $\mathbf{E}[O]_{i_1,j_1}^{i_2,j_2}$ on the optimal resource cost of type O can be written in the form

$$\mathbf{E}[O]_{i_1,j_1}^{i_2,j_2} = \mathbf{A}[O]_{i_1}^{i_2} + \mathbf{B}[O]_{j_1}^{j_2}$$

where $\mathbf{A}[O]_{i_1}^{i_2} = \mathbf{E}[O]_{i_1,0}^{i_2,0}$ and $\mathbf{B}[O]_{j_1}^{j_2} = \mathbf{E}[O]_{0,j_1}^{0,j_2}$. In the above equation, $\mathbf{A}[O]_{i_1}^{i_2}$ ($\mathbf{B}[O]_{j_1}^{j_2}$) means the additional effect of Alice's QSI $A_{i_1+1} \cdots A_{i_2}$ (Bob's QSI $B_{j_1+1} \cdots B_{j_2}$). This together with Theorem 6 gives us the following theorem which explains the effects of the more QSI $A_{i_1+1} \cdots A_{i_2}$ and $B_{j_1+1} \cdots B_{j_2}$ in the state transfer with QSI of ρ.

Theorem 8. *In the state transfer with QSI of ρ, the additional effects of Alice's QSI $A_{i_1+1} \cdots A_{i_2}$ are given by $\boldsymbol{A}[e]_{i_1}^{i_2} = 0$ and*

$$\boldsymbol{A}[c]_{i_1}^{i_2} = 2\boldsymbol{A}[Q]_{i_1}^{i_2} = -2\boldsymbol{A}[E]_{i_1}^{i_2}$$
$$= I(C_A; A_{i_1+1} \cdots A_{i_2} | A_1 \cdots A_{i_1}).$$

For Bob's QSI $B_{j_1+1} \cdots B_{j_2}$, the additional effects are

$$\boldsymbol{B}[c]_{j_1}^{j_2} = \boldsymbol{B}[e]_{j_1}^{j_2} = 2\boldsymbol{B}[Q]_{j_1}^{j_2} = 2\boldsymbol{B}[E]_{j_1}^{j_2}$$
$$= I(C_A; B_{j_1+1} \cdots B_{j_2} | B_1 \cdots B_{j_1}).$$

9

Remark that in Theorem 8 only the additional effect $\mathbf{A}[E]_{i_1}^{i_2}$ is non-positive, while the other additional effects are non-negative. Moreover, by comparing Theorem 6 and Theorem 8, it is verified that the effect $\mathbf{A}[O]_{i_1}$ ($\mathbf{B}[O]_{j_1}$) and the additional effect $\mathbf{A}[O]_{i_1}^{i_2}$ ($\mathbf{B}[O]_{j_1}^{j_2}$) on the optimal resource cost of type O can have the same sign, since the QCMI is always non-negative [3]. This means that the use of more QSI $A_{i_1+1} \cdots A_{i_2}$ and $B_{j_1+1} \cdots B_{j_2}$ can enhance the effects of QSI $A_1 \cdots A_{i_1}$ and $B_1 \cdots B_{j_1}$ in the state transfer with QSI of ρ.

IV. NEW OPERATIONAL MEANINGS OF QMI AND QCMI IN TERMS OF QSI

In this section, we present new operational meanings of the QMI, the QCMI, and the chain rule for the QMI.

From the effects of Alice's QSI provided in Theorem 6, we can obtain the following new operational meanings of the QMI, which have never been considered before.

Corollary 9 (Operational meanings of QMI). *Let ρ_{CS} be a quantum state. Consider the state merging with QSI of ρ_{CS}, in which C is merged from Alice to Bob.*

(i) If Alice has S and uses it as QSI, then $I(C;S)$ can be interpreted as how much the classical communication cost can be reduced compared to the case that Alice uses no QSI.

(ii) If Bob has S and uses it as QSI, then $I(C;S)$ can be interpreted as how much both classical communication and entanglement costs can be reduced compared to the case that Bob uses no QSI.

The additional effects of Alice's more QSI in Theorem 8 provides us new operational meanings of the QCMI, which have never appeared in any previous literature.

Corollary 10 (Operational meanings of QCMI). *Let $\rho_{CS_1 S_2}$ be a quantum state. Consider the state merging with QSI of $\rho_{CS_1 S_2}$, in which C is merged from Alice to Bob.*

(i) If Alice has $S_1 S_2$ and uses it as QSI, then $I(C;S_2|S_1)$ means how much the classical communication cost can be more reduced compared to the case that Alice uses QSI S_1 only.

(ii) If Bob has $S_1 S_2$ and uses it as QSI, then $I(C;S_2|S_1)$ means how much both classical communication and entanglement costs can be more reduced compared to the case that Bob uses QSI S_1 only.

We note that other operational meanings of the QMI and the QCMI have been found in literature [6, 7]. In both meanings, one argument of the QMI and the QCMI is interpreted

10

as the reference system. This means that the operational meanings are explained in terms of the reference system which has nothing to do with the corresponding operational tasks. On the other hand, our operational meanings in Corollary 9 and Corollary 10 are intuitive and natural since they only involve Alice's and Bob's systems without mentioning the reference.

In addition, there is one more difference between our operational meanings and the others. We first note that each of the operational meanings for the quantum conditional entropy [6], the QCMI [7], and the min- and max-entropies [4] is obtained from one concrete operation. However, the state transfer with QSI can describe various operational situations in which more (or less) QSI can be used. From comparing these situations, we can see that the effects of QSI can be naturally derived, and hence the QMI and the QCMI can be operationally interpreted with respect to the effects, even though each of them does not correspond to any concrete operation.

We furthermore remark that if QSI $S_1 S_2$ can be almost produced from QSI S_1 then the optimal cost of the state merging with QSI $S_1 S_2$ is almost the same as one of the state merging with QSI S_1 only. Recently, it has been shown that there is an important relation between the QCMI and the recovery map through the Markov chain condition [16], that is, for any state $\rho = \rho_{CS_1S_2}$, there exists a quantum operation $R_{S_1 \to S_1 S_2}$ such that

$$F(\rho, R_{S_1 \to S_1 S_2}(\rho_{CS_1})) \geq 2^{-\frac{1}{2} I(C; S_2 | S_1)_\rho}. \tag{5}$$

This implies that the converse of our above remark is also true. Thus we can obtain the following corollary.

Corollary 11. *In the state merging with QSI $S_1 S_2$, the amount of the reduced cost by adding QSI S_2 to QSI S_1 is close to zero if and only if the QSI $S_1 S_2$ can be almost recovered from the QSI S_1.*

Moreover, the inequality (5) also implies that if the fidelity of its left-hand side decreases then the QCMI $I(C; S_2 | S_1)$ increases. This means that if QSI $S_1 S_2$ cannot be properly recovered from QSI S_1 then the state merging with QSI $S_1 S_2$ can have the more reduced optimal cost than that of the state merging with QSI S_1.

The chain rule [3] for the QMI is that

$$I(C; S_1 \cdots S_n)$$
$$= I(C; S_1) + I(C; S_2|S_1) + \cdots + I(C; S_n|S_1 \cdots S_{n-1})$$
$$= I(C; S_1 \cdots S_i) + I(C; S_{i+1} \cdots S_n|S_1 \cdots S_i) \tag{6}$$

for $1 \le i \le n$, where the first equality is the original chain rule but it can be simply rewritten by exploiting the rightmost side in Eq. (6). From the concept of the state merging with QSI, we can interpret the chain rule in Eq. (6) as follows. In the state merging with QSI, the cost reduced by using the whole QSI $S_1 \cdots S_n$ is equal to the sum of the cost reduced by using the partial QSI $S_1 \cdots S_i$ and the cost more reduced by using the additional QSI $S_{i+1} \cdots S_n$.

V.   EXAMPLES OF STATE TRANSFER WITH QSI

Our protocol includes many well-known protocols of quantum information theory in the sense that their optimal resource costs directly obtained from Eqs. (1) and (2). We present four protocols which exploit qubit channels and other four protocols using bit channels. Denote Q and E by the optimal quantum communication and entanglement costs.

(i) *Schumacher compression* (SC): In the state redistribution with QSI of $|\psi\rangle_{A_1 \cdots A_m C_A B_1 \cdots B_n R}$, if any QSI does not exist, that is, $m = n = 0$, then the protocol becomes the SC [2] as depicted in Fig. 2 (a). From Eq. (1), we have $Q = H(C_A)$ and $E = 0$, which are the optimal resource costs for SC.

(ii) *Fully quantum Slepian-Wolf* (FQSW): FQSW [9, 10] is described in Fig. 2 (b), which is a special case of our state redistribution with QSI if Alice does not have any QSI but Bob can use his QSI, that is, $m = 0$ and $n = 1$. $Q = H(C_A) - \frac{1}{2}I(C_A; B_1)$ and $E = -\frac{1}{2}I(C_A; B_1)$ computed from Eq. (1) are identical to the optimal costs of FQSW.

(iii) *Fully quantum reverse Shannon* (FQRS): FQRS [9, 10] can be considered as the state redistribution with QSI when Alice has QSI A_1 but Bob does not as in Fig. 2 (c), that is, $m = 1$ and $n = 0$. Using Eq. (1), its optimal costs are given by $Q = H(C_A) - \frac{1}{2}I(C_A; A_1)$ and $E = \frac{1}{2}I(C_A; A_1)$, which are equivalent to the optimal costs of FQRS.

(iv) *State redistribution* (SR): In SR [7, 8], both Alice and Bob have QSI A_1 and B_1 as (d) in Fig. 2, that is, $m = 1$ and $n = 1$. Its optimal resource costs $Q = H(C_A) - \frac{1}{2}I(C_A; A_1) - \frac{1}{2}I(C_A; B_1)$ and $E = \frac{1}{2}I(C_A; A_1) - \frac{1}{2}I(C_A; B_1)$ can be achieved from Eq. (1).

FIG. 2: The initial and final states for protocols with qubit/bit channels which can be classified into (a) SC/QT, (b) FQSW/SM, (c) FQRS/GQT, (d) SR/GSM, according to the use of QSI. C_A is transferred from Alice to Bob, R is the reference, and A_1 and B_1 are Alice's and Bob's QSI, respectively.

As mentioned earlier, we continue to see the protocols with bit channels, which are contained in the state merging with QSI of $|\psi\rangle_{A_1 \cdots A_m C_A B_1 \cdots B_n R}$. Let us now define c and e as the optimal classical communication and entanglement costs, respectively.

(v) *Quantum teleportation* (QT): In the original QT [1], Alice and Bob can teleport only one qubit unknown to them. However, we here assume that they asymptotically teleport an initial state known to themselves. Then its optimal costs can be obtained as $c = 2H(C_A)$ and $e = H(C_A)$ from Eq. (2). This is described in (a) of Fig. 2, as in the case of SC.

(vi) *State merging* (SM): In SM [5, 6], Alice has no QSI but Bob has QSI, as depicted in (b) of Fig. 2. This is equivalent to FQSW except for using different kind of channels. From Eq. (2), its optimal costs $c = 2H(C_A) - I(C_A; B_1)$ and $e = H(C_A) - I(C_A; B_1)$ can be obtained.

(vii) *Generalized quantum teleportation* (GQT) and *Generalized state merging* (GSM): In QT and SM, if Alice has QSI A_1 and exploits it for teleporting and merging C_A, then we call these protocols GQT and QSM, which are seen in (c) and (d) of Fig. 2, respectively. We note that the concepts of the GQT and the GSM have been known in literature [7, 11–13], but the optimal resource costs have not precisely been mentioned. By using Eq. (2), it can be shown that $c = 2H(C_A) - I(C_A; A_1)$ and $e = H(C_A)$ are the optimal costs for GQT, and $c = 2H(C_A) - I(C_A; A_1) - I(C_A; B_1)$ and $e = H(C_A) - I(C_A; B_1)$ for GSM.

So far, we have seen that the state transfer with QSI includes many quantum information protocols to transfer Alice's information to Bob, and our protocol is the most generalized one when taking account of Alice's and Bob's QSI.

VI. CONCLUSION

We have considered the state transfer with QSI as a general quantum communication protocol, and have determined its optimal resource costs when Alice and Bob use their QSI. We also have investigated the effects of (additional) QSI on the optimal resource costs in the state transfer with QSI. Based on this study, we have provided new operational meanings of the QMI and the QCMI, in addition to a new operational interpretation of the chain rule for the QMI, which is naturally understandable with respect to the state transfer with QSI. In addition, we expect that our state transfer with QSI provides further understandings of specific multipartite quantum states, such as the Greenberger-Horne-Zeilinger state [17] and the Werner state [18].

Throughout this paper, we have assumed that the initial states of the protocols are independent and identically distributed (i.i.d.). However, there have been some results [19–23]

which do not take into account the i.i.d. assumption. Since these results have provided theoretical bases for the proofs of some practical applications, such as quantum key distribution with finite resources [24, 25], it can be helpful to devise the one-shot version of our work. For this, recent results about resource costs for the one-shot quantum state redistribution [23, 26] might be useful.

Furthermore, it would be interesting to investigate the optimal resource costs of the state transfer with QSI under various conditions. For instance, we can assume that Alice and Bob can consume non-local noisy resources [14, 27], or they can use a local resource, such as maximally coherent states [28–30], as in the incoherent quantum state merging [30] and the coherence distillation [31].

[1] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).

[2] B. Schumacher, Phys. Rev. A **51**, 2738 (1995).

[3] M. M. Wilde, *Quantum Information Theory* (Cambridge University Press, 2013).

[4] R. Knig, R. Renner, and C. Schaffner, IEEE Trans. Inf. Theory **55**, 4337 (2009).

[5] M. Horodecki, J. Oppenheim, and A. Winter, Nature **436**, 673 (2005).

[6] M. Horodecki, J. Oppenheim, and A. Winter, Commun. Math. Phys. **269**, 107 (2007).

[7] I. Devetak and J. Yard, Phys. Rev. Lett. **100**, 230501 (2008).

[8] J. T. Yard and I. Devetak, IEEE Trans. Inf. Theory **55**, 5339 (2009).

[9] I. Devetak, Phys. Rev. Lett. **97**, 140503 (2006).

[10] A. Abeyesinghe, I. Devetak, P. Hayden, and A. Winter, Proc. R. Soc. A **465**, 2537 (2009).

[11] R. Filip, Phys. Rev. A **69**, 052301 (2004).

[12] J. Lee, M. S. Kim, Y. J. Park, and S. Lee, J. Mod. Opt. **47**, 2151 (2009).

[13] J. Oppenheim, arXiv:0805.1065v1 (2008).

[14] I. Devetak, A. W. Harrow, and A. Winter, Phys. Rev. Lett. **93**, 230504 (2004).

[15] A. Harrow, Phys. Rev. Lett. **92**, 097902 (2004).

[16] O. Fawzi and R. Renner, Commun. Math. Phys. **340**, 575 (2015).

[17] D. M. Greenberger, M. A. Horne, and A. Zeilinger, *Bells Theorem, Quantum Theory, and Conceptions of the Universe* (Kluwer Academics, Dordrecht, The Netherlands, 1989).

[18] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).

[19] M. Berta, arXiv:0912.4495v1 (2008).

[20] M. Berta, M. Christandl, and R. Renner, Commun. Math. Phys. **306**, 579 (2011).

[21] J. M. Renes and R. Renner, IEEE Trans. Inf. Theory **58**, 1985 (2012).

[22] N. Datta and M.-H. Hsieh, IEEE Trans. Inf. Theory **59**, 1929 (2013).

[23] M. Berta, M. Christandl, and D. Touchette, IEEE Trans. Inf. Theory **62**, 1425 (2016).

[24] M. Tomamichel, C. C. W. Lim, N. Gisin, and R. Renner, Nat. Commun. **3**, 634 (2012).

[25] M. Curty, F. Xu, W. Cui, C. C. W. Lim, K. Tamaki, and H.-K. Lo, Nat. Commun. **5**, 3732 (2014).

[26] A. Anshu, R. Jain, and N. A. Warsi, arXiv:1702.02396v3 (2017).

[27] I. Devetak, A. W. Harrow, and A. J. Winter, IEEE Trans. Inf. Theory **54**, 4587 (2008).

[28] T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. **113**, 140401 (2014).

[29] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, Phys. Rev. Lett. **115**, 020403 (2015).

[30] A. Streltsov, E. Chitambar, S. Rana, M. N. Bera, A. Winter, and M. Lewenstein, Phys. Rev. Lett. **116**, 240405 (2016).

[31] A. Winter and D. Yang, Phys. Rev. Lett. **116**, 120404 (2016).