Quantum Information Processing (2018) 17:281
https://doi.org/10.1007/s11128-018-2050-3

Quantum histories and correlations in quantum measurements

Marcelo Losada[1] · Roberto Laura[2]

Received: 9 May 2018 / Accepted: 31 August 2018 / Published online: 8 September 2018

Abstract
In this paper, the measurement process is described as an interaction between two quantum systems: the system to be measured and the measuring instrument. The time evolution is determined only by the Schrödinger equation, i.e., there is no collapse postulate. The description of the measurement process is performed using the quantum histories approach, which allows to express the correlation between the properties of the measured system, before the measurement, and the properties of the pointer variable of the instrument, after the measurement. From the point of view of this approach, we explore the possibility of obtaining information of incompatible observables with the same instrument, using different families of histories.

Keywords Consistent histories · Quantum histories · Quantum measurement · Quantum foundations

1 Introduction

In the standard formulation of quantum mechanics, there are two types of physical processes: ordinary physical processes and measurements processes [1,2]. Some authors [3–8] consider that this distinction is not satisfactory from a theoretical point of view, and it is desirable a quantum theory which describes both types of processes in the same way, without appealing to the collapse postulate. These theoretical difficulties led to consider the measurement process as an ordinary interaction. In the modern view of quantum mechanics, a quantum measurement is described as an interaction between two quantum systems: the system to be measured and the measuring apparatus. This way of describing the measurement process seems to be a necessary step

✉ Marcelo Losada
marcelolosada@yahoo.com

1 Universidad de Buenos Aires and Consejo Nacional de Investigaciones Científicas y Técnicas, Buenos Aires, Argentina

2 Universidad Nacional de Rosario and Instituto de Física Rosario, Santa Fe, Argentina

in order to have an interpretation of quantum mechanics independent of the notion of external observer, suitable for dealing with quantum cosmology problems.

Since the measurement process establishes a relation between a property of the measured system at one time and a property of the measuring instrument at a later time, the quantum histories approach for quantum mechanics [5,7–18] seems to be an appropriate tool for describing this process.

Three aspects of the quantum histories approach are of fundamental importance for describing the quantum measurement process. First, in this approach the measurement process is considered as ordinary physical processes, i.e., as an interaction between a system to be measured and a measuring device. Second, it is abandoned the idea that the physical observables have a definite value only if the quantum state coincides with some eigenvector of the observable. According to this approach, the quantum state does not describe the actual properties of the system, but only their probabilities of actualization. Finally, it extends the standard formalism of quantum mechanics in order to be able to define logical operations between properties at different times.

In the quantum histories approach, the probability measure cannot be defined for all quantum histories. In order to have a well-defined probability, only subsets of histories, which satisfy a consistency condition, can be considered. Different conditions have been proposed in the literature, in this work, we will consider two of them: the weak consistency condition, proposed by Griffiths [9], Gell-Mann and Hartle [5,16] and the global consistency condition, proposed by Laura and Vanni [17]. The subsets of histories which satisfy any consistency condition are called consistent families of histories.

In this paper, we are going to apply this approach to the measurement process, focusing on the possibility to obtain information of incompatible observables, using different families of histories. In Sect. 2, we summarize the basic ideas of the quantum histories approach and we present two possible consistency conditions: the weak consistency condition and the global consistency condition. In Sect. 3, we describe the quantum measurement process from the point of view of the quantum histories approach, and we show the correlation between the properties of the pointer variable and the measured properties of the system. In Sect. 4, we consider the possibility to correlate incompatible observables with the same pointer variable of an instrument. We analyze this situation using two consistency conditions: the weak consistency condition and the global consistency condition. Finally, in Sect. 5 we present some conclusions.

2 The quantum histories formalism

In this section, we review the quantum histories approach to quantum mechanics [5,7,8,10,14,19]. The central idea of this approach is to represent quantum histories of n times with orthogonal projectors defined on the Hilbert space $\breve{\mathcal{H}} = \mathcal{H} \otimes \overset{n}{\cdots} \otimes \mathcal{H}$, given by the tensor product of n Hilbert spaces of the physical system. A history $\breve{F} = P_1 \otimes \cdots \otimes P_n$ represents a sequence of properties $P_1, \ldots, P_n$, at times $t_1, \ldots, t_n$.

In order to define probabilities for quantum histories, it is necessary to define a family of histories. For this purpose, first we have to choose a basis of projectors of

$\mathcal{H}$ at each time t_i $(1 \leq i \leq n)$, i.e., a set of projectors $\mathcal{B}_i = \{P_{k_i}\}_{k_i \in \sigma_i}$ (σ_i is a index set) which sum the identity of $\mathcal{H}$ and which are mutually orthogonal:

$$P_{k_i} P_{k_i'} = \delta_{k_i k_i'} \, P_{k_i}, \quad \sum_{k_i} P_{k_i} = I, \quad k_i, k_i' \in \sigma_i, \quad i = 1, \ldots, n;$$

where I is the identity of the Hilbert space $\mathcal{H}$.

Second, we define the product histories $\breve{F}_{k_1,\ldots,k_n}$, choosing one projector P_{k_i} at each time t_i:

$$\breve{F}_{k_1,\ldots,k_n} = P_{k_1} \otimes \cdots \otimes P_{k_n}, \quad (k_1, \ldots, k_n) \in \breve{\sigma}, \quad \breve{\sigma} = \sigma_1 \times \cdots \times \sigma_n.$$

We also define the histories $\breve{F}_\Lambda$ summing the histories $\breve{F}_{k_1,\ldots,k_n}$ with $(k_1, \ldots, k_n) \in \Lambda \subseteq \breve{\sigma}$, i.e., $\breve{F}_\Lambda = \sum_{(k_1,\ldots,k_n) \in \Lambda} \breve{F}_{k_1,\ldots,k_n}$. These histories represent disjunctions of the histories $\breve{F}_{k_1,\ldots,k_n}$.

Finally, the set $\mathcal{F}$ of all histories $\breve{F}_\Lambda$ is called a *family of histories*,

$$\mathcal{F} = \left\{ \breve{F}_\Lambda \in P(\breve{\mathcal{H}}) \,\middle|\, \breve{F}_\Lambda = \textstyle\sum_{(k_1,\ldots,k_n) \in \Lambda} \breve{F}_{k_1,\ldots,k_n}, \quad \Lambda \subseteq \breve{\sigma} \right\}.$$

Since the family $\mathcal{F}$ is obtained from histories $\breve{F}_{k_1,\ldots,k_n}$, we say that they generate the family $\mathcal{F}$, and we called them *atomic histories*.

Given a quantum system with an initial state ρ_0 at time t_0, the probability of a product history $\breve{F}_{k_1,\ldots,k_n} = P_{k_1} \otimes \cdots \otimes P_{k_n}$ is given by

$$\Pr_{\rho_0}(\breve{F}_{k_1,\ldots,k_n}) = \mathrm{Tr}(P_{k_n}(t_0) \cdots P_{k_1}(t_0) \rho_0 P_{k_1}(t_0) \cdots P_{k_n}(t_0)).$$

where $U(t', t) = e^{-iH(t'-t)/\hbar}$ is the time-evolution operator from time t to t', and $P_i(t_0) = U(t_0, t_i) P_i U(t_i, t_0)$ are the time-translated projectors.

The probability of a general history $\breve{F}_\Lambda$ is defined in the following way:

$$\Pr_{\rho_0}(\breve{F}_\Lambda) = \mathrm{Tr}\{C^\dagger(\breve{F}_\Lambda) \rho_0 C(\breve{F}_\Lambda)\}. \tag{1}$$

We have introduced the chain operator $C(\breve{F}_\Lambda) = \sum_{(k_1,\ldots,k_n) \in \Lambda} C(\breve{F}_{k_1,\ldots,k_n})$, where

$$C(\breve{F}_{k_1,\ldots,k_n}) = P_{k_1}(t_0) P_{k_2}(t_0) \cdots P_{k_n}(t_0), \quad \text{and} \quad P_{k_i}(t_0) = U(t_0, t_i) P_{k_i} U(t_i, t_0).$$

For $n = 2$, i.e., for histories of two times, we will use a simpler notation. The indexes k_1 and k_2 are replaced by the indexes i and j, respectively. Therefore, the product histories take the following form

$$\breve{F}_{i,j} = P_i \otimes P_j, \quad i \in \sigma_1, \ j \in \sigma_2.$$

The chain operator for a history $\breve{F}_{i,j}$ is given by

$$C(\breve{F}_{i,j}) = U(t_0, t_1) P_i U(t_1, t_0) U(t_0, t_2) P_j U(t_2, t_0), \tag{2}$$

and the probability of $\breve{F}_{i,j}$ is

$$\Pr_{\rho_0}(\breve{F}_{i,j}) = \mathrm{Tr}\{C^\dagger(\breve{F}_{i,j})\rho_0 C(\breve{F}_{i,j})\}. \tag{3}$$

In general, the probability definition given in Eq. (1) does not satisfy the axiom of additivity. Therefore, to have a well-defined probability, families of histories must satisfy an additional condition, usually called *consistency condition*. A family of quantum histories satisfying this additional condition is called a *consistent family of histories*. It is worth noticing that different consistent families of histories can be considered for a single physical system.

Several consistency conditions were proposed in the literature in order to have well-defined probabilities of histories. In this paper, we consider two of them.

Weak consistency condition

$$\mathrm{Re}\left\{\mathrm{Tr}\left[C^\dagger(\breve{F}_{k_1,\ldots,k_n})\rho_0 C(\breve{F}_{k'_1,\ldots,k'_n})\right]\right\} = 0, \quad \forall\,(k_1,\ldots,k_n) \neq (k'_1,\ldots,k'_n). \tag{4}$$

The weak consistency condition was proposed by Griffiths [9], Gell-Mann and Hartle [5,16]. A family of histories satisfying this condition will be called a *weakly consistent family of histories*.

In some works, it was criticized for not being physically adequate [20] and in others it was criticized for allowing to many families of histories and for not describing satisfactorily quasi-classical domains [16,21,22]. Moreover, an important objection was raised by Kent in [23]. He proved that this condition admits retrodictions of contrary properties. For some proponents of the quantum histories approach, this is not a problem of the theory, because retrodictions are obtained using different families which cannot be considered simultaneously [24,25]. In any case, this fact is considered by some authors as a serious failure of this approach [23,26–28]. In order to overcome some of these difficulties, alternative consistent conditions have been proposed.

Global consistency condition

$$\mathrm{Re}\left\{\mathrm{Tr}\left[C^\dagger(\breve{F}_{k_1,\ldots,k_n})\rho_0 C(\breve{F}_{k'_1,\ldots,k'_n})\right]\right\} = 0, \quad \forall\,(k_1,\ldots,k_n) \neq (k'_1,\ldots,k'_n),\ \forall\,\rho_0. \tag{5}$$

The global consistency condition is equivalent to the weak consistency condition (4) imposed for all the physical states. A family of histories satisfying this condition will be called a *globally consistent family of histories*.

The global consistency condition was first proposed by Laura and Vanni [17,29] in an equivalent form, consisting in the commutation of the time-translated projectors of the histories of the family:

$$[P_{k_i}(t_0), P_{k_j}(t_0)] = 0, \quad \forall i, j = 1, \ldots n, \ \forall\, k_i \in \sigma_i, \ \forall\, k_j \in \sigma_j. \tag{6}$$

The choice of the time t_0 is arbitrary, we can choose any time.

In [29], it was proved that this condition is consistent with having families of histories independent of the initial state of the system, in analogy with the basis of compatible properties of standard quantum mechanics [29]. Moreover, it was shown that the global consistency condition does not allow contrary retrodictions [30–32] and that it can be applied for describing single measurements and sequences of incompatible measurements [33,34]. Finally, the decay process [35] and the double slit experiment [18] were described using this condition.

In the next section, we describe the measurement process from the point of view of the quantum history approach, and we explore the possibility of obtaining information of incompatible observables with the same measurement instrument.

3 The measurement process as a correlation in the quantum histories approach

In this section, we follow the standard description of the measurement process as a quantum interaction [1,36–38] and we show that it is possible to describe this process using the quantum histories approach. The measurement process is considered as an interaction between two quantum systems: the measured system S and the measuring instrument M. Both systems compose the total system $S + M$. The time evolution is determined only by the Schrödinger equation, and no collapse postulate is considered.

The Hilbert spaces of the system S and the measuring instrument M are $\mathcal{H}_S$ and $\mathcal{H}_M$, respectively. The Hilbert space of the total system is the tensor product of the four Hilbert spaces, i.e., $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$.

The measuring instrument measures an observable of the system S, represented by the operator Q, with eigenvalues q_i. In order to measure the observable Q, the measurement instrument has a macroscopically distinguishable observable represented by an operator A with eigenvalues a_j, which is usually called the pointer variable. Any vector of $\mathcal{H}$ can be written as a linear combination of the orthonormal vectors $|q_i\rangle \otimes |a_j\rangle$, with $|q_i\rangle \in \mathcal{H}_S$ and $|a_j\rangle \in \mathcal{H}_M$.

The interaction of the measured system with the measurement instrument is represented by a unitary transformation satisfying

$$U(t_2, t_1)|q_i\rangle \otimes |a_0\rangle = |\phi_i\rangle \otimes |a_i\rangle, \tag{7}$$

where $|\phi_i\rangle$ is an arbitrary vector of the Hilbert space $\mathcal{H}_S$ and $|a_0\rangle$ is the reference state of the pointer variable [38]. This expression shows the relation between the value a_j of the pointer variable at time t_2 and the value q_j of the measured observable Q at a previous time t_1.

To avoid a heavier notation, we do not consider the degrees of freedom of the measurement instrument M different from the pointer variable, nor the variables of the system S which are not involved in the measurement. However, these non-relevant variables can be easily added into the description [33].

The measurement process is an interaction between quantum systems, which correlates the properties of the measured system S at a time t_1 with the observed properties of the instrument M at a time t_2. In order to describe the measurement process from the point of view of the quantum histories approach, first we have to choose an appropriate family of histories which includes the properties of the system S at a time t_1 and the properties of the instrument M at a time t_2.

In the Hilbert space of the composite system $\mathcal{H}$, the properties associated with the values a_j of the pointer variable are represented by the projectors

$$A_j = \sum_i |q_i\rangle\langle q_i| \otimes |a_j\rangle\langle a_j| = I_S \otimes |a_j\rangle\langle a_j|,$$

where I_S is the identity of $\mathcal{H}_S$. It should be noted that $\sum_j A_j = I$, with I the identity of the whole Hilbert space $\mathcal{H}$. In particular, the property associated with the reference value a_0 of the pointer variable is represented by the projector

$$A_0 = I_S \otimes |a_0\rangle\langle a_0|.$$

We also consider the negation of property A_0, given by $\bar{A}_0 = I - A_0$.

The properties associated with the values q_i of observable Q are represented by the projectors

$$Q_i = \sum_j |q_i\rangle\langle q_i| \otimes |a_j\rangle\langle a_j| = |q_i\rangle\langle q_i| \otimes I_M,$$

where I_M is the identity of $\mathcal{H}_M$. It would be useful to consider the conjunction between projectors Q_i and the projector A_0,

$$Q_i A_0 = |q_i\rangle\langle q_i| \otimes |a_0\rangle\langle a_0|,$$

which represents the following property of the composite system: the observable Q of the system S has the value q_i and the pointer variable of the measuring instrument M is in reference value a_0.

Now we are going to choose a family of histories of two times. At time t_1, we consider the properties associated with the conjunction between projectors Q_i and the projector A_0, i.e., the projectors $Q_i A_0$. Also, we consider the projector $\bar{A}_0$ to complete the basis of projectors. Projectors $Q_i A_0$ and $\bar{A}_0$ are orthogonal to each other and they sum the identity,

$$Q_i A_0 \bar{A}_0 = 0, \quad Q_i A_0 Q_{i'} A_0 = \delta_{ii'}, \quad \sum_i Q_i A_0 + \bar{A}_0 = I, \tag{8}$$

therefore, $\mathcal{B}_1 = \{Q_i A_0, \bar{A}_0\}$ is a basis of projectors for time t_1.

At time t_2, we consider the properties represented by projectors A_j. They are orthogonal to each other, and they sum the identity,

$$A_j A_{j'} = \delta_{jj'}, \quad \sum_j A_j = I. \tag{9}$$

Hence, $\mathcal{B}_2 = \{A_j\}$ is a basis of projectors for time t_2.

Finally, we define the family $\mathcal{F}$, generated by the following atomic histories

$$\breve{F}_{i,j} = Q_i A_0 \otimes A_j, \quad \breve{F}_{\bar{0},j} = \bar{A}_0 \otimes A_j. \tag{10}$$

In order to check that $\mathcal{F}$ is a globally consistent family, we have to prove that the projectors of the basis for time t_1, given in Eq. (8) and the projectors of the basis for time t_2, given in Eq. (9), commute when they are translated to a common time, see Eq. (6).

If we choose to translate all the projectors to the time t_2, the conditions given in (6) take the form

$$\left[U(t_2,t_1)Q_iA_0U(t_1,t_2),A_j\right]=0,\quad \left[U(t_2,t_1)\bar{A}_0U(t_1,t_2),A_j\right]=0.$$

The projectors Q_iA_0 and $\bar{A}_0$ translated from time t_1 to time t_2, according to Eq. (7), are given by

$$\begin{aligned}U(t_2,t_1)Q_iA_0U(t_1,t_2)&=|\phi_i\rangle\langle\phi_i|\otimes|a_i\rangle\langle a_i|,\\ U(t_2,t_1)\bar{A}_0U(t_1,t_2)&=I-\sum_i|\phi_i\rangle\langle\phi_i|\otimes|a_i\rangle\langle a_i|.\end{aligned}$$

Therefore, the commutation relations are satisfied, and $\mathcal{F}$ satisfies the global consistency condition. Since this condition implies the weak consistency condition for all states [29], $\mathcal{F}$ is also a weakly consistent family of histories.

Now, we are going to use the family of histories $\mathcal{F}$ to describe the correlation between the observed properties of the instrument M at time t_2 and the properties of the measured system S at the time t_1. In order to do that, we consider the histories represented by $Q_iA_0\otimes I$ and $A_0\otimes A_j$. It is easy to see that they belong to family $\mathcal{F}$, because they are obtained adding atomic histories of $\mathcal{F}$, i.e.,

$$\begin{aligned}Q_iA_0\otimes I&=\sum_j\breve{F}_{i,j}\in\mathcal{F},\\ A_0\otimes A_j&=\sum_i\breve{F}_{i,j}\in\mathcal{F}.\end{aligned}$$

The projectors $Q_iA_0\otimes I$ and $A_0\otimes A_j$ represent the following histories:

$$\begin{aligned}Q_iA_0\otimes I &\longrightarrow \text{At time } t_1,\text{ the observable } Q \text{ of the system } S \text{ has the value } q_i \text{ and}\\ &\qquad\ \text{the pointer variable of } M \text{ is in reference value } a_0.\\ A_0\otimes A_j &\longrightarrow \text{At time } t_1,\text{ the pointer variable of } M \text{ is in reference value } a_0 \text{ and,}\\ &\qquad\ \text{at time } t_2,\text{ the pointer variable of } M \text{ measures the value } a_j.\end{aligned}$$

The correlation between the values of the measured observable Q, at time t_1, and the values of the pointer variable, at time t_2, can be obtained by computing the conditional probability of histories $Q_iA_0\otimes I$, given histories $A_0\otimes A_j$. If $\Pr_\rho(A_0\otimes A_j)>0$, the conditional probability is given by:

$$\Pr_{\rho_0}(Q_i A_0 \otimes I | A_0 \otimes A_j) = \frac{\Pr_{\rho_0}\left[(Q_i A_0 \otimes I)(A_0 \otimes A_j)\right]}{\Pr_{\rho_0}(A_0 \otimes A_j)} = \frac{\Pr_{\rho_0}(Q_i A_0 \otimes A_j)}{\Pr_{\rho_0}(A_0 \otimes A_j)}, \tag{11}$$

with ρ_0 an arbitrary initial state of the composed system. In "Appendix A", we prove that the numerator is given by

$$\Pr_{\rho_0}(Q_i A_0 \otimes A_j) = \delta_{i,j} \Pr_{\rho_0}(Q_i A_0 \otimes A_i), \tag{12}$$

Then, the denominator is given by

$$\begin{aligned} \Pr_{\rho_0}(A_0 \otimes A_j) &= \sum_i \Pr_{\rho_0}(Q_i A_0 \otimes A_j) = \sum_i \delta_{i,j} \Pr_{\rho_0}(Q_i A_0 \otimes A_i) \\ &= \Pr_{\rho_0}(Q_j A_0 \otimes A_j). \end{aligned} \tag{13}$$

Therefore, from Eqs. (11), (12) and (13), we obtain

$$\Pr_{\rho_0}(Q_i A_0 \otimes I | A_0 \otimes A_j) = \delta_{i,j}. \tag{14}$$

Equation (14) explicitly shows the correlation provided by the measurement process: if the pointer variable is in its reference position a_0 at time t_1 and it is registered the value a_j at time t_2, then the measured observable Q had the value q_j at time $t_1 < t_2$. Therefore, the conditional probability (14) describes the correlation between the values of the pointer variable and the values of the measured observable.

It should be stressed that this result was obtained using the global consistency condition. A similar result was obtained by Griffiths using the weak consistency condition [39,40]. Since the global condition is stronger than the weak condition, our result is also valid for the weak consistency condition.

Finally, it is worth noting that this result is valid for arbitrary initial states of the composed system, even for states that before the measurement involve linear combinations of vectors with different values of the variable to be measured. These initial states are a problem for the standard interpretation of the quantum theory, but they are not problematic for obtaining the correlation given by the conditional probability equation. This is the way in which the quantum histories approach avoids the definite outcome problem of quantum measurements.

4 Different correlations in quantum measurements

The previous result was obtained considering the basis of projectors $\mathcal{B}_1 = \{Q_i A_0, \bar{A}_0\}$ at time t_1 and the basis of projectors $\mathcal{B}_2 = \{A_j\}$ at time t_2. However, different basis can be considered at each time. This freedom is responsible for what has been called the preferred basis problem. In ordinary quantum mechanics, this problem refers to a single time, but in the quantum histories approach, different basis can be chosen at each time.

Different solutions has been proposed to this problem. Some authors consider the possibility of providing quantum mechanics with an additional postulate for a preferred basis, as it is the case for the modal interpretations [41–44]. Another solution for the problem is given by Zurek [45], arguing that decoherence selects the basis of the pointer of the measurement instrument. Moreover, there are some authors who consider that the freedom of choice of the basis of properties is an essential characteristic of quantum mechanics that should be accepted [8,39] and that this choice should be made according to the utility for describing a given physical process. These different opinions coexist and the discussion on the subject is open.

In Sect. 3, the choice of the basis for time t_2 was made taking into account the utility for the description of the measurement process. The pointer variable should be macroscopic and accessible to the observer [38]. As a defined value of the pointer variable is obtained at the end of the measurement process, the basis for time t_2 should include properties corresponding to well-defined values of the pointer variable. However, a question can still be made for the possibility to choose a different basis for time t_1, which includes properties of the measured system which do not correspond with the observable Q.

In what follows, we are going to analyze if it is possible to consider families of histories with the basis $\mathcal{B}_2$ for time t_2, but with a basis $\mathcal{B}'_1$ for time t_1, different from the basis $\mathcal{B}_1$ already defined. In other words, we want to study the possibility of providing information about incompatible observables with the same pointer variable of the instrument.

4.1 Weak consistency condition

We are going to define an alternative basis $\mathcal{B}'_1$ for time t_1, different from $\mathcal{B}_1 = \{Q_i A_0, \bar{A}_0\}$. For simplicity, we consider that the dimension of Hilbert space $\mathcal{H}_S$ is finite, and we call it s. Using the vectors $|q_i\rangle$, we define other basis of vectors $|p_i\rangle$, given by

$$|p_i\rangle = \sum_{l=1}^{s} \alpha_{il} |q_l\rangle \quad \text{with} \quad \langle p_i | p_j \rangle = \delta_{ij} \tag{15}$$

and we also define the projectors $P_i = |p_i\rangle\langle p_i| \otimes I_M$.

In order to define the basis $\mathcal{B}'_1$, we use the product of projectors P_i and A_0, and we complete the basis with the projector $\bar{A}_0 = I - A_0$, i.e.,

$$\mathcal{B}'_1 = \left\{ P_i A_0,\ \bar{A}_0 \right\}$$

is a basis of projectors.

Using basis $\mathcal{B}'_1$ for time t_1 and $\mathcal{B}_2$ for time t_2, we define the family of histories $\mathcal{G}$, generated by the following atomic histories

$$\breve{G}_{i,j} = P_i A_0 \otimes A_j, \quad \breve{G}_{\bar{0},j} = \bar{A}_0 \otimes A_j.$$

We consider at time t_1 an initial state

$$\rho = \frac{I_S}{s} \otimes |a_0\rangle\langle a_0| = \frac{1}{s} A_0.$$

In "Appendix B", we prove that the family $\mathcal{G}$ is weakly consistent for the initial state ρ.

Now, we are going to use the family of histories $\mathcal{G}$ to describe the correlation between the observed properties of the instrument M at time t_2 and the properties of the system S represented by projectors P_i at the time t_1. In order to do that, we consider the histories represented by $P_i A_0 \otimes I$ and $A_0 \otimes A_j$. It is easy to see that they belong to family $\mathcal{G}$,

$$P_i A_0 \otimes I = \sum_j \breve{G}_{i,j} \in \mathcal{G},$$

$$A_0 \otimes A_j = \sum_i \breve{G}_{i,j} \in \mathcal{G}.$$

The two projectors $P_i A_0 \otimes I$ and $A_0 \otimes A_j$ represent the following histories:

$$\begin{aligned} P_i A_0 \otimes I &\longrightarrow \text{At time } t_1,\ \text{the system } S \text{ has the property represented by } P_i \text{ and} \\ &\qquad \text{the pointer variable of } M \text{ is in reference value } a_0. \\ A_0 \otimes A_j &\longrightarrow \text{At time } t_1,\ \text{the pointer variable of } M \text{ is in reference value } a_0 \text{ and,} \\ &\qquad \text{at time } t_2,\ \text{the pointer variable of } M \text{ measures the value } a_j. \end{aligned}$$

Using family $\mathcal{G}$, we can calculate the conditional probability of history $P_i A_0 \otimes I$, given $A_0 \otimes A_j$. In "Appendix C", we prove that this probability is given by

$$\Pr(P_i A_0 \otimes I | A_0 \otimes A_j) = |\alpha_{ij}|^2. \tag{16}$$

Therefore, the weakly consistent family $\mathcal{G}$ allows to describe correlations between properties A_j of the pointer variable, given in basis $\mathcal{B}_2$, and properties P_i of the measured system, given in basis $\mathcal{B}'_1$.

On the other hand, in Sect. 3, we have considered the family of histories $\mathcal{F}$, generated by the basis $\mathcal{B}_1$ at time t_1 and $\mathcal{B}_2$ at time t_2. We have proved that $\mathcal{F}$ satisfies the global consistency condition, i.e., the weakly consistency condition for all states. Within this family, in Eq. (14), we calculated the correlations between properties A_j of the pointer variable, and properties Q_i of the measured system, given in $\mathcal{B}_1$.

Both results evidence that the quantum histories approach, with the weak consistency condition, allows that the measuring instrument provides information about different incompatible properties, Q_i and P_i, of the measured system. Since the properties Q_i and P_i belong to different basis, the information obtained from the instrument depends on the family of histories which is chosen. The empirical property A_j revealed by the measuring instrument is not enough to determine the properties of the system at a previous time, it is also necessary to choose a family of histories.

As it was said previously, in the quantum histories approach the choice of the basis for time t_2 is based on the utility for the description of the measurement process. As a defined value of the pointer variable is obtained at the end of the measurement process, the basis for time t_2 should include properties corresponding to well-defined values of the pointer variable. However, for time t_1, the previous criterion does not determine a unique basis. While basis $\mathcal{B}_2$ is chosen in order to describe the different pointer values that are obtained after the measurement, there is freedom to choose the basis for time t_1. In the next section, we will show that this is not the case if we use the global consistency condition.

4.2 Global consistency condition

We are going to show that it is not possible to consider a globally consistent family of histories $\mathcal{G}$, with basis $\mathcal{B}_1' = \{P_i A_0,\ \bar{A}_0\}$ for time t_1 and $\mathcal{B}_2 = \{A_j\}$ for time t_2.

If $\mathcal{G}$ is a globally consistent family of histories, then the corresponding projectors from $\mathcal{B}_1'$ and $\mathcal{B}_2$ must commute when translated to a common time. In particular,

$$\left[U(t_2, t_1) P_i A_0 U(t_1, t_2), A_j\right] = 0,$$

or equivalently,

$$U(t_2, t_1) P_i A_0 U(t_1, t_2) A_j = A_j U(t_2, t_1) P_i A_0 U(t_1, t_2).$$

If we multiply both members by $A_{j'}$, with $j' \neq j$, we obtain $A_{j'} U(t_2, t_1) P_i A_0 U(t_1, t_2) A_j = 0$. Taken into account Eq. (22), we have

$$\begin{aligned} A_{j'} U(t_2, t_1) P_i A_0 U(t_1, t_2) A_j &= A_{j'} \sum_{l,m}^{s} \alpha_{il} \alpha_{im}^* |\phi_l\rangle\langle\phi_m| \otimes |a_l\rangle\langle a_m| A_j \\ &= \alpha_{ij'} \alpha_{ij}^* |\phi_{j'}\rangle\langle\phi_j| \otimes |a_{j'}\rangle\langle a_j| = 0. \end{aligned}$$

Then, $\alpha_{ij'} \alpha_{ij}^* = 0$ for $j \neq j'$. Therefore, for each index i, there is only one index j_i such that $\alpha_{ij_i} \neq 0$. This implies that $|p_i\rangle = \alpha_{ij_i} |q_{j_i}\rangle$, for all $1 \leq i \leq s$, which is equivalently to say that $\mathcal{B}_1' = \mathcal{B}_1$. Then, it is not possible to consider a globally consistent family $\mathcal{G}$, with basis $\mathcal{B}_2$ at time t_2 and basis $\mathcal{B}_1' \neq \mathcal{B}_1$ at time t_1. Once we have chosen the basis $\mathcal{B}_2$ at time t_2, there is no freedom to choose a basis at time t_1 different from $\mathcal{B}_1$.

This implies that the measuring instrument M cannot provide information about properties of the system S which do not belong to the basis $\mathcal{B}_1$. The empirical properties A_j, revealed by the measuring instrument at time t_2, are enough to determine the properties of the system at a previous time t_1.

Therefore, in the quantum histories approach with the global consistency condition, the choice of the basis $\mathcal{B}_2$ at time t_2 is based on the utility for the description of the measurement process, and the basis $\mathcal{B}_1$ at time t_1 is determined by the global consistency condition itself. This is an important difference with the weak consistency condition, which allows the measuring instrument to provide information about different incompatible properties of the measured system and this information depends on the family of histories which is chosen.

However, it should be noted that the quantum history approach, independently of the consistency condition, does not provide with a criterion for choosing at time t_2 the basis of properties of the pointer variable. This fact was considered as a limitation for developing an interpretation of quantum mechanics independent of the notion of external observer, suitable for dealing with quantum cosmology problems [46–48].

5 Conclusions

In this work, we analyzed the quantum measurement process from the point of view of the quantum histories approach, focusing on the possibility of correlating incompatible observables of a system with the same measurement instrument, using different families of histories.

In Sect. 2, we summarized the main ideas of the quantum histories approach. Three aspects of the this approach are of fundamental importance for describing the quantum measurement process. First, the measurement process is described as an ordinary physical interaction between a measured system and a measuring instrument. Second, the state vector does not describe the actual properties of the system, but determines the probabilities that the possible properties will be updated. Third, it is possible to define logical operations between properties at different times.

In the quantum histories approach, the probability measure cannot be defined for all quantum histories. In order to have a well-defined probability, only consistent families of histories can be considered. We considered two possible consistency conditions: the weak consistency condition and the global consistency condition.

In Sect. 3, we applied the quantum histories approach to the measurement process. We defined a family of histories $\mathcal{F}$, suitable for describing the correlations between the value of the measured observable and the value of the pointer variable of the measuring instrument. We proved that this family satisfies both consistency conditions, and we obtained the expected correlations showing that the conditional probability is equal to one.

Finally, in Sect. 4, we analyzed the possibility of obtaining information about incompatible observables with the same pointer variable of the measurement instrument. In the quantum histories approach, the choice of the basis for time t_2 is based on the utility for the description of the measurement process. Since at the end of the measurement a definite value of the pointer variable is obtained, the basis for time t_2 should be $\mathcal{B}_2$, which includes properties corresponding to well-defined values of the pointer variable. With respect to the basis for time t_1, we considered two cases: weakly consistent families and globally consistent families. In the first case, we found that the previous criterion does not determine a unique basis, there is freedom to choose the basis for time t_1. This implies that the measuring instrument provides information about different incompatible properties of the measured system, depending on the chosen basis. In the second case, we found that basis $\mathcal{B}_1$ is the only possible basis for time t_1, compatible with basis $\mathcal{B}_2$ at time t_2. This implies that the same pointer variable of the instrument cannot provide information about properties of the system which do not belong to basis $\mathcal{B}_1$, i.e., the pointer variable of the instrument can only measure compatible observables of the system.

From a conceptual point of view, the fact that the same pointer variable of an instrument could provide information about incompatible observables of a system is not satisfactory. Therefore, in regard to this matter, the global consistency condition seems to be in a better position than weak consistency condition.

Acknowledgements This research was founded by the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), the Universidad de Buenos Aires (UBA), and the Universidad Nacional de Rosario (UNR).

A Deduction of Eq. (12)

We are going to show that

$$\mathrm{Pr}_{\rho_0}(Q_i A_0 \otimes A_j) = \delta_{ij} \mathrm{Pr}_{\rho_0}(Q_i A_0 \otimes A_i). \tag{17}$$

First, we remember that

$$U(t_2, t_1)|q_i, a_0\rangle = |\phi_i\rangle \otimes |a_i\rangle, \tag{18}$$

$$Q_i A_0 = \sum_{s,r} |q_i, a_0\rangle\langle q_i, a_0|. \tag{19}$$

Then, taken into account Eqs. (18) and (19), we obtain

$$U(t_2, t_1) Q_i A_0 U(t_1, t_2) = |\phi_i\rangle\langle\phi_i| \otimes |a_i\rangle\langle a_i|. \tag{20}$$

In order to prove Eq. (17), we calculate the chain operator of the history $Q_i A_0 \otimes A_j$, see Eq. (2),

$$\begin{aligned}
C(Q_i A_0 \otimes A_j) &= U(t_0, t_1) Q_i A_0 U(t_1, t_0) U(t_0, t_2) A_j U(t_2, t_0) \\
&= U(t_0, t_1) U(t_1, t_2) U(t_2, t_1) Q_i A_0 U(t_1, t_2) A_j U(t_2, t_0) \\
&= U(t_0, t_2) |\phi_i\rangle\langle\phi_i| \otimes |a_i\rangle\langle a_i| A_j U(t_2, t_0) \\
&= \delta_{ij} C(Q_i A_0 \otimes A_i),
\end{aligned}$$

where we have used $U(t_1, t_2)U(t_2, t_1) = I$ and Eq. (20). Finally, taking into account Eq. (3), we compute the probability of the history,

$$\begin{aligned}
\mathrm{Pr}_{\rho_0}(Q_i A_0 \otimes A_j) &= \mathrm{Tr}\left[C^\dagger(Q_i A_0 \otimes A_j)\rho_0 C(Q_i A_0 \otimes A_j)\right] \\
&= \delta_{ij} \mathrm{Tr}\left[C^\dagger(Q_i A_0 \otimes A_i)\rho_0 C(Q_i A_0 \otimes A_i)\right] \\
&= \delta_{ij} \mathrm{Pr}_{\rho_0}(Q_i A_0 \otimes A_i).
\end{aligned}$$

B Weakly consistency of family $\mathcal{G}$

We are going to show that the family $\mathcal{G}$, generated by the atomic histories

$$\breve{G}_{i,j} = P_i A_0 \otimes A_j, \quad \breve{G}_{\bar{0},j} = \bar{A}_0 \otimes A_j,$$

is a weakly consistent family, for an initial state $\rho = \frac{1}{s} A_0$ at time t_1.

First, we consider histories $\breve{G}_{i,j}$ and $\breve{G}_{i',j'}$, with $i \neq i'$ or $j \neq j'$, and we prove that they satisfy the weak consistency condition given in Eq. (4):

$$\begin{aligned}
\mathrm{Tr}\left[C^\dagger(\breve{G}_{i,j})\rho C(\breve{G}_{i',j'})\right] &= \mathrm{Tr}\left[A_j U(t_2,t_1) P_i A_0 \frac{1}{s} A_0 P_{i'} A_0 U(t_1,t_2) A_{j'}\right] \\
&= \frac{\delta_{ii'}\delta_{jj'}}{s} \mathrm{Tr}\left[A_j U(t_2,t_1) P_i A_0 U(t_1,t_2) A_j\right] = 0, \\
&\quad \forall i \neq i' \text{ or } \forall j \neq j'.
\end{aligned}$$

Second, we show that histories $\breve{G}_{i,j}$ and $\breve{G}_{\bar{0},j'}$ satisfy the weak consistency condition:

$$\mathrm{Tr}\left[C^\dagger(\breve{G}_{i,j})\rho C(\breve{G}_{\bar{0},j'})\right] = \frac{1}{s}\mathrm{Tr}\left[A_j U(t_2,t_1) P_i A_0 A_0 \bar{A}_0 U(t_1,t_2) A_{j'}\right] = 0, \quad \forall i, j, j'.$$

where we have used that $A_0 \bar{A}_0 = 0$. The same argument is valid for histories $\breve{G}_{\bar{0},j}$ and $\breve{G}_{\bar{0},j'}$. Therefore, family $\mathcal{G}$ is a weakly consistent family of histories.

C Deduction of Eq. (16)

The conditional probability of history $P_i A_0 \otimes A_j$, given $A_0 \otimes A_j$, is

$$\Pr(P_i A_0 \otimes I | A_0 \otimes A_j) = \frac{\Pr\left[(P_i A_0 \otimes I)(A_0 \otimes A_j)\right]}{\Pr(A_0 \otimes A_j)} = \frac{\Pr(P_i A_0 \otimes A_j)}{\Pr(A_0 \otimes A_j)}. \quad (21)$$

Using Eq. (3), we obtain the numerator of expression (21)

$$\begin{aligned}
\Pr(P_i A_0 \otimes A_j) &= \mathrm{Tr}\left[A_j U(t_2,t_1) P_i A_0 \frac{1}{s} A_0 P_i A_0 U(t_1,t_2) A_j\right] \\
&= \frac{1}{s}\mathrm{Tr}\left[A_j U(t_2,t_1) P_i A_0 U(t_1,t_2) A_j\right].
\end{aligned}$$

Taking into account the definition of projectors P_i and A_0, and Eq. (15), we obtain

$$\begin{aligned}
P_i A_0 &= |p_i\rangle\langle p_i| \otimes |a_0\rangle\langle a_0| \\
&= \sum_{l,m}^{s} \alpha_{il}\alpha_{im}^{*} |q_l\rangle\langle q_m| \otimes |a_0\rangle\langle a_0|.
\end{aligned}$$

Then, using the time evolution given in Eq. (7), we obtain

$$U(t_2, t_1) P_i A_0 U(t_1, t_2) = \sum_{l,m}^{s} \alpha_{il}\alpha^{*}_{im} |\phi_l\rangle\langle\phi_m| \otimes |a_l\rangle\langle a_m|. \tag{22}$$

Therefore,

$$\begin{aligned}\Pr(P_i A_0 \otimes A_j) &= \frac{1}{s}\sum_{l,m}^{s} \alpha_{il}\alpha^{*}_{im} \mathrm{Tr}\left[A_j|\phi_l\rangle\langle\phi_m| \otimes |a_l\rangle\langle a_m|A_j\right] \\ &= \frac{1}{s}\sum_{l,m}^{s} \alpha_{il}\alpha^{*}_{im}\delta_{jm}\delta_{jl} = \frac{|\alpha_{ij}|^2}{s}.\end{aligned} \tag{23}$$

The denominator of expression (21) is given by

$$\Pr(A_0 \otimes A_j) = \sum_{i=1}^{s} \Pr(P_i A_0 \otimes A_j) = \sum_{i=1}^{s} \frac{|\alpha_{ij}|^2}{s} = \frac{1}{s}. \tag{24}$$

Finally, from Eqs. (21), (23) and (24), we obtain

$$\Pr(P_i A_0 \otimes I | A_0 \otimes A_j) = |\alpha_{ij}|^2.$$

References

1. von Neumann, J.: Mathematische Grundlagen der Quantenmechanik. Springer, Berlin (1932)
2. Dirac, P.A.M.: The Principles of Quantum Mechanics. Oxford University Press, Oxford (1930)
3. Dieks, D.: Quantum mechanics without the projection postulate and its realistic interpretation. Found. Phys. **19**, 1397–1423 (1989)
4. Ballentine, L.E.: Limitations of the projection postulate. Found. Phys. **20**, 1329–1343 (1990)
5. Gell-Mann, M., Hartle, J.B.: Quantum mechanics in the light of quantum cosmology. In: Zurek, W. (ed.) Complexity, Entropy and the Physics of Information. Addison-Wesley, Reading (1990)
6. Goldstein, S.: Quantum theory without observers—part one. Phys. Today **51**, 42–46 (1998)
7. Omnés, R.: Understanding Quantum Mechanics. Princeton University Press, Princeton (1999)
8. Griffiths, R.B.: A consistent quantum ontology. Stud. Hist. Philos. Mod. Phys. **44**, 93–114 (2013)
9. Griffiths, R.B.: Consistent histories and the interpretation of quantum mechanics. J. Stat. Phys. **36**, 219–272 (1984)
10. Griffiths, R.B.: Consistent Quantum Theory. Cambridge University Press, Cambridge (2002)
11. Griffiths, R.B.: The new quantum logic. Found. Phys. **44**, 610–640 (2014)
12. Omnès, R.: Interpretation of quantum mechanics. Phys. Lett. A **125**, 169–172 (1987)
13. Omnès, R.: Logical reformulation of quantum mechanics. I. Foundations. J. Stat. Phys. **53**, 893–932 (1988)
14. Omnès, R.: The Interpretation of Quantum Mechanics. Princeton University Press, Princeton (1994)
15. Hartle, J.B.: The quantum mechanics of cosmology. In: Hartle, S.J., Piran, T. (eds.) Quantum Cosmology and Baby Universes. World Scientific, Coleman, Singapore (1991)
16. Gell-Mann, M., Hartle, J.B.: Classical equations for quantum systems. Phys. Rev. D **47**, 3345 (1993)
17. Laura, R., Vanni, L.: Time translation of quantum properties. Found. Phys. **39**, 160–173 (2009)
18. Losada, M., Vanni, L., Laura, R.: Probabilities for time-dependent properties in classical and quantum mechanics. Phys. Rev. A **87**, 052128 (2013)

19. Isham, C.J.: Quantum logic and the histories approach to quantum theory. J. Math. Phys. **35**, 2157 (1994)
20. Diosi, L.: Anomalies of weakened decoherence criteria for quantum histories. Phys. Rev. Lett. **92**, 170401 (2004)
21. Laloë, F.: Do we really understand quantum mechanics? Strange correlations, paradoxes, and theorems. Am. J. Phys. **69**, 655 (2001)
22. Dowker, F., Kent, A.: On the consistent histories approach to quantum mechanics. J. Stat. Phys. **82**, 1575–1646 (1996)
23. Kent, A.: Consistent sets yield contrary inferences in quantum theory. Phys. Rev. Lett. **78**, 2874–2877 (1997)
24. Griffiths, R.B., Hartle, J.B.: Comment on consistent sets yield contrary inferences in quantum theory. Phys. Rev. Lett. **81**, 1981 (1998)
25. Hartle, J.B.: Quantum physics and human language. J. Phys. A **40**, 3101–3121 (2007)
26. Kent, A.: Consistent sets and contrary inferences in quantum theory: reply to Griffiths and Hartle. Phys. Rev. Lett. **81**, 1982 (1998)
27. Okon, E., Sudarsky, D.: On the consistency of the consistent histories approach to quantum mechanics. Found. Phys. **44**, 19–33 (2014)
28. Bassi, A., Ghirardi, G.: Decoherent histories and realism. J. Stat. Phys. **98**, 457–494 (2000)
29. Losada, M., Laura, R.: Generalized contexts and consistent histories in quantum mechanics. Ann. Phys. **344**, 263–274 (2014)
30. Losada, M., Laura, R.: Quantum histories without contrary inferences. Ann. Phys. **351**, 418–425 (2014)
31. Losada, M.: Contrary quantum histories and contrary inferences. Physica A **503**, 379–389 (2018)
32. Losada, M., Lombardi, O.: Histories in quantum mechanics: distinguishing between formalism and interpretation. Eur. J. Philos. Sci. **8**, 367–394 (2018)
33. Vanni, L., Laura, R.: The logic of quantum measurements. Int. J. Theor. Phys. **52**, 2386–2394 (2013)
34. Losada, M., Vanni, L., Laura, R.: The measurement process in the generalized contexts formalism for quantum histories. Int. J. Theor. Phys. **55**, 817–824 (2016)
35. Losada, M., Laura, R.: The formalism of generalized contexts and decay processes. Int. J. Theor. Phys. **52**, 1289–1299 (2013)
36. Schlosshauer, M.: Decoherence, the measurement problem, and interpretations of quantum mechanics. Rev. Mod. Phys. **76**, 1267 (2005)
37. Schlosshauer, M.: Decoherence and the Quantum-to-Classical Transition (The Frontiers Collection). Springer, New York (2007)
38. Ballentine, L.E.: Quantum Mechanics. World Scientific, Singapur (1998). A modern Development
39. Griffiths, R.B.: Consistent quantum measurements. Stud. Hist. Philos. Mod. Phys. **52**, 188–197 (2015)
40. Griffiths, R.B.: What quantum measurements measure. Phys. Rev. A **96**, 032110 (2017)
41. Bacciagaluppi, G., Hemmo, M.: Modal interpretations, decoherence and measurements. Stud. Hist. Philos. Mod. Phys. **27**, 239–277 (1996)
42. Dieks, D., Vermaas, P.E.: The Modal Interpretation of Quantum Mechanics. Kluwer Academic Publishers, Dordrecht (1998)
43. Bene, G., Dieks, D.: A perspectival version of the modal interpretation of quantum mechanics and the origin of macroscopic behavior. Found. Phys. **32**, 645–671 (2002)
44. Lombardi, O., Castagnino, M.: A modal-Hamiltonian interpretation of quantum mechanics. Stud. Hist. Philos. Mod. Phys. **39**, 380–443 (2008)
45. Zurek, W.H.: Pointer basis of quantum apparatus: Into what mixture does the wave packet collapse? Phys. Rev. D **24**, 1516–1525 (1981)
46. Okon, E., Sudarsky, D.: Measurements according to consistent histories. Stud. Hist. Philos. Mod. Phys. **48**, 7–12 (2014)
47. Okon, E., Sudarsky, D.: The consistent histories formalism and the measurement problem. Stud. Hist. Philos. Mod. Phys. **52**, 217–222 (2015)
48. Castagnino, M., Fortin, S., Laura, R., Sudarsky, D.: Interpretations of quantum theory in the light of modern cosmology. Found. Phys. **47**, 1387–1422 (2017)