Unextendible Maximally Entangled Bases in $\mathbb{C}^{pd}\otimes\mathbb{C}^{qd}$

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Abstract

The construction of unextendible maximally entangled bases is tightly related to quantum information processing like local state discrimination. We put forward two constructions of UMEBs in $\mathbb{C}^{pd}\otimes\mathbb{C}^{qd}(p\leq q)$ based on the constructions of UMEBs in $\mathbb{C}^{d}\otimes\mathbb{C}^{d}$ and in $\mathbb{C}^{p}\otimes\mathbb{C}^{q}$, which generalizes the results in [Phys. Rev. A. 94, 052302 (2016)] by two approaches. Two different 48-member UMEBs in $\mathbb{C}^{6}\otimes\mathbb{C}^{9}$ have been constructed in detail.

1 Introduction

It is well known that the quantum states are divided into two parts: separable states and entanglement states. Quantum entanglement, as a potential resource, is widely applied into many quantum information process, such as quantum computation [1], quantum teleportation [2], quantum cryptography [3] as well as nonlocality [4]. Nonlocality is a very useful concept in quantum mechanics [4, 5, 6] and is tightly related to entanglement. However, it is proved that the unextendible product bases (UPBs) reveal some nolocality without entanglement [7, 8]. The UPB is a set of incomplete orthogonal product states in bipartite quantum system $\mathbb{C}^{d}\otimes\mathbb{C}^{d'}$ consisting of fewer than dd' vectors which have no additional product states orthogonal to each element of the set [9].

In 2009, S. Bravyi and J. A. Smolin [10] first proposed the notion of unextendible maximally entangled basis(UMEB): a set of incomplete orthogonal maximally entangled states in $\mathbb{C}^{d}\otimes\mathbb{C}^{d'}$ consisting of fewer than dd' vectors which have no additional maximally entangled vectors that are orthogonal to all of them. These incomplete bases have some special properties. In bipartite space $\mathbb{C}^{d}\otimes\mathbb{C}^{d}$, one can get a state on the UMEB's complementary subspace, whose entanglement of assistance (EoA) is strictly smaller than $\log d$, the asymptotic EoA [10]. As for in $\mathbb{C}^{d}\otimes\mathbb{C}^{d'}$, one can also get a state on the complementary subspace of UMEB, corresponding to a quantum channel, which would not be unital. Besides, it cannot be convex mixtures of unitary operators too [11]. In addition, for a given mixed state, its Schmidt number is hard to calculate. If we can get a n-member UMEB $\{|\phi_i\rangle\}$ in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, the Schmidt number of the following state

$$\rho^{\perp} = \frac{1}{dd' - n}(I - \sum_{i=1}^{n} |\phi_i\rangle\langle\phi_i|),$$

is smaller than d [12]. Therefore, different UMEBs can be used to construct different mixed entangled states with limited Schmidt number, even state with different Schmidt number.

In [11], B. Chen and S. M. Fei provided a way to construct UMEBs in some special cases of bipartite system. Then H. Nan et al. [13], M. S. Li et al. [14], Y. L. Wang et al. [15, 16], Y. Guo [17], G. J. Zhang et al. [18] developed some new results of UMEB in bipartite system. Later, Y.J. Zhang et al. [19] and Y. Guo et al. [20] generalized the notion of UMEB from bipartite systems to multipartite quantum systems. In [20], Y. Guo showed that if there exists an $N-$member UMEB $\{|\psi_j\rangle\}$ in $\mathbb{C}^d \otimes \mathbb{C}^d$, then there exists a $qd^2 - q(d^2 - N)$-member UMEB in $\mathbb{C}^{qd} \otimes \mathbb{C}^{qd}$ for any $q \in \mathbb{N}$. Y. Guo et al. [21] also proposed the definition of entangled bases with fixed Schmidt number.

In this paper, we study UMEBs in bipartite system $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd}$ $(p \leq q)$. A systematic way of constructing UMEBs in $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd}$ from that in $\mathbb{C}^d \otimes \mathbb{C}^d$ is presented firstly, and a construction of 48-member UMEB in $\mathbb{C}^6 \otimes \mathbb{C}^9$ is given as an example. Furthermore, a explicit method to construct UMEBs containing $pqd^2 - d(pq - N)$ maximally entangled vectors in $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd}$ from an N-member UMEB in $\mathbb{C}^p \otimes \mathbb{C}^q$ is presented. Moreover, another construction of 48-member UMEB in $\mathbb{C}^6 \otimes \mathbb{C}^9$ is obtained, thus generalized the results in Yu Guo[phys.Rev.A.94,052302(2016)] by two approaches.

2 Preliminaries

Throughout the paper, we denote $[d]' = \{0, 1, \cdots, d-1\}$ and $[d]^* = \{1, 2, \cdots, d\}$.

A pure state $|\psi\rangle$ is said to be a maximally entangled state in $\mathbb{C}^d \otimes \mathbb{C}^{d'} (d \leq d')$ if and only if for a arbitrary given orthonormal basis $\{|i\rangle\}$ of $\mathbb{C}^d$, there exists an orthonormal basis $\{|i'\rangle\}$ of $\mathbb{C}^{d'}$ such that $|\psi\rangle$ can be written as $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \otimes |i'\rangle$[6].

A set of pure states $\{|\phi_i\rangle\}_{i=0}^{n-1} \in C^d \otimes C^{d'}$ with the following conditions is called an unextendible maximally entangled bases(UMEB) [10]:

(i)$|\phi_i\rangle, i \in [n]'$ are all maximally entangled states,

(ii)$\langle\phi_i|\phi_j\rangle = \delta_{ij}, i, j \in [n]'$,

(iii)$n < dd'$, and if a pure state $|\psi\rangle$ meets that $\langle\phi_i|\psi\rangle = 0, i \in [n]'$, then $|\psi\rangle$ can not be maximally entangled.

Let $\mathcal{M}_{d'\times d}$ be the Hilbert space of all $d' \times d$ complex matrices equipped with the inner product defined by $\langle A|B\rangle = Tr(A^\dagger B)$ for any $A, B \in \mathcal{M}_{d'\times d}$. If $\{A_i\}_{i=0}^{dd'-1}$ constitutes a Hilbert-Schmidt basis of $\mathcal{M}_{d'\times d}$, where $\langle A_i|A_j\rangle = d\delta_{ij}$, then there is a one-to-one correspondence between an orthogonal basis in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ $\{|\phi_i\rangle\}$ and $\{A_i\}$ as follows [20, 21]:

$$|\phi_i\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \sum_{\ell'=0}^{d'-1} a^{(i)}_{\ell' k} |k\rangle|\ell'\rangle \in \mathbb{C}^d \otimes \mathbb{C}^{d'} \quad \Leftrightarrow \quad A_i = [a^{(i)}_{\ell' k}] \in \mathcal{M}_{d'\times d},$$

$$Sr(|\phi_i\rangle) = rank(A_i), \qquad \langle\phi_i|\phi_j\rangle = \frac{1}{d} Tr(A_i^\dagger A_j), \tag{1}$$

where $Sr(|\phi_i\rangle)$ denotes the Schmidt number of $|\phi_i\rangle$. Obviously, $|\phi_i\rangle$ is a maximally entangled pure state in $C^d \otimes C^{d'}$ iff A_i is a $d' \times d$ singular-value-1 matrix (a matrix whose singular values all equal to 1). Specially, A_i is a unitary matrix when $d = d'$.

For simplicity we adopt the following definitions [17]. We call a Hilbert-Schmidt basis $\Omega = \{A_i\}_{i=0}^{d^2-1}$ in $\mathcal{M}_{d\times d}$ a unitary Hilbert-Schmidt basis (UB) of $\mathcal{M}_{d\times d}$ if A_is are unitary matrices, and a Hilbert-Schmidt basis $\Omega = \{A_i\}_{i=0}^{dd'-1}$ in $\mathcal{M}_{d'\times d}$ a singular-value-1 Hilbert-Schmidt basis (SV1B) of $\mathcal{M}_{d\times d}$ if A_is are singular-value-1 matrices. A set of $d \times d$ unitary matrices $\Omega = \{A_i\}_{i=0}^{n-1}$ $(n < d^2)$ is called an unextendible unitary Hilbert-Schmidt basis (UUB) of $\mathcal{M}_{d\times d}$ if (i) $\mathrm{Tr}(A_i^\dagger A_j)$=$d\delta_{ij}$; (ii) if $\mathrm{Tr}(A_i^\dagger X)$=0, $i \in [n]'$, then X is not unitary.

[Definition] A set of $d \times d'$ $(d < d')$ singular-value-1 matrices $\Omega = \{A_i\}_{i=1}^{dd'}$ $(n < dd')$ is called an unextendible singular-value-1 Hilbert-Schmidt basis (USV1B) of $\mathcal{M}_{d\times d}$ if (a) $\mathrm{Tr}(A_i^\dagger A_j)$=$d\delta_{ij}$; (b) if $\mathrm{Tr}(A_i^\dagger X)$=0, $i \in [n]'$, then X is not a singular-value-1 matrix.

According to the Eq.(1), it is obvious that $\Omega = \{A_i\}_{i=0}^{d^2-1}$ is a UB iff $\{|\phi_i\rangle\}$ is a maximally entangled basis (MEB) of $\mathbb{C}^d \otimes \mathbb{C}^d$ while $\Omega = \{A_i\}_{i=0}^{dd'-1}$ is a SV1B iff $\{|\phi_i\rangle\}$ is a MEB of $\mathbb{C}^d \otimes \mathbb{C}^{d'}$. And $\Omega = \{A_i\}_{i=0}^{n-1}$ $(n < d^2)$ is a UUB iff $\{|\phi_i\rangle\}$ is a UMEB of $\mathbb{C}^d \otimes \mathbb{C}^d$ while $\Omega = \{A_i\}_{i=0}^{n-1}$ $(n < dd')$ is a USV1B iff $\{|\phi_i\rangle\}$ is a UMEB of $\mathbb{C}^d \otimes \mathbb{C}^{d'}$[17].

3 UMEBs in $\mathbb{C}^{pd}\otimes\mathbb{C}^{qd}$ $(p\le q)$ from UMEBs in $\mathbb{C}^{d}\otimes\mathbb{C}^{d}$

Theorem 1. If there is an N-member UMEB $\{|\psi_j\rangle\}$ in $\mathbb{C}^d\otimes\mathbb{C}^d$, then there exists a $pqd^2-p(d^2-N)$-member UMEB in $\mathbb{C}^{pd}\otimes\mathbb{C}^{qd}(p\leq q)$.

Proof. Let $\{W_j=[w^j_{i'i}]\}_{j=0}^{N-1}$ be a UUB of $\mathcal{M}_{d\times d}$ corresponding to $\{|\psi_j\rangle\}$,

$$|\psi_j\rangle=\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}\sum_{i'=0}^{d-1}w^j_{i'i}|i\rangle\otimes|i'\rangle,\quad j\in[N]'.$$

Denote

$$U_{nm}=\sum_{a=0}^{d-1}e^{\frac{2\pi na\sqrt{-1}}{d}}|a\oplus_d m\rangle\langle a|,$$

$$V_{kl}=\sum_{a=0}^{p-1}e^{\frac{2\pi ka\sqrt{-1}}{p}}|a\oplus_q l\rangle\langle a|,$$

where $m,n\in[d]'$; $\ l\in[q]'$; $\ k\in[p]'$; $\ j\in[N]'$, and

$$B^j_{k0}=V_{k0}\otimes W_j,\quad k\in[p]';\quad j\in[N]',$$

$$B^{nm}_{kl}=V_{kl}\otimes U_{nm},\quad k\in[p]';l\in[q-1]^*;m,n\in[d]'.$$

Set $C_1=\{B^j_{k0}\}$ and $C_2=\{B^{nm}_{kl}\}$. Then $C_1\cup C_2$ is exactly a USV1B in $\mathcal{M}_{qd\times pd}$.

Firstly, all B^j_{k0} and B^{nm}_{kl} are $qd\times pd$ singular-value-1 matrices, which satisfy the conditions in the Definition:

(a) $Tr[(B^j_{k0})^\dagger B^{j'}_{k'0}]=pd\delta_{kk'}\delta_{jj'}$, $Tr[(B^{nm}_{kl})^\dagger B^{n'm'}_{k'l'}]=pd\delta_{kk'}\delta_{ll'}\delta_{nn'}\delta_{mm'}$ and $Tr[(B^{nm}_{kl})^\dagger B^j_{k'0}]=0$, where $j,j'\in[N]'$; $k,k'\in[p]'$; $l,l'\in[q-1]^*$; $n,n',m,m'\in[d]'$.

(b) Denote S the matrix space of $(I_p,O_{p\times(q-p)})^t\otimes R$, where t stands for matrix transpose, I_p is the $p\times p$ identity matrix, $O_{p\times(q-p)}$ is the $p\times(q-p)$ zero matrix and $R\in\mathcal{M}_{d\times d}$. Obviously the dimension of $S^\perp$ is $p(q-1)d^2$. Thus C_2 is an SV1B of $S^\perp$ with $p(q-1)d^2$ elements.

Assume that D is a singular-value-1 matrix in $\mathcal{M}_{qd\times pd}$, which is orthogonal to all matrices in $C_1\cup C_2$. Since C_2 is a SV1B of $S^\perp$, then $D\in S$. No loss of generality, set

$$D=\begin{pmatrix}A\\O_1\end{pmatrix},\quad A=diag(A_1,A_2,\cdots,A_p),\quad O_1=O_{(q-p)d\times pd},$$

where $A_h(h\in[p]^*)$ are all $d\times d$ matrices. Note that D is orthogonal to each B^j_{k0} in C_1, i.e.,

$$Tr(D^\dagger B^j_{k0})=0,\quad k=[p]';\quad j\in[N]'.$$

Then

$$Tr\left[\left(\begin{array}{cc} A^{\dagger} & O_1^{\dagger} \end{array}\right)_{pd\times qd}\cdot\left(\begin{array}{c} G \\ O_1 \end{array}\right)_{qd\times pd}\right]=0,$$

where $G=diag(\omega_p^{0k}W_j,\omega_p^{1k}W_j,\cdots,\omega_p^{(p-1)k}W_j)$, i.e.,

$$\omega_p^{0k}Tr(A_1^{\dagger}W_j)+\omega_p^{-1k}Tr(A_2^{\dagger}W_j)+\cdots+\omega_p^{(1-p)k}Tr(A_p^{\dagger}W_j)=0.$$

Hence,

$$HX_j=0,$$

where

$$H=\left(\begin{array}{ccccc} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_p^{p-1} & \omega_p^{p-2} & \dots & \omega_p^{1} \\ 1 & \omega_p^{p-2} & \omega_p^{p-4} & \dots & \omega_p^{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_p^{1} & \omega_p^{2} & \dots & \omega_p^{p-1} \end{array}\right),\quad X_j=\left(\begin{array}{c} Tr(A_1^{\dagger}W_j) \\ Tr(A_2^{\dagger}W_j) \\ Tr(A_3^{\dagger}W_j) \\ \vdots \\ Tr(A_p^{\dagger}W_j) \end{array}\right).$$

Obviously, $X_j=O$ for $j\in[N]'$ since $detH\neq 0$. That is to say, $Tr(A_h^{\dagger}W_0)=Tr(A_h^{\dagger}W_1)=\cdots=Tr(A_h^{\dagger}W_{N-1})=0$, $h\in[p]^*$. As every A_n is orthogonal to each W_j, whereas $\{W_j\}$ is a UUB in $\mathcal{M}_{d\times d}$, none of A_h is unitary. Moreover, all the singular values of A_hs are also the singular values of D. Therefore, D is not a singular-value-1 matrix, which contradicts to the assumption. Thus, $C_1\cup C_2$ is a USV1B in $\mathcal{M}_{qd\times pd}$. $\square$

Example 1. A 48-member UMEB in $\mathbb{C}^6\otimes\mathbb{C}^9$ from a 6-member UMEB in $\mathbb{C}^3\otimes\mathbb{C}^3$.

A 6-member UMEB in $\mathbb{C}^3\otimes\mathbb{C}^3$ from Ref.[10] is as follows:

$$W_j=I-(1-e^{\sqrt{-1}\theta})|\psi_j\rangle\langle\psi_j|, j=[6]',$$

where

$$|\psi_{0,1}\rangle=\frac{1}{\sqrt{1+\phi^2}}(|0\rangle\pm\phi|1\rangle),$$
$$|\psi_{2,3}\rangle=\frac{1}{\sqrt{1+\phi^2}}(|1\rangle\pm\phi|2\rangle),$$
$$|\psi_{4,5}\rangle=\frac{1}{\sqrt{1+\phi^2}}(|2\rangle\pm\phi|0\rangle),$$

with $\phi=(1+\sqrt{5})/2$.

Then, denote

$$U_{nm}=\left(\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)^m\cdot\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \omega_3 & 0 \\ 0 & 0 & \omega_3^2 \end{array}\right)^n,\quad m,n\in[3]',$$

$$V_{kl}=\left(\begin{array}{ccc}0&0&1\\1&0&0\\0&1&0\end{array}\right)^{l}\cdot\left(\begin{array}{cc}1&0\\0&1\\0&0\end{array}\right)\cdot\left(\begin{array}{cc}1&0\\0&-1\end{array}\right)^{k},\quad k\in[2]';\quad l\in[3]',$$

where $\omega_3=e^{\frac{2\pi\sqrt{-1}}{3}}$.

Let

$$B_{01}^{nm}=\left(\begin{array}{cc}0&0\\U_{nm}&0\\0&U_{nm}\end{array}\right),\quad B_{11}^{nm}=\left(\begin{array}{cc}0&0\\U_{nm}&0\\0&-U_{nm}\end{array}\right),$$

$$B_{02}^{nm}=\left(\begin{array}{cc}0&U_{nm}\\0&0\\U_{nm}&0\end{array}\right),\quad B_{12}^{nm}=\left(\begin{array}{cc}0&U_{nm}\\0&0\\-U_{nm}&0\end{array}\right),$$

$$B_{00}^{j}=\left(\begin{array}{cc}W_j&0\\0&W_j\\0&0\end{array}\right),\quad B_{10}^{j}=\left(\begin{array}{cc}W_j&0\\0&-W_j\\0&0\end{array}\right),$$

where $n,m\in[3]'$; $j\in[6]'$. Set $C_1=\{B_{k0}^{j}\}$, $C_2=\{B_{kl}^{nm}\}$, for $k\in[2]'$; $j\in[6]'$; $l\in[2]^*$; $n,m\in[3]'$. According to Theorem 1, we have that $C_1\cup C_2$ is a 48-number UMEB in $\mathbb{C}^6\otimes\mathbb{C}^9$.

Remark 1. Theorem 1 in Ref.[15] is a special case of the above Theorem 1 for $p=q$.

4 UMEBs in $\mathbb{C}^{pd}\otimes\mathbb{C}^{qd}(p\leq q)$ from UMEBs in $\mathbb{C}^p\otimes\mathbb{C}^q$

Next, we will present a general approach to construct UMEBs in $\mathbb{C}^{pd}\otimes\mathbb{C}^{qd}$ from UMEBs in $\mathbb{C}^p\otimes\mathbb{C}^q$.

Theorem 2. If there is an N-member UMEB $\{|\psi_j\rangle\}$ in $\mathbb{C}^p\otimes\mathbb{C}^q$, then there exists a $pqd^2-d(pq-N)$-member UMEB in $\mathbb{C}^{pd}\otimes\mathbb{C}^{qd}(p\leq q)$.

Proof. Let $\{W_j=[w_{i'i}^{j}]\}_{j=0}^{N-1}$ be a USV1B of $\mathcal{M}_{q\times p}$ corresponding to $\{|\psi_j\rangle\}$, then

$$|\psi_j\rangle=\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}\sum_{i'=0}^{d'-1}w_{i'i}^{j}|i\rangle\otimes|i'\rangle,\quad j\in[N]'.$$

Denote

$$U_{nm}=\sum_{a=0}^{d-1}e^{\frac{2\pi na\sqrt{-1}}{d}}|a\oplus_d m\rangle\langle a|,$$

$$V_{kl}=\sum_{a=0}^{p-1}e^{\frac{2\pi ka\sqrt{-1}}{p}}|a\oplus_q l\rangle\langle a|,$$

where $m,n\in[d]';\ \ l\in[q]';\ \ k\in[p]';\ \ j\in[N]'$. Let

$$B_{n0}^{j}=U_{n0}\otimes W_j,\quad n\in[d]';\ \ j\in[N]',$$

$$B_{nm}^{kl}=U_{nm}\otimes V_{kl},\quad m\in[d-1]^*;n\in[d]';l\in[q]';k\in[p]',$$

and

$$C_1=\{B_{n0}^{j}\},\quad C_2=\{B_{nm}^{kl}\}.$$

then, $C_1\cup C_2$ is exactly a USV1B in $\mathcal{M}_{qd\times pd}$.

Firstly, all B_{n0}^{j} and B_{nm}^{kl} are $qd\times pd$ singular-value-1 matrices, satisfying the conditions in the Definition:

(a) $Tr[(B_{n0}^{j})^{\dagger}B_{n'0}^{j'}]=qd\delta_{nn'}\delta_{jj'}$, $Tr[(B_{nm}^{kl})^{\dagger}B_{n'm'}^{k'l'}]=qd\delta_{nn'}\delta_{mm'}\delta_{kk'}\delta_{ll'}$ and $Tr[(B_{nm}^{kl})^{\dagger}B_{n'0}^{j}]=0$, where $j,j'\in[N]';\ k,k'\in[q]';\ l,l'\in[p]';\ n,n'\in[d]';\ m,m'\in[d-1]^*$.

(b) Denote S the matrix space of $I_d\otimes R$, where $R\in\mathcal{M}_{q\times p}$. Obviously, the dimension of $S^{\perp}$ is $pq(d-1)d$.

Setting $C_1=\{B_{n0}^{j}\}$ and $C_2=\{B_{nm}^{kl}\}$, we have that C_2 with $pq(d-1)d$ elements is an SV1B of $S^{\perp}$, and $C_1\cup C_2$ is just a USV1B in $\mathcal{M}_{qd\times pd}$.

Assume that D is a singular-value-1 matrix in $\mathcal{M}_{qd\times pd}$, which is orthogonal to all matrices in $C_1\cup C_2$. Since C_2 is a SV1B of $S^{\perp}$, then $D\in S$. No loss of generality, set

$$D=diag(A_1,A_2,\cdots,A_d)_{qd\times pd},$$

where $A_h(h\in[d]^*)$ are all $q\times p$ matrices. Similar to the proof of Theorem 1, we can prove that none of A_h is singular-value-1 matrices. Moreover, all the singular values of all A_hs are also the singular values of D, namely, D is not a singular-value-1 matrix, which contradicts to the assumption. Thus, $C_1\cup C_2$ is a USV1B in $\mathcal{M}_{qd\times pd}$. $\square$

Example 2. A 48-member UMEB in $\mathbb{C}^6\otimes\mathbb{C}^9$ from a 4-member UMEB in $\mathbb{C}^2\otimes\mathbb{C}^3$.

A 4-member UMEB in $\mathbb{C}^2\otimes\mathbb{C}^3$ is as follows:

$$W_{0,1}=|0'\rangle\langle 0|\pm|1'\rangle\langle 1|,$$

$$W_{2,3}=|0'\rangle\langle 1|\pm|1'\rangle\langle 0|.$$

Denote

$$U_{nm}=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}^{m}\cdot\begin{pmatrix}1&0&0\\0&\omega_3&0\\0&0&\omega_3^2\end{pmatrix}^{n},\quad m,n\in[3]',$$

$$V_{kl}=\left(\begin{array}{ccc}0&0&1\\1&0&0\\0&1&0\end{array}\right)^{l}\cdot\left(\begin{array}{cc}1&0\\0&1\\0&0\end{array}\right)\cdot\left(\begin{array}{cc}1&0\\0&-1\end{array}\right)^{k},\quad k\in[3]';\quad l\in[2]',$$

where $\omega_3=e^{\frac{2\pi\sqrt{-1}}{3}}$. Let

$$B_{01}^{kl}=\left(\begin{array}{ccc}0&0&V_{kl}\\V_{kl}&0&0\\0&V_{kl}&0\end{array}\right),B_{11}^{kl}=\left(\begin{array}{ccc}0&0&\omega_3^2V_{kl}\\V_{kl}&0&0\\0&\omega_3V_{kl}&0\end{array}\right),B_{21}^{kl}=\left(\begin{array}{ccc}0&0&\omega_3V_{kl}\\V_{kl}&0&0\\0&\omega_3^2V_{kl}&0\end{array}\right),$$

$$B_{02}^{kl}=\left(\begin{array}{ccc}0&V_{kl}&0\\0&0&V_{kl}\\V_{kl}&0&0\end{array}\right),B_{12}^{kl}=\left(\begin{array}{ccc}0&\omega_3V_{kl}&0\\0&0&\omega_3^2V_{kl}\\V_{kl}&0&0\end{array}\right),B_{22}^{kl}=\left(\begin{array}{ccc}0&\omega_3^2V_{kl}&0\\0&0&\omega_3V_{kl}\\V_{kl}&0&0\end{array}\right),$$

$$B_{00}^{j}=\left(\begin{array}{ccc}W_j&0&0\\0&W_j&0\\0&0&W_j\end{array}\right),B_{10}^{j}=\left(\begin{array}{ccc}W_j&0&0\\0&\omega_3W_j&0\\0&0&\omega_3^2W_j\end{array}\right),B_{20}^{j}=\left(\begin{array}{ccc}W_j&0&0\\0&\omega_3^2W_j&0\\0&0&\omega_3W_j\end{array}\right),$$

where $k\in[3]'$; $l\in[2]'$; $j\in[4]'$. Set $C_1=\{B_{n0}^{j}\}$, $\quad C_2=\{B_{nm}^{kl}\}$, for $k\in[3]'$; $l\in[2]'$; $j\in[4]'$; $n\in[3]'$; $m\in[2]^*$. Then according to Theorem 2, $C_1\cup C_2$ is a 48-member UMEB in $\mathbb{C}^6\otimes\mathbb{C}^9$.

Remark 2. The Constructions of UMEB in Theorem 1 and Theorem 2 are different, which can be easily seen from the Examples 1 and 2. We can give a state with Schmidt number 4 in the subspace of the UMEB in Example 1. While what we can get in the subspace of the UMEB in Example 2 are the states with Schmidt number no more than 3. In fact, according to Theorem 2, one can construct a UMEB in $\mathbb{C}^4\otimes\mathbb{C}^6$ from the UMEB in $\mathbb{C}^2\otimes\mathbb{C}^3$, while one can not do this way from the Theorem 1. Here Theorem 1 in [15] is a also special case of the above Theorem 2 for $p=q$.

Remark 3. By using Theorem 2 in [18], we can give a $p(q-r)$-member UMEB in $\mathbb{C}^p\otimes\mathbb{C}^q$. According to Theorem 2 in this paper, we can obtain a $pd(qd-r)$-member UMEB in $\mathbb{C}^{pd}\otimes\mathbb{C}^{qd}$, in whose subspace we can get some states with Schmidt number dr. We can also get a $pd(qd-r)$-member UMEB directly by Theorem 2 in [18], nevertheless, in the associated subspace, one can only attain the states with Schmidt number no greater than r. Therefore, they are different constructions. Actually, there are many N-number UMEBs in $\mathbb{C}^p\otimes\mathbb{C}^q$, where $p\nmid N$. In this case, it doesn't hold that $pd|(pqd^2-d(pq-N))$. Namely, we can not even get a UMEB with the same number of members by Theorem 2 in [18].

5 Conclusion

We have provided an explicit way of constructing a $pqd^2 - p(d^2 - N)$-member UMEB in $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd}$ from an N-member UMEB in $\mathbb{C}^d \otimes \mathbb{C}^d$, and constructed a 48-number UMEB in $\mathbb{C}^6 \otimes \mathbb{C}^9$ as a detailed example. We have also established a method to construct a $pqd^2 - d(pq - N)$-member UMEB in $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd}$ from an N-member UMEB in $\mathbb{C}^p \otimes \mathbb{C}^q$, and presented another 48-member UMEB in $\mathbb{C}^6 \otimes \mathbb{C}^9$. These results may highlight the further investigations on the construction of unextendible bases and the theory of quantum entanglement.

References

[1] A. Barenco, A. K. Ekert. Dense Coding Based on Quantum Entanglement[J]. Journal of Modern Optics, 1995, 42(6): 1253-1259

[2] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki. Quantum entanglement[J]. Reviews of Modern Physics, 2009, 81: 865-942

[3] W. Tittel, J. Brendel, H. Zbinden, N. Gisin. Quantum cryptography using entangled photons in energy-time bell states[J]. Physical Review Letters, 2000, 84(20): 4737

[4] S. B. Zheng. Quantum nonlocality for a three-particle nonmaximally entangled state without inequalities[J]. Physical Review A, 2002, 66(1): 90-95

[5] W. J. Guo, D. H. Fan, L. F. Wei. Experimentally testing Bells theorem based on Hardys nonlocal ladder proofs[J]. Science China Physics, Mechanics Astronomy. 2015, 58: 024201.

[6] H. X. Meng, H. X. Cao, W. H. Wang, L. Chen, Y. J. Fan. Continuity of the robustness of contextuality and the contextuality cost of empirical models[J]. Science China Physics, Mechanics & Astronomy. 2016, 59(4): 640303

[7] D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, B. M. Terhal. Unextendible product bases, uncompletable product bases and bound entanglement[J]. Commun. Math. Phys. 2013, 238, 379

[8] S. D. Rinaldis. Distinguishability of complete and unextendible product bases[J]. Phys. Rev. A. 2004, 70: 022309

[9] C. H. Bennett et al. Unextendible product bases and bound entanglement[J]. Phys. Rev. Lett. 1999, 82: 5385

[10] S. Bravyi, J. A. Smolin. Unextendible maximally entangled bases[J]. Physical Review A, 2011, 84: 042306

[11] B. Chen, S. M. Fei. Unextendible maximally entangled bases and mutually unbiased bases[J]. Physical Review A, 2013, 88: 034301

[12] Y. Guo, S. J. Wu. Unextendible entangled bases with fixed Schmidt number[J]. Physical Review A, 2014, 48(24): 245301

[13] H. Nan, Y. H. Tao, L. S. Li, J. Zhang. Unextendible Maximally Entangled Bases and Mutually Unbiased Bases in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$[J]. International Journal of Theoretical Physics, 2015, 54: 927

[14] M. S. Li, Y. L. Wang, S. M. Fei, Z. J. Zheng. Unextendible maximally entangled bases in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$[J]. Physical Review A, 2014, 89: 062313

[15] Y.L. Wang, M. S. Li, S.M. Fei. Unextendible maximally entangled bases in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$[J]. Physical Review A, 2014, 90: 034301

[16] Y. L. Wang, M. S. Li, S. M. Fei. Connecting the UMEB in $\mathbb{C}^d \otimes \mathbb{C}^d$ with partial Hadamard matrices[J]. Quantum Information Processing, 2017, 16(3): 84

[17] Y. Guo. Constructing the unextendible maximally entangled basis from the maximally entangled basis[J]. Physical Review A, 2016, 94: 052302

[18] G. J. Zhang, Y. H. Tao, Y. F. Han, X. L. Yong, S. M. Fei. Constructions of Unextendible Maximally Entangled Bases in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$[J]. Scientific Reports, 2018, 8(1): 319

[19] Y. J. Zhang, H. Zhao, N. Jing, S. M. Fei. Multipartite unextendible entangled basis[J]. International Journal of Theoretical Physics, 2017, 56(11): 3425-3430

[20] Y. Guo, Y. P. Jia, X. L. Li. Multipartite unextendible entangled basis[J]. Quantum Inf Process. 2015, 14: 3553

[21] Y. Guo, S. P. Du, X. L. Li, S. J. Wu. Entangled bases with fixed Schmidt number[J]. Journal of Physics A: Mathematical and Theoretical, 2015, 48: 245301

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