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The unextendible product bases of four qubits: Hasse diagrams

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We consider the unextendible product bases (UPBs) of fixed cardinality m in quantum systems of n qubits. These UPBs are divided into finitely many equivalence classes with respect to an equivalence relation introduced by N. Johnston. There is a natural partial order " \leq " on the set of these equivalence classes for fixed m, and we use this partial order to study the topological closure of an equivalence class of UPBs. In the case of four qubits, for m = 8, 9, 10, we construct explicitly the Hasse diagram of this partial order.

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I. INTRODUCTION

Multiqubit systems of quantum physics have been physically realized in recent years. Investigating the properties of multiqubit states is both physically and mathematically meaningful. In particular, the multiqubit positive-partial-transpose (PPT) entangled states can be constructed using multiqubit unextendible product bases (UPBs) [4]. UPBs indicate the quantum nonlocality in the discrimination of product states by local operations and classical communication (LOCC). Further, the three-qubit PPT entangled state obtained from a UPB is a biseparable entangled state, i.e., it is separable w.r.t. any bipartition of the three systems [3]. It exhibits a fundamental difference between the bipartite and multipartite quantum systems.

We consider the UPBs of fixed cardinality m in quantum systems of n qubits. These UPBs are divided into finitely many equivalence classes with respect to an equivalence relation introduced by N. Johnston [11]. There is a natural partial order " \leq " on the set of these equivalence classes for fixed m, and we use this partial order to study the topological closure of an equivalence class of UPBs. A UPB is called *proper* if it does not span the whole Hilbert space of the system.

Let \mathcal{U} and \mathcal{U}' be two *m*-state UPBs of an *n*-qubit system. We raise the question of deciding whether \mathcal{U}' lies in the closure of the equivalence class \mathcal{E} which contains \mathcal{U} . Roughly speaking this means whether \mathcal{U}' can be approximated (to an arbitrary small precision) by UPBs belonging to \mathcal{E} . We solve this problem by using the above mentioned partial order, see Eq. (6) and Proposition 10.

In the case of four qubits, it is known that the cardinality m of an UPB is one of the numbers 6, 7, 8, 9, 10, 12, 16. The construction of the proper UPBs (those with m < 16) has been completed in 2014 by N. Johnston. He also classified these UPBs up to equivalence [11, 12]. There are in total 1446 equivalence classes of proper UPBs of four qubits. For $m \leq 10$ we describe the corresponding partial order " \leq " explicitly by constructing all arrows of its Hasse diagram. Due to a large number of equivalence classes, only a partial result is obtained for m = 12. In all cases except m = 16 we have determined the number of connected components of the Hasse diagram as well as the maximal and minimal equivalence classes.

In our recent paper [19] we have introduced a novel method to study the *n*-qubit UPBs. In the next section we recall this method, give the definitions and cite some facts that we need. In particular, we define there the unextendible orthogonal matrices (UOMs). Every $m \times n$ UOM, say X, generates an infinite family of m-state n-qubit UPBs which we denote by $\mathcal{F}_X^{\#}$. In Sec. III we define the partial order " \leq ", introduce its Hasse diagram, recall the definition of decomposable orthogonal matrices and state some of their properties. In Sec. IV we investigate the closure of equivalence classes of UPBs. In Sec. V we describe the four-qubit Hasse diagrams of m-state UOMs for m = 8, 9, 10, 12, and identify the maximal and minimal equivalence classes of UOMs. In the Appendix A we list the representatives of the equivalence classes of UOMs of four qubits that we use. The arrows of the Hasse diagrams are listed in Appendix B.

II. PRELIMINARIES

Let $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ be the Hilbert space representing a quantum system A_1, \ldots, A_n consisting of n qubits. Each \mathcal{H}_j is a 2-dimensional Hilbert space. We fix an orthonormal basis $|0\rangle_j, |1\rangle_j$ of \mathcal{H}_j . Usually, the subscript j will be suppressed. We say that a vector $|v\rangle \in \mathcal{H}$ is a unit vector if ||v|| = 1. For any nonzero vector $|v_j\rangle \in \mathcal{H}_j$ we denote by $[v_j]$ the 1-dimensional subspace of \mathcal{H}_j spanned by this vector. As a rule, we shall not distinguish two unit vectors which differ only in the phase. By using this convention, we can say that for any unit vector $|v_j\rangle \in \mathcal{H}_j$ there exists a unique unit vector $|v_j\rangle^{\perp} \in \mathcal{H}_j$ which is perpendicular to $|v_j\rangle$.

A product vector is a nonzero vector $|x\rangle = |x_1\rangle \otimes \cdots \otimes |x_n\rangle$, which will be written also as $|x\rangle = |x_1, \ldots, x_n\rangle$. If ||x|| = 1 we shall assume (as we may) that each $||x_j|| = 1$. Two product vectors $|x\rangle = |x_1, \ldots, x_n\rangle$ and $|y\rangle = |y_1, \ldots, y_n\rangle$ are orthogonal if and only if $|y_j\rangle = |x_j\rangle^{\perp}$ for at least one index j. An orthogonal product set (OPS) is a set of pairwise orthogonal unit product vectors in \mathcal{H} . The cardinality of an OPS cannot exceed 2^n , the dimension of \mathcal{H} . We say that an OPS is an orthogonal product basis (OPB), if its cardinality is 2^n . As an example, the 2^n product vectors $|x_s\rangle = |s_1, \ldots, s_n\rangle$, where $s := (s_1, \ldots, s_n)$ runs through all binary $\{0, 1\}$ -sequences of length n, is an OPB. We refer to this OPB as the standard OPB. However, there are many other OPBs. A set of unit product vectors is called an unextendible product basis (UPB) if these vectors are orthogonal to each other and there is no product vector orthogonal to all of them [4, 5]. Note that we allow the possibility that UPB spans the whole Hilbert space \mathcal{H} .

Our method uses formal matrices whose entries are vector variables represented by letters of an infinite countable alphabet **X**. Each $x \in \mathbf{X}$ has its companion $x' \in \mathbf{X}$ such that $x \neq x'$ and (x')' = x. We say that x' is the *perpendicular* of x. We say that a subset of **X** is *independent* if it does not contain any pair of the form $\{x, x'\}$. We say that two vector variables x and y are *independent* if $x \notin \{y, y'\}$.

By $\mathcal{M}(m,n)$ we denote the set of $m \times n$ matrices $X = [x_{ij}], x_{ij} \in \mathbf{X}$, such that if x occurs in some column of X then neither x nor x' occurs in any other column of X.

Let $x = [x_1 \ x_2 \ \cdots \ x_n]$ and $y = [y_1 \ y_2 \ \cdots \ y_n]$ be two row vectors with $x_j, y_j \in \mathbf{X}$. We say that x and y are *orthogonal* to each other, $x \perp y$, if $y_j = x'_j$ for at least one index j.

We say that a matrix $X \in \mathcal{M}(m, n)$ is orthogonal if any two of its rows are orthogonal to each other. For two matrices X and Y with the same number of columns we say that they are orthogonal to each other $(X \perp Y)$ if each row of X is orthogonal to each row of Y.

We denote by $\mathcal{O}(m,n)$ the subset of $\mathcal{M}(m,n)$ consisting of all orthogonal matrices. We also set $\mathcal{O}(n) := \mathcal{O}(2^n,n)$ for the special case $m = 2^n$.

The objects that we are interested in are the unextendible orthogonal matrices (UOM). We say that a matrix $X \in \mathcal{O}(m,n)$ is unextendible if there is no row which is orthogonal to X. The reader can easily verify that the following four small matrices are UOMs:

$$\begin{bmatrix} a \\ a' \end{bmatrix}, \begin{bmatrix} a & b \\ a & b' \\ a' & c \\ a' & c' \end{bmatrix}, \begin{bmatrix} a & c & e \\ a' & d' & f \\ b & c' & f' \\ b' & d & e' \end{bmatrix}, \begin{bmatrix} x & y & z & w \\ x' & b & d & e \\ a & y' & d' & f \\ a' & c & z' & e' \\ a & b' & d & w' \\ x' & c' & d' & f' \end{bmatrix}.$$
(1)

We say that two matrices $X, Y \in \mathcal{M}(m, n)$ are *equivalent* if X can be transformed to Y by permuting the rows, permuting the columns, and by renaming the vector variables. The renaming must respect the orthogonality, i.e., we require that if a vector variable x is renamed to y then x' has to be renamed to y'. For $X \in \mathcal{M}(m, n)$ we shall denote by [X] its equivalence class. Note that if $X \in \mathcal{O}(m, n)$ then $[X] \subseteq \mathcal{O}(m, n)$. Since **X** is infinite, there are infinitely many matrices in $\mathcal{M}(m, n)$. On the other hand, there are only finitely many equivalence classes in $\mathcal{M}(m, n)$.

many matrices in $\mathcal{M}(m, n)$. On the other hand, there are only finitely many equivalence classes in $\mathcal{M}(m, n)$. For example, for the UOM $X = \begin{bmatrix} a \\ a' \end{bmatrix} \in \mathcal{M}(2, 1)$ the equivalence class [X] consists of all matrices $\begin{bmatrix} x \\ x' \end{bmatrix}, x \in \mathbf{X}$.

The equivalence of two multiqubit UPBs has been defined in [11, p. 4]. Let us recall that definition. Two multiqubit UPBs are *equivalent* if they have the same orthogonality graphs up to permuting the qubits and relabeling of vertices. For the definition of orthogonality graphs see [5, Definition 4] or [11, p. 3].

There is a natural one-to-one correspondence between the equivalence classes of UOMs and the equivalence classes of UPBs. To explain this correspondence, we need the concept of evaluations.

Given $X \in \mathcal{O}(m, n)$, we define an *evaluation* α of X to be a mapping which, for each $j = 1, 2, \ldots, n$, replaces each vector variable x in column j by a unit vector $\alpha(x)$ in the 2-dimensional Hilbert space \mathcal{H}_j of the jth qubit. It is mandatory that $\alpha(x') = \alpha(x)^{\perp}$ whenever x' also occurs in X. As a result of applying an evaluation α on X, we obtain an $m \times n$ matrix whose entries are unit vectors in the corresponding Hilbert spaces \mathcal{H}_j . We denote this matrix by $\alpha(X)$. After that we can form m product vectors by simply taking the tensor product of the unit vectors in a row of

 $\alpha(X)$. In this way we obtain an orthogonal set, \mathcal{U} , of product vectors (OPS) in \mathcal{H} . We refer to this OPS as the OPS of $\alpha(X)$.

In general, even when X is a UOM, \mathcal{U} is not necessarily a UPB. To ensure that \mathcal{U} is a UPB, we have to require that if two independent vector variables, say x and y, occur in the same column of X, then $\alpha(y) \neq \alpha(x)$ and $\alpha(y) \neq \alpha(x)^{\perp}$. For an arbitrary $X \in \mathcal{O}(m, n)$, we say that an evaluation α of X satisfying this additional condition is generic. It is proved in [19, Lemma 2] that if X is a UOM and α is a generic evaluation of X then the OPS of $\alpha(X)$ is in fact a UPB.

We denote by \mathcal{F}_X the set of all OPS of $\alpha(X)$ where α ranges over all evaluations of X, and by $\mathcal{F}_X^{\#}$ we denote the set of all OPS of $\alpha(X)$ where α ranges only over all generic evaluations of X. If X is a UOM then it is easy to see that any two UPBs in $\mathcal{F}_X^{\#}$ are equivalent to each other according to the definition of equivalence given above.

The equivalence class of UPBs which corresponds to the equivalence class [X] of a UOM X is the equivalence class which contains the set $\mathcal{F}_X^{\#}$. Explicitly, this equivalence class is the union

$$\bigcup_{\sigma \in S_n} \mathcal{F}_{X^{\sigma}}^{\#} \tag{2}$$

where σ runs over all permutations in the symmetric group S_n , and X^{σ} is the UOM obtained by permuting the columns of X by σ .

For $X \in \mathcal{M}(m,n)$ we denote by $\nu_j(X)$ the cardinality of a maximal independent subset of the set of all entries in column j of X. (All maximal independent subsets have the same cardinality.) We also set $\nu(X) = \sum_j \nu_j(X)$. If $\nu(X) = l$ we say that X lies on level l.

We define a binary relation " \prec " on $\mathcal{M}(m, n)$ which is a slight modification but equivalent to the definition given in [19]. Let $X \in \mathcal{M}(m, n)$ and $1 \leq j \leq n$. Further, let x, y be independent vector variables such that x or x' and y or y'occur in column j of X. Denote by Y the matrix obtained from X by replacing all occurrencies (if any) of x and x' in X by y and y', respectively. Then we write $Y \prec X$ and we say that Y is obtained from X by the *identification* y = x. Note that X and Y differ only in column j and that $\nu_j(Y) = \nu_j(X) - 1$. We warn the reader that the expression "identification y = x" does not mean that the vector variables x and y are the same, it just means that we are getting rid of the variables y and y' in X and replacing them with x and x', respectively.

If $Y \prec X$ and X is a UOM then Y is orthogonal but does not have to be a UOM. For instance this is the case for

$$X = \begin{bmatrix} a & c & e \\ a' & d' & f \\ b & c' & f' \\ b' & d & e' \end{bmatrix}, \quad Y = \begin{bmatrix} a & c & e \\ a' & d' & e \\ b & c' & e' \\ b' & d & e' \end{bmatrix}.$$
 (3)

Indeed $Y \prec X$ by the identification f = e, and X is a UOM while Y is not as $[a c' e] \perp Y$.

The relation \prec extends naturally to equivalence classes of UOMs. If \mathcal{X}, \mathcal{Y} are two equivalence classes of UOMs, we write $\mathcal{Y} \prec \mathcal{X}$ if $Y \prec X$ for some $X \in \mathcal{X}$ and some $Y \in \mathcal{Y}$.

III. PARTIAL ORDER

If $X = [x_{i,j}] \in \mathcal{M}(m,n)$ and $x \in \mathbf{X}$ we define the *multiplicity*, $\mu(x, X)$, of x in X to be the number of pairs (i, j) such that $x_{i,j} = x$. Thus if x does not occur in X then $\mu(x, X) = 0$. When X is known from the context we shall simplify this notation by writing just $\mu(x)$. Finally, we set $\mu(X) = \max_{i,j} \mu(x_{i,j}, X)$.

Let $X \in \mathcal{M}(m, n)$. If $\mu(x, X) = \mu(x', X)$ for all vector variables x in column j of X, then we say that the column j of X is *balanced*, and otherwise we say that it is *imbalanced*. We say that X is *balanced* if each of its columns is balanced, and otherwise we say that X is *imbalanced*. It is obvious that X is *imbalanced* if m is odd. We have shown in [6] that all UOM in $\mathcal{O}(n)$ are necessarily balanced.

Next we recall from [19] the definition of the partial order " \leq " in $\mathcal{M}(m, n)$. For two matrices $X, Y \in \mathcal{M}(m, n)$, we say that $Y \leq X$ if there exists a finite chain

$$Y = Z_0 \prec Z_1 \prec \dots \prec Z_k = X, \quad k \ge 0. \tag{4}$$

We write Y < X if $Y \leq X$ and $Y \neq X$. Note that if $X \in \mathcal{O}(m, n)$ then all the Z_i in the chain (4) belong to $\mathcal{O}(m, n)$. Further, if X and Y in (4) are UOM then so are all the Z_i . This follows immediately from [19, Lemma 17].

Let us also recall the definition of maximal and minimal UOMs.

Definition 1 We say that a UOM $X \in \mathcal{O}(m, n)$ is maximal if there is no UOM $Y \in \mathcal{O}(m, n)$ such that $X \prec Y$. Similarly, we say that a UOM $X \in \mathcal{O}(m, n)$ is minimal if there is no UOM $Y \in \mathcal{O}(m, n)$ such that $Y \prec X$. Further we say that a UOM is isolated if it is both maximal and minimal.

The definition of the partial order " \leq " on $\mathcal{M}(m, n)$ extends naturally to equivalence classes of matrices in $\mathcal{M}(m, n)$. If \mathcal{X} and \mathcal{Y} are two equivalence classes of matrices in $\mathcal{M}(m, n)$ and $Y \leq X$ for some $X \in \mathcal{X}$ and some $Y \in \mathcal{Y}$, then we shall write $\mathcal{Y} \leq \mathcal{X}$. If $\mathcal{X} \leq \mathcal{Y}$ and $\mathcal{X} \neq \mathcal{Y}$ then we write $\mathcal{X} < \mathcal{Y}$. Further, we write $\mathcal{Y} \prec \mathcal{X}$ if $Y \prec X$ for some $X \in \mathcal{X}$ and some $Y \in \mathcal{Y}$.

We can also extend the definition of maximal and minimal UOM to the equivalence classes of UOMs. E.g. we say that an equivalence class $\mathcal{X} \subseteq \mathcal{O}(m, n)$ of UOMs is *maximal* if there is no equivalence class of UOMs $\mathcal{Y} \subseteq \mathcal{O}(m, n)$ such that $\mathcal{X} < \mathcal{Y}$.

Next, we say that a UOM X and its equivalence class [X] are *reducible* if $\nu_j(X) = 1$ for at least one j. Otherwise we say that X, and [X], are *irreducible*.

For convenience, we denote by UOM[m, n] the set of equivalence classes of UOMs in $\mathcal{O}(m, n)$. This is a finite partially ordered set with partial order " \leq ".

Definition 2 If $\mathcal{X}, \mathcal{Y} \in \text{UOM}[m, n]$ and $\mathcal{Y} \prec \mathcal{X}$ we shall write $\mathcal{X} \rightarrow \mathcal{Y}$ and refer to it as an arrow. The set UOM[m, n] equipped with all arrows that exist between its members is the Hasse diagram of the partially ordered set $(\text{UOM}[m, n], \leq)$.

The Hasse diagram of UOM[m, n] can be viewed as a graph by ignoring the direction of arrows. We shall refer to the connected components of this graph also as the connected components of the Hasse diagram.

The following construction has been introduced in [19]. Let $A \in \mathcal{M}(r, s)$ and $B \in \mathcal{M}(m, n-s)$, n > s, and assume that any vector variables x and y of A and B, respectively, are independent. Further, let B be partitioned into blocks $B_k \in \mathcal{M}(m_k, n-s)$, $k = 1, \ldots, r$,

$$B = \begin{bmatrix} B_1 \\ \vdots \\ B_r \end{bmatrix}.$$

Then we denote by $A \models (B_1, B_2, \ldots, B_r)$ the matrix $[\tilde{A} B] \in \mathcal{M}(m, n)$, where \tilde{A} is the $m \times s$ matrix obtained from A by replacing, for each k, the row k of A by m_k copies of that row. Note that if a vector variable x occurs in one of the blocks B_k then x or x' may occur in another block but necessarily in the same column.

For instance we have

$$\begin{bmatrix} x\\x' \end{bmatrix} \models \left(\begin{bmatrix} a\\a' \end{bmatrix}, \begin{bmatrix} b\\b' \end{bmatrix} \right) = \begin{bmatrix} x&a\\x&a'\\x'&b\\x'&b' \end{bmatrix}.$$
(5)

It is easy to see that if A and the B_k are orthogonal matrices, then the matrix $A \models (B_1, B_2, \ldots, B_r)$ is also orthogonal. It is shown in [19, Proposition 7] that the matrix $A \models (B_1, B_2, \ldots, B_r)$ is a UOM if and only if A and all the B_k are UOMs.

Definition 3 We say that a matrix $X \in \mathcal{M}(m, n)$ is decomposable if it is equivalent to a matrix $A \models (B_1, B_2, \ldots, B_r)$, $r \ge 1$.

Let X be a decomposable UOM, say $X = A \models (B_1, \ldots, B_r)$ where $A \in \mathcal{O}(r, s)$ and $B_k \in \mathcal{O}(m_k, n-s)$, $k = 1, 2, \ldots, r$, are UOMs and $\sum m_k = m$. Then we can easily decide whether X is maximal or minimal. The first lemma below is proved in [19, Lemma 18].

Lemma 4 X is maximal if and only if A and all B_k are maximal and no two of the blocks B_k have a vector variable in common. If X is not maximal then there exists a UOM Y such that $X \prec Y$ and Y is obtained from X by modifying a single column in either A or just one of the blocks B_k .

Lemma 5 X is minimal if and only if A and all B_k are minimal and for each $j \in \{1, 2, ..., n-s\}$ all vector variables which occur in column j of all the B_k s already occur in just one of the B_k s. If X is not minimal then there exists a UOM Y such that $Y \prec X$ and Y is obtained from X by making a single identification in either A or just one of the blocks B_k .

We omit the proof of this lemma as it is similar to the proof of [19, Lemma 18].

For convenience, let us denote by \mathbf{OPS}_m the set of OPSs of cardinality m in \mathcal{H} , and by \mathbf{UPB}_m the set of UPBs of cardinality m in \mathcal{H} . As mentioned in the introduction, for $X \in \mathcal{O}(m, n)$, we define

$$\mathcal{F}_X = \{ \text{OPS of } \alpha(X) : \alpha \text{ evaluation of } X \},$$

$$\mathcal{F}_X^{\#} = \{ \text{OPS of } \alpha(X) : \alpha \text{ generic evaluation of } X \}$$

For an UOM $X \in \mathcal{O}(m, n)$, we have that $\mathcal{F}_X \subseteq \mathbf{OPS}_m$ and $\mathcal{F}_X^{\#} \subseteq \mathbf{UPB}_m$. By [19, Lemma 3], each $\mathcal{U} \in \mathbf{UPB}_m$ belongs to some $\mathcal{F}_X^{\#}$. Moreover, from the proof of that lemma it follows easily that X can be recovered, uniquely up to row permutations and renaming of vector variables, from any $\mathcal{U} \in \mathcal{F}_X^{\#}$. Thus we have

Proposition 6 When X runs through the set of representatives of the equivalence classes of UOMs in $\mathcal{O}(m,n)$, then the sets (2) form a partition of \mathbf{UPB}_m .

Consequently, for UOMs X and Y we have [X] = [Y] if and only if $\mathcal{F}_{X^{\sigma}}^{\#} = \mathcal{F}_{Y}^{\#}$ for some $\sigma \in S_{n}$. One can also show (see Corollary 9) that [X] = [Y] if and only if $\mathcal{F}_{X^{\sigma}} = \mathcal{F}_{Y}$ for some $\sigma \in S_{n}$.

Since we are identifying two unit product vectors if they differ only in phase, such product vectors are in fact points in the projective space \mathcal{PH} associated with \mathcal{H} . More precisely, these points lie on the Segre subvariety $\Sigma_n := \mathcal{PH}_1 \times \cdots \times \mathcal{PH}_n$ of \mathcal{PH} . Consequently, after ordering the product vectors of a $\mathcal{U} \in \mathbf{OPS}_m$, we obtain a point in Σ_n^m , the product of m copies of Σ_n . As an example, if $\{|\psi_i\rangle : i = 1, 2, 3, 4\}$ is an OPS in \mathcal{H} then the ordered quadruple $(|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, |\psi_4\rangle)$ can be viewed as a point of Σ_n^4 . Since the product vectors of \mathcal{U} can be ordered in m!ways and they are orthogonal to each other, we obtain in fact m! different points of Σ_n^m .

The set \mathbf{OPS}_m is closed in Σ_n^m because it is defined by the equations saying that its product vectors are orthogonal to each other. However, its subset \mathbf{UPB}_m is not closed in general. For instance, in the case m = 4, n = 3 consider the set \mathcal{U}_t of the four pure product states

$$\begin{aligned} |\psi_1(t)\rangle &= |0\rangle \otimes |0\rangle, \\ |\psi_2(t)\rangle &= |1\rangle \otimes |e_-\rangle \otimes (\cos t |0\rangle + \sin t |1\rangle), \\ |\psi_3(t)\rangle &= |e_+\rangle \otimes |1\rangle \otimes (-\sin t |0\rangle + \cos t |1\rangle), \\ |\psi_4(t)\rangle &= |e_-\rangle \otimes |e_+\rangle \otimes |1\rangle, \end{aligned}$$

where $|e_{\pm}\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, and t is a real parameter. For $t \in (0, \pi/2)$ the set \mathcal{U}_t is a UPB while for t = 0 it is an OPS which is not a UPB. As $\mathcal{U}_0 = \lim_{t \to 0} \mathcal{U}_t$, we see that \mathcal{U}_0 belongs to the closure of **UPB**₄.

Our main goal in this section is to describe the closure of any equivalence class of *m*-state UPBs viewed as a subset of the variety Σ_n^m . We first make the following observation.

Lemma 7 For any $X \in \mathcal{O}(m, n)$, the set $\mathcal{F}_X^{\#}$ is open in \mathcal{F}_X (with respect to the relative topology of \mathcal{F}_X).

Proof. Let (x, y) be a pair of independent vector variables which occur in the same column of X. Denote by $\mathcal{F}_X^{(x,y)}$ the subset of \mathcal{F}_X consisting of the OPS of $\alpha(X)$ where α runs through all evaluations of X subject to the condition that $\alpha(y) = \alpha(x)$ or $\alpha(y) = \alpha(x)^{\perp}$. Clearly, this is a closed subset of \mathcal{F}_X . The assertion now follows from the observation that

$$\mathcal{F}_X^{\#} = \mathcal{F}_X \setminus \bigcup_{(x,y)} \mathcal{F}_X^{(x,y)}$$

where (x, y) runs through the finite set of all pairs of the kind mentioned above.

Theorem 8 If $X \in \mathcal{O}(m, n)$ then $\overline{\mathcal{F}_X^{\#}} = \mathcal{F}_X$.

Proof. First we note that the set \mathcal{F}_X is a closed subset of Σ_n^m . This follows from the observation that \mathcal{F}_X , considered as a subset of Σ_n^m , is the image of a direct product of a finite number of copies of the projective spaces \mathcal{PH}_j , $1 \leq j \leq n$, under a continuous map to Σ_n^m . Indeed let us select a maximal set V of independent vector variables that occur in the matrix X. If $x \in V$ occurs in column j of X, we denote by S_x a copy of \mathcal{PH}_j . Finally we set $S := \times_{x \in V} S_x$. To any point $v := (|v_x\rangle)_{x \in V} \in S$ we attach an evaluation α_v of X as follows: if $x \in V$ then we set $\alpha_v(x) = |v_x\rangle$ and if x' occurs in X we also set $\alpha_v(x') = |v_x\rangle^{\perp}$. We can now define a continuous map $f : S \to \Sigma_n^m$ as follows: f(v) is the OPS

of the matrix $\alpha_v(X)$. The image of f is precisely the subset \mathcal{F}_X of Σ_n^m . Since S is compact, its image $f(S) = \mathcal{F}_X$ is closed in Σ_n^m . As $\mathcal{F}_X^{\#} \subseteq \mathcal{F}_X$, it follows that $\overline{\mathcal{F}_X^{\#}} \subseteq \mathcal{F}_X$.

We shall now prove the opposite inclusion $\mathcal{F}_X \subseteq \overline{\mathcal{F}_X^{\#}}$. Let $\mathcal{U} \in \mathcal{F}_X$ be arbitrary. Then \mathcal{U} is the OPS of the matrix $\alpha(X)$ for some evaluation α of X. It is easy to see that there exists a finite chain (4) such that \mathcal{U} is the OPS of $\beta(Y)$ where the evaluation β of Y is the restriction of α . Therefore it suffices to prove the assertion in the case where $Y \prec X$. Thus Y can be obtained from X by an identification x = y where x and y are independent vector variables such that x and at least one of y and y' occur in the same column of X, say in column j. We choose a continuous path σ : $[0,1] \rightarrow \mathcal{PH}_j$ such that $\sigma(0) = \alpha(x)$ and $\sigma(1) = \alpha(y)$, and moreover $\sigma(t) \neq \alpha(z), \alpha(z)^{\perp}$ for all $t \in (0,1)$ and all vector variables z in column j of X. Let us now define a continuous one-parameter family $\alpha_t, 0 \leq t \leq 1$, of evaluations of X by setting $\alpha_t(x) = \sigma(t), \alpha_t(x') = \sigma(t)^{\perp}$, and $\alpha_t(z) = \alpha(z)$ for all $z \neq x, x'$. Then $\alpha_0 = \alpha, \alpha_1 = \beta$ and α_t is generic for $0 \leq t < 1$. Thus $\alpha_t(X) \in \mathcal{F}_X^{\#}$ for $0 \leq t < 1$. Clearly we have $\beta(Y) = \lim_{t \to 1} \alpha_t(X)$. Since this limit belongs to the closure of $\mathcal{F}_X^{\#}$, our assertion is proved.

Corollary 9 For $X, Y \in \mathcal{O}(m, n)$, the equalities $\mathcal{F}_X^{\#} = \mathcal{F}_Y^{\#}$ and $\mathcal{F}_X = \mathcal{F}_Y$ are equivalent to each other.

Proof. Theorem 8 shows that the first equality implies the second one. We shall prove the converse.

Assume that $\mathcal{F}_X = \mathcal{F}_Y$. By Lemma 7, $\mathcal{F}_X^{\#}$ is open in \mathcal{F}_X and by the theorem it is also dense in \mathcal{F}_X . Hence, both $\mathcal{F}_X^{\#}$ and $\mathcal{F}_Y^{\#}$ are dense open subsets of \mathcal{F}_X . Therefore $\mathcal{F}_X^{\#}$ and $\mathcal{F}_Y^{\#}$ cannot be disjoint, i.e., there exists $\mathcal{U} \in \mathcal{F}_X^{\#} \cap \mathcal{F}_Y^{\#}$. As mentioned in the beginning of this section, Y can be obtained from X by permuting the rows and renaming the vector variables. Consequently $\mathcal{F}_X^{\#} = \mathcal{F}_Y^{\#}$.

Now let $X \in \mathcal{O}(m, n)$ be any UOM. The corresponding equivalence class of *m*-state UPBs in \mathcal{H} is given by the expression (2). As explained above, we can consider this equivalence class as a subset of Σ_n^m . It follows immediately from Theorem 8 that the closure of this equivalence class is the union

$$\bigcup_{\sigma \in S_n} \mathcal{F}_{X^{\sigma}}.$$
(6)

Proposition 10 For $X, Y \in \mathcal{O}(m, n)$ we have

(i) $Y \leq X \Longrightarrow \mathcal{F}_Y \subseteq \mathcal{F}_X;$

(ii) $Y < X \Longrightarrow \mathcal{F}_Y \subseteq \mathcal{F}_X \setminus \mathcal{F}_X^{\#}$.

Proof. (i) Since there is a chain (4) from Y to X, it suffices to prove the assertion in the case where $Y \prec X$. Then Y can be obtained from X by performing a single identification, say y = x. For $\mathcal{U} \in \mathcal{F}_Y$ there exists an evaluation β of Y such that \mathcal{U} is the OPS of the matrix $\beta(Y)$. We can extend β to an evaluation α of X by setting $\alpha(y) = \beta(x)$ and $\alpha(y') = \beta(x)^{\perp}$. Then \mathcal{U} is also the OPS of the matrix $\alpha(X)$, i.e., we have $\mathcal{U} \in \mathcal{F}_X$.

(ii) We shall use again the chain (4). As Y < X the length k of this chain is at least 1. Thus we have $Y \leq Z_{k-1}$. By using (i) we deduce that $\mathcal{F}_Y \subseteq \mathcal{F}_{Z_{k-1}}$. Hence, it suffices to prove (ii) in the case where $Y \prec X$. By (i) we know that $\mathcal{F}_Y \subseteq \mathcal{F}_X$. It remains to show that $\mathcal{F}_Y \cap \mathcal{F}_X^{\#} = \emptyset$. Let $\mathcal{U} \in \mathcal{F}_Y$ be arbitrary. Then \mathcal{U} is the OPS of $\beta(Y)$ for some evaluation β of Y. Since $Y \prec X$, there is an

Let $\mathcal{U} \in \mathcal{F}_Y$ be arbitrary. Then \mathcal{U} is the OPS of $\beta(Y)$ for some evaluation β of Y. Since $Y \prec X$, there is an identification y = x which transforms X into Y. Let α be any generic evaluation of X. Since x and y are independent vector variables, we must have $\alpha(y) \neq \alpha(x)$. Consequently $\beta(Y) \neq \alpha(X)$, and so $\mathcal{U} \notin \mathcal{F}_X^{\#}$.

V. THE FOUR-QUBIT HASSE DIAGRAMS

We shall construct all the arrows of the Hasse diagram of UOM[m, 4] for m = 8, 9, 10. In the case m = 12 we construct only the arrows connecting the irreducible equivalence classes. We shall also determine the maximal and minimal equivalence classes of UOMs in all four cases m = 8, 9, 10, 12.

The cases m = 6, 7 are omitted because each of them contains only one equivalence class. The case m = 16 is also omitted though for a different reason. In that case the equivalence classes have not been enumerated so far, only the maximal equivalence classes are known [6].

In the table below we show the distribution of the equivalence classes of UOMs in $\mathcal{O}(m,4)$ over various levels $\nu = 7, 8, \ldots, 14$.

For a fixed value of m, the classes on the top level are maximal and those on the bottom level are minimal. For instance, in $\mathcal{O}(10, 4)$ both classes on level 11 are maximal and all 24 classes on level 8 are minimal.

Most of the proofs rely on the results of Johnston [11]. In particular we have used his classification of four qubit UPBs in order to construct the above table.

$\nu \backslash m$	6	7	8	9	10	12
14						2
13			1			11
12			1			46
11			6		2	154
10	1		29	1	17	332
9		1	46	4	37	392
8			43	6	24	227
7			18			45
Total	1	1	144	11	80	1209

Table I: The distribution of the equivalence classes of UOMs in $\mathcal{O}(m, 4)$ over various levels.

For a given UOM X, we refer to the quadruple $[\nu_1(X), \ldots, \nu_4(X)]$ as ν -numbers of X. We shall also use the μ -numbers $[\mu_1(X), \ldots, \mu_4(X)]$ of X, where $\mu_j(X)$ is the largest multiplicity of the vector variables in column j of X.

In Appendix A we have listed the representatives of the equivalence classes in UOM[m, 4] for m = 8, 9, 10, 12. For m = 12 we list only the representatives of the irreducible classes. These lists are extracted from [12]. (They are presented in a different format, suitable for this paper.) Our listing is by levels, starting from the top level and ending with the bottom level. We record one matrix per line by using the following conventions:

(i) the rows are listed one after the other and separated by white space;

(ii) we only use the vector variables $a_j, b_j, c_j, d_j, j = 1, 2, 3, 4$, and their perpendiculars which we denote here by capital letters A_j, B_j, C_j, D_j , respectively;

(iii) we omit the subscripts as they can be easily recovered: each vector variable in column j should have the subscript j;

(iv) we end each line by listing the ν -numbers of the matrix.

We denote by $X_{l,k}$ the kth matrix in the list of representatives on level l. Note that the number, m, of rows of the UOM is supressed, it does not appear in the symbol $X_{l,k}$.

As an example, in the case m = 8, the last matrix on level 11 is

$$X_{11,6} = \begin{bmatrix} a_1 & c_2 & c_3 & a_4 \\ a_1 & c_2' & a_3 & c_4 \\ a_1 & a_2 & c_3' & c_4' \\ b_1 & a_2' & a_3' & a_4' \\ a_1' & b_2 & b_3 & a_4 \\ a_1' & b_2' & a_3 & b_4 \\ a_1' & a_2 & b_3' & b_4' \\ b_1' & a_2' & a_3' & a_4' \end{bmatrix}$$

and its ν -numbers are [2,3,3,3]. Their sum is 11 and so this matrix lies on level 11. This is recorded as the first subscript of $X_{11,6}$. The second subscript 6 means that this matrix occupies the sixth place in our list of representatives on level 11.

In view of the large number of equivalence classes of UOMs, we had to examine many cases to find the arrows of the Hasse diagram. For that purpose we used computer programs that we wrote in Maple. We shall only sketch here the main steps of the search.

For a given UOM say $X = X_{l,r}$ on level l there are only finitely many matrices Y such that $Y \prec X$. A computer program can easily generate all such matrices Y and it can also test whether Y is a UOM. If the test is negative, then Y is discarded. Otherwise Y is a UOM and we use another computer program which tests whether Y is equivalent to some UOM on level l - 1. In fact we know that this must be the case, there is a unique s such that Y is equivalent to $X_{l-1,s}$. Then we have found the arrow $X_{l,r} \to X_{l-1,s}$. The latter program is described in [19, Section 4]. Unfortunately when m = 12 or m = 10 this program fails in many cases because it uses too much time. Such cases have to be dealth with separately. After processing in this way all UOMs on level l, we can decide which UOMs on level l - 1 are maximal and which UOMs on level l are minimal.

The Appendix B consists of four subsections, one for each of the cases m = 8, 9, 10, 12. Each subsection contains several tables where we record the arrows of the Hasse diagram of UOM[m, 4]. Each of these tables contains the list of all arrows from level l to level l-1 for fixed l. For instance in the case m = 8 the first table lists the arrows from level 11 to 10. The second line of that table is:

$$2 \to 4, 5$$
 $d_4 = b_4, d_4 = B_4$

The meaning of $2 \to 4, 5$ is that we have two arrows: $2 \to 4$ and $2 \to 5$. More precisely, these arrows are $X_{11,2} \to X_{10,4}$ and $X_{11,2} \to X_{10,5}$ and they belong to the Hasse diagram of UOM[8,4] since m = 8. The rest of that line contains two identifications: $d_4 = b_4$ and $d_4 = B_4$. These identifications allow us to verify the existence of the two arrows mentioned above. The first identification $d_4 = b_4$ justifies the first arrow $2 \to 4$, and $d_4 = B_4$ justifies $2 \to 5$. After performing the identification $d_4 = b_4$ on the matrix $X_{11,2}$ we obtain say the matrix Y. The verification of the arrow $2 \to 4$ is completed by showing that Y is equivalent to $X_{10,4}$.

We now state the main results of our computations for each of the cases m = 8, 9, 10, 12.

A. UOM[8,4]

UOM[8, 4] has cardinality 144 and occupies the levels 7 – 13. We start with some facts that can be proved directly without using the computer. Let us first show that there is an arrow from the class represented by $X_{13,1}$ to the one represented by $X_{12,1}$. Note that these two matrices have the same first column, and that the columns 2 and 3 of $X_{13,1}$ are the same as the columns 3 and 4 of $X_{12,1}$, respectively. Next we identify the variable c_4 to a_4 , i.e., we set $c_4 = a_4$ and $C_4 = A_4$, in the last column of $X_{13,1}$. (For the sake of clarity we use here subscripts which were omitted in the tables of Appendix A.) It is easy now to verify that this new matrix, let us call it Y, is equivalent to $X_{12,1}$. We just have to rename d_4 to c_4 (and D_4 to C_4) in the last column of Y and then by permuting the columns we obtain the matrix $X_{12,1}$. We conclude that $X_{13,1}$ is not minimal and $X_{12,1}$ is not maximal.

Similarly one can verify that we have arrows $X_{12,1} \to X_{11,k}$ for k = 1, 2, 3, 4, 5. On the other hand we claim that there is no arrow from $X_{12,1}$ to $X_{11,6}$. To prove this claim, it suffices to inspect their ν numbers, [1,3,4,4] and [2,3,3,3], respectively. If $Y \prec X_{12,1}$ is a UOM, then 3 of the ν -numbers of Y must be the same as the corresponding ν -numbers of $X_{12,1}$. It follows that Y and $X_{11,6}$ have different ν -numbers, and so they are not equivalent. This proves our claim. We conclude that $X_{12,1}$ is not minimal, the UOMs $X_{11,k}$ are not maximal for k = 1, 2, 3, 4, 5, and that $X_{11,6}$ is maximal.

The arrows from level l = 11 to level 10 are listed in the first table of Appendix B. The first line of that table indicates that there are four arrows emanating from the first UOM on level 11. Namely there exist arrows from $X_{11,1}$ to $X_{10,k}$, k = 1, 2, 3, 6. The existence of the first of these arrows can be verified as follows. First we identify the independent variables d_4 and a_4 in column 4 of $X_{11,1}$ to obtain a matrix, say Y. This means that we have to replace d_4 and D_4 with a_4 and A_4 , respectively. After this is done we have to verify that Y is equivalent to $X_{10,1}$. In this case this is very easy, we just switch the columns 3 and 4 of Y to obtain the matrix $X_{10,1}$. Hence they are equivalent. (In this case one can identify two independent variables in the third column of $X_{11,1}$, to obtain another proof that the arrow $X_{11,1} \rightarrow X_{10,1}$ exists.) For the remaining three arrows emanating from $X_{11,1}$ we use the identifications $d_4 = B_4$, $d_4 = A_4$, $d_4 = b_4$, respectively.

The following proposition follows from the above comments and the list of arrows given in Appendix B for m = 8.

Proposition 11 UOM[8,4] has cardinality 144 and occupies the levels 7 to 13. There are 16 maximal and 23 minimal classes.

The representatives of the maximal classes are $X_{13,1}$, $X_{11,6}$, $X_{10,[17-19]}$, $X_{10,[22-29]}$, $X_{9,46}$, $X_{8,24}$ and $X_{8,30}$.

The minimal classes are the 18 classes on level 7 and the classes with representatives $X_{8,25}$, $X_{8,34}$ and $X_{9,[26-28]}$.

The Hasse diagram has four connected components. The three small components (shown on Fig. 1) have as vertices the classes of

$$\begin{split} &X_{10,17}, X_{9,26};\\ &X_{10,18}, X_{10,19}, X_{9,27}, X_{9,28};\\ &X_{10,[22-23]}, X_{9,29}, X_{9,[34-36]}, X_{8,25}, X_{8,34}, \end{split}$$

respectively.

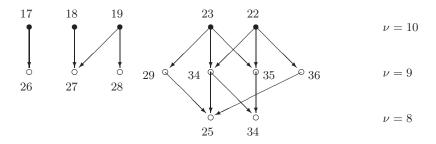


Figure 1: UOM[8,4], Hasse diagrams of the three small connected components. The maximal classes are marked by bullets.

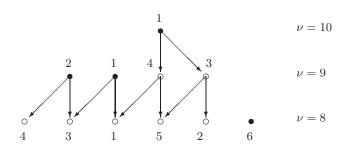


Figure 2: UOM[9,4], Hasse diagram of equivalence classes. This is the whole Hasse diagram.

B. UOM[9, 4]

UOM[9,4] has cardinality 11 and occupies only the levels 8-10.

Proposition 12 The Hasse diagram of UOM[9,4] has exactly 10 arrows, see Appendix B and Fig. 2. It has two connected components, one of them consists of the isolated class $[X_{8,6}]$. The representatives of the maximal classes are: $X_{10,1}$, $X_{9,1}$, $X_{9,2}$ and $X_{8,6}$. The minimal classes are all 6 classes on level 8.

C. UOM[10, 4]

UOM[10, 4] has cardinality 80 and occupies the levels 8-11.

Proposition 13 UOM[10,4] has exactly 14 maximal equivalence classes. Their representatives are $X_{11,1}$ and $X_{11,2}$ on level 11, $X_{10,8-17}$ on level 10, and $X_{9,17}$ and $X_{9,24}$ on level 9. The minimal equivalence classes are all 24 classes on level 8. The Hasse diagram has two connected components, the smaller component consisting of 20 classes is sketched on Figure 3.

Proof. All the assertions follow from the tables in Appendix B for m = 10 where all arrows of the Hasse diagram are listed.

D. UOM[12, 4]

UOM[12, 4] has cardinality 1209 and occupies the levels 7 – 14. Among the 1209 classes only 161 are irreducible. The latter occupy the levels 8-12. There are 26,64,51,18,2 irreducible equivalence classes on these levels, respectively. For each of these levels l, the representatives of the irreducible equivalence classes are listed in Appendix A for m = 12.

Proposition 14 We consider here the equivalence classes of UOMs in $\mathcal{O}(12, 4)$.

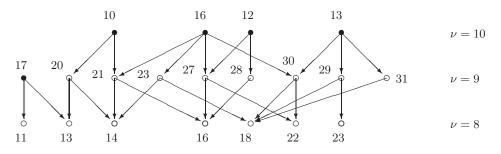


Figure 3: UOM[10, 4], Hasse diagram of the small connected component. There are two connected components, the one shown is the smaller one. The bigger component has 60 vertices and we omit the picture.

(i) There are exactly 45 minimal classes, namely all the classes lying on level 7.

(ii) There are exactly 11 maximal classes. Three of them are reducible, two on level 14 and one on level 13. The eight irreducible maximal classes have as representatives the UOMs: $X_{12,1}$ and $X_{12,2}$ on level 12; $X_{11,1}$, $X_{11,2}$, $X_{11,17}$ and $X_{11,18}$ on level 11; and $X_{10,50}$ and $X_{10,51}$ on level 10.

Proof. Let $X \in \mathcal{O}(12,4)$ be a reducible UOM. Up to equivalence, we may assume that

$$X = \begin{bmatrix} x \\ x' \end{bmatrix} \models (U, V),$$

where U and V are UOMs of 3 qubits. Since the UPBs of 3 qubits have cardinality 4 or 8, we may assume that $U \in \mathcal{O}(8,3)$ and $V \in \mathcal{O}(4,3)$. Note that V is unique up to equivalence and that its ν -numbers are [2,2,2].

(i) Assume now that the above reducible UOM X is minimal. By Lemma 5, U must be minimal. By [6, Corollary 15 and Figure 1], U is unique up to equivalence and its ν -numbers are [1,1,1]. By the same lemma, each vector variable in U must occur in V. Hence the ν -numbers of X are [1,2,2,2] and so $\nu(X) = 7$. Since there are no UOMs on level 6, all UOMs on level 7 are minimal.

The tables in Appendix B show that there are no irreducible minimal UOMs on levels higher than 8, and by using a computer program we have verified that the same is true for irreducible UOMs on level 8. This completes the proof of (i).

(ii) Let X be a reducible UOM given by the above formula and assume that it is maximal. Then, by Lemma 4, U and V must be maximal and they have no vector variable in common. Hence, we can assume that U is a representative of one of the three maximal equivalence classes of OPBs in $\mathcal{O}(8,3)$ (see [6, Fig. 1]). Two of them, classes 1 and 2, have $\nu(U) = 7$ and the third one, class 6, has $\nu(U) = 6$. By permuting the last four rows of X, we may also assume that V is the third matrix displayed in (1). Since $\nu(X) = 1 + \nu(U) + \nu(V) = \nu(U) + 7$, we have $\nu(X) = 14$ in the first two cases and $\nu(X) = 13$ in the third case.

Note that there are no arrows from a reducible UOM to an irreducible UOM. Hence, the assertion about the irreducible maximal equivalence classes follows from the tables in Appendix B. \Box

Lemma 4 (ii) implies that all reducible classes of UOMs in $\mathcal{O}(12, 4)$ can be obtained from just the 3 maximal ones by applying the relation " \prec ".

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1. Representatives of UOM[8, 4]

Level 13 1 addd aDcc acDC aCCD Abbb AaBa ABaA AAAB [1,4,4,4] Level 12 1 acdd aaDc aAcd aCCC Abbb AaaB AABa ABAA [1,3,4,4] Level 11 $1 \ abdd \ aaDc \ aAcD \ aBCC \ Abbb$ AaBa AAaB ABAA [1, 2, 4, 4] $2 \ abdd \ aaDc \ aAcD \ aBCC \ Abbb$ AaaB AABa ABAA [1, 2, 4, 4] $3 \ acad \ aacD \ aAAc \ aCCC \ AbAb \ Aaaa$ AAbB ABAA [1,3,3,4]4 acad aaAc aAcD aCCC Abbb AaaB AABa ABAA [1,3,3,4]5~acad~aaAc~aAcD~aCCC~AbbbAaAB AABa ABaA [1,3,3,4]6 acca aCac aaCC bAAA Abba ABab AaBB BAAA [2,3,3,3] Level 10 $aAAc \ aBCC \ Abbb$ $1 \ abad$ aacDAaaB $AABa \ ABAA \ [1,2,3,4]$ $2 \ abad$ aacD $aAAc \ aBCC \ AbAb$ AabBAAaaABBA [1, 2, 3, 4] $aAcD \ aBCC \ Abbb$ 3 abad aaAcAaBaAAAB ABaA [1, 2, 3, 4] $4 \ abad$ aaAcaAcD aBCC Abab AabBAAAa ABBA [1, 2, 3, 4] $aAcD \ aBCC \ AbAb$ $5 \ abad$ aaAcAabBAAaa $ABBA \ [1, 2, 3, 4]$ 6 abad aaAcaAcD aBCC Abab AaAa AAbB $ABBA \ [1, 2, 3, 4]$ $7 \ acca$ aCacaaAA aACC AAbb AbaBAaAA $ABBa \ [1,3,3,3]$ $ABBA \ [1, 3, 3, 3]$ aaAA aACC Aabb AbAB8 accaaCacAAaaAbAB $9 \ acca$ aCacaaAA aACC AAbb Aaaa $ABBA \ [1, 3, 3, 3]$ $10 \ acca$ aCacaaAA aACC AAbb AaaAAbABABBa[1, 3, 3, 3]aAAc aCCC Abbb AAAB ABaa aacAAaBA[1, 3, 3, 3] $11 \ acaa$ aAAc aCCC AbbA AABbABaB $12 \ acaa$ AaAa[1, 3, 3, 3]aacAaAAc aCCC Abab AAAa $ABBA \ [1, 3, 3, 3]$ $13 \ acaa$ aacAAabBaAAc aCCC AbAa AAabABBB [1, 3, 3, 3] $14 \ acaa$ aacAAabA $15 \ acaa$ aacAaAAc aCCC AbAA Aaba AAabABBB [1, 3, 3, 3]ABBB [1, 3, 3, 3]aAAc aCCC AbAA AAba 16 acaa aacAAaab aaaC $aAbB \ baAa$ bABDA b a dABbABbAb[2, 2, 2, 4] $17 \ aaac$ 18 aaad aaaDaAabaaAAbbAaABBC BAbB [2, 2, 2, 4]Aaac $19 \ aaac$ aaAaaaaCaABabAbbAaca AbCBBBAA [2, 2, 3, 3] $20 \ aaCc$ aABbbacabAbaAbAAABAA BaaC BAaB[2, 2, 3, 3] $21 \ aaCc$ aAbBbAabAbAA ABAA BaaC BABabaca [2, 2, 3, 3]22 aAbbabABbaad bBBD AAba AbAABaac BaaC[2, 2, 2, 4] $23 \ aaca$ aabAaaCabAacbAAbAbaCAbABBBBA[2, 2, 3, 3] $24 \ aBAb$ aBAB bbbd bbbDAABA Baac AABaBaaC[2, 2, 2, 4] $25 \ aBaC \ aBAB \ bbca$ bbCaAAbAAABA Baac BaAb[2, 2, 3, 3]ABaC $26 \ aABA \ aACa$ bbbAABAB Baac [2, 2, 3, 3]bbcaBaAb $27 \ aBAb$ $aBAB \ bbbd$ AABa AABA Bbbc [2, 2, 2, 4]baaDBaaC $28 \ aBAb \ aABB \ bbbd$ ABAa AABA Bbbc baaDBaaC[2, 2, 2, 4] $29 \ aAAD \ aBaC$ baBcAABB ABbA Bbab bbbdBaAa[2, 2, 2, 4]

L	evel 9								
	abbd	aBac	aaAD	aABC	Abbb	ABaa	AaAB	AABA	[1, 2, 2, 4]
2	abbd	aBac	aaAD	aABC	Abbb	ABaa	AAAB	AaBA	[1, 2, 2, 4]
3	abbd	aaBc	aAaD	aBAC	AbBb	Aaba	AAAB	ABaA	[1, 2, 2, 4]
4	abbd	aaBc	aAaD	aBAC	Abab	AaAa	AAbB	ABBA	[1, 2, 2, 4]
5	abbd	aaBc	aAaD	aBAC	Abab	AaBB	AAAa	ABbA	[1, 2, 2, 4]
6	abbd	aaBc		aBAC	Abab	AaAa	AABB	ABbA	[1, 2, 2, 4]
	abca	aaaA	aACc	aBAC		AaaA	AAbB	ABBa	[1, 2, 3, 3]
	abaa	aacA	aAAc	aBCC		AaBa	AAaB	ABAA	[1, 2, 3, 3]
9	abaa	aacA	aAAc	aBCC		AaBA	AAaB	ABAa	[1, 2, 3, 3]
	abaa	aacA	aAAc	aBCC		AaBA	AAAB	ABaa	[1, 2, 3, 3]
	abaa	aacA	aAAc	aBCC		AaaB	AABa	ABAA	[1, 2, 3, 3]
	abaa	aacA	aAAc	aBCC		AaAB	AABa	ABaA	[1, 2, 3, 3]
	abaa	aacA	aAAc	aBCC		AaAB	AABA	ABaa	[1, 2, 3, 3]
	abaa	aacA	aAAc	aBCC aBCC		AabB AabB	AAAa A A A A	ABBA	[1, 2, 3, 3]
	abaa abaa	aacA	aAAc aAAc	aBCC			AAAA AAAb	ABBa ADDD	[1, 2, 3, 3]
	abaa	aacA aacA	aAAc aAAc	aBCC		AabA Aaba	AAAb AAAb	ABBB ABBB	[1, 2, 3, 3]
	abaa	aacA	aAAc	aBCC		AabB	AAaa	ABBA	[1, 2, 3, 3] [1, 2, 3, 3]
	abaa	aacA	aAAc	aBCC		AabB	AAaA	ABBa	[1, 2, 3, 3] [1, 2, 3, 3]
	abaa	aacA	aAAc	aBCC		Aaba	AAab	ABBB	[1, 2, 3, 3] [1, 2, 3, 3]
	abaa	aacA	aAAc	aBCC		AaAA	AAbB	ABBa	[1, 2, 3, 3] [1, 2, 3, 3]
	abaa	aacA	aAAc	aBCC	AbaA	AaAb	AAba	ABBB	[1, 2, 3, 3]
	abaa	aacA	aAAc	aBCC	AbAb	Aaaa	AAbB	ABBA	[1, 2, 3, 3]
	abaa	aacA	aAAc	aBCC	AbAb	AaaA	AAbB	ABBa	[1, 2, 3, 3]
25	abaa	aacA	aAAc	aBCC	AbAA	Aaab	AAba	ABBB	[1, 2, 3, 3]
26	aaaa	aaaA	aAbB	baAa	bABC	Abac	ABbA	BbAb	[2, 2, 2, 3]
27	aaaa	aaaA	aAaA	aaAB	bbAb	Aaac	ABBC	BAba	[2, 2, 2, 3]
28	aaaa	aAbb	baaA	baAc	bAaB	Aaaa	AbAC	BBBA	$\left[2,2,2,3\right]$
29	abAc	aaaa	aAbC	baaA	AbAb	Aaaa	AAbB	BBBA	[2, 2, 2, 3]
	aBaC	aBAB	bbaa	bbAa	AAbA		Baac	BaAb	[2, 2, 2, 3]
31	aaaC	aAAB	bbac	bbAb	ABbA	ABBA		BaAa	[2, 2, 2, 3]
	aaaC	aaAB	bbac	bbAb	ABAa	ABBA		BAbA	[2, 2, 2, 3]
	aBaa	aaAB		bBAb			BAbA		[2, 2, 2, 3]
	aAba	abAA		baaC	AAbA	AbAa	Baab		[2, 2, 2, 3]
	aAbc	abAC	baaa	baaA	AAbb	AbAB	BaaA	BBBa	[2, 2, 2, 3]
	aAbb	abAB		bBBC	AAbA	AbAa	Baaa	BaaA	[2, 2, 2, 3]
	aBAc	aBAC aBAa	bbba	bbbA	AABb	AABB		BaaA BaAA	[2, 2, 2, 3]
	aBaC aBAb	aBAa aBAB		bbbA bbBA	AABb AAbA	AABB AABa	Baac	BaAA BaaC	[2, 2, 2, 3]
	aBaC	aBAB		bbBa	AAbA AAbA	AABA AABA		BaAb	[2, 2, 2, 3]
	aBAc	aABC		bbbA	ABAb	AABB		BaaA	[2, 2, 2, 3] [2, 2, 2, 3]
	aBAb	aABB		baaA	ABAa	AABA		BaaC	[2, 2, 2, 3] [2, 2, 2, 3]
	aBAc	aABC		baaA	ABAb	AABB		Baaa	[2, 2, 2, 3] [2, 2, 2, 3]
	aABC		bbba	baBa	AAAA	ABaA	Bbac	BaAb	[2, 2, 2, 3] [2, 2, 2, 3]
	aAAA		bbba	baBc	AABB		Bbab	BaAa	[2, 2, 2, 3]
	aAbc	aABb	bbbC	baac		ABAA		Baab	[2, 2, 2, 3]

L	evel 8								
1	abba	aBac	aaAA	aABC	Abbb	ABaA	AaAB	AABa	[1, 2, 2, 3]
2	abba	aBac	aaAA	aABC	Abba	ABab	AaAA	AABB	[1, 2, 2, 3]
3	abba	aBac	aaAA	aABC	Abbb	AAaB	ABAa	AaBA	[1, 2, 2, 3]
4	abba	aBac	aaAA	aABC	AbbA	AAaa	ABAb	AaBB	[1, 2, 2, 3]
5	abbc	aaaC	aBAa	aABA	Abbb	ABaa	AaAB	AABA	[1, 2, 2, 3]
6	abbc	aaaC	aBAa	aABA	Abbb	AaaB	ABAA	AABa	[1, 2, 2, 3]
7	abbc	aaaC	aBAa	aABA	Abbb	AAaB	ABAA	AaBa	[1, 2, 2, 3]
8	abba	aaBc	aAaA	aBAC	AbBb	Aaba	AAaB	ABAA	[1, 2, 2, 3]
9	abba	aaBc	aAaA	aBAC	AbBb	AabA	AAaB	ABAa	[1, 2, 2, 3]
10	abba	aaBc	aAaA	aBAC	AbBb	AabA	AAAB	ABaa	[1, 2, 2, 3]
11	abba	aaBc	aAaA	aBAC	AbAb	AabB	AAaA	ABBa	[1, 2, 2, 3]
12	abba	aaaA	aABc	aBAC	A baa	AabA	AAAb	ABBB	[1, 2, 2, 3]
13	abba	aaaA	aABc	aBAC	AbAa	AabA	AAab	ABBB	[1, 2, 2, 3]
14	abba	aaaA	aABc	aBAC	AbAA	Aaba	AAab	ABBB	[1, 2, 2, 3]
15	abba	aaBc	aAaA	aBAC	AbBa	AaaA	AAbb	ABAB	[1, 2, 2, 3]
16	abba	aaBc	aAaA	aBAC	AbBA	Aaaa	AAbb	ABAB	[1, 2, 2, 3]
17	abba	aaBc	aAaA	aBAC	AbBa	AaAA	AAbb	ABaB	[1, 2, 2, 3]
18	abba	aaBc	aAaA	aBAC	AbBA	AaAa	AAbb	ABaB	[1, 2, 2, 3]
19	abba	aaBc	aAaA	aBAC	A b a b	AaAa	AAbB	ABBA	[1, 2, 2, 3]
20	abba	aaBc	aAaA	aBAC	A b a b	AaAA	AAbB	ABBa	[1, 2, 2, 3]
21	abba	aaBc	aAaA	aBAC	AbAb	Aaaa	AABB	ABbA	[1, 2, 2, 3]
22	abba	aaBc	aAaA	aBAC	AbAb	AaaA	AABB	ABba	[1, 2, 2, 3]
23	abba	aaBc	aAaA	aBAC	AbAa	Aaab	AABA	ABbB	[1, 2, 2, 3]
24	aBab	aBAa	bbbb	baaB	AaBb	AABa	Bbba	BAAA	[2, 2, 2, 2]
25	aaaa	aAaa	baAb	bAbA	A baa	ABBB	BabA	BAAb	[2, 2, 2, 2]
26	aBAa	aBAA	bbba	bbbA	AABb	AABB	Baaa	BaaA	[2, 2, 2, 2]
27	aABa	aAbA	bbba	bbBA	ABAb	ABAB	Baaa	BaaA	[2, 2, 2, 2]
28	aABa	aabA	bbba	bbBA	ABAb	ABAB	Baaa	BAaA	[2, 2, 2, 2]
29	aABa	aAbA	bbba	bbBA	ABaA	ABAB	Baaa	BaAb	[2, 2, 2, 2]
30	aaBa	aBBA	bbba	bbAA	AAaA	ABAB	BAaa	Babb	[2, 2, 2, 2]
31	aaBa	aBbb	bbba	bABA	ABaa	AaAA	BbaA	BAAB	[2, 2, 2, 2]
32	aaBa	abaA	bbba	bBBA	ABaa	AabA	BAAb	BAAB	[2, 2, 2, 2]
33	aBaa	aBaA	bbba	bAAA	AaBa	AaBA	Bbbb	BAAB	[2, 2, 2, 2]
34	aabA	aAAb	bbaa	bBaa	AAbA	AaAb	BBaa	BbBB	[2, 2, 2, 2]
35	aBaa	aBaA	bbba	baAA	AaBa	AABb	Bbbb	BAAB	[2, 2, 2, 2]
36	aBaA	aBBa	bbbb	bbaB	AAAB	AABb	Baba	BaAA	[2, 2, 2, 2]
37	aaBA	aAAB	bbbb	bBaa	AaBA	AAAB	Bbbb	BBaa	[2, 2, 2, 2]
38	aBAa	aABA	bbbb	baaB	ABAa	AABA	BbbB	Baab	[2, 2, 2, 2]
39	aBAa	aABA	bbbb	baaB	ABAA	AABa	BbbB	Baab	[2, 2, 2, 2]
40	aAbB	aBab	bbbb	baaB	AABA	ABAa	BbBa	BaAA	[2, 2, 2, 2]
41	aAAB	aBaA	bbbb	baBa	AAAB	ABbb	BbaA	BaBa	[2, 2, 2, 2]
42	aAAB	aBaA	bbbb	baBa	AABa	ABbb	BbaA	BaAB	[2, 2, 2, 2]
43	aAbB	aABA	bbbb	baaB	ABab	ABAa	BbBa	BaAA	[2, 2, 2, 2]

L	evel 7	,							
1	abbb	aBaa	aaBA	aAAB	Abbb	ABaa	AaBA	AAAB	[1, 2, 2, 2]
2	abbb	aBaa	aaAB	aABA	AbbB	ABaa	AaAb	AABA	[1, 2, 2, 2]
3	abbb	aBaa	aaAB	aABA	AbbB	ABaA	AaAb	AABa	[1, 2, 2, 2]
4	abbb	aBaa	aaAB	aABA	AbbB	Aaab	ABAA	AABa	[1, 2, 2, 2]
5	abbb	aBaa	aaAB	aABA	AbbB	AAab	ABAa	AaBA	[1, 2, 2, 2]
6	abbb	aaBa	aAaB	aBAA	AbBB	Aaba	AAab	ABAA	[1, 2, 2, 2]
7	abbb	aaBa	aAaB	aBAA	AbBB	AabA	AAAb	ABaa	[1, 2, 2, 2]
8	abbb	aaBa	aAaB	aBAA	AbBA	AabB	AAAa	ABab	[1, 2, 2, 2]
9	abbb	aaBa	aAaB	aBAA	AbaB	Aabb	AAAA	ABBa	[1, 2, 2, 2]
10	abbb	aaBa	aAaB	aBAA	AbAB	Aabb	AAaa	ABBA	[1, 2, 2, 2]
11	abbb	aaBa	aAaB	aBAA	AbAB	Aabb	AAaA	ABBa	[1, 2, 2, 2]
12	abbb	aaaB	aABa	aBAA	AbBB	Aaba	AAab	ABAA	[1, 2, 2, 2]
13	abbb	aaaB	aABa	aBAA	AbBB	AabA	AAab	ABAa	[1, 2, 2, 2]
14	abbb	aaaB	aABa	aBAA	AbBB	AabA	AAAb	ABaa	[1, 2, 2, 2]
15	abbb	aaaB	aABa	aBAA	A baa	AabA	AAAb	ABBB	[1, 2, 2, 2]
16	abbb	aaBa	aAaB	aBAA	AbaB	AaAa	AAbb	ABBA	[1, 2, 2, 2]
17	abbb	aaBa	aAaB	aBAA	AbaB	AaAA	AAbb	ABBa	[1, 2, 2, 2]
18	abbb	aaBa	aAaB	aBAA	AbAB	AaaA	AAbb	ABBa	[1, 2, 2, 2]

2. Representatives of UOM[9, 4]

Level 10 1 aaca aaCa abBA bAaa bBAA Aaac AaaC AbAB BAbb [2,2,3,3]

Level 9 1 aaac aaaC aaAb aAaA bAbaAaaa AABb AbbA BBAB [2, 2, 2, 3]2 aaAa aAbb baac baaC bAAB AAaa AbAb Baaa BBBA [2,2,2,3] 3 aaac aaaC aAbB baAa bABA Abaa ABaa AbbA BBAb [2, 2, 2, 3]4 aaac aaaC aAbb baAa bABA Aaaa AAaa AbbA BBAB [2,2,2,3] Level 8 1 aaaa aAaa abAb baaA bAAB Aaaa AaAa AAbbBBBA [2, 2, 2, 2]2 abaa aBaa aBAb baaA bAAB Aaba AaBa AAbb[2, 2, 2, 2]BbBA3 aaAa aAbb baaabaaA bAAB AAaa AbAb Baaa BBBA [2, 2, 2, 2] $4 \ abAa \ aabA \ baaa$ bAaa bAAA AbaA AaAb Bbaa BBBB [2, 2, 2, 2]5 aaaa aaAa aAbbbaaA bAAB Abaa ABaa ABAb BbBA [2, 2, 2, 2]6 abAa aabA aAab baaa bAAA AAba AbaA AaAb BBBB [2,2,2,2]

3. Representatives of UOM[10, 4]

Level 11 1 aabd aabD aaBc aaBC aAAb bAaa Aaaa AbAb ABCA BAcB [2,2,3,4] 2 aaad aaaD aaAc aaAC aAcA bAba Aaba AbBB ABcA BACb [2,2,3,4]

L	evel 10)									
1	aaad	aaaD	aaAb	aAbb	bBAB	Aaac	AaaC	AAaa	AbAa	BABA	[2, 2, 2, 4]
2	aaad	aaaD	aAaA	abAA	baAa	bABa	Aaac	AaaC	AAbb	BBAB	[2, 2, 2, 4]
3	aaad	aaaD	abAA	baAa	bABa	Aaac	AaaC	AAbB	BAab	BBAb	[2, 2, 2, 4]
4	aaac	aaAa	aaAA	aaaC	aAca	bAbA	AabA	Abca	ABBB	BACb	$\left[2,2,3,3\right]$
5	aabc	aaBa	aaBA	aabC	aAAa	bAcA	Aaab	AbAa	ABCB	BAab	$\left[2,2,3,3\right]$
6	aaba	aaBa	aacA	aaCA	aAAc	bAab	Aaab	AbAc	ABbB	BABC	$\left[2,2,3,3\right]$
7	aaba	aaBa	aabA	aaBA	aAAc	bAab	Aaab	AbAc	ABCB	BAcC	$\left[2,2,3,3\right]$
8	aaca	aaCa	aAba	abbA	bABb	Aaac	AaAa	AaaC	AAaB	BBAA	$\left[2,2,3,3\right]$
9	aaac	aaAa	aaaC	aAab	bAAa	bbAA	Aaca	AaCa	ABBA	BAbB	[2, 2, 3, 3]
10	abbA	aaca	aaCa	aAba	bABB	AbAb	Aaac	AaaC	AAab	BBAA	[2, 2, 3, 3]
11	aAac	aAaC	aAAb	bacA	baba	baBa	AAaa	ABCA	Baaa	BbAB	[2, 2, 3, 3]
12	aaAc	aaAC	aAAb	bbad	bbaD	bBaa	AaAa	ABBA	Bbab	BAbB	[2, 2, 2, 4]
13	aBab	aaAc	aaAC	bbad	bbaD	bAAa	AaAa	ABBA	Bbab	BAbB	[2, 2, 2, 4]
14	aaAb	aaAB	bbad	bbaD	bBaa	bAAa	AaAa	ABBA	Bbac	BAbC	[2, 2, 2, 4]
15	aAab	aAaB	baca	baCa	babA	bAAa	AAaa	ABBA	Babc	BbAC	[2, 2, 3, 3]
16	aAac	aAaC	aAAb	baca	babA	baCa	AAbb	AbBA	Baaa	BBAB	$\left[2,2,3,3\right]$
17	abaa	aaAa	aBac	aBaC	baAA	bAAB	Aaca	AaCa	AAbb	BbBA	$\left[2,2,3,3\right]$

L	evel 9										
1	aaaa	aaaA	aaAb	aAbb	bBAB	Aaac	AaaC	AAaa	AbAa	BABA	[2, 2, 2, 3]
2	aaaa	aaaA	aaAa	aAba	bBAA	Aaac	AaaC	AAab	AbAb	BABB	[2, 2, 2, 3]
3	aaaa	aaaA	aaAc	aAbc	bBAC	Aaaa	AaaA	AAab	AbAb	BABB	[2, 2, 2, 3]
4	aaac	aaaC	aAaa	abAa	baAA	bABA	Aaaa	AaaA	AAbb	BBAB	[2, 2, 2, 3]
5	aaac	aaaC	aAaA	abAA	baAa	bABa	Aaab	AaaB	AAbB	BBAb	[2, 2, 2, 3]
6	aaaa	aaaA	aAaa	abAa	baAA	bABA	Aaac	AaaC	AAbb	BBAB	[2, 2, 2, 3]
7	aaab	aaaB	aAaA	abAA	baAa	bABa	Aaac	AaaC	AAbB	BBAb	[2, 2, 2, 3]
8	aaab	aaaB	aAaA	abAA	baAa	bABa	Aaab	AaaB	AAbc	BBAC	[2, 2, 2, 3]
9	aaaA	aAac	abAc	baaa	bBAB	AaaA	AaAb	AAbb	Baaa	BABC	[2, 2, 2, 3]
10	aaab	aAaA	abAA	baaB	baAa	bABa	Aaab	AAbc	BaaB	BBAC	[2, 2, 2, 3]
11	aaaa	aaaA	abAA	baAa	bABa	Aaac	AaaC	AAbB	BAab	BBAb	[2, 2, 2, 3]
12	aaaa	aaaA	abAB	baAb	bABb	Aaac	AaaC	AAbA	BAaa	BBAa	[2, 2, 2, 3]
13	aaaa	aaaA	abAB	baAb	bABb	Aaaa	AaaA	AAbC	BAac	BBAc	[2, 2, 2, 3]
14	aaaa	aAaa	abAa	abbA	bBAb	Aaac	AaaC	AAaa	AaAB	BABA	[2, 2, 2, 3]
15	aAac	aAaC	aAAB	baaa	baAa	babA	AAaa	ABBA	Baaa	BbAb	[2, 2, 2, 3]
16	aaac	aaaC	aaAb	aAbb	bBAB	A baa	AaaA	ABaa	AbAa	BABA	[2, 2, 2, 3]
17	aaac	aaaC	aAaa	aaAA	bbAa	bAbA	A baa	AabA	ABBB	BAAb	[2, 2, 2, 3]
18	abaa	aaaA	aBaa	aBAa	baAA	bAbA	Aaac	AaaC	AABB	BbAb	[2, 2, 2, 3]
19	aaac	aaaC	aaAb	bAaa	bABA	Aaaa	AbAa	AbbA	BAaa	BBAB	[2, 2, 2, 3]
20	aAbb	aaac	aaaC	aaAb	bBAB	AbbA	Aaaa	AAaa	AbAa	BABA	[2, 2, 2, 3]
21	aAac	aAaC	aAAb	babA	baaa	baAa	AAbb	ABBA	Baaa	BbAB	[2, 2, 2, 3]
22	aaba	aaBc	aaBC	aAaa	bAAB	AbaA	AbAb	Aaaa	AAaa	BBbA	[2, 2, 2, 3]
23	aAbA	aAaa	aAAa	baac	baaC	baAb	AAbb	AbAB	Baaa	BBBA	[2, 2, 2, 3]
24	abab	aaAA	aAbB	bBab	baAa	bAAb	A bac	AbaC	AaAA	BBBa	[2, 2, 2, 3]
25	aaAa	aaAA	bbac	bBaa	bbaC	bAAa	AaAa	ABBA	Bbab	BAbB	[2, 2, 2, 3]
26	aaAb	aaAB	bbaa	bBaa	bbaA	bAAa	AaAa	ABBA	Bbac	BAbC	[2, 2, 2, 3]
27	aaAc	aaAC	aAAa	bbaa	bbaA	bBab	AaAb	ABBB	Bbaa	BAbA	[2, 2, 2, 3]
28	aaAa	aaAA	aAAc	bbaa	bbaA	bBab	AaAb	ABBB	Bbac	BAbC	[2, 2, 2, 3]
29	aBab	aaAc	aaAC	bbaa	bbaA	bAAa	AaAa	ABBA	Bbab	BAbB	[2, 2, 2, 3]
30	aBaa	aaAc	aaAC	bbaa	bbaA	bAAb	AaAb	ABBB	Bbaa	BAbA	[2, 2, 2, 3]
31	aBac	aaAa	aaAA	bbaa	bbaA	bAAb	AaAb	ABBB	Bbac	BAbC	[2, 2, 2, 3]
32	aaAa	aaAA	bbac	bbaC	bBab	bAAb	AaAb	ABBB	Bbaa	BAbA	[2, 2, 2, 3]
33	aaAa	aaAA	bbaa	bbaA	bBab	bAAb	AaAb	ABBB	Bbac	BAbC	[2, 2, 2, 3]
34	aaAc	aaAC	bbaa	bbaA	bBab	bAAb	AaAb	ABBB	BbaA	BAba	[2, 2, 2, 3]
35	aBab	aaAa	bbac	bbaC	bbAA	bAAa	AaAa	ABbA	Bbab	BABB	[2, 2, 2, 3]
36	abaa	aaaA	aBba	aBBa	baAA	bAbA	Aaac	AaaC	AABb	BbAB	[2, 2, 2, 3]
37	abaa	aBac	aBaC	baAa	bbBA	Aaaa	AabA	AAbB	BaAa	BAAb	[2, 2, 2, 3]

L	evel 8										
1	aaaa	aAaa	aaAa	aabA	bAAB	Aaaa	AAaa	AbaA	AbAb	BBBA	[2, 2, 2, 2]
2	aAaa	aAaA	aAAb	baaa	baAa	babA	AAaa	ABBA	Baaa	BbAB	[2, 2, 2, 2]
3	aAaa	aaAa	aabA	baaa	bAaA	bAAb	AAaa	AbAB	Baaa	BBBA	[2, 2, 2, 2]
4	aAaa	aaaA	aaAb	baaa	bBAB	AAaa	AbAa	AbBA	Baaa	BAbA	[2, 2, 2, 2]
5	aaaa	aaaA	aaAb	bAaa	bBAB	Aaaa	AbAa	AbBA	BAaa	BAbA	[2, 2, 2, 2]
6	aaaa	aAaa	aBAB	bbAa	bbBA	Aaaa	AAaa	AAbA	BaaA	BaAb	[2, 2, 2, 2]
7	abaa	aBaa	aAbA	bbAa	bbBA	Aaaa	AAaa	ABAB	BaaA	BaAb	[2, 2, 2, 2]
8	abaa	aBaa	aBAB	bbAa	bbBA	Aaaa	AAaa	AAbA	BaaA	BaAb	[2, 2, 2, 2]
9	aaaa	aAaa	abaA	abAb	bBBA	Aaba	AaBa	AAaa	AabA	BAAB	[2, 2, 2, 2]
10	aAba	aABa	aAbA	baaa	baaA	baAb	AAaa	ABAB	Baaa	BbBA	[2, 2, 2, 2]
11	abaA	aAaa	abAb	baaa	bBBA	AaAa	AabA	AAbb	Baaa	BAAB	[2, 2, 2, 2]
12	abaa	aBaa	aBaA	baAa	bAAB	Aaaa	AabA	AAbb	BaAa	BbBA	[2, 2, 2, 2]
13	aaaa	aaAa	aAba	abbA	bBAA	AbAb	AAab	Aaaa	AaaA	BABB	[2, 2, 2, 2]
14	aAaa	aAaA	aAAb	babA	baaa	baAa	AAbb	ABBA	Baaa	BbAB	[2, 2, 2, 2]
15	abba	aaaA	aBAB	bbBa	bbAA	bAaA	Aaba	AAba	AaaA	BABb	[2, 2, 2, 2]
16	aAAb	aaAa	aaAA	bbaa	bBaa	bbaA	AaAa	ABbA	Bbab	BABB	[2, 2, 2, 2]
17	aaba	aaBb	aaBB	aAaa	bAAb	AbaA	AbAB	Aaaa	AAaa	BBbA	[2, 2, 2, 2]
18	aAAb	aaAa	aaAA	bbaa	bBaa	bbaA	AbAb	ABbA	Baaa	BABB	[2, 2, 2, 2]
19	abab	aaAa	aBbA	bBaa	bbaB	bAAa	A b a b	AaAa	AaAA	BABB	[2, 2, 2, 2]
20	aaAa	aBBA	bbab	bBaa	bbaB	bAAa	AaAa	AaAA	BbaB	BAbb	[2, 2, 2, 2]
21	aBba	aaaA	bbba	bbBa	bbAA	bAaA	AaaA	ABAb	Bbba	BABB	[2, 2, 2, 2]
22	aBba	aaaA	aaAA	bbba	bbBa	bAaA	AaaA	ABAb	Bbba	BABB	[2, 2, 2, 2]
23	aBba	aaBa	aaaA	bbba	bAaA	bbAA	AaaA	ABAb	Bbba	BABB	[2, 2, 2, 2]
24	abaa	aBba	aBBa	baaA	bbAB	Aaaa	AaAb	AAbb	BaaA	BABA	[2, 2, 2, 2]

4. Representatives of irreducible classes in UOM[12, 4]

In $\mathcal{O}(12, 4)$ there are 1209 equivalence classes of UOMs of which only 161 are irreducible. While all the equivalence classes occupy the levels 7-14, the irreducible ones occupy only the levels 8-12. We list below the representatives of the 161 irreducible classes.

Level 12

1 aaaa aaAe aaAE abaA aBaA bAAb Aabd AabD AaBc AaBC AAcB BACa [2,2,3,5] 2 aaba aadA aaDA abBa aBBa bAbb Aaad AaAc AaAC AaaD AAcB BACA [2,2,4,4]

L	evel 11												
1	aAaa	aAcA	aACA	baab	babB	baBB	AAac	AAaC	AbAb	Baad	BaaD	BBAa	[2, 2, 3, 4]
2	acaa	abAa	aCaa	aBAb	aBAB	bAaA	Aaba	AAca	AACa	AaBc	AaBC	BbbA	[2, 3, 3, 3]
3	aaaa	aAae	aAaE	aabA	aaBA	bAAb	Aaad	AaaD	AAac	AAaC	AbAB	BBAa	[2, 2, 2, 5]
4	aAad	aAaD	aAAb	baaa	babA	baBA	AAac	AAaC	AbAa	Baae	BaaE	BBAB	[2, 2, 2, 5]
5	aaca	aaaA	aaCa	abAA	aBAA	bAab	Aaad	AaaD	AaAc	AaAC	AAbB	BABa	[2, 2, 3, 4]
6	aaca	aabA	aaCa	abBA	aBBA	bAAa	Aaad	AaaD	AaAc	AaAC	AAab	BAbB	[2, 2, 3, 4]
7	aaab	aaAd	aaAD	abaB	aBaB	bAAc	Aaca	AaaA	AaAA	AaCa	AAbC	BABb	[2, 2, 3, 4]
8	aaab	aaAd	aaAD	abaB	aBaB	bAAc	Aaca	AabA	AaBA	AaCa	AABC	BAbb	[2, 2, 3, 4]
9	aaba	aaaA	aaAA	abBa	aBBa	bAbb	Aaad	AaaD	AaAc	AaAC	AAcB	BACA	[2, 2, 3, 4]
10	aaaa	aaAd	aaAD	abaA	aBaA	bAAb	Aaba	AabA	AaBc	AaBC	AAcB	BACa	[2, 2, 3, 4]
11	aaab	aaAd	aaAD	abaB	aBaB	bAAA	Aaba	AabA	AaBc	AaBC	AAca	BACb	[2, 2, 3, 4]
12	aaab	aaAd	aaAD	abaB	aBaB	bAAc	Aaba	AabA	AaBa	AaBA	AAcC	BACb	[2, 2, 3, 4]
13	aaaa	aaAa	aaAA	abaA	aBaA	bAAb	Aabd	AabD	AaBc	AaBC	AAcB	BACa	[2, 2, 3, 4]
14	aaab	aaAa	aaAA	abaB	aBaB	bAAA	Aabd	AabD	AaBc	AaBC	AAca	BACb	[2, 2, 3, 4]
15	aaab	aaAa	aaAA	abaB	aBaB	bAAc	Aaba	AabA	AaBd	AaBD	AAcC	BACb	[2, 2, 3, 4]
16	aaca	aabA	aaBA	abCa	aBCa	bABA	Aaad	AaAc	AaAC	AaaD	AAbb	BAcB	[2, 2, 3, 4]
17	acaa	aacA	aaCA	aAaA	aAAc	aAAC	aCaa	baAa	Abab	AAbB	ABBA	BaAa	[2, 3, 3, 3]
18	aaac	aAaa	abAa	aAcA	aaAA	aACA	aaaC	bBAa	A cab	ACbA	AbBB	BBAa	$\left[2,3,3,3\right]$

Ι	Level 10												
1	abaB	aaAA	bbab	bBad	bBaD	baAa	bAAc	bAAC	AbaB	AaAA	BAbb	BBBa	[2, 2, 2, 4]
2	aaBA	aAaB	baca	babA	baCa	bAab	bAAc	bAAC	AaBA	AAaB	BbAa	BBbb	[2, 2, 3, 3]
3	aaba	aAAa	aaBb	aaBB	bAaa	bBbA	Aaca	AaCa	AAAa	AbAA	BAac	BAaC	$\left[2,2,3,3\right]$
4	aAAa	aAac	aAaC	baba	baBb	baBB	AAca	AACa	AbbA	Baaa	BaAa	BBAA	[2, 2, 3, 3]
5	aAba	aABc	aABC	baaa	baAb	baAB	AAca	AACa	AbaA	Baaa	BaAa	BBbA	[2, 2, 3, 3]
6	aAaa	aAAc	aAAC	baca	baCb	baCB	AAba	AABa	AbcA	Baba	BaBa	BBaA	[2, 2, 3, 3]
7	aaaa	aAad	aAaD	aabA	aaBA	bAAb	Aaac	AaaC	AAaa	AAaA	AbAB	BBAa	[2, 2, 2, 4]
8	aaab	aAad	aAaD	aabB	aaBB	bAAA	Aaac	AaaC	AAaa	AAaA	AbAa	BBAb	[2, 2, 2, 4]
9	aaaa	aAaa	aAaA	aabA	aaBA	bAAb	Aaad	AaaD	AAac	AAaC	AbAB	BBAa	[2, 2, 2, 4]
10	aaab	aAaa	aAaA	aabB	aaBB	bAAA	Aaad	AaaD	AAac	AAaC	AbAa	BBAb	[2, 2, 2, 4]
11	aaab	aAaa	aAaA	aabB	aaBB	bAAc	Aaad	AaaD	AAaa	AAaA	AbAC	BBAb	[2, 2, 2, 4]
12	aaab	aAaa	aAaA	aabB	aaBB	bAAc	Aaaa	AaaA	AAad	AAaD	AbAC	BBAb	[2, 2, 2, 4]
13	aaaA	aAad	aAaD	abAb	baaa	bBAa	AAac	AAaC	AabA	AaBA	Baaa	BAAB	[2, 2, 2, 4]
14	aaaA	aAad	aAaD	aAAb	baaa	bBAB	AaaA	AAac	AAaC	AbAA	Baba	BaBa	[2, 2, 2, 4]
15	aaab	aAaA	aabB	aaBB	bAaa	bBAb	Aaad	AaaD	AAaA	AbAc	BAaa	BAAC	[2, 2, 2, 4]
16	aAac	aAaC	aAAb	baaa	babA	baBA	AAaa	AAaA	AbAa	Baad	BaaD	BBAB	[2, 2, 2, 4]
17	aAac	aAaC	aAAA	baab	babB	baBB	AAaa	AAaA	AbAb	Baad	BaaD	BBAa	[2, 2, 2, 4]
18	aAaa	aAaA	aAAb	baaa	babA	baBA	AAac	AAaC	AbAa	Baad	BaaD	BBAB	[2, 2, 2, 4]
19	aAaa	aAaA	aAAA	baab	babB	baBB	AAac	AAaC	AbAb	Baad	BaaD	BBAa	[2, 2, 2, 4]
20	aAaa	aAaA	aAAc	baab	babB	baBB	AAaa	AAaA	AbAb	Baad	BaaD	BBAC	[2, 2, 2, 4]
21	aAad	aAaD	aAAb	baaa	babA	baBA	AAac	AAaC	AbAa	Baaa	BaaA	BBAB	[2, 2, 2, 4]
22	aAad	aAaD	aAAA	baab	babB	baBB	AAac	AAaC	AbAb	Baaa	BaaA	BBAa	[2, 2, 2, 4]
23	aAad	aAaD	aAAc	baab	babB	baBB	AAaa	AAaA	AbAb	Baaa	BaaA	BBAC	[2, 2, 2, 4]
24	aAaa	aAaA	aAAc	baab	babB	baBB	AAad	AAaD	AbAb	Baaa	BaaA	BBAC	[2, 2, 2, 4]
25	aaca	aaaA	aaCa	abAA	aBAA	bAab	Aaac	AaaC	AaAa	AaAA	AAbB	BABa	[2, 2, 3, 3]
26	aaac	aaAa	aaaC	abAA	aBAA	bABa	Aaca	AaCa	AabA	AaBA	AAbb	BAaB	[2, 2, 3, 3]
27	aaca	aaaA	aaCa	abAA	aBAA	bAab	Aaaa	AaaA	AaAc	AaAC	AAbB	BABa	[2, 2, 3, 3]
28	aaac	aaAa	aaaC	abAA	aBAA	bABa	Aaba	AaBa	AacA	AaCA	AAbb	BAaB	[2, 2, 3, 3]
29	aaac	aaAa	aaaC	abAA	aBAA	bAca	Aaba	AaBa	AabA	AaBA	AACb	BAaB	[2, 2, 3, 3]
30	aaca	aabA	aaCa	abBA	aBBA	bAAa	Aaac	AaaC	AaAb	AaAB	AAaB	BAbb	[2, 2, 3, 3]
31	aaca	aabA	aaCa	abBA	aBBA	bAAa	Aaab	AaaB	AaAc	AaAC	AAaB	BAbb	[2, 2, 3, 3]
32	aaca	aabA	aaCa	abBA	aBBA	bAAa	Aaab	AaaB	AaAb	AaAB	AAac	BAbC	[2, 2, 3, 3]
33	aaca	aaaA	aaCa	abAA	aBAA	bAac	Aabb	AaaB	AaAB	AaBb	AABC	BAba	[2, 2, 3, 3]
34	aaba	aacA	aaCA	abBa	aBBa	bAaA	Aaac	AaAa	AaAA	AaaC	AAAb	BAbB	[2, 2, 3, 3]
35	aaba	aacA	aaCA	abBa	aBBa	bAAA	Aaac	AaAa	AaAA	AaaC	AAab	BAbB	[2, 2, 3, 3]
36	aaba	aaaA	aaAA	abBa	aBBa	bAcA	AaAc	Aaaa	AaaA	AaAC	AACb	BAbB	[2, 2, 3, 3]
37	aaaa	aaaA	aaAA	abAa	aBAa	bAcA	Aabc	AaBa	AaBA	AabC	AACb	BAaB	[2, 2, 3, 3]
38	aaca	aabA	aaBA		aBCa		Aaac	AaAa	AaAA	AaaC	AAbb	BAcB	[2, 2, 3, 3]
39	aaba	aaaA	aaBa	abAA	aBAA	bAac	Aacb	AaaB	AaAB	AaCb	AABC	BAba	[2, 2, 3, 3]
40	aaaa	aaAa	aabA		aBBA		Aaab	AaAb	AacB	AaCB		BAbC	[2, 2, 3, 3]
41	aaab	aaAa			aBaB				AacA		AABC		[2, 2, 3, 3]
42	aaaa	aaAa	aaAA	abaA	aBaA	bAAc	Aabb	AaBb	AacB	AaCB	AABC	BAba	[2, 2, 3, 3]

43	aaab	aaAa	aaAA	abaB	aBaB	bAcb	Aaba	AaBa	AabA	AaBA	AACc	BAAC	[2, 2, 3, 3]
44	aaaa	aaAa	aaAA	abaA	aBaA	bAca	Aabb	AaBb	AabB	AaBB	AACc	BAAC	$\left[2,2,3,3\right]$
45	aaab	aaAa	aaaB	abAA	aBAA	bAca	Aabb	AaBa	AaBA	AabB	AACc	BAaC	$\left[2,2,3,3\right]$
46	aacA	aaCA	abAa	aBAa	baaa	bAab	AaaA	AaAc	AaAC	AAbB	Baaa	BABA	$\left[2,2,3,3\right]$
47	aaAa	aabA	abBA	aBBA	baaa	bAca	AaAc	AaaA	AaAC	AACb	Baaa	BAbB	$\left[2,2,3,3\right]$
48	aaAa	aacA	aaCA	aABA	baaa	bAbb	AaAc	AaaA	AaAC	AAAB	Bbaa	BBaa	[2, 2, 3, 3]
49	aaAa	aacA	aaCA	aAAB	baaa	bAbb	AaAc	AaaA	AaAC	AABA	Bbaa	BBaa	[2, 2, 3, 3]
50	aAad	abAd	aBAb	aBAB	baad	bbbD	AAbd	AaAd	AABc	AABC	Baaa	BaaA	[2, 2, 2, 4]
51	abAa	aAaa	aBAc	aBAC	baaa	bbbA	AaAa	AAca	AACa	AABA	Baab	BaaB	[2, 2, 3, 3]

Level 9												
1 a a a B	aBAa	baab	bAac	bAaC	bbAa	baAA	bAAA	AaaB	ABAa	BAbA	BbBb	[2, 2, 2, 3]
2 abaB	aaAa	bbab	bBaa	bBaA	baAA	bAAc	bAAC	AbaB	AaAa	BAbb	BBBA	[2, 2, 2, 3]
$3 \ abaA$	aaAB	bbaa	bBaa	bBaA	baAb	bAAc	bAAC	AbaA	AaAB	BAba	BBBb	[2, 2, 2, 3]
$4 \ abaC$	aaAA	bbac	bBab	bBaB	baAa	bAAb	bAAB	AbaC	AaAA	BAbc	BBBa	[2, 2, 2, 3]
5 aaaa	aaAa	aAAa	aabA	bAaa	bbBA	Aaaa	AbAa	ABAc	ABAC	BAab	BAaB	[2, 2, 2, 3]
$6 \ abAa$	aBAc	aBAC	baaa	bAaa	bbBA	AaAa	AAAa	AAbA	BAaa	Baab	BaaB	[2, 2, 2, 3]
7 aaAa	aAAc	aAAC	bbaa	bBaa	baBA	AaAa	AAAa	ABbA	BBaa	Bbab	BbaB	[2, 2, 2, 3]
$8 \ aBaa$	aAAc	aAAC	bbaa	baAa	baBA	ABaa	AAAa	ABbA	BaAa	Bbab	BbaB	[2, 2, 2, 3]
9 aaaa	aAaa	aAaA	aabA	aaBA	bAAb	Aaac	AaaC	AAaa	AAaA	AbAB	BBAa	[2, 2, 2, 3]
10 $aaab$	aAaa	aAaA	aabB	aaBB	bAAA	Aaac	AaaC	AAaa	AAaA	AbAa	BBAb	[2, 2, 2, 3]
$11 \ aaaa$	aAab	aAaB	aabA	aaBA	bAAB	Aaac	AaaC	AAaa	AAaA	AbAb	BBAa	[2, 2, 2, 3]
12 $aaaa$	aAaa	aAaA	aabA	aaBA	bAAB	Aaac	AaaC	AAab	AAaB	AbAb	BBAa	[2, 2, 2, 3]
13 $aaaa$	aAaa	aAaA	aabA	aaBA	bAAb	Aaaa	AaaA	AAac	AAaC	AbAB	BBAa	[2, 2, 2, 3]
$14 \ aaab$	aAaa	aAaA	aabB	aaBB	bAAA	Aaaa	AaaA	AAac	AAaC	AbAa	BBAb	[2, 2, 2, 3]
15 $aaaa$	aAaa	aAaA	aabA	aaBA	bAAB	Aaab	AaaB	AAac	AAaC	AbAb	BBAa	[2, 2, 2, 3]
$16 \ aaab$	aAaa	aAaA	aabB	aaBB	bAAc	Aaaa	AaaA	AAaa	AAaA	AbAC	BBAb	[2, 2, 2, 3]
$17 \ aaaa$	aAab	aAaB	aabA	aaBA	bAAc	Aaab	AaaB	AAaa	AAaA	AbAC	BBAa	[2, 2, 2, 3]
18 $aaaa$	aAaa	aAaA	aabA	aaBA	bAAc	Aaab	AaaB	AAab	AAaB	AbAC	BBAa	[2, 2, 2, 3]
$19 \ aaac$	aAab	aAaB	aabC	aaBC	bAAA		AaaB	AAaa	AAaA	AbAa	BBAc	[2, 2, 2, 3]
20 $aaac$	aAab	aAaB	aabC	aaBC	bAAB	Aaaa	AaaA	AAaa	AAaA	AbAb	BBAc	[2, 2, 2, 3]
$21 \ aaaA$	aAac	aAaC	abAb	baaa	bBAa		AAaA		AaBA	Baaa	BAAB	[2, 2, 2, 3]
$22 \ aaaA$	aAac	aAaC	abAB	baaa	bBAa				AaBA	Baaa	BAAb	[2, 2, 2, 3]
$23 \ aaaA$		aAaA	abAb	baaa	bBAa		AAaC			Baaa	BAAB	[2, 2, 2, 3]
	aAab	aAaB	abAB	baaa	bBAa		AAaC		AaBA	Baaa	BAAb	[2, 2, 2, 3]
	aAab	aAaB	abAc	baaa	bBAa		AAaB		AaBA	Baaa	BAAC	[2, 2, 2, 3]
	aAac	aAaC	aAAb	baaa	bBAB		AAaa	AAaA		Baba	BaBa	[2, 2, 2, 3]
	aAac	aAaC	aAAB	baaa	bBAb		AAab	AAaB		Baba	BaBa	[2, 2, 2, 3]
		aAaA	aAAb	baaa	bBAB	AaaA		AAaC		Baba	BaBa	[2, 2, 2, 3]
	aAab	aAaB	aAAB		bBAb		AAac	AAaC		Baba	BaBa	[2, 2, 2, 3]
	aAab	aAaB	aAAc	baaa	bBAC			AAaB		Baba	BaBa	[2, 2, 2, 3]
	aAaA			bAaa	bBAa	Aaac	AaaC	AAaA		BAaa	BAAB	[2, 2, 2, 3]
32 aaaa			aaBA		bBAa		AaaC		AbAb		BAAB	[2, 2, 2, 3]
33 aaab			aaBB				AaaC					[2, 2, 2, 3]
34 aaab	aAaA		aaBB		bBAb	Aaac		AAaA		BAaa		[2, 2, 2, 3]
35 aaab				bAaA			AaaA				BAAC	[2, 2, 2, 3]
36 <i>aaaa</i>			aaBA				AaaA				BAAC	[2, 2, 2, 3]
$37 \ aaac$			aaBC	bAaa	bBAc		AaaB			BAaa	BAAb	[2, 2, 2, 3]
38 aAaa				baaC			AAaA		Baaa	BabA	BaBA	[2, 2, 2, 3]
$39 \ aaaA$				bAab			AAaB		BAab	Baba D C	BaBa	[2, 2, 2, 3]
		aAAA		babB			AAaA		Baac	BaaC	BBAa	[2, 2, 2, 3]
41 aAab				babA	baBA		AAaA		Baac			[2, 2, 2, 3]
42 aAaa				babA	baBA		AAaB		Baac	BaaC		[2, 2, 2, 3]
43 aAac				babB	baBB		AAaA A A a A		Baaa Baab	BaaA BaaB		[2, 2, 2, 3]
44 aAac				babA			AAaA		Baab	BaaB		[2, 2, 2, 3]
45 aAac	aAaC	aaab	oada	babA	0 <i>a</i> BA	AA40	AAaB	A0Aa	Baaa	BaaA	DDA0	[2, 2, 2, 3]

46	aAaa	aAaA	aAAA	baab	babB	baBB	AAac	AAaC	AbAb	Baaa	BaaA	BBAa	[2, 2, 2, 3]
47	aAaa	aAaA	aAAB	baaa	babA	baBA	AAac	AAaC	AbAa	Baab	BaaB	BBAb	[2, 2, 2, 3]
48	aAab	aAaB	aAAB	baaa	babA	baBA	AAac	AAaC	AbAa	Baaa	BaaA	BBAb	[2, 2, 2, 3]
49	aAaa	aAaA	aAAc	baab	babB	baBB	AAaa	AAaA	AbAb	Baaa	BaaA	BBAC	[2, 2, 2, 3]
50	aAab	aAaB	aAAc	baaa	babA	baBA	AAaa	AAaA	AbAa	Baab	BaaB	BBAC	[2, 2, 2, 3]
51	aAaa	aAaA	aAAc	baaa	babA	baBA	AAab	AAaB	AbAa	Baab	BaaB	BBAC	[2, 2, 2, 3]
52	aAab	aAaB	aAAc	baaa	babA	baBA	AAab	AAaB	AbAa	Baaa	BaaA	BBAC	[2, 2, 2, 3]
53	aAab	aAaB	aAAA	baac	babC	baBC	AAaa	AAaA	AbAc	Baab	BaaB	BBAa	[2, 2, 2, 3]
54	aAaa	aAaA	aAAA	baac	babC	baBC	AAab	AAaB	AbAc	Baab	BaaB	BBAa	[2, 2, 2, 3]
55	aAaa	aAaA	aAAB	baac	babC	baBC	AAaa	AAaA	AbAc	Baab	BaaB	BBAb	[2, 2, 2, 3]
56	aAac	aAaC	abAa	baaa	babA	baBA	AAaa	AAaA	AAAb	Baaa	BaaA	BBAB	[2, 2, 2, 3]
57	aAac	aAaC	aAAb	baaa	babA	baBA	AAaa	AAaA	AbAa	Baaa	BaaA	BBAB	[2, 2, 2, 3]
58	aaaA	aAab	aAbB	aABB	baaa	baAc	AbaA	AAaa	ABaA	ABAC	Baaa	BbAb	[2, 2, 2, 3]
59	aAac	aAaC	aabA	aaBA	baaa	bAAb	AbaA	AAaa	ABaA	ABAB	Baaa	BbAa	[2, 2, 2, 3]
60	aaaA	aAac	aAaC	abAA	baaa	bBAb	AbaA	AAaa	ABaA	AAAB	Baba	BaBa	[2, 2, 2, 3]
61	aaaA	aAac	aAaC	aAAB	baaa	bBAb	AbaA	AAaa	ABaA	AbAA	Baba	BaBa	[2, 2, 2, 3]
62	aAac	abAc	aBAa	aBAA	baac	bbbC	AAbc	AaAc	AABa	AABA	Baab	BaaB	[2, 2, 2, 3]
63	aBaA	aaBa	baba	bbaA	bAaa	baAA	bAAc	bAAC	ABaA	AaBa	Bbbb	BAAB	[2, 2, 2, 3]
64	aAaa	abAa	aBAc	aBAC	baaa	bbBA	AAba	AaAa	AABa	AAbA	Baab	BaaB	[2, 2, 2, 3]

Level 8

1	aaBA	aAAB	aBaa	bbaa	baAa	babA	bAAb	bAaA	AaBA	AAAB	ABaa	Bbbb	[2, 2, 2, 2]
2	aABa	aBAA	baaa	baAa	bAba	bbAA	baaA	bAaA	AABa	ABAA	Baab	BbbB	[2, 2, 2, 2]
3	aBaa	aaBA	bbaa	bbAa	bBAa	babA	bAaA	bAAA	ABaa	AaBA	BAAb	BbbB	[2, 2, 2, 2]
4	aBba	aaAA	bbba	bbBa	bBBa	baaA	bAaA	bAAA	ABba	AaAA	BABb	BbaB	[2, 2, 2, 2]
5	aBba	aaAA	bbba	baBa	bABa	bbaA	bAAA	bBaA	ABba	AaAA	BABb	BbaB	[2, 2, 2, 2]
6	aaaa	aAaa	aAAa	aaAb	aaAB	bbaA	Aaaa	AaAa	AAaa	AAAa	ABBA	BAbA	[2, 2, 2, 2]
7	aAaa	aaAa	aAAa	abAA	baaa	bBBA	AAaa	AaAa	AAAa	AAbA	Baab	BaaB	[2, 2, 2, 2]
8	aaaa	aAaa	aAAa	aBBA	baAa	bbAA	AAaa	AAAa	Aaab	AaaB	BaAa	BAbA	[2, 2, 2, 2]
9	aaaa	aAaa	aaAa	aBBA	bAAa	bbAA	AAaa	AaAa	Aaab	AaaB	BAAa	BAbA	[2, 2, 2, 2]
10	aAaa	aAAa	abAA	baaa	baAa	bBBA	AAaa	AAAa	AAbA	BaAa	Baab	BaaB	[2, 2, 2, 2]
11	aAaa	aaAa	abAA	baaa	bAAa	bBBA	AAaa	AaAa	AAbA	BAAa	Baab	BaaB	[2, 2, 2, 2]
12	aAaa	abAa	aBAa	aBbA	baaa	baBA	AaAa	AAAa	AAab	AAaB	Baaa	BbAA	[2, 2, 2, 2]
13	aAaa	aaAa	aAAa	aBbA	baaa	baBA	AbAa	ABAa	AAab	AAaB	Baaa	BbAA	[2, 2, 2, 2]
14	aaAa	aAAa	aBAA	bbaa	bBaa	bbBA	AaAa	AAAa	AabA	Baaa	BAab	BAaB	[2, 2, 2, 2]
15	aaAa	aAAa	abBA	bbaa	bBab	bBaB	AaAa	AAAa	AaAA	Baaa	BAaa	BAbA	[2, 2, 2, 2]
16	aaaa	aaAa	aAAa	aabA	bAaa	bbBA	Aaaa	AbAa	ABAa	ABAA	BAab	BAaB	[2, 2, 2, 2]
17	aaaa	aaAa	aAAa	aBAA	bAaa	bbBA	Aaaa	AbAa	ABAa	AabA	BAab	BAaB	[2, 2, 2, 2]
18	abAa	aBAa	aAbA	baaa	bAaa	bbBA	AaAa	AAAa	ABAA	BAaa	Baab	BaaB	[2, 2, 2, 2]
19	abAa	aBAa	aBAA	baaa	bAaa	bbBA	AaAa	AAAa	AAbA	BAaa	Baab	BaaB	[2, 2, 2, 2]
20	aaAa	aAAa	aAAA	bbaa	bBaa	baBA	AaAa	AAAa	ABbA	BBaa	Bbab	BbaB	[2, 2, 2, 2]
21	aBaa	aAAa	aaAA	bbaa	baAa	bABA	ABaa	AAAa	ABbA	BaAa	Bbab	BbaB	[2, 2, 2, 2]
22	aBaa	aAAa	aAAA	bbaa	baAa	baBA	ABaa	AAAa	ABbA	BaAa	Bbab	BbaB	[2, 2, 2, 2]
23	abAa	aBAa	aAab	aAaB	baaa	baBA	AaAa	AAba	AABa	ABbA	Baaa	BbAA	[2, 2, 2, 2]
24	aAaa	abAa	aBAa	abAA	baaa	bBBA	AAba	AaAa	AABa	AAbA	Baab	BaaB	[2, 2, 2, 2]
25	aAaa	abAa	aBAa	aAbA	baaa	bBBA	AAba	AaAa	AABa	AbAA	Baab	BaaB	[2, 2, 2, 2]
26	aBAa	aaaB	baab	bAaa	bbAa	bAbA	baAA	bABA	ABAa	AaaB	BbBb	BAbA	[2, 2, 2, 2]

5. The case
$$m = 8$$

Arrows from level 11 to 10:

$1 \rightarrow 1, 2, 3, 6$	$d_4 = a_4, d_4 = B_4, d_4 = A_4, d_4 = b_4$
$2 \rightarrow 4, 5$	$d_4 = b_4, d_4 = B_4$
$3 \rightarrow 1, 5, 8, 9,$	$c_3 = B_3, c_3 = b_3, c_4 = A_4, c_4 = a_4,$
$3\rightarrow 13, 15, 16$	$d_4 = a_4, d_4 = b_4, d_4 = B_4$
$4 \to 2, 3, 4, 5, 9,$	$c_2 = B_2, c_3 = b_3, c_3 = B_3, c_2 = b_2, c_4 = b_4,$
$4 \to 10, 12, 13, 14, 15$	$c_4 = B_4, d_4 = b_4, c_4 = A_4, d_4 = A_4, d_4 = a_4$
$5 \to 4, 6, 7, 10,$	$c_3 = b_3, c_3 = B_3, c_4 = B_4, c_4 = b_4,$
$5\rightarrow 11, 13, 14$	$d_4 = A_4, c_4 = a_4, d_4 = a_4$
$6 \rightarrow 7, 20, 21$	$b_1 = a_1, c_4 = b_4, c_4 = B_4$

Arrows from level 10 to 9:

$1\to 4, 6, 11, 12, 14, 18$	$c_3 = b_3, c_3 = B_3, d_4 = a_4, d_4 = A_4, d_4 = b_4, d_4 = B_4$
$2 \rightarrow 2, 3, 17, 18, 19, 20$	$c_3 = b_3, c_3 = B_3, d_4 = b_4, d_4 = a_4, d_4 = A_4, d_4 = B_4$
$3 \rightarrow 3, 6, 12, 13, 15, 19$	$c_3 = B_3, c_3 = b_3, d_4 = a_4, d_4 = A_4, d_4 = b_4, d_4 = B_4$
$4 \rightarrow 2, 5, 7, 9, 10, 21, 22, 24$	$c_3 = b_3, c_3 = B_3, d_4 = b_4, c_4 = A_4, c_4 = a_4, d_4 = A_4, d_4 = B_4, d_4 = a_4$
$5 \rightarrow 5, 6, 8, 9, 22, 23, 24, 25$	$c_3 = b_3, d_4 = a_4, c_4 = A_4, c_4 = a_4, d_4 = b_4, d_4 = a_4, d_4 = A_4, d_4 = B_4$
$6 \rightarrow 1, 2, 14, 15, 16, 17$	$c_3 = b_3, c_3 = B_3, d_4 = a_4, d_4 = A_4, d_4 = b_4, d_4 = B_4$
$7 \rightarrow 7, 16$	$c_2 = b_2, c_4 = B_4$
$8 \rightarrow 8, 11$	$c_2 = B_2, c_2 = b_2$
$9\rightarrow9,12,20,25$	$c_3 = B_3, c_3 = b_3, c_2 = B_2, c_2 = b_2$
$10 \to 10, 13, 17, 22$	$c_3 = B_3, c_3 = b_3, c_2 = B_2, c_2 = b_2$
$11 \rightarrow 10, 16$	$c_2 = b_2, c_2 = B_2$
$12 \rightarrow 19, 24$	$c_2 = b_2, c_2 = B_2$
$13 \to 14, 15, 18, 21, 23, 24$	$c_2 = b_2, c_3 = b_3, c_4 = b_4, c_3 = B_3, c_4 = B_4, c_2 = B_2$
$14\rightarrow7,9,13,17$	$c_3 = b_3, c_2 = B_2, c_3 = B_3, c_2 = b_2$
$15\rightarrow 8, 12, 20, 22$	$c_2 = B_2, c_3 = B_3, c_2 = b_2, c_3 = b_3$
$16 \rightarrow 11, 25$	$c_2 = B_2, c_2 = b_2$
$17 \rightarrow 26$	$c_4 = A_4$
$18 \rightarrow 27$	$d_4 = A_4$
$19 \rightarrow 27, 28$	$c_4 = A_4, c_4 = b_4$
$20 \rightarrow 16, 30, 31$	$b_2 = a_2, c_3 = b_3, c_3 = B_3$
$21 \rightarrow 7, 31, 33$	$b_2 = a_2, c_3 = B_3, c_3 = b_3$
$22 \rightarrow 34, 35, 36$	$b_4 = A_4, d_4 = c_4, c_4 = a_4$
$23 \rightarrow 29, 34, 35$	$c_3 = a_3, c_4 = B_4, c_3 = b_3$
$24 \rightarrow 37$	$b_4 = a_4$
$25 \to 30, 37, 39, 40$	$b_3 = a_3, c_4 = b_4, c_4 = B_4, c_3 = b_3$
$26 \rightarrow 32, 41, 43$	$b_4 = a_4, c_4 = b_4, c_4 = B_4$
$27 \rightarrow 38, 39, 41$	$c_4 = a_4, d_4 = C_4, b_4 = a_4$
$28\rightarrow 1, 40, 42, 43$	$b_1 = a_1, b_4 = a_4, c_4 = a_4, b_4 = A_4$
$29 \rightarrow 3, 44, 45$	$b_1 = a_1, b_4 = a_4, b_4 = A_4$

Arrows from level 9 to 8:

$1 \rightarrow 1, 2, 6$	$d_4 = a_4, c_4 = a_4, c_4 = A_4$
$2 \to 5, 7, 8, 9$	$c_4 = a_4, c_4 = A_4, d_4 = a_4, d_4 = A_4$
$3 \rightarrow 10, 21, 22, 23$	$d_4 = A_4, c_4 = A_4, c_4 = a_4, d_4 = a_4$
$4 \rightarrow 12, 19, 20$	$c_4 = a_4, d_4 = a_4, d_4 = A_4$
$5 \rightarrow 3, 4, 11, 15, 16$	$c_4 = a_4, c_4 = b_4, c_4 = B_4, d_4 = A_4, d_4 = a_4$
$6 \rightarrow 13, 14, 17, 18$	$c_4 = a_4, c_4 = A_4, d_4 = A_4, d_4 = a_4$
$7 \to 5, 11$	$c_3 = B_3, c_3 = b_3$
$8 \rightarrow 4, 18$	$c_3 = B_3, c_3 = b_3$
$9 \to 7, 11, 16, 17$	$c_4 = B_4, c_3 = B_3, c_4 = b_4, c_3 = b_3$
$10 \rightarrow 5, 15$	$c_3 = B_3, c_3 = b_3$
$11 \rightarrow 18, 19$	$c_3 = B_3, c_3 = b_3$
$12 \rightarrow 17, 18, 20, 21$	$c_4 = B_4, c_3 = b_3, c_4 = b_4, c_3 = B_3$
$13 \rightarrow 17, 22$	$c_3 = b_3, c_3 = B_3$
$14 \to 1, 8, 12, 13$	$c_3 = b_3, c_3 = B_3, c_4 = b_4, c_4 = B_4$
$15 \to 1, 9, 13, 23$	$c_3 = b_3, c_3 = B_3, c_4 = b_4, c_4 = B_4$
$16 \rightarrow 2, 5$	$c_3 = b_3, c_3 = B_3$
$17 \to 5, 6, 7, 22$	$c_4 = b_4, c_3 = b_3, c_3 = B_3, c_4 = B_4$
$18 \rightarrow 8, 12, 14, 23$	$c_3 = b_3, c_4 = b_4, c_4 = B_4, c_3 = B_3$
$19 \to 9, 10, 14$	$c_3 = b_3, c_3 = B_3, c_4 = b_4$
$20 \rightarrow 7, 21$	$c_3 = b_3, c_3 = B_3$
$21 \rightarrow 3, 8$	$c_3 = B_3, c_3 = b_3$
$22 \rightarrow 4,7,15,17$	$c_3 = B_3, c_3 = b_3, c_4 = b_4, c_4 = B_4$
$23 \rightarrow 3, 13$	$c_3 = b_3, c_3 = B_3$
$24 \rightarrow 3, 9, 14$	$c_3 = b_3, c_4 = B_4, c_3 = B_3$
$25 \rightarrow 16, 18$	$c_3 = b_3, c_3 = B_3$
$29 \rightarrow 25$	$c_4 = B_4$
$30 \rightarrow 2, 26, 27$	$b_3 = a_3, c_4 = b_4, c_4 = B_4$
$31 \rightarrow 5, 27, 28$	$b_3 = a_3, c_4 = b_4, c_4 = B_4$
$32 \rightarrow 6, 29, 33$	$b_3 = a_3, c_4 = B_4, c_4 = b_4$
$33 \rightarrow 11, 28, 32$	$b_3 = a_3, c_4 = b_4, c_4 = B_4$
$34 \rightarrow 25, 34$	$c_4 = a_4, c_4 = b_4$
$35 \rightarrow 34$	$c_4 = B_4$
$36 \rightarrow 25$	$b_4 = A_4$
$37 \rightarrow 26, 37$	$b_4 = a_4, c_4 = b_4$
$38 \rightarrow 27, 33, 41$	$c_4 = a_4, b_4 = a_4, c_4 = b_4$
$39 \rightarrow 27, 38$	$b_4 = a_4, c_4 = b_4$
$40 \rightarrow 2, 37, 38$	$b_3 = a_3, c_4 = b_4, c_4 = B_4$
$41 \rightarrow 33, 38$	$b_4 = a_4, c_4 = B_4$
$42 \to 1, 29, 36, 42$	$b_1 = a_1, b_4 = A_4, c_4 = B_4, c_4 = b_4$
$43 \to 6, 29, 38, 39$	$b_1 = a_1, b_4 = a_4, c_4 = b_4, c_4 = B_4$
	$b_2 = A_2, b_3 = A_3, c_4 = b_4, c_4 = B_4$
	$b_2 = A_2, b_1 = a_1, b_4 = A_4, c_4 = b_4, c_4 = B_4$
$46 \rightarrow 28,35$	$c_4 = B_4, b_4 = a_4$

$1 \rightarrow 9, 17$	$c_4 = b_4, c_4 = B_4$	$22 \rightarrow 3, 4$	$c_4 = B_4, c_4 = b_4$
$2 \rightarrow 1, 2$	$c_4 = b_4, c_4 = B_4$	$23 \rightarrow 17, 18$	$c_4 = b_4, c_4 = B_4$
$3 \rightarrow 10, 11$	$c_4 = B_4, c_4 = b_4$	$24 \rightarrow 5$	$b_4 = A_4$
$4 \rightarrow 5, 13$	$c_4 = b_4, c_4 = B_4$	$26 \rightarrow 1$	$b_4 = a_4$
$5 \rightarrow 2, 6$	$c_4 = b_4, c_4 = B_4$	$27 \rightarrow 2$	$b_4 = a_4$
$6 \rightarrow 2, 3$	$c_4 = b_4, c_4 = B_4$	$28 \rightarrow 6$	$b_4 = a_4$
$7 \rightarrow 4, 6$	$c_4 = B_4, c_4 = b_4$	$29 \rightarrow 3$	$b_4 = A_4$
$8 \rightarrow 11, 16$	$c_4 = B_4, c_4 = b_4$	$30 \rightarrow 10$	$b_4 = A_4$
$9 \rightarrow 11, 18$	$c_4 = b_4, c_4 = B_4$	$31 \rightarrow 8$	$b_4 = A_4$
$10 \rightarrow 8, 18$	$c_4 = B_4, c_4 = b_4$	$32 \rightarrow 12$	$b_4 = a_4$
$11 \rightarrow 6, 12$	$c_4 = B_4, c_4 = b_4$	$33 \rightarrow 2$	$b_4 = A_4$
$12 \rightarrow 16, 17$	$c_4 = B_4, c_4 = b_4$	$35 \rightarrow 6$	$b_4 = A_4$
$13 \rightarrow 9, 11$	$c_4 = b_4, c_4 = B_4$	$36 \rightarrow 9$	$b_4 = A_4$
$14 \rightarrow 11, 18$	$c_4 = b_4, c_4 = B_4$	$37 \rightarrow 1$	$b_1 = a_1$
$15 \rightarrow 5, 6$	$c_4 = B_4, c_4 = b_4$	$38 \rightarrow 2$	$b_1 = a_1$
$16 \rightarrow 12, 13$	$c_4 = b_4, c_4 = B_4$	$39 \rightarrow 3$	$b_4 = a_4$
$17 \rightarrow 4, 13$	$c_4 = b_4, c_4 = B_4$	$40 \rightarrow 7, 8$	$b_4 = A_4, b_1 = A_1$
$18 \rightarrow 13, 14$	$c_4 = b_4, c_4 = B_4$	$41 \rightarrow 2$	$b_1 = a_1$
$19 \rightarrow 14, 15$	$c_4 = B_4, c_4 = b_4$	$42 \rightarrow 3, 17$	$b_1 = a_1, b_2 = a_2$
$20 \rightarrow 7, 14$	$c_4 = b_4, c_4 = B_4$	$43 \rightarrow 4, 18$	$b_3 = A_3, b_4 = A_4$
$21 \to 4,7$	$c_4 = b_4, c_4 = B_4$		

6. The case m = 9

The UOMs occupy the levels 8 - 10. There are only two arrows from level 10 to 9:

$$1 \to 3, 4$$
 $c_3 = b_3, c_3 = a_3$

and eight arrows from level 9 to 8:

$1 \rightarrow 1, 3$	$c_4 = a_4, c_4 = b_4$	$3 \rightarrow 2, 5$	$c_4 = b_4, c_4 = a_4$
$2 \to 3,4$	$c_4 = a_4, c_4 = b_4$	$4 \rightarrow 1, 5$	$c_4 = a_4, c_4 = b_4$

7. The case m = 10

Arrows from level 11 to 10:

 $\begin{array}{l} 1 \rightarrow 1, 3, 5, 7 \ b_3 = a_3, c_3 = B_3, c_4 = a_4, d_4 = c_4 \\ 2 \rightarrow 2, 4, 6, 7 \ b_3 = a_3, c_4 = a_4, c_4 = b_4, d_4 = c_4 \end{array}$

Arrows from level 10 to 9:

$1 \rightarrow 1, 2, 3$	$d_4 = a_4, d_4 = b_4, d_4 = c_4$
$2\rightarrow 4,5,6,7,8$	$c_4 = a_4, c_4 = b_4, d_4 = a_4, d_4 = b_4, d_4 = c_4$
$3 \rightarrow 11, 12, 13$	$d_4 = a_4, d_4 = b_4, d_4 = c_4$
$4\rightarrow 3,4,6,9$	$c_4 = a_4, b_3 = a_3, b_3 = A_3, c_4 = b_4$
$5\to 1,2,8,10,11,12$	$b_3 = A_3, b_3 = a_3, c_4 = a_4, c_4 = b_4, c_3 = B_3, c_3 = b_3$
$6\rightarrow5,7,9,13$	$b_4 = A_4, b_4 = a_4, c_3 = a_3, c_3 = b_3$
$7 \rightarrow 3, 8, 13$	$b_3 = a_3, b_4 = a_4, c_3 = B_3$
$8\rightarrow2,14,15,16$	$c_4 = a_4, c_3 = a_3, c_3 = b_3, c_4 = b_4$
$9\to 6, 14, 18, 19$	$c_4 = a_4, c_3 = A_3, c_4 = b_4, c_3 = b_3$
$10 \rightarrow 20, 21$	$c_4 = a_4, c_3 = b_3$
$11 \rightarrow 15, 25$	$b_3 = a_3, c_3 = b_3$
$12 \rightarrow 27, 28$	$d_4 = b_4, d_4 = c_4$
$13 \rightarrow 29, 30, 31$	$d_4 = a_4, d_4 = b_4, d_4 = c_4$
$14\to 25, 26, 32, 33, 34$	$b_4 = a_4, d_4 = a_4, c_4 = b_4, d_4 = b_4, d_4 = c_4$
$15 \to 16, 22, 26, 35$	$b_4 = a_4, c_3 = a_3, c_3 = b_3, c_4 = b_4$
$16\to 21, 23, 27, 30$	$c_3 = a_3, c_4 = a_4, c_4 = b_4, c_3 = b_3$
$17 \to 18, 22, 36, 37$	$c_4 = a_4, c_3 = a_3, c_4 = b_4, c_3 = B_3$

Arrows from level 9 to 8:

$1 \rightarrow 1, 3$	$c_4 = a_4, c_4 = b_4$	$20 \rightarrow 13, 14$	$c_4 = a_4, c_4 = b_4$
$2 \rightarrow 1, 2$	$c_4 = a_4, c_4 = b_4$	$21 \rightarrow 14, 16$	$c_4 = a_4, c_4 = b_4$
$3 \rightarrow 1$	$c_4 = a_4$	$22 \rightarrow 9, 17$	$c_4 = a_4, c_4 = b_4$
$4 \rightarrow 1, 4$	$c_4 = a_4, c_4 = b_4$	$23 \rightarrow 14, 18$	$c_4 = a_4, c_4 = b_4$
$5 \rightarrow 5, 6$	$c_4 = a_4, c_4 = b_4$	$24 \rightarrow 15, 19$	$c_4 = a_4, c_4 = b_4$
$6 \rightarrow 1, 5$	$c_4 = a_4, c_4 = b_4$	$25 \rightarrow 10, 20$	$c_4 = a_4, c_4 = b_4$
$7 \rightarrow 4, 6$	$c_4 = a_4, c_4 = b_4$	$26 \rightarrow 10, 21$	$b_4 = a_4, c_4 = b_4$
$8 \rightarrow 1, 6$	$b_4 = a_4, c_4 = b_4$	$27 \rightarrow 16, 22$	$c_4 = a_4, c_4 = b_4$
$9 \rightarrow 4, 5$	$c_4 = a_4, c_4 = A_4$	$28 \rightarrow 16$	$b_4 = a_4$
$10 \rightarrow 2, 3, 7, 8$	$b_4 = A_4, b_4 = a_4, c_4 = B_4, c_4 = b_4$	$29 \rightarrow 18, 23$	$c_4 = a_4, c_4 = b_4$
$11 \to 6,8$	$c_4 = a_4, c_4 = b_4$	$30 \rightarrow 18, 22$	$c_4 = a_4, b_4 = a_4$
$12 \rightarrow 6, 7$	$c_4 = a_4, c_4 = b_4$	$31 \rightarrow 18$	$c_4 = a_4$
$13 \rightarrow 6$	$c_4 = a_4$	$32 \rightarrow 19, 21$	$c_4 = a_4, c_4 = b_4$
$14 \rightarrow 1,9$	$c_4 = a_4, c_4 = b_4$	$33 \rightarrow 10, 19$	$b_4 = a_4, c_4 = A_4$
$15 \rightarrow 2, 10$	$c_4 = a_4, c_4 = b_4$	$34 \rightarrow 19, 20$	$c_4 = a_4, c_4 = b_4$
$16 \rightarrow 9, 10$	$c_4 = a_4, c_4 = b_4$	$35 \rightarrow 17, 21$	$c_4 = a_4, c_4 = b_4$
$17 \rightarrow 11, 13$	$c_4 = b_4, c_4 = a_4$	$36 \rightarrow 17, 24$	$c_4 = a_4, c_4 = b_4$
$18 \rightarrow 9, 12$	$c_4 = b_4, c_4 = a_4$	$37 \rightarrow 12, 24$	$c_4 = a_4, c_4 = b_4$
$19 \to 5, 12$	$c_4 = a_4, c_4 = b_4$		

8. The case m = 12

Arrows from level 12 to 11:

 $\begin{array}{ll} 1 \rightarrow 3, 4, 10, 11 & b_3 = a_3, c_3 = b_3, d_4 = a_4, d_4 = b_4, \\ 1 \rightarrow 12, 13, 14, 15 & d_4 = c_4, e_4 = a_4, e_4 = b_4, e_4 = d_4 \\ 2 \rightarrow 5, 6, 7, 8, & b_3 = a_3, c_3 = a_3, c_4 = a_4, c_4 = b_4, \\ 2 \rightarrow 9, 12, 13, 16 & d_3 = a_3, d_4 = c_4, d_3 = b_3, d_3 = c_3 \end{array}$

 $1 \rightarrow 1, 4, 5, 6, 19$ $c_3 = b_3, d_4 = A_4, d_4 = b_4, d_4 = c_4, c_3 = a_3$ $2 \rightarrow 2, 3, 5, 31$ $c_4 = b_4, c_2 = a_2, c_2 = b_2, c_4 = a_4$ $3 \to 7, 8, 9,$ $c_4 = a_4, c_4 = b_4, e_4 = a_4,$ $3 \rightarrow 10, 11, 12$ $e_4 = b_4, e_4 = c_4, e_4 = d_4$ $4 \rightarrow 16, 17, 18, 19, 20, c_4 = a_4, c_4 = b_4, d_4 = a_4, d_4 = b_4, d_4 = c_4,$ $4 \rightarrow 21, 22, 23, 24$ $e_4 = a_4, e_4 = b_4, e_4 = c_4, e_4 = d_4$ $5 \rightarrow 9, 13, 25, 26,$ $c_3 = a_3, c_3 = b_3, c_4 = a_4, c_4 = b_4,$ $5 \rightarrow 27, 28, 29$ $d_4 = a_4, d_4 = b_4, d_4 = c_4,$ $6 \rightarrow 14, 21, 30, 31, 32$ $c_3 = a_3, c_3 = b_3, c_4 = b_4, d_4 = b_4, d_4 = c_4$ $c_3 = a_3, c_3 = b_3, b_3 = a_3, c_4 = a_4,$ $7 \rightarrow 11, 15, 25, 34,$ $7 \rightarrow 35, 36, 37, 38$ $c_4 = A_4, d_4 = a_4, d_4 = b_4, d_4 = c_4$ $8 \rightarrow 15, 20, 26, 30,$ $b_3 = a_3, c_3 = b_3, b_4 = a_4, c_4 = A_4,$ $8 \rightarrow 31, 41, 42$ $c_4 = a_4, d_4 = a_4, d_4 = b_4$ $9 \rightarrow 9, 14, 36, 41, 43$ $b_3 = A_3, c_3 = a_3, d_4 = a_4, d_4 = b_4, d_4 = c_4$ $b_3 = A_3, b_3 = a_3, c_3 = B_3, c_3 = b_3,$ $10 \rightarrow 7, 9, 16, 18,$ $c_4 = b_4, d_4 = b_4, d_4 = c_4$ $10 \rightarrow 33, 39, 45$ $11 \rightarrow 8, 10, 17, 19,$ $b_3 = A_3, b_3 = a_3, c_3 = B_3, c_3 = b_3,$ $c_4 = a_4, c_4 = b_4, d_4 = a_4, d_4 = b_4$ $11 \rightarrow 32, 33, 40, 42$ $12 \rightarrow 11, 12, 20, 29,$ $b_3 = a_3, b_3 = A_3, c_3 = b_3, b_4 = a_4,$ $12 \rightarrow 32, 44$ $c_4 = A_4, d_4 = b_4$ $13 \rightarrow 9, 37, 42, 44$ $b_3 = a_3, c_4 = a_4, d_4 = b_4, d_4 = c_4$ $b_3 = a_3, c_3 = b_3, d_4 = b_4, d_4 = a_4$ $14 \rightarrow 10, 22, 39, 40$ $b_3 = A_3, b_3 = a_3, c_3 = B_3, c_3 = b_3,$ $15 \rightarrow 11, 12, 23, 24,$ $15 \rightarrow 37, 40, 43, 45$ $b_4 = a_4, c_4 = A_4, d_4 = a_4, d_4 = b_4$ $16 \rightarrow 13, 14, 38$ $c_3 = a_3, b_3 = a_3, c_4 = a_4$ $17 \to 27, 46$ $c_2 = a_2, c_2 = b_2$ $18 \rightarrow 25, 36, 47, 48, 49$ $b_2 = a_2, c_4 = a_4, c_3 = b_3, c_2 = a_2, c_2 = A_2$

10 00 5.	
$1 \to 2, 3, 4$	$d_4 = a_4, d_4 = b_4, d_4 = c_4$
$2 \rightarrow 1, 3$	$c_4 = a_4, c_3 = b_3$
$3 \rightarrow 1, 5, 6, 29, 34$	$c_4 = b_4, c_3 = a_3, c_3 = b_3, b_4 = a_4, c_4 = a_4$
$4 \rightarrow 2, 6, 8, 41, 46$	$c_4 = b_4, c_3 = a_3, c_3 = b_3, b_4 = a_4, c_4 = a_4$
$5 \rightarrow 3, 6, 7, 40, 48$	$c_4 = b_4, c_3 = a_3, c_3 = b_3, b_4 = a_4, c_4 = a_4$
$6 \to 4, 6, 54$	$c_4 = b_4, c_3 = b_3, c_4 = a_4$
$7 \to 9, 11, 17$	$c_4 = b_4, c_3 = b_3, c_4 = a_4$ $d_4 = a_4, d_4 = b_4, d_4 = c_4$
$8 \to 10, 12, 19$	$a_4 = a_4, a_4 = b_4, a_4 = c_4$ $d_4 = a_4, d_4 = b_4, d_4 = c_4$
$9 \rightarrow 9, 12, 13, 15, 18$	$a_4 = a_4, a_4 = b_4, a_4 = b_4$ $c_4 = a_4, c_4 = b_4, d_4 = a_4, d_4 = b_4, d_4 = c_4$
$10 \rightarrow 10, 11, 14, 15, 20$	$c_4 = a_4, c_4 = b_4, a_4 = a_4, a_4 = b_4, a_4 = c_4$ $c_4 = a_4, c_4 = b_4, d_4 = a_4, d_4 = b_4, d_4 = c_4$
$11 \rightarrow 9, 10, 16, 18, 20$	$b_4 = a_4, c_4 = a_4, d_4 = a_4, d_4 = b_4, d_4 = c_4$
$12 \rightarrow 13, 14, 16, 17, 19$	$b_4 = a_4, c_4 = a_4, d_4 = a_4, d_4 = b_4, d_4 = c_4$
$13 \rightarrow 21, 22, 23, 24, 25$	$c_4 = a_4, c_4 = b_4, d_4 = a_4, d_4 = b_4, d_4 = c_4$
$14 \rightarrow 26, 27, 28, 29, 30$	$c_4 = a_4, c_4 = b_4, d_4 = a_4, d_4 = b_4, d_4 = c_4$
$15 \to 31, 32, 33, 34,$	$b_4 = a_4, b_4 = A_4, c_4 = a_4, c_4 = A_4,$
$15 \rightarrow 35, 36, 37$	$d_4 = a_4, d_4 = b_4, d_4 = c_4$
$16 \to 38, 41, 44, 50, 57$	$c_4 = a_4, c_4 = b_4, d_4 = b_4, d_4 = c_4, d_4 = a_4$
$17 \to 40, 42, 43, 45, 53$	$c_4 = a_4, c_4 = b_4, d_4 = a_4, d_4 = b_4, d_4 = c_4$
$18 \rightarrow 38, 42, 47, 51, 56$	$c_3 = a_3, c_4 = b_4, d_4 = b_4, d_4 = c_4, d_4 = a_4$
$19 \to 40, 41, 46, 48, 54$	$c_4 = a_4, c_4 = b_4, d_4 = a_4, d_4 = b_4, d_4 = c_4$
$20 \to 38, 40, 49, 52, 55$	$b_4 = a_4, c_4 = a_4, d_4 = a_4, d_4 = b_4, d_4 = c_4$
$21 \to 45, 48, 52, 56, 57$	$c_4 = b_4, d_4 = b_4, d_4 = c_4, d_4 = a_4, c_4 = a_4$
$22 \to 43, 44, 46, 47, 55$	$c_4 = a_4, c_4 = b_4, d_4 = a_4, d_4 = b_4, d_4 = c_4$
$23 \to 43, 49, 51, 54, 57$	$c_4 = a_4, d_4 = a_4, d_4 = b_4, d_4 = c_4, b_4 = a_4$
$24 \to 46, 49, 50, 53, 56$	$c_4 = a_4, d_4 = a_4, d_4 = b_4, d_4 = c_4, b_4 = a_4$
$25 \rightarrow 9, 13, 21, 31$	$c_4 = a_4, c_3 = a_3, c_3 = b_3, c_4 = b_4$
$26 \rightarrow 15, 22, 32, 38$	$c_3 = a_3, c_4 = a_4, c_4 = b_4, c_3 = b_3$
$27 \to 9, 23, 32$	$c_3 = a_3, c_3 = b_3, c_4 = b_4$
$28 \rightarrow 12, 24, 31, 38$	$c_4 = a_4, c_4 = b_4, c_3 = a_3, c_3 = b_3$
$29 \to 13, 18, 24, 38$	$b_3 = a_3, c_4 = a_4, c_4 = b_4, c_3 = b_3$
$30 \rightarrow 27, 33, 40, 45$	$c_3 = a_3, c_4 = a_4, c_4 = b_4, c_3 = b_3$
$31 \rightarrow 29, 34, 40, 48$	$c_3 = a_3, c_4 = a_4, c_4 = b_4, c_3 = b_3$
$32 \rightarrow 10, 30, 40, 52$	$b_4 = a_4, c_3 = a_3, c_4 = b_4, c_3 = b_3$
$33 \rightarrow 12, 39, 41, 42, 52$	$b_4 = a_4, c_3 = b_3, c_4 = B_4, c_4 = b_4, c_3 = a_3$
$34 \rightarrow 10, 28, 33, 56$	$c_4 = a_4, c_3 = a_3, c_4 = b_4, c_3 = b_3$
$35 \rightarrow 10, 26, 34, 57$	$c_4 = a_4, c_3 = a_3, c_4 = b_4, c_3 = b_3$
	$b_3 = a_3, b_3 = A_3, c_4 = a_4, c_3 = a_3, c_3 = A_3, c_4 = b_4$
	$b_3 = a_3, b_3 = a_3, c_4 = a_4, c_3 = a_3, c_3 = a_3, c_4 = b_4$ $b_3 = A_3, b_3 = a_3, c_4 = a_4, c_4 = b_4, c_3 = b_3, c_3 = B_3$
	$c_4 = a_4, c_3 = a_3, c_4 = a_4, c_4 = b_4, c_3 = b_3, c_3 = b_3$
$39 \rightarrow 11, 25, 30, 39, 44$	$b_4 = a_4, c_3 = a_3, c_4 = b_4, c_3 = b_3, c_4 = b_4$
	$b_4 = a_4, b_4 = A_4, c_3 = a_3, c_4 = b_4, c_3 = b_3, c_4 = B_4, c_4 = b_4$ $b_4 = a_4, b_4 = A_4, c_3 = a_3, c_3 = b_3, c_4 = B_4, c_4 = b_4$
	$b_4 = a_4, b_4 = A_4, c_4 = a_4, c_4 = A_4, c_3 = a_3$
	$b_4 = a_4, b_4 = A_4, c_4 = b_4, c_3 = b_3, c_3 = a_3$
$43 \rightarrow 16, 18, 30, 49$	$b_3 = a_3, b_4 = a_4, c_4 = a_4, c_3 = b_3$
$44 \rightarrow 18,52$	$b_3 = a_3, c_3 = b_3$
	$b_3 = A_3, b_4 = A_4, c_4 = b_4, c_3 = B_3, c_3 = b_3$
	$c_4 = a_4, c_3 = a_3, c_3 = b_3$
	$b_3 = A_3, b_3 = a_3, c_4 = a_4, c_4 = b_4, c_3 = A_3, c_3 = a_3$
$48 \rightarrow 26, 61$	$c_4 = a_4, c_4 = b_4$
$49 \rightarrow 28,60$	$c_3 = a_3, c_4 = b_4$
$50 \rightarrow 62, 64$	$c_4 = b_4, d_4 = c_4$
$51 \to 5, 61, 63, 64$	$c_3 = a_3, c_4 = a_4, c_4 = b_4, c_3 = b_3$

In the table below all identifications take place in column 4 of the matrices $X_{9,i}$ and so we shall omit the subscript 4.

Arrows from level 9 to 8:

$1 \rightarrow 2, 3$	c = a, c = b	$33 \rightarrow 17, 18$	b = a, c = a
$2 \rightarrow 3, 5$	c = a, c = b	$34 \rightarrow 18, 19$	c = a, c = b
$3 \rightarrow 3, 4$	c = a, c = b	$35 \rightarrow 8, 16$	b = a, c = a
$4 \rightarrow 3$	c = a	$36 \rightarrow 9, 18$	b = a, c = b
$5 \rightarrow 2, 16$	c = b, c = a	$37 \rightarrow 12, 16$	c = a, b = a
$6 \to 3, 15, 19$	c = b, b = a, c = a	$38 \rightarrow 10, 14$	c = a, c = b
$7 \rightarrow 4, 20$	c = b, c = a	$39 \rightarrow 10, 21$	b = a, c = b
$8 \rightarrow 5,22$	c = b, c = a	$40 \rightarrow 15, 20$	c = a, c = b
$9 \rightarrow 6, 8$	c = a, c = b	$41 \rightarrow 19, 22$	c = a, c = b
$10 \rightarrow 7, 10$	c = a, c = b	$42 \rightarrow 18, 21$	c = a, c = b
$11 \rightarrow 8, 11$	c = a, c = b	$43 \rightarrow 15, 21$	c = a, c = b
$12 \rightarrow 9, 11$	c = a, c = b	$44 \rightarrow 14, 22$	c = a, c = b
$13 \rightarrow 6,9$	c = a, c = b	$45 \rightarrow 18, 20$	c = a, c = b
$14 \rightarrow 7, 11$	c = a, c = b	$46 \rightarrow 15, 22$	c = a, c = b
$15 \rightarrow 8, 10$	c = a, c = b	$47 \rightarrow 14, 21$	c = a, c = b
$16 \rightarrow 6, 7$	b = a, c = a	$48 \rightarrow 19, 20$	c = a, c = b
$17 \rightarrow 6, 11$	b = a, c = b	$49 \rightarrow 10, 15$	b = a, c = a
$18 \rightarrow 6, 10$	b = a, c = b	$50 \rightarrow 10, 22$	b = a, c = b
$19 \rightarrow 7, 9$	b = a, c = b	$51 \rightarrow 10, 21$	b = a, c = b
$20 \rightarrow 7, 8$	b = a, c = a	$52 \rightarrow 10, 20$	b = a, c = b
$21 \rightarrow 7, 8$	c = b, c = a	$53 \rightarrow 15, 18$	b = a, c = b
$22 \rightarrow 13, 14$	c = a, c = b	$54 \rightarrow 15, 19$	b = a, c = b
$23 \rightarrow 8, 13$	c = a, c = b	$55 \rightarrow 14, 15$	c = a, b = a
$24 \rightarrow 12, 14$	c = a, c = b	$56 \rightarrow 10, 18$	c = a, c = b
$25 \rightarrow 8, 14$	b = a, c = b	$57 \rightarrow 10, 19$	c = a, c = b
$26 \rightarrow 7, 16$	c = a, c = b	$58 \rightarrow 12, 24$	b = a, c = a
$27 \rightarrow 15, 17$	c = b, c = a	$59 \rightarrow 13, 23$	c = a, c = b
$28 \rightarrow 7, 17$	c = a, c = b	$60 \rightarrow 17, 25$	c = a, c = b
$29 \rightarrow 15, 16$	c = b, c = a	$61 \rightarrow 16, 24$	c = a, c = b
$30 \rightarrow 7, 15$	b = a, c = b	$62 \rightarrow 1, 24, 26$	b = a, c = a, c = b
$31 \rightarrow 9, 12$	c = a, c = b	$63 \rightarrow 2, 26$	c = a, c = b
$32 \rightarrow 8, 13$	c = a, c = b	$64 \rightarrow 24, 26$	b = a, c = b