# Tighter monogamy and polygamy relations using Rényi- $\alpha$ entropy

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We investigate monogamy relations related to the Rényi- $\alpha$  entanglement and polygamy relations related to the Rényi- $\alpha$  entanglement of assistance. We present new entanglement monogamy relations satisfied by the  $\mu$ -th power of Rényi- $\alpha$  entanglement with  $\alpha \in [(\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$  for  $\mu \geq 2$ , and polygamy relations satisfied by the  $\mu$ -th power of Rényi- $\alpha$  entanglement of assistance with  $\alpha \in [(\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$  for  $0 \leq \mu \leq 1$ . These relations are shown to be tighter than the existing ones.

PACS numbers: 03.67.Mn, 03.65.Ud

### I. INTRODUCTION

One fundamental property of quantum entanglement is in its limited shareability in multi-party quantum systems [1]. For example, if the two subsystems are more entangled with each other, then they will share a less amount of entanglement with the other subsystems with specific entanglement measures. This restricted shareability of entanglement is named as the monogamy of entanglement (MoE). The concept of monogamy is an essential feature allowing for security in quantum key distribution [2]. It also plays an important role in many field of physics such as foundations of quantum mechanics [3–5], condensed matter physics [6, 7], statistical mechanics [3], and even black-hole physics [8, 9]. Monogamy inequality was first built for three-qubit systems using tangle as the bipartite entanglement measure [10], and generalized into multi-qubit systems in terms of various entanglement measures [11].

On the other hand, the assisted entanglement, which is a dual concept to bipartite entanglement measures, is known to have a dually monogamous or polygamous property in multiparty quantum systems. The polygamous property can be regarded as another kind of entanglement constraints in multi-qubit systems, and Gour et~al~[12] established the first dual monogamy inequality or polygamy inequality for multi-qubit systems using concurrence of assistance (CoA). For a three-qubit pure state  $|\psi\rangle_{A_1A_2A_3}$ , a polygamy inequality was introduced as:

$$C^{2}\left(|\psi\rangle_{A_{1}|A_{2}A_{3}}\right) \leq \left[C^{a}\left(\rho_{A_{1}A_{2}}\right)\right]^{2} + \left[C^{a}\left(\rho_{A_{1}A_{3}}\right)\right]^{2},\tag{1}$$

where CoA for a bipartite state  $\rho_{AB}$  is defined as:  $C^a\left(\rho_{AB}\right) = \max_{\{p_i,|\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle_{AB})$ , with the maximum is taken over all possible pure state decompositions of  $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i|$  and  $C\left(|\psi_i\rangle_{AB}\right)$  denotes the concurrence [13] of  $|\psi_i\rangle_{AB}$ . Furthermore, it is shown that for any pure state  $|\psi\rangle_{A_1A_2...A_n}$  in a n-qubit system [14], we have

$$C^{2}\left(|\psi\rangle_{A_{1}|A_{2}\cdots A_{n}}\right) \leq \left[C^{a}\left(\rho_{A_{1}A_{2}}\right)\right]^{2} + \cdots + \left[C^{a}\left(\rho_{A_{1}A_{n}}\right)\right]^{2}.$$
 (2)

Rényi- $\alpha$  entanglement (R $\alpha$ E) [15] is a well-defined entanglement measure which is the generalization of entanglement of formation (EOF) and has the merits for characterizing quantum phases with differing computational power [16], ground state properties in many-body systems [17], and topologically ordered states [18, 19]. Therefore, it is natural to study the monogamy inequality of R $\alpha$ E and the polygamy inequality of R $\alpha$ E of assistance in multipartite entanglement detection.

In this paper, we show that the monogamy inequality of  $R\alpha E$  and the polygamy inequality of  $R\alpha E$  of assistance obtained so far can be made tighter. When  $\alpha \in [(\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$ , we establish entanglement monogamy relations for the  $\mu$ -th power of  $R\alpha E$  with  $\mu \geq 2$  and polygamy relations for the  $\mu$ -th power of  $R\alpha E$  of assistance with  $0 \leq \mu \leq 1$  which are tighter than those in [20, 21].

## II. TIGHTER MONOGAMY RELATIONS FOR RÉNYI- $\alpha$ ENTANGLEMENT

Let  $\mathbf{H}_X$  denote a discrete finite-dimensional complex vector space associated with a quantum subsystem X. For a bipartite pure state  $|\psi\rangle_{AB}$  in vector space  $\mathbf{H}_A\otimes\mathbf{H}_B$ , the  $\mathrm{R}\alpha\mathrm{E}$  is defined as [22]

$$E_{\alpha}(|\psi\rangle_{AB}) = S_{\alpha}(\rho_A) = \frac{1}{1-\alpha}\log_2(\operatorname{tr}\rho_A^{\alpha}),$$
 (3)

where the Rényi- $\alpha$  entropy is  $S_{\alpha}(\rho_A) = [\log_2(\sum_i \lambda_i^{\alpha})]/(1-\alpha)$  with  $\alpha$  being a nonnegative real number and  $\lambda_i$  being the eigenvalue of reduced density matrix  $\rho_A$ . The Rényi- $\alpha$  entropy  $S_{\alpha}(\rho)$  converges to the von Neumann entropy when the order  $\alpha$  tends to 1. For a bipartite mixed state  $\rho_{AB}$ , the R $\alpha$ E is defined via the convex-roof extension

$$E_{\alpha}(\rho_{AB}) = \min \sum_{i} p_{i} E_{\alpha}(|\psi_{i}\rangle_{AB}), \tag{4}$$

where the minimum is taken over all possible pure state decompositions of  $\rho_{AB} = \sum_{i} p_{i} |\psi_{i}\rangle_{AB} \langle \psi_{i}|$ .

In particular, for a bipartite  $2 \otimes d$  mixed state  $\rho_{AB}$ , the Rényi- $\alpha$  entanglement has an analytical expression [20]

$$E_{\alpha}(\rho_{AB}) = f_{\alpha} \left[ C^{2}(\rho_{AB}) \right], \tag{5}$$

where the order  $\alpha$  ranges in the region  $[(\sqrt{7}-1)/2,(\sqrt{13}-1)/2]$  and the function  $f_{\alpha}(x)$  has the form

$$f_{\alpha}(x) = \frac{1}{1-\alpha} \log_2 \left[ \left( \frac{1-\sqrt{1-x}}{2} \right)^{\alpha} + \left( \frac{1+\sqrt{1-x}}{2} \right)^{\alpha} \right]. \tag{6}$$

For any two-qubit state  $\rho_{AB}$  with  $\alpha \geq (\sqrt{7}-1)/2$ , there also exist an analytic formula of R $\alpha$ E [22]

$$E_{\alpha}(\rho_{AB}) = f_{\alpha}[C(\rho_{AB})], \tag{7}$$

where the function  $f_{\alpha}(x)$  has the form (6).

In Ref.[20], we have known that for an arbitrary three-qubit mixed state  $\rho_{A_1A_2A_3}$ , the  $\mu$ -th power Rényi- $\alpha$  entanglement obeys the monogamy relation

$$E^{\mu}_{\alpha}\left(\rho_{A_1|A_2A_3}\right) \ge E^{\mu}_{\alpha}\left(\rho_{A_1A_2}\right) + E^{\mu}_{\alpha}\left(\rho_{A_1A_3}\right),$$
 (8)

where the order  $\alpha \ge (\sqrt{7}-1)/2 \simeq 0.823$  and the power  $\mu \ge 2$ . Moreover, in N-qubit systems, the following monogamy relation is also satisfied

$$E_{\alpha}^{\mu}(\rho_{A|B_{1}B_{2}\cdots B_{N-1}}) \ge \sum_{i=1}^{k-1} E_{\alpha}^{\mu}(\rho_{AB_{i}}) + E_{\alpha}^{\mu}(\rho_{A|B_{k}\cdots B_{N-1}}), \tag{9}$$

where the power  $\mu \geq 2$  and the order  $\alpha \in [(\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$ .

In fact, we can prove the following results for Rényi- $\alpha$  entanglement. Before this, we need to consider a Lemma for concurrence.

**Lemma 1** [23] For any  $2 \otimes 2 \otimes 2^{n-2}$  mixed state  $\rho \in \mathbf{H}_A \otimes \mathbf{H}_B \otimes \mathbf{H}_C$ , if  $C_{AB} \geqslant C_{AC}$ , we have

$$C_{A|BC}^{\alpha} \geqslant C_{AB}^{\alpha} + (2^{\frac{\alpha}{2}} - 1)C_{AC}^{\alpha},$$
 (10)

for all  $\alpha \geqslant 2$ .

[Proof] Since it has been shown that  $C_{A|BC}^2 \ge C_{AB}^2 + C_{AC}^2$  for arbitrary  $2 \otimes 2 \otimes 2^{n-2}$  tripartite state  $\rho_{ABC}$  [24]. Then, if  $C_{AB} \ge C_{AC}$ , we have

$$\begin{split} C_{A|BC}^{\alpha} &\geqslant (C_{AB}^2 + C_{AC}^2)^{\frac{\alpha}{2}} \\ &= C_{AB}^{\alpha} \left( 1 + \frac{C_{AC}^2}{C_{AB}^2} \right)^{\frac{\alpha}{2}} \\ &\geqslant C_{AB}^{\alpha} \left[ 1 + (2^{\frac{\alpha}{2}} - 1) \left( \frac{C_{AC}^2}{C_{AB}^2} \right)^{\frac{\alpha}{2}} \right] \\ &= C_{AB}^{\alpha} + (2^{\frac{\alpha}{2}} - 1) C_{AC}^{\alpha} \end{split}$$

where the second inequality is due to  $(1+t)^x \ge 1 + (2^x-1)t^x$  for any real number x and t,  $0 \le t \le 1$ ,  $x \in [1, \infty]$ . As the subsystems A and B are equivalent in this case, we have assumed that  $C_{AB} \ge C_{AC}$  without loss of generality. Moreover, if  $C_{AB} = 0$  we have  $C_{AB} = C_{AC} = 0$ . That is to say the lower bound becomes trivially zero.

**Theorem 1** For any N-qubit mixed state  $\rho \in \mathbf{H}_A \otimes \mathbf{H}_{B_1} \otimes \cdots \otimes \mathbf{H}_{B_{N-1}}$ , if  $C_{AB_i} \geqslant C_{A|B_{i+1}\cdots B_{N-1}}$  for  $i=1,2,\cdots,m$ , and  $C_{AB_j} \leqslant C_{A|B_{j+1}\cdots B_{N-1}}$  for  $j=m+1,\cdots,N-2, \ \forall \ 1 \leqslant m \leqslant N-3, \ N \geqslant 4$ , the Rényi- $\alpha$  entanglement  $E_{\alpha}(\rho)$  satisfies

$$E_{\alpha}^{\mu}(\rho_{A|B_{1}B_{2}\cdots B_{N-1}})$$

$$\geqslant E_{\alpha}^{\mu}(\rho_{AB_{1}}) + (2^{\mu} - 1)E_{\alpha}^{\mu}(\rho_{AB_{2}}) + \dots + (2^{\mu} - 1)^{m-1}E_{\alpha}^{\mu}(\rho_{AB_{m}})$$

$$+ (2^{\mu} - 1)^{m+1} \left(E_{\alpha}^{\mu}(\rho_{AB_{m+1}}) + \dots + E_{\alpha}^{\mu}(\rho_{AB_{N-2}})\right)$$

$$+ (2^{\mu} - 1)^{m}E_{\alpha}^{\mu}(\rho_{AB_{N-1}}), \tag{11}$$

for  $\mu \ge 2$  and  $\alpha \in [(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$ .

[Proof] For  $\mu \geqslant 2$ , we have

$$f_{\alpha}^{\mu}(x^{2} + y^{2}) \geqslant (f_{\alpha}(x^{2}) + f_{\alpha}(y^{2}))^{\mu}$$
  
 
$$\geqslant f_{\alpha}^{\mu}(x^{2}) + (2^{\mu} - 1)f_{\alpha}^{\mu}(y^{2}), \tag{12}$$

where the first inequality is due to the convex property of  $f_{\alpha}(x)$  for  $\alpha \geq (\sqrt{7} - 1)/2$  [22], and the second inequality is obtained from a similar consideration in the proof of the second inequality in Lemma 1.

Let  $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i| \in \mathbf{H}_A \otimes \mathbf{H}_{B_1} \otimes \cdots \otimes \mathbf{H}_{B_{N-1}}$  be the optimal decomposition of  $E_{\alpha}(\rho_{A|B_1B_2\cdots B_{N-1}})$  for the N-qubit mixed state  $\rho$ ; then we have

$$E_{\alpha}^{2}(\rho_{A|B_{1}B_{2}\cdots B_{N-1}}) = \left[\sum_{i} p_{i}E_{\alpha}(|\psi_{i}\rangle_{A|B_{1}B_{2}\cdots B_{N-1}})\right]^{2}$$

$$= \left\{\sum_{i} p_{i}E_{\alpha}[C_{A|B_{1}B_{2}\cdots B_{N-1}}(|\psi_{i}\rangle)]\right\}^{2}$$

$$\geq \left\{E_{\alpha}\left[\sum_{i} p_{i}C_{A|B_{1}B_{2}\cdots B_{N-1}}(|\psi_{i}\rangle)\right]\right\}^{2}$$

$$\geq \left\{E_{\alpha}\left[C_{A|B_{1}B_{2}\cdots B_{N-1}}(\rho)\right]\right\}^{2}$$

$$= E_{\alpha}^{2}\left[C_{A|B_{1}B_{2}\cdots B_{N-1}}^{2}(\rho)\right], \tag{13}$$

here we have used in the second equality the pure state formula of the R $\alpha$ E and taken the  $E_{\alpha}(C)$  as a function of the concurrence C for  $\alpha \geq (\sqrt{7}-1)/2$ ; in the third inequality we have used the monotonically increasing and convex properties of  $E_{\alpha}(C)$  as a function of the concurrence [22]; in the forth inequality we have used the convex property of concurrence for mixed states. Then from (13) we have

$$E_{\alpha}^{\mu}(\rho_{A|B_{1}B_{2}\cdots B_{N-1}}) \geqslant f_{\alpha}^{\mu}(C_{AB_{1}}^{2} + C_{AB_{2}}^{2} + \cdots + C_{AB_{m-1}}^{2})$$

$$\geqslant f_{\alpha}^{\mu}(C_{AB_{1}}^{2}) + (2^{\mu} - 1)f_{\alpha}^{\mu}(C_{AB_{2}}^{2} + \cdots + C_{AB_{m-1}}^{2})$$

$$\geqslant f_{\alpha}^{\mu}(C_{AB_{1}}^{2}) + (2^{\mu} - 1)f_{\alpha}^{\mu}(C_{AB_{2}}^{2}) + (2^{\mu} - 1)^{2}f_{\alpha}^{\mu}(C_{AB_{3}}^{2} + \cdots + C_{AB_{m-1}}^{2})$$

$$\geqslant \cdots$$

$$\geqslant f_{\alpha}^{\mu}(C_{AB_{1}}^{2}) + (2^{\mu} - 1)f_{\alpha}^{\mu}(C_{AB_{2}}^{2}) + \cdots + (2^{\mu} - 1)^{m-1}f_{\alpha}^{\mu}(C_{AB_{m}}^{2})$$

$$+ (2^{\mu} - 1)^{m}f_{\alpha}^{\mu}(C_{A|B_{m-1}\cdots B_{N-1}}^{2}), \qquad (14)$$

where we have used the monogamy inequality  $C^x(\rho_{A|B_1B_2\cdots B_{N-1}}) \ge C^x(\rho_{AB_1}) + C^x(\rho_{AB_2}) + \cdots + C^x(\rho_{AB_{N-1}})$  with  $x \ge 2$  for N-qubit states  $\rho$  and the monotonically increasing property of  $f_{\alpha}(C^2)$  to obtain the first inequality. By using (12) repeatedly, we get the other inequalities.

Since  $C_{AB_i} \geqslant C_{A|B_{i+1}\cdots B_{N-1}}$  for  $i=1,2,\cdots,m$ , and  $C_{AB_j} \leqslant C_{A|B_{j+1}\cdots B_{N-1}}$  for  $j=m+1,\cdots,N-2, \ \forall \ 1 \leqslant m \leqslant N-3, \ N\geqslant 4$ , by using (12) and the similar consideration in the proof of the second inequality in Lemma 1, then we have

$$f_{\alpha}^{\mu}(C_{A|B_{m+1}\cdots B_{N-1}}^{2}) \geqslant (2^{\mu} - 1)f_{\alpha}^{\mu}(C_{AB_{m+1}}^{2}) + f_{\alpha}^{\mu}(C_{A|B_{m+2}\cdots B_{N-1}}^{2})$$

$$\geqslant (2^{\mu} - 1)\left(f_{\alpha}^{\mu}(C_{AB_{m+1}}^{2}) + \cdots + f_{\alpha}^{\mu}(C_{AB_{N-2}}^{2})\right)$$

$$+ f_{\alpha}^{\mu}(C_{AB_{N-1}}^{2}). \tag{15}$$

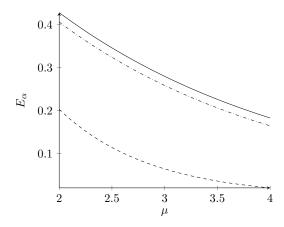


FIG. 1: Behavior of the Rényi- $\alpha$  entanglement of  $|\psi\rangle$  and its lower bound, which are functions of  $\mu$  plotted. The solid line represents the Rényi- $\alpha$  entanglement of  $|\psi\rangle$  in Example 1, the dot-dashed line represents the lower bound from our result, and the dashed line represents the lower bound from the result in (9) of [20].

Since for any  $2 \otimes 2$  quantum state  $\rho_{AB_i}$ ,  $\alpha \in [(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$ ,  $E_{\alpha}(\rho_{AB_i}) = f_{\alpha}[C^2(\rho_{AB_i})]$ , therefore combining (14) and (15), we have Theorem 1.

Moreover, for the case that  $C_{AB_i} \ge C_{A|B_{i+1}\cdots B_{N-1}}$  for all  $i=1,2,\cdots,N-2$ , we have a simple tighter monogamy relation for the Rényi- $\alpha$  entanglement:

**Theorem 2** If  $C_{AB_i} \geqslant C_{A|B_{i+1}\cdots B_{N-1}}$  for all  $i=1,2,\cdots,N-2$ , we have

$$E_{\alpha}^{\mu}(\rho_{A|B_{1}B_{2}\cdots B_{N-1}})$$

$$\geq E_{\alpha}^{\mu}(\rho_{AB_{1}}) + (2^{\mu} - 1)E_{\alpha}^{\mu}(\rho_{AB_{2}})\cdots + (2^{\mu} - 1)^{N-2}E_{\alpha}^{\mu}(\rho_{AB_{N-1}}), \tag{16}$$

for  $\mu \geqslant 2$  and  $\alpha \in [(\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$ .

As an example, let us consider the three-qubit state  $|\psi\rangle$  in the generalized Schmidt decomposition form [25, 26],

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle,$$
(17)

where  $\lambda_i \geq 0$ , i = 0, 1, 2, 3, 4 and  $\sum_{i=0}^4 \lambda_i^2 = 1$ . Set  $\lambda_0 = \lambda_1 = \frac{1}{2}$ ,  $\lambda_2 = \lambda_3 = \lambda_4 = \frac{\sqrt{6}}{6}$ . Since  $\alpha \in [(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$ , we choose  $\alpha = (\sqrt{7} - 1)/2 \approx 0.823$ , we have  $E_{\alpha}(|\psi\rangle_{A|B}) = E_{\alpha}(|\psi\rangle_{A|C}) = 0.318620$ ,  $E_{\alpha}(|\psi\rangle_{A|BC}) = 0.654205$ , and then  $E_{\alpha}^{\mu}(|\psi\rangle_{A|BC}) = (0.654205)^{\mu}$ ,  $E_{\alpha}^{\mu}(|\psi\rangle_{A|B}) + E_{\alpha}^{\mu}(|\psi\rangle_{A|C}) = 2(0.318620)^{\mu}$ ,  $E_{\alpha}^{\mu}(|\psi\rangle_{A|B}) + (2^{\mu} - 1)E_{\alpha}^{\mu}(|\psi\rangle_{A|C}) = 2^{\mu}(0.318620)^{\mu}$ . It is easily verified that our result is better than the result in (9) for  $\mu \geq 2$ ; see Fig 1.

## III. TIGHTER POLYGAMY RELATIONS FOR RÉNYI- $\alpha$ ENTANGLEMENT OF ASSISTANCE

As a dual concept to Rényi- $\alpha$  entanglement, we define the Rényi- $\alpha$  entanglement of assistance (REoA) as

$$E_{\alpha}^{a}(\rho_{AB}) = \max \sum_{i} p_{i} E_{\alpha}(|\psi_{i}\rangle_{AB}), \qquad (18)$$

where the maximum is taken over all possible pure state decompositions of  $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i|$ . In Ref. [21], we know that for any two-qubit state  $\rho_{AB}$  and  $\alpha \geq (\sqrt{7} - 1)/2$ , we have

$$E_{\alpha}^{a}(\rho_{AB}) \ge f_{\alpha}\left(C^{a}\left(\rho_{AB}\right)\right),\tag{19}$$

where  $E_{\alpha}^{a}(\rho_{AB})$  and  $C^{a}(\rho_{AB})$  are the REoA and CoA of  $\rho_{AB}$ , respectively. And for any  $(\sqrt{7}-1)/2 \leq \alpha \leq (\sqrt{13}-1)/2$  and the function  $f_{\alpha}(x)$  defined on the domain  $\mathcal{D} = \{(x,y) | 0 \leq x, y \leq 1, 0 \leq x^{2} + y^{2} \leq 1\}$ , we have

$$f_{\alpha}(\sqrt{x^2 + y^2}) \le f_{\alpha}(x) + f_{\alpha}(y). \tag{20}$$

From Ref. [21], it has been shown that for  $(\sqrt{7}-1)/2 \le \alpha \le (\sqrt{13}-1)/2, 0 \le \mu \le 1$ , and any N-qubit state  $\rho_{A|B_1B_2\cdots B_{N-1}}$ , we have

$$\left[E_{\alpha}^{a}\left(\rho_{A|B_{1}B_{2}\cdots B_{N-1}}\right)\right]^{\mu} \leq \left[E_{\alpha}^{a}\left(\rho_{A|B_{1}}\right)\right]^{\mu} + \dots + \left[E_{\alpha}^{a}\left(\rho_{A|B_{N-1}}\right)\right]^{\mu}.\tag{21}$$

In the following, we study the polygamy relations of REoA for N-qubit generalized W-class state. For N-qubit generalized W-class state,  $|\psi\rangle_{AB_1\cdots B_{N-1}} \in H_A \otimes H_{B_1} \otimes \cdots \otimes H_{B_{N-1}}$  defined by

$$|\psi\rangle_{AB_1\cdots B_{N-1}} = a|10\cdots 0\rangle + b_1|01\cdots 0\rangle + \cdots + b_{N-1}|00\cdots 1\rangle, \tag{22}$$

with  $|a|^2 + \sum_{i=1}^{N-1} |b_i|^2 = 1$ , one has [27],

$$C(\rho_{AB_i}) = C^a(\rho_{AB_i}), \quad i = 1, 2, ..., N - 1,$$
 (23)

where  $\rho_{AB_i} = Tr_{B_1 \cdots B_{i-1} B_{i+1} \cdots B_{N-1}}(|\psi\rangle_{AB_1 \cdots B_{N-1}}\langle\psi|).$ 

**Theorem 3** Let  $\rho_{AB_1\cdots B_{N-1}}$  denote the N-qubit reduced density matrix of the N-qubit generalized W-class state  $|\psi\rangle_{AB_1\cdots B_{N-1}}\in H_A\otimes H_{B_1}\otimes\cdots\otimes H_{B_{N-1}}, \text{ if } C_{AB_i}\geqslant C_{A|B_{i+1}\cdots B_{N-1}} \text{ for } i=1,2,\cdots,m, \text{ and } C_{AB_j}\leqslant C_{A|B_{j+1}\cdots B_{N-1}} \text{ for } j=m+1,\cdots,N-2, \ \forall \ 1\leqslant m\leqslant N-3, \ N\geqslant 4, \text{ the Rényi-}\alpha \text{ entanglement of assistance } E^\alpha_\alpha(\rho) \text{ satisfies}$ 

$$\begin{aligned}
&\left[E_{\alpha}^{a}\left(\rho_{A|B_{1}B_{2}\cdots B_{N-1}}\right)\right]^{\mu} \\
&\leqslant \left[E_{\alpha}^{a}\left(\rho_{A|B_{1}}\right)\right]^{\mu} + (2^{\mu} - 1)\left[E_{\alpha}^{a}\left(\rho_{A|B_{2}}\right)\right]^{\mu} + \dots + (2^{\mu} - 1)^{m-1}\left[E_{\alpha}^{a}\left(\rho_{A|B_{m}}\right)\right]^{\mu} \\
&\quad + (2^{\mu} - 1)^{m+1}\left(\left[E_{\alpha}^{a}\left(\rho_{A|B_{m+1}}\right)\right]^{\mu} + \dots + \left[E_{\alpha}^{a}\left(\rho_{A|B_{N-2}}\right)\right]^{\mu}\right) \\
&\quad + (2^{\mu} - 1)^{m}\left[E_{\alpha}^{a}\left(\rho_{A|B_{N-1}}\right)\right]^{\mu},
\end{aligned} (24)$$

for  $0 \le \mu \le 1$  and  $\alpha \in [(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$ .

[Proof] For  $0 \le \mu \le 1$ , we have

$$\left[ f_{\alpha}(\sqrt{x^2 + y^2}) \right]^{\mu} \leq \left[ f_{\alpha}(x) + f_{\alpha}(y) \right]^{\mu} 
\leq f_{\alpha}^{\mu}(x) + (2^{\mu} - 1) f_{\alpha}^{\mu}(y),$$
(25)

where the first inequality is due to inequality (20) and the monotonically increasing property of  $x^{\mu}$  for  $0 \le \mu \le 1$ , and the second equality is obtained from a similar consideration in the proof of the second inequality in Lemma 1. Here we note that  $(1+t)^x \le 1 + (2^x - 1)t^x$  with  $0 \le t \le 1$ ,  $x \in [0,1]$ .

For the N-qubit generalized W-class state  $|\psi\rangle_{A|B_1B_2\cdots B_{N-1}}$ , from Eq.(2), we have

$$C^{2}\left(|\psi\rangle_{A|B_{1}B_{2}\cdots B_{N-1}}\right) \leq \left[C^{a}\left(\rho_{A|B_{1}}\right)\right]^{2} + \cdots + \left[C^{a}\left(\rho_{A|B_{N-1}}\right)\right]^{2}.$$
 (26)

Assuming that  $C^2\left(\rho_{A|B_1B_2\cdots B_{N-1}}\right) \leq \left[C^a\left(\rho_{A|B_1}\right)\right]^2 + \cdots + \left[C^a\left(\rho_{A|B_{N-1}}\right)\right]^2 \leq 1$ , then

$$\begin{split} & \left[ E_{\alpha}^{a} \left( \rho_{A|B_{1}B_{2}\cdots B_{N-1}} \right) \right]^{\mu} \\ & = f_{\mu}^{\mu} \left( C \left( \rho_{A|B_{1}B_{2}\cdots B_{N-1}} \right) \right) \\ & \leq f_{\alpha}^{\mu} \left( \sqrt{\left[ C^{a} \left( \rho_{A|B_{1}} \right) \right]^{2} + \dots + \left[ C^{a} \left( \rho_{A|B_{N-1}} \right) \right]^{2}} \right) \\ & = f_{\alpha}^{\mu} \left( \sqrt{\left[ C \left( \rho_{A|B_{1}} \right) \right]^{2} + \dots + \left[ C \left( \rho_{A|B_{N-1}} \right) \right]^{2}} \right) \\ & \leq f_{\alpha}^{\mu} \left( C \left( \rho_{A|B_{1}} \right) \right) + \left( 2^{\mu} - 1 \right) f_{\alpha}^{\mu} \left( C \left( \rho_{A|B_{2}} \right) \right) + \dots + \left( 2^{\mu} - 1 \right)^{m-1} f_{\alpha}^{\mu} \left( C \left( \rho_{A|B_{m}} \right) \right) \\ & + \left( 2^{\mu} - 1 \right)^{m+1} \left( f_{\alpha}^{\mu} \left( C \left( \rho_{A|B_{m+1}} \right) \right) + \dots + f_{\alpha}^{\mu} \left( C \left( \rho_{A|B_{N-2}} \right) \right) \right) \\ & + \left( 2^{\mu} - 1 \right)^{m} f_{\alpha}^{\mu} \left( C \left( \rho_{A|B_{N-1}} \right) \right) \\ & \leq \left[ E_{\alpha}^{a} \left( \rho_{A|B_{1}} \right) \right]^{\mu} + \left( 2^{\mu} - 1 \right) \left[ E_{\alpha}^{a} \left( \rho_{A|B_{m+1}} \right) \right]^{\mu} + \dots + \left[ E_{\alpha}^{a} \left( \rho_{A|B_{N-2}} \right) \right]^{\mu} \right) \\ & + \left( 2^{\mu} - 1 \right)^{m} \left[ E_{\alpha}^{a} \left( \rho_{A|B_{N-1}} \right) \right]^{\mu}, \end{split} \tag{27}$$

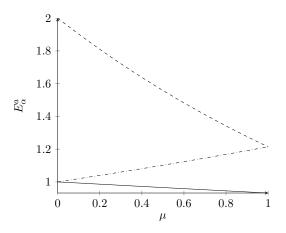


FIG. 2: Behavior of the Rényi- $\alpha$  entanglement of assistance of  $|W\rangle$  and its upper bound, which are functions of  $\mu$  plotted. The solid line represents the Rényi- $\alpha$  entanglement of assistance of  $|W\rangle$  in Example 2, the dot-dashed line represents the upper bound from our result, and the dashed line represents the upper bound from (21) in [21].

where in the second inequality we have used the monotonically increasing property of  $f_{\alpha}(x)$  for  $\alpha \geq (\sqrt{7} - 1)/2$ , and the third equality is due to (23). By using (25) repeatedly and the similar consideration in the proof of Theorem 1, we get the forth inequality. The last inequality is due to (19) and (23).

Then we consider the case  $C^2\left(\rho_{A|B_1B_2...B_{N-1}}\right) \le 1 \le \left[C^a\left(\rho_{A|B_1}\right)\right]^2 + \dots + \left[C^a\left(\rho_{A|B_{N-1}}\right)\right]^2$ . There must exist  $k \in \{1, \dots, N-2\}$  such that  $\left[C^a\left(\rho_{A|B_1}\right)\right]^2 + \dots + \left[C^a\left(\rho_{A|B_k}\right)\right]^2 \le 1$ ,  $\left[C^a\left(\rho_{A|B_1}\right)\right]^2 + \dots + \left[C^a\left(\rho_{A|B_{k+1}}\right)\right]^2 > 1$ . By defining  $T = \left[C^a\left(\rho_{A|B_1}\right)\right]^2 + \dots + \left[C^a\left(\rho_{A|B_{k+1}}\right)\right]^2 - 1 > 0$ , we can derive

$$\left[E_{\alpha}^{a}\left(\rho_{A|B_{1}B_{2}\cdots B_{N-1}}\right)\right]^{\mu} = f_{\alpha}^{\mu}\left(C\left(\rho_{A|B_{1}B_{2}\cdots B_{N-1}}\right)\right) \leq f_{\alpha}^{\mu}\left(1\right) 
= f_{\alpha}^{\mu}\left(\sqrt{\left[C^{a}\left(\rho_{A|B_{1}}\right)\right]^{2} + \cdots + \left[C^{a}\left(\rho_{A|B_{k+1}}\right)\right]^{2} - T}\right) 
\leq f_{\alpha}^{\mu}\left(\sqrt{\left[C\left(\rho_{A|B_{1}}\right)\right]^{2} + \cdots + \left[C\left(\rho_{A|B_{k+1}}\right)\right]^{2}}\right),$$
(28)

where we have used the monotonically increasing property of  $f_{\alpha}(x)$  in the second inequality, in the forth inequality we have used (23) and the monotonically increasing property of  $f_{\alpha}(x)$ .

When  $k+1 \leq m$ , we have

$$f_{\alpha}^{\mu} \left( \sqrt{\left[ C\left( \rho_{A|B_{1}} \right) \right]^{2} + \dots + \left[ C\left( \rho_{A|B_{k+1}} \right) \right]^{2}} \right)$$

$$\leq f_{\alpha}^{\mu} \left( C\left( \rho_{A|B_{1}} \right) \right) + (2^{\mu} - 1) f_{\alpha}^{\mu} \left( C\left( \rho_{A|B_{2}} \right) \right) + \dots + (2^{\mu} - 1)^{k} f_{\alpha} \left( C\left( \rho_{A|B_{k+1}} \right) \right)$$

$$\leq \left[ E_{\alpha}^{a} \left( \rho_{A|B_{1}} \right) \right]^{\mu} + (2^{\mu} - 1) \left[ E_{\alpha}^{a} \left( \rho_{A|B_{2}} \right) \right]^{\mu} + \dots + (2^{\mu} - 1)^{m-1} \left[ E_{\alpha}^{a} \left( \rho_{A|B_{m}} \right) \right]^{\mu}$$

$$+ (2^{\mu} - 1)^{m+1} \left( \left[ E_{\alpha}^{a} \left( \rho_{A|B_{m+1}} \right) \right]^{\mu} + \dots + \left[ E_{\alpha}^{a} \left( \rho_{A|B_{N-2}} \right) \right]^{\mu} \right)$$

$$+ (2^{\mu} - 1)^{m} \left[ E_{\alpha}^{a} \left( \rho_{A|B_{N-1}} \right) \right]^{\mu},$$

$$(29)$$

where we have used (25) repeatedly and the similar consideration in the proof of Theorem 1 in the first inequality, and the second inequality is due to (19) and (23).

When k+1 > m, we have

$$f_{\alpha}^{\mu}\left(\sqrt{\left[C\left(\rho_{A|B_{1}}\right)\right]^{2}+\cdots+\left[C\left(\rho_{A|B_{k+1}}\right)\right]^{2}}\right)$$

$$\leq f_{\alpha}^{\mu}\left(C\left(\rho_{A|B_{1}}\right)\right)+\left(2^{\mu}-1\right)f_{\alpha}^{\mu}\left(C\left(\rho_{A|B_{2}}\right)\right)+\cdots+\left(2^{\mu}-1\right)^{m-1}f_{\alpha}^{\mu}\left(C\left(\rho_{A|B_{m}}\right)\right)$$

$$+\left(2^{\mu}-1\right)^{m+1}\left(f_{\alpha}^{\mu}\left(C\left(\rho_{A|B_{m+1}}\right)\right)+\cdots+f_{\alpha}^{\mu}\left(C\left(\rho_{A|B_{k}}\right)\right)\right)$$

$$+\left(2^{\mu}-1\right)^{m}f_{\alpha}^{\mu}\left(C\left(\rho_{A|B_{k+1}}\right)\right)$$

$$\leq \left[E_{\alpha}^{a}\left(\rho_{A|B_{1}}\right)\right]^{\mu}+\left(2^{\mu}-1\right)\left[E_{\alpha}^{a}\left(\rho_{A|B_{2}}\right)\right]^{\mu}+\cdots+\left(2^{\mu}-1\right)^{m-1}\left[E_{\alpha}^{a}\left(\rho_{A|B_{m}}\right)\right]^{\mu}$$

$$+\left(2^{\mu}-1\right)^{m+1}\left(\left[E_{\alpha}^{a}\left(\rho_{A|B_{m+1}}\right)\right]^{\mu}+\cdots+\left[E_{\alpha}^{a}\left(\rho_{A|B_{N-2}}\right)\right]^{\mu}\right)$$

$$+\left(2^{\mu}-1\right)^{m}\left[E_{\alpha}^{a}\left(\rho_{A|B_{N-1}}\right)\right]^{\mu},$$

$$(30)$$

where we have used (25) repeatedly and the similar consideration of the proof of Theorem 1 in the first inequality, and the second inequality is due to (19) and (23).

Combing (28), (29) and (30), we have completed the proof of Theorem 3.

Moreover, for the case that  $C_{AB_i} \ge C_{A|B_{i+1}\cdots B_{N-1}}$  for all  $i=1,2,\cdots,N-2$ , we have a simple tighter monogamy relation for the Rényi- $\alpha$  entanglement of assistance:

**Theorem 4** If  $C_{AB_i} \geqslant C_{A|B_{i+1}\cdots B_{N-1}}$  for all  $i=1,2,\cdots,N-2$ , we have

$$\begin{aligned}
& \left[ E_{\alpha}^{a} \left( \rho_{A|B_{1}B_{2}\cdots B_{N-1}} \right) \right]^{\mu} \\
& \leq \left[ E_{\alpha}^{a} \left( \rho_{A|B_{1}} \right) \right]^{\mu} + \left( 2^{\mu} - 1 \right) \left[ E_{\alpha}^{a} \left( \rho_{A|B_{2}} \right) \right]^{\mu} + \dots + \left( 2^{\mu} - 1 \right)^{N-2} \left[ E_{\alpha}^{a} \left( \rho_{A|B_{N-1}} \right) \right]^{\mu}, \tag{31}
\end{aligned}$$

for  $0 \le \mu \le 1$  and  $\alpha \in [(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$ .

As an example, let us consider the W state,  $|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$ . Set  $\alpha = (\sqrt{7} - 1)/2 \approx 0.823$ , then we have  $E^a_{\alpha}(|W\rangle_{A|B}) = E^a_{\alpha}(|W\rangle_{A|C}) = 0.607218$ ,  $E^a_{\alpha}(|W\rangle_{A|BC}) = 0.932108$ , and then  $\left[E^a_{\alpha}(|W\rangle_{A|BC})\right]^{\mu} = (0.932108)^{\mu}$ ,  $\left[E^a_{\alpha}(|W\rangle_{A|B})\right]^{\mu} + \left[E_{\alpha}(|W\rangle_{A|C})\right]^{\mu} = 2(0.607218)^{\mu}$ ,  $\left[E^a_{\alpha}(|W\rangle_{A|B})\right]^{\mu} + \left[(2^{\mu} - 1)E^a_{\alpha}(|W\rangle_{A|C})\right]^{\mu} = 2^{\mu}(0.607218)^{\mu}$  for  $0 \leq \mu \leq 1$ . It is easily verified that our results are better than the results in (21) for  $0 \leq \mu \leq 1$ ; see Fig 2.

### IV. CONCLUSION

Entanglement monogamy and polygamy relations are not only fundamental property of entanglement in multiparty systems but also provide us an efficient way of characterizing multipartite entanglement. We have presented monogamy relations satisfied by the  $\mu$ -th power of Rényi- $\alpha$  entanglement for  $\mu \geq 2$  and  $\alpha \in [(\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$ , and polygamy relations satisfied by the  $\mu$ -th power of Rényi- $\alpha$  entanglement of assistance for  $0 \leq \mu \leq 1$  and  $\alpha \in [(\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$ . They are tighter , at least for some classes of quantum states, than the existing entanglement monogamy and polygamy relations. Tighter monogamy and polygamy relations imply finer characterizations of the entanglement distribution. Our approach may also be used to further study the monogamy and polygamy properties related to other quantum correlations.

**Acknowledgments** This work is supported by the NSFC 11571119 and NSFC 11475178.

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