4×4 unextendible product basis and genuinely entangled space

Kai Wang,^{1,*} Lin Chen,^{1,2,†} Lijun Zhao,¹ and Yumin Guo³

¹School of Mathematics and Systems Science, Beihang University, Beijing 100191, China

²International Research Institute for Multidisciplinary Science, Beihang University, Beijing 100191, China

³School of Mathematical Sciences, Capital Normal University, Beijing 100048, China

(Dated: January 21, 2019)

We show that there are six inequivalent 4×4 unextendible product bases (UPBs) of size eight, when we consider only 4-qubit product vectors. We apply our results to construct Positive-Partial-Transpose entangled states of rank nine. They are at the same 4-qubit, $2 \times 2 \times 4$ and 4×4 states, and their ranges have product vectors. One of the six UPBs turns out to be orthogonal to an almost genuinely entangled space, in the sense that the latter does not contain 4×4 product vector in any bipartition of 4-qubit systems. We also show that the multipartite UPB orthogonal to a genuinely entangled space exists if and only if the $n \times n \times n$ UPB orthogonal to a genuinely entangled space exists for some n. These results help understand an open problem in [Phys. Rev. A 98, 012313, 2018].

PACS numbers: 03.65.Ud, 03.67.Mn

I. INTRODUCTION

The unextendible product basis (UPB) has been extensively useful in the study of positive-partial-transpose (PPT) entangled states, symmetric PPT states, Bell inequalities and fermionic system [1–6]. Recently it has been shown that there exists a non-orthogonal UPB orthogonal to a genuinely entangled (GE) subspace [7]. In the same paper, an open problem was proposed to ask whether the multipartite UPB orthogonal to a GE subspace exists. In this paper, we shall construct the 4×4 UPBs using the 4-qubit system. We apply the UPBs to construct PPT entangled states and an almost GE space, so as to approach the open problem.

The multiqubit system can be reliably constructed in experiments [8, 9]. The multiqubit UPBs have been more and more studied theoretically [10–13]. Nevertheless, quantum-information tasks often deal with entangled states of high dimensions, and we need to construct UPBs of high dimensions using multiqubit UPBs. The traditional idea [14] relies on the assumption that the range of constructed PPT entangled states is orthogonal to a UPB, and thus has no product state. It is an interesting problem to construct PPT entangled states using a proper subset of UPBs, so that the range of PPT entangled states has product states. Compared to the traditional idea, the construction would help create more PPT entangled states of high dimensions and more complex properties, and shows the power of UPBs we have not realized so far. This is the first motivation of this paper.

Next, the GE state is a mixed state, which is not the convex sum of product states with respect to any bipartition of systems [15, 16]. Physically, the GE state need be constructed using at least one GE pure state. The GE states such as the Greenberger-Horne-Zeilinger (GHZ) states, W states and their copies [17] play a key role in quantum communication and computing. However it is a hard problem to determine whether a given *n*-partite state is a GE state. For n = 2, the problem reduces to the well-known separability problem [18]. The problem has received much attentions in theory and experiment [19–25]. Very recently, Ref. [7] constructed the notion of multipartite GE spaces containing only GE states. In other word, any pure state in the GE space is not a product vector with respect to any bipartition of systems. Ref. [7] constructed a non-orthogonal UPB [31] orthogonal to a GE space. It remains an open problem whether there exists a UPB orthogonal to a GE space. The positive answer of this problem would connect the two important notions, and thus motivate progress on the study of both of them theoretically and experimentally. This is the second motivation of this paper.

In this paper we show that there are six 4×4 UPBs of size eight consisting of 4-qubit product vectors. It turns out to be much harder than the construction of 4×4 UPBs of size 6, 7 and 9 consisting of 4-qubit product vectors [26]. We do not rely on the classification of 4-qubit UPBs by programming in [11]. We apply our results to construct PPT entangled states of rank nine. They are at the same 4-qubit, $2 \times 2 \times 4$ and 4×4 states, and their ranges have product vectors. We further show that a family of UPB is orthogonal to an almost GE space, in the sense that the latter does not contain any $2 \times 2 \times 4$ and 4×4 product vector. We also show that the multipartite UPB orthogonal to a GE space exists if and only if the $n \times n \times n$ UPB orthogonal to a GE space exists for some integer n. These results help understand the answer to the open problem in [7].

The rest of this paper is structured as follows. In Sec. II we introduce the notions and facts such as UPBs and UOMs. For the convenience of readers, we summary our results of six 4×4 UPBs of size eight in Sec. III. We

^{*}kaywong@buaa.edu.cn

[†]linchen@buaa.edu.cn (corresponding author)

present two applications of our results in Sec. IV and V, respectively. Finally we conclude in Sec. VI.

II. PRELIMINARIES

We refer to the 4-qubit subspace as $\mathcal{H}_{ABCD} = \mathcal{H}_A \otimes$ $\mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. For X = A, B, C, D, we refer to $|\psi_i\rangle \in \mathcal{H}_X$ as a 2-dimensional vector. The product vector in \mathcal{H}_{ABCD} is a 4-partite nonzero vector of the form $|\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle \otimes |\psi_4\rangle := |\psi_1, \psi_2, \psi_3, \psi_4\rangle.$ Suppose $\{|0\rangle, |1\rangle\}$ is the computational basis in \mathbb{C}^2 . For any alphabet say a, we shall refer to $|a\rangle, |a'\rangle$ as a different orthonormal basis in \mathbb{C}^2 , i.e., $|a\rangle$ is not orthogonal to $|0\rangle$ and $|1\rangle$. One can similarly define the *n*-partite product vectors in the space $\mathcal{H} = \mathcal{H}_1 \otimes ... \otimes \mathcal{H}_n$. The set of *n*-partite orthonormal product vectors $\{|a_{i,1}\rangle, ..., |a_{i,n}\rangle\}$ is a UPB in \mathcal{H} if there is no *n*-partite product vector orthogonal to the set. We shall use the following two properties of UPBs in the body and appendices of this paper. If we obtain one UPB from another by using the properties, then we say that the two UPBs are equivalent. The properties will greatly simplify the determination of UPBs.

Lemma 1 (i) If $\{|a_{i,1},...,a_{i,n}\rangle\}_{i=1,...,m}$ is an n-partite UPB of size m then so is $\{|a_{i,\sigma(1)},...,a_{i,\sigma(n)}\rangle\}_{i=1,...,m}$, where σ is an index permutation. That is, if we switch arbitrarily the systems of a UPB then we obtain another UPB.

(ii) If $\{|a_{i,1},...,a_{i,n}\rangle\}_{i=1,...,m}$ is an n-partite UPB of size m then so is $\{U_1|a_{i,1}\rangle \otimes ... \otimes U_n|a_{i,n}\rangle\}_{i=1,...,m}$, where $U_1,...,U_n$ are arbitrary unitary matrices. That is, performing any product unitary transformation $U_1 \otimes ... \otimes U_n$ on a UPB produces another UPB.

We further need the notion of unextendible orthogonal matrix (UOM) [27, p1]. To understand the notion, we refer to product vectors of an *n*-qubit UPB of size *m* as row vectors of an $m \times n$ matrix. The matrix is known as the UOM of the UPB. For orthogonal qubit states $|x\rangle$ and $|x'\rangle$ we shall refer to them as the vector variables *x* and *x'* in UOMs, and vice versa. For example, the three-qubit UPB $|0, 0, 0\rangle, |1, y, z\rangle, |x, 1, z'\rangle, |x', y', 1\rangle$ can be expressed as the UOM

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & y & z \\ x & 1 & z' \\ x' & y' & 1 \end{bmatrix}.$$
 (1)

where $x, y, z \neq 0, 1$. The first column of this UOM has only one *independent* vector variable x, since x' represents the qubit orthogonal to $|x\rangle$ up to global factors. Since the product vectors in the UPB are orthogonal, we say that the rows of UOM are also orthogonal. Further more we refer to the k'th column of a UOM as the counterpart of the k'th qubit of the corresponding UPB, and vice versa. So we can simply refer to the qubits of UPBs or UOMs throughout the paper. Furthermore, if the four-qubit UPB is still a UPB in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^4$ or $\mathbb{C}^4 \otimes \mathbb{C}^4$, then we shall refer to the corresponding UOM as a UOM in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^4$ or $\mathbb{C}^4 \otimes \mathbb{C}^4$, respectively. These notations will simplify our arguments in this paper.

III. THE SUMMARY OF 4×4 UPBS OF SIZE EIGHT

We present the six matrices $F_1, F_2, ..., F_6$ in (A1)-(A25) in Appendix ??. The inequalities for entries in F_i 's are satisfied if F_i corresponds to a UPB of size eight in $\mathbb{C}^4 \otimes \mathbb{C}^4$. To explain the details, we construct the matrices such as $F_1(i_3 = i'_4)$ in (A2), when the two vector variables i_3 and i'_4 are the same. We will explain with details why only $F_1, F_2, ..., F_6$ may be UOMs in $\mathbb{C}^4 \otimes \mathbb{C}^4$ using 4-qubit systems in Appendix C.

This section consists of two parts. First we prove that $F_1, F_2, ..., F_6$ are indeed UOMs in $\mathbb{C}^4 \otimes \mathbb{C}^4$. Lemma 1 (i) allows the operation of permuting column 1, 3 and 2, 4 at the same time, Lemma 1 (ii) allows the operations (ii.a) permuting column 1, 2, and (ii.b) permuting column 3, 4. Second we show $F_1, F_2, ..., F_6$ are inequivalent in terms of the above three operations.

First, we assume that

$$\mathcal{F}_{j} = \{ |a_{ji}, b_{ji}, c_{ji}, d_{ji} \rangle, i = 1, 2, ..., 8 \},$$
(2)

is the set of product vectors defined by the matrices F_j , j = 1, 2, ..., 6. We have the following observation.

Lemma 2 (i) For any j, any five two-qubit product vectors of the set $\{|a_{ji}, b_{ji}\rangle, i = 1, 2, ..., 8\}$ span \mathbb{C}^4 ; any four of the set span a subspace of dimension three or four.

(ii) For any j, any five two-qubit product vectors of the set $\{|c_{ji}, d_{ji}\rangle, i = 1, 2, ..., 8\}$ span \mathbb{C}^4 ; any four of the set span a subspace of dimension three or four.

(iii) Suppose the set of four distinct two-qubit product vectors $|a_{ji}, b_{ji}\rangle$, i = 1, 2, 3, 4 spans a 3-dimensional subspace in \mathbb{C}^4 . Then $a_{j\sigma(1)} = a_{j\sigma(2)}$ and $b_{j\sigma(3)} = b_{j\sigma(4)}$ for an index permutation σ .

(iv) Suppose the set of four distinct two-qubit product vectors $|c_{ji}, d_{ji}\rangle$, i = 1, 2, 3, 4 spans a 3-dimensional subspace in \mathbb{C}^4 . Then $c_{j\sigma(1)} = c_{j\sigma(2)}$ and $d_{j\sigma(3)} = d_{j\sigma(4)}$ for an index permutation σ .

Proof. (i) The first claim of assertion (i) can be proven by checking the first two columns of matrices $F_1, F_2, ..., F_6$. In fact, there exist four linearly independent two-qubit product vectors in any five. The second claim of assertion (i) is a corollary of the first claim.

(ii) Using the similar argument to the first two columns, the first claim of assertion (ii) can be proven by checking the last two columns of matrices $F_1, F_2, ..., F_6$. The second claim of assertion (ii) is a corollary of the first claim.

(iii), (iv) The assertions can be verified directly or by programming. $\hfill \Box$

We are now in a position to prove that $F_1, F_2, ..., F_6$ in (A1)-(A25) are UOMs. Suppose there exists a product vector $|x, y\rangle \in \mathbb{C}^4 \otimes \mathbb{C}^4$ orthogonal to \mathcal{F}_1 . Lemma 2 (i) and (ii) show that $|x\rangle$ is orthogonal to four states of $|a_{1i}, b_{1i}\rangle, i = 1, 2, ..., 8$, and $|y\rangle$ is orthogonal to four states of $|c_{1i}, d_{1i}\rangle, i = 1, 2, ..., 8$. Using F_1 , Lemma 2 (iii) and (iv), one can show that such two sets of four states do not exist. So $|x, y\rangle$ does not exist, and F_1 is a UOM. One can similarly prove that $F_2, F_3, ..., F_6$ are UOMs.

In the second part of this section, we explain the inequivalence of $F_1, F_2, ..., F_6$. We refer readers to Table I for the maximum number of independent vector variables in each columns of the UOMs. Since only the UOMs with identical number may be equivalent, we obtain that F_4 is not equivalent to any other F_j 's. Further, F_2, F_6 are not equivalent to any one of F_1, F_3 and F_5 . Next, F_3 and F_6 are not equivalent because column 1 and 3 of F_2 have the submatrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and column 1 (or 2) and 3 of F_6 do not. One can similarly show that F_1, F_3, F_5 are not equivalent each other.

UOM maximum number of independent variables

00111	mannam namber of macpendent variables
F_1	[2, 2, 2, 3]
F_2	[2, 2, 2, 4]
F_3	[2, 2, 3, 2]
F_4	[2, 3, 2, 3]
F_5	[3, 2, 2, 2]
F_6	[2, 2, 2, 4]

TABLE I: For any x, we say that the pair x, x'contributes only one independent vector variable. For the UOM F_1 , the array [2, 2, 2, 3] means that each of column 1, 2, 3 of F_1 has exactly 2 independent vector variables, and column 4 of F_1 has exactly 3 independent vector variables. The arrays for other F_j 's are similarly defined.

IV. APPLICATION 1: CONSTRUCTING PPT ENTANGLED STATES USING A PROPER SUBSET OF UPB

In this section we construct PPT entangled states using the six UPBs in Sec. III. The traditional idea is that $\rho = \frac{1}{|S|} (I - \sum_{j \in S} |x_j\rangle \langle x_j|)$ is a PPT entangled state if the set of orthogonal product vectors $\{|x_j\rangle, j \in S\}$ is a UPB, then the range of ρ has no product vectors. Different from the idea, we show that every UPB of the UOMs F_1, \ldots, F_6 has a proper subset of cardinality seven, such that they span a subspace whose orthogonal complement space is the range of a PPT entangled state of rank nine. It sheds novel light on the construction of PPT entangled states using UPBs. This is a corollary of the following observation.

Lemma 3 Suppose $d = d_1 d_2 ... d_n$, and $|x_1\rangle, ..., |x_m\rangle \in \mathbb{C}^d = \mathbb{C}^{d_1} \otimes ... \otimes \mathbb{C}^{d_n}$'s are m orthonormal product vectors. (i) If the range of $\rho = \frac{1}{d-m} (I_d - \sum_{j=1}^m |x_j\rangle\langle x_j|)$ contains at most d - m - 1 linearly independent product vectors,

then ρ is a PPT entangled state of rank d - m. (ii) Suppose $|y_1\rangle, ..., |y_m\rangle \in \mathbb{C}^d$ are m orthonormal product vectors. If the UOMs of the two sets $|x_i\rangle$'s and $|y_i\rangle$'s are locally equivalent, then the numbers of product vectors orthogonal to the two sets are the same, or are both infinite.

Proof. (i) We prove the assertion by contradiction. Suppose ρ is a separable state. Let $\rho = \sum_i p_i |a_i\rangle\langle a_i|$ where $|a_i\rangle \in \mathbb{C}^d$ are product vectors. So the set $\{|a_i\rangle\}$ has exactly d - m linearly independent vectors. Since $\{|a_i\rangle\}$ spans the range of ρ , the latter also has exactly d - m linearly independent vectors. It is a contradiction with the hypothesis that $\mathcal{R}(\rho)$ has at most d - m - 1 linearly independent vectors. So assertion (i) holds.

(ii) Let $|b\rangle$ be a product vector orthogonal to $|x_i\rangle$'s. Since the latter is locally equivalent to $|y_i\rangle$'s, there is a local unitary matrix U such that $U|x_i\rangle = P|y_{\sigma(i)}\rangle$, $\forall i$ up to a vector permutation matrix P and an index permutation σ . So $P^{\dagger}U|b\rangle$ is a product vector orthogonal to $|y_i\rangle$'s.

Let X, Y be the numbers of product vectors orthogonal to the two sets $\{|x_i\rangle\}$ and $\{|y_i\rangle\}$, respectively. The last paragraph shows that $X \leq Y$. If we switch x_i and y_i in the last paragraph, then the argument still holds. We have $X \geq Y$. So we have X = Y, and the assertion holds.

In the following we apply the above lemma to constructing PPT entangled states. By deleting the *i*'th product vector $|a_{ji}, b_{ji}, c_{ji}, d_{ji}\rangle \in \mathcal{F}_j$, we refer to \mathcal{S}_{ji} as the set of remaining seven product vectors for i = 1, 2, ..., 8 and j = 1, 2, ..., 6. That is

$$\mathcal{S}_{ji} = \mathcal{F}_j \setminus \{ |a_{ji}, b_{ji}, c_{ji}, d_{ji} \rangle \}.$$
(3)

Let S_{ji} be the UOM of S_{ji} , \mathcal{T}_{ji} the set of 4×4 product vectors orthogonal to S_{ji} , and T_{ji} the UOM of \mathcal{T}_{ji} . We present the following observation.

Lemma 4 (i) $|\mathcal{T}_{11}| = 4$ or 6. The latter holds if and only if $i_3 = i'_4$.

(ii) $|\hat{\mathcal{T}}_{21}(i_2 = i_3, i_4 = 0)| = 6.$

(iii) $|\mathcal{T}_{31}| = 4$ or 5. The latter holds if and only if $i_3 = i'_4$.

 $\begin{array}{l} (iv) \ |\mathcal{T}_{41}| = 4. \\ (v) \ |\mathcal{T}_{51}| = 4 \ or \ 6. \end{array} \ The \ latter \ holds \ if \ and \ only \ if \\ i_5 = i_6'. \\ (vi) \ |\mathcal{T}_{61}| = 6. \end{array}$

We refer readers to its proof in Appendix B. Now we present the main theorem of this section. The first part of following theorem from the fact that $F_1, ..., F_6$ are UOMs

Theorem 5 (i) Suppose \mathcal{F} is one of the six sets $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6$. The state

$$\alpha = \frac{1}{8} \left(I - \sum_{|\psi_j\rangle \in \mathcal{F}} |\psi_j\rangle \langle \psi_j | \right) \tag{4}$$

is at the same time a 4-qubit, $2 \times 2 \times 4$, and 4×4 PPT entangled state of rank eight.

(ii) Suppose S is one of the six sets $S_{11}, S_{21}(i_2 = i_3, i_4 = 0), S_{31}, S_{41}, S_{51}$ and $S_{61}(i_2 = i_3)$. The state

$$\beta = \frac{1}{9} \left(I - \sum_{|\psi_j\rangle \in \mathcal{S}} |\psi_j\rangle \langle \psi_j | \right) \tag{5}$$

is at the same time a 4-qubit, $2 \times 2 \times 4$, and 4×4 PPT entangled state of rank nine.

We have demonstrated the idea of constructing a PPT entangled state using a proper subset of a UPB. We have used the subset in (3) by removing the first vector in \mathcal{F}_i . One may construct more PPT entangled states in the same way, by removing one of the other vectors in \mathcal{F}_i for some *i*. We have found that some \mathcal{S}_{ji} has more than nine product vectors and thus Lemma 3 does not work here.

Next, the states α and β in Theorem 8 are both of robust entanglement in the sense that they are entangled w.r.t. different partitions of systems. It is not close to the genuine entanglement, as the state may become unentangled if we switch the systems. This is a problem we will tackle in the next section. On the other hand from Lemma 4, the range of states β constructed by \mathcal{F}_i 's has product vectors. We are not sure whether the smaller subset of \mathcal{F}_i would generate PPT entangled states too.

V. APPLICATION 2: THE UPB ORTHOGONAL TO AN ALMOST GE SPACE

The multipartite GE space contains no bipartite product vectors w.r.t. any bipartition of systems [7]. The GE space exists. For example, the one-dimensional subspace spanned by the multiqubit GHZ state is a GE space. Further, the GE space remains a GE space if it is multiplied by a local unitary transformation [28]. Just like the determination of GE states, characterizing the GE space of arbitrary dimension turns out to be a hard problem.

The paper [7] has constructed nonorthogonal UPBs orthogonal to a GE space. It remains an open problem whether the UPB orthogonal to a GE space exists. In this section we construct a 4-qubit UPB whose orthogonal space \mathcal{G} has no 4×4 product vectors w.r.t. any 4×4 bipartition of the 4-qubit space. It is known that the $2 \times N$ UPB does not exist [14]. So \mathcal{G} contains $2 \times N$ product vectors, and it is not a GE space. Nevertheless, \mathcal{G} is still close to a GE space, and we refer to \mathcal{G} as an *almost GE space*. In the following, the first main result of this section shows that the space orthogonal to the UPB \mathcal{F}_6 in (??) is an almost GE space.

Lemma 6 \mathcal{F}_6 is a bipartite UPB across any one of the three partitions of systems, namely AB : CD, AC : BD and AD : BC.

Proof. The bipartite partition of four-qubit system has only two cases, namely 2×8 and 4×4 . Here we investigate only system 4×4 since there does not exist bipartite UPB of $2 \times n$ systems. Further, the 4×4 partition for \mathcal{F}_6 occurs in the system AB : CD, AC : BD and AD : BC. We have proved \mathcal{F}_6 in $\mathcal{H}_{AB} \otimes \mathcal{H}_{CD}$ is a UOM in section III. The remaining task is to prove that \mathcal{F}_6 is a UPB in $\mathcal{H}_{AC} \otimes \mathcal{H}_{BD}$ and $\mathcal{H}_{AD} \otimes \mathcal{H}_{BC}$. We show by contradiction that \mathcal{F}_6 is a UPB in $\mathcal{H}_{AC} \otimes \mathcal{H}_{BD}$, and one can similarly prove the other case.

Assume that $|v\rangle = |\alpha, \beta\rangle \in \mathcal{H}_{AC} \otimes \mathcal{H}_{BD}$ is orthogonal to the states $|v_1\rangle, ..., |v_7\rangle$, and $|v_8\rangle$ in \mathcal{F}_6 , see Fig. 1. First, there must be three row vectors of \mathcal{F}_6 orthogonal to $|v\rangle$ on system AC. Then we denote them by $|v_1\rangle, |v_2\rangle, |v_3\rangle$. Similarly, there must be three row vectors $|v_4\rangle, |v_5\rangle, |v_6\rangle$ orthogonal to $|v\rangle$ on system BD. Next, the space spanned by any four of $|v_1\rangle, |v_2\rangle, \cdots, |v_8\rangle$ of system AC or BDhas dimension three or four from Lemma 2. That is to say, there is at most one of the remaining two vectors of \mathcal{F}_6 orthogonal to $|v\rangle$. If there is one among the two row vectors orthogonal to $|v\rangle$ then we denote it by $|v_7\rangle$. Further, $|v_8\rangle$ is not orthogonal to $|v\rangle$. It is a contradiction with the assumption the beginning of this paragraph. \Box

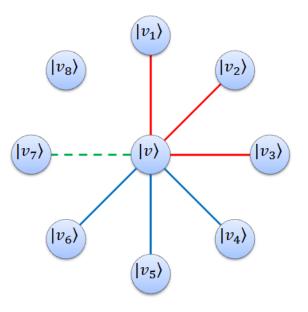


FIG. 1: The red line between the two product states $|v\rangle$ and $|v_i\rangle$ means that they are orthogonal on system AC.

The blue line implies that the product vectors are orthogonal on system BD. The dashed line means that $|v\rangle$ may be orthogonal to $|v_7\rangle$. There is no line between $|v\rangle$ and $|v_8\rangle$, and it means that $|v\rangle$ and $|v_8\rangle$ are not

orthogonal.

In the remaining of this section, we investigate the properties of UPBs orthogonal to a GE space, if such UPBs exist. For simplicity we shall refer to such UPBs as *GEUPB*. We present the following observation.

Lemma 7 (i) The multipartite UPB is a GEUPB if and only if it is a bipartite UPB across any patition of systems.

(ii) The tensor product of two m-partite UPBs is still an m-partite UPB, and vice versa.

(iii) The tensor product of two m-partite GEUPBs is still an m-partite GEUPB, and vice versa.

That is, suppose

$$S = \{ |a_{i1}, ..., a_{im} \rangle \}, \tag{6}$$

$$T = \{ |b_{j1}, ..., b_{jm} \rangle \}, \tag{7}$$

are two m-partite GEUPBs in the space $\otimes_{i=1}^{m} \mathcal{H}_i$ and $\otimes_{i=1}^{m} \mathcal{K}_j$, respectively. Then the set

$$\{|a_{i1}, b_{j1}\rangle \otimes \dots \otimes |a_{im}, b_{jm}\rangle\}$$

$$\tag{8}$$

is an m-partite GEUPB of $|S| \cdot |T|$ product vectors in the space $\otimes_{i=1}^{m} (\mathcal{H}_i \otimes \mathcal{K}_i)$.

Proof. (i) The assertion follows from [7, Remark 5].

(ii) The first part of assertion (ii) follows from [2, Theorem 8]. The "vice versa" part follows from the definition of UPBs.

(iii) The "vice versa" part follows from the definition of GEUPBs. We prove the first part of assertion (iii) as follows.

By the definition of GEUPBs, S and T are bipartite UPBs across any cut of systems, say $U : \overline{U}$. Assertion (ii) implies that the set in (8) is a bipartite UPB in the system $(U_S, U_T) : (\overline{U}_S, \overline{U}_T)$. So assertion (iii) follows from assertion (i).

Using the above lemmas, we present the second main result of this section.

Theorem 8 The multipartite GEUPB exists if and only if the $n \times n \times n$ GEUPB exists for some integer n.

Proof. (i) It suffices to prove the "if" part. It follows from the definition of GE spaces that the tripartite GE-UPB $S_{ABC} \subseteq \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ exists. So S_{BCA} and S_{CAB} are both tripartite GEUPBs. It follows from Lemma 7

(iii) that the tensor product of S_{ABC} , S_{BCA} and S_{CAB} is also a tripartite GEUPB. Since its local systems have equal dimensions, the assertion holds.

To further investigate the problem in [7], the theorem shows that it suffices to find the $n \times n \times n$ GEUPB or prove its nonexistence. It is known that the 3-qubit UPB has cardinality four [29], just like those in (1). So the 3qubit UPB is orthogonal to a bipartite product vector in the system A : BC. Thus the 3-qubit UPB is not a GEUPB, and we have n > 2 in terms of Theorem 8. In particular a three-qutrit UPB of size seven has been constructed [2], though it is evidently not a GEUPB. We need construct three-qutrit UPBs of larger size, say close to the upper bound 23, because any multipartite PPT states of rank at most three are separable states [30].

VI. CONCLUSIONS

We have shown that there are six inequivalent 4×4 UPB of size eight, when we only consider 4-qubit product vectors. We have constructed entangled states that are at the same 4-qubit, $2 \times 2 \times 4$ and 4×4 entangled states of rank nine. The results have been obtained using the UOM. We further have shown that the UPB \mathcal{F}_6 is orthogonal to a space containing no 4×4 product vector w.r.t any bipartition of 4-qubit system. We also have shown that the multipartite UPB orthogonal to a GE space exists if and only if the $n \times n \times n$ UPB orthogonal to a GE space for some integer n. In spite of these results, we are still unable to show whether the UPB orthogonal to a GE space exists. Another problem is to characterize PPT entangled states of rank eight or nine that are not generated by orthogonal 4-qubit product vectors.

Acknowledgments

This work was supported by the NNSF of China (Grant No. 11871089), and the Fundamental Research Funds for the Central Universities (Grant Nos. KG12040501, ZG216S1810 and ZG226S18C1).

Appendix A: The description of six four-qubit UOMs $F_1, F_2, ..., F_6$

$$\begin{split} F_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & g_3 & 0, 1 & h_3 \neq 0, 1 & i_4 \neq 0, 1 \\ f_5 &= g_0, 1 & h_3 \neq 0, 1 & i_4 \neq 0, 1 \\ f_5 &= g_3, 1 & i_4' \\ f_5 &= g_3, 1 & i_4' \\ f_5 &= 0 & h_3' & 1 \\ f_5 &= 0 & h_3' & 1 \\ f_5 &= 0, 1 & h_3 \neq 0, 1 & h_3 \neq 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 & i_5 \neq 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 & i_5 \neq 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 & i_5 \neq 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 & i_5 \neq 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 & i_5 \neq 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 & i_5 \neq 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 & i_5 \neq 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 & i_5 \neq 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 & i_5 \neq 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 & i_5 \neq 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 & h_3' \neq 0, 1 & i_5 \neq 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 & h_3' \neq 0, 1 & h_3' \neq 0, 1 & i_5 \\ f_5 &= 0, 1 & h_3' = 0, 1 & h_3' \neq 0, 1 & h_3' \neq 0, 1 & i_2 \\ f_5 &= 0, 1 & h_3' = 0, 1 & h_3' \neq 0, 1 & h_3' \neq 0, 1 & i_2 \\ f_5 &= 0, 1 & h_3' = 0, 1 & h_3' = 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 & h_3' = 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 & h_3' = 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 & h_3' = 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 & h_3' = 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 & h_3' = 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 & h_3' = 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 \\ f_5 &= 0, 1 & h_3' = 0, 1 \\ f_5 &= 0, 1 & h_3' = h_3' = 1 \\ f_5 &= 0, 1 & h_3' = h_3' = 1 \\ f_5 &= 0, 1 & h_3' = h_3' = 1 \\ f_5 &= 0, 1 & h_3' = h_3' = 1 \\ f_5 &= 0, 1 & h_3' = h_3' = 1 \\ f_5 &= 0, 1 & h_3' = h_3' = 1 \\ f_5 &= 0, 1 & h_3' = h_3' = 1 \\ f_5 &= 0, 1 & h_3' = h_3' = 1 \\ f_5 &= 0, 1 & h_3' = h_3' = 1 \\ f_5 &= 0, 1 & h_3' = h_3' = 1 \\ f_5 &= 0, 1 & h_3' = h_3' = 1 \\ f_5 &= 0, 1 & h_3' = h_3' = 1 \\ f_5 &= 0, 1 & h_3' = h_3' = 1 \\ f_5 &= 0, 1 & h_3' = h_3' = h_3' = h_3' \\ f_5 &= 0, 1 & h_3' = h_3' = h_3' \\ f_5 &= 0, h_3' = 1 \\ f_5 &= 0$$

$$\begin{split} (ii)F_2(i_2 = i'_5, i_1 \neq 0, 1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i_2 \neq 0, 1, i_4, i'_4 \\ 1 & g_5 \neq 0, 1 & h_3 & i_4 \\ f_5 & g_3 & 1 & i'_4 \\ f_5 & g_3 & 1 & i'_2 \\ f_5 & 0 & h'_3 & 1 \end{bmatrix}, \end{split}$$
(A7)
$$F_2(i_2 = i'_3, i_1 = 0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i_2 \neq 0, 1 \\ 1 & g_5 \neq 0, 1 & h'_2 \neq 0, 1 \\ g_5 \neq 0, 1 & g_5 & h_3 & 0 \\ f_5 \neq 0, 1 & g_5 & h_3 & 0 \\ f_5 \neq 0, 1 & g_5 & h_3 & 1 \\ f_5 & g_3 & 1 & i_2 \\ f_5 & 1 & h'_3 & i'_2 \\ f_5 & 0 & h'_3 & 1 \end{bmatrix},$$
(A8)
$$F_2(i_2 = i'_5, i_4 = 1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i_2 \neq 0, 1 \\ f_5 & g_3 & 1 & i_2 \\ f_5 \neq 0, 1 & g_5 & h_3 & 1 \\ f_5 & g_3 & 1 & i_2 \\ f_5 \neq 0, 1 & g_5 & h_3 & 1 \\ f_5 & g_3 & 1 & i_2 \\ f_5 & 0 & h'_3 & 1 \end{bmatrix},$$
(A9)
$$F_2(i_2 = i_4, i_3 = 0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i_2 \neq 0, 1, i_3, i'_3 \\ f_5 & g_3 & 1 & i'_2 \\ f_5 & 0 & h'_3 & 1 \end{bmatrix},$$
(A10)
$$F_2(i_2 = i_4, i_3 = 0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i_2 \neq 0, 1 \\ g_5 & g_3 & 1 & i'_2 \\ f_5 & g_3 & 1 & 0 \\ f_$$

$$\begin{split} F_2(i_2=i'_4,i_3=0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i_2 \neq 0,1 \\ 1 & g_3 \neq 0,1 & h_3 \neq 0,1 & 0 \\ 1 & g_3' & h_3 & i'_2 \\ f_5 & g_3 & 1 & 1 & i_2 \\ f_5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i_2 \neq 0,1 \\ 1 & g_3 \neq 0,1 & h_3 \neq 0,1 & 1 \\ 1 & g_3 \neq 0,1 & h_3 \neq 0,1 & 1 \\ 1 & g_3 \neq 0,1 & h_3 \neq 0,1 & 1 \\ 1 & g_3 \neq 0,1 & h_3 \neq 0,1 & 1 \\ 1 & g_3 \neq 0,1 & h_3 \neq 0,1 & 1 \\ f_5 \neq 0,1 & g_3' & 1 & i_2 \\ f_5 & g_3 & 1 & 0 \\ f_5' & 1 & h_3' & i_2' \\ f_5' & 0 & h_3' & 1 \end{bmatrix}, \quad (A15) \\ \\ F_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i_2 \neq 0,1 \\ 1 & g_3 \neq 0,1 & h_3 \neq 0,1 & 1 \\ f_5 \neq 0,1 & g_3' & 1 & i_2' \\ f_5 & 0 & h_3' & 1 \end{bmatrix}, \quad (A16) \\ \\ F_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i_2 \neq 0,1 \\ f_5 & g_3 & 1 & 1 \\ f_5' & 1 & h_3' & i_2' \\ f_5' & 0 & h_3' & 1 \end{bmatrix}, \\ \\ F_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i_2 \neq 0,1 \\ f_5 & 0,1 & g_3' & 1 & i_3 \\ f_5 & g_4 & 1 & i_3 \\ f_5 & g_4 & 1 & i_3 \\ f_5 & 0,1 & g_3' & 1 & i_3 \\ f_5 & 0,1 & g_3' & 1 & i_3 \\ f_5 & 0,1 & g_3' & 1 & i_3 \\ f_5 & g_4 & 1 & i_3 \\ f_5 & 0 & h_3' & 1 \end{bmatrix}, \\ \\ F_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i_2 \neq 0,1 \\ 1 & g_3 \neq 0,1 & h_3 \neq 0,1 & 1 \\ 1 & g_3 \neq 0,1 & h_3 \neq 0,1 \\ 1 & g_3 \neq 0,1 & h_3 \neq 0,1 \\ 1 & g_3 \neq 0,1 & h_3 \neq 0,1 \\ f_5 & 0 & h_3' & 1 \end{bmatrix}, \\ \\ F_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & h_2 \neq 0,1 & 1 \\ f_5 & 0 & 0 & i_2 \neq 0,1 \\ f_5 & 0 & h_3' & 1 \end{bmatrix}, \\ \\ (i) F_6(i_2 = 1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & h_2 \neq 0,1 & 1 \\ 1 & g_3' & h_3 & 1 \\ f_5 & g_3 & 1 & i_2' \\ f_5' & g_3 & 1 & i_2' \\ f_$$

$$\begin{split} F_{0}(i_{2}=1,i_{3}=i'_{4}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & h_{2} \neq 0, 1 & 1 & 1 \\ 1 & g_{3} \neq 0, 1 & 0 & i_{3} \\ f_{5} \neq 0, 1 & 1 & h'_{2} & i'_{5} \\ f_{5} & 0 & 1 & i_{3} \\ f'_{5} & g_{3} & 1 & 0 \\ f'_{5} & g'_{3} & h'_{2} & 1 \end{bmatrix}, \quad (A21) \end{split}$$

$$(ii) F_{6}(i_{2}=i_{3}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & h_{2} \neq 0, 1 & i_{2} \neq 0, 1, i_{4}, i'_{4} \\ 1 & g_{3} \neq 0, 1 & 0 & i_{2} \\ f_{5} & 0 & 1 & i'_{4} \\ f'_{5} & g'_{3} & h'_{2} & 1 \end{bmatrix}, \quad (A22)$$

$$(iii) F_{6}(i_{2}=i'_{3}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & h_{2} \neq 0, 1 & i_{2} \neq 0, 1, i_{4}, i'_{4} \\ 1 & g_{3} \neq 0, 1 & 0 & i'_{2} \\ f_{5} & 0 & 1 & i'_{4} \\ f'_{5} & g'_{3} & h'_{2} & 1 \end{bmatrix}, \quad (A23)$$

$$F_{6}(i_{2}=i'_{3}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & h_{2} \neq 0, 1 & i_{2} \neq 0, 1, i_{4}, i'_{4} \\ 1 & g'_{3} \neq 0, 1 & 0 & i'_{2} \\ f_{5} & 0 & 1 & i'_{4} \\ f'_{5} & g'_{3} & 1 & i'_{2} \\ f'_{5} & g'_{3} & h'_{2} & 1 \\ f'_{5} & g'_{3} & h'_{2} & 1 \\ f'_{5} & g'_{3} & h'_{2} & 1 \\ f_{5} \neq 0, 1 & 1 & h'_{2} & i'_{2} \\ f_{5} & 0 & 1 & 0 \\ g'_{5} & g'_{3} & h'_{2} & 1 \\ f_{5} \neq 0, 1 & 1 & h'_{2} & i'_{2} \\ f_{5} & g'_{3} & h'_{2} & 1 \\ f_{5} \neq 0, 1 & 1 & h'_{2} & i'_{2} \\ f'_{5} & g'_{3} & h'_{2} & 1 \\ (iv) F_{6}(i_{2} = i'_{4}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & h_{2} \neq 0, 1, i_{2} \neq 0, 1, i_{3}, i'_{3} \\ 1 & g'_{3} & h_{2} & 1 \\ f_{5} \neq 0, 1 & 1 & h'_{2} & i'_{3} \\ f'_{5} & g'_{3} & h'_{2} & 1 \\ \end{bmatrix}, \quad (A24)$$

Appendix B: The proof of Lemma 4

(i) Using F_1 in Appendix A we have

$$S_{11} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & g_3 \neq 0, 1 & h_3 \neq 0, 1 & i_3 \neq 0, 1, i_4 \\ 1 & g'_3 & h_3 & i_4 \neq 0, 1 \\ f_5 \neq 0, 1 & g'_3 & 1 & i'_4 \\ f_5 & g_3 & 1 & i'_3 \\ f'_5 & 0 & h'_3 & 1 \\ f'_5 & 1 & h'_3 & 0 \end{bmatrix}$$
(B1)

In this section we prove Lemma 4.

=

Let the 4×4 product state $|x, y\rangle \in \mathcal{T}_{11}$. Lemma 2 (i) and (ii) imply that we have two cases. First, $|x\rangle$ is orthogonal to four product vectors in $\{|a_{1i}, b_{1i}\rangle, i = 2, ..., 8\}$, and $|y\rangle$ is orthogonal to three product vectors in $\{|c_{1i}, d_{1i}\rangle, i = 2, ..., 8\}$. Second, $|x\rangle$ is orthogonal to three product vectors in $\{|a_{1i}, b_{1i}\rangle, i = 2, ..., 8\}$, and $|y\rangle$ is orthogonal to four product vectors in $\{|c_{1i}, d_{1i}\rangle, i = 2, ..., 8\}$.

In the first case using Lemma 2 (iii) we obtain

$$|0,0,0,0\rangle, |f_5',0,h_3',0\rangle, |f_5,g_3,\beta_3\rangle, |f_5,g_3',\beta_4\rangle \in \mathcal{T}_{21}, (B2)$$

where β_3, β_4 are two-qubit states in $\mathbb{H}_C \otimes \mathbb{H}_D$. In the second case, we still use Lemma 2 (iv) and exclude the same states in (B2). If $i_3 = i'_4$ then we obtain

$$|\beta_5, h_3, i_3\rangle, |\beta_6, h_3, i_3'\rangle \in \mathcal{T}_{21}.$$
 (B3)

Hence $|\mathcal{T}_{11}| = 4$ or 6. The latter holds if and only if $i_3 = i'_4$.

(ii) Using $F_2(i_2 = i_3, i_4 = 0)$ we have

$$S_{21}(i_2 = i_3, i_4 = 0) = \begin{bmatrix} 0 & 1 & 0 & i_3 \neq 0, 1 \\ 1 & g_3 \neq 0, 1 & h_3 \neq 0, 1 & i_3 \\ 1 & g'_3 & h_3 & 0 \\ f_5 \neq 0, 1 & g'_3 & 1 & 1 \\ f_5 & g_3 & 1 & i'_3 \\ f'_5 & 1 & h'_3 & i'_3 \\ f'_5 & 0 & h'_3 & 1 \end{bmatrix}.$$

Similar to case (i), we obtain $|0,0,0,0\rangle, |f'_5,0,h'_3,0\rangle, |f_5,g_3,0,i'_3\rangle, |f_5,g'_3,\gamma_1\rangle,$

 $|\gamma_2, h'_3, i_3\rangle, |f'_5, g_3, h_3, i'_3\rangle \in \mathcal{T}_{21}(i_2 = i_3, i_4 = 0).$ Hence $|\mathcal{T}_{21}(i_2 = i_3, i_4 = 0)| = 6.$

(iii) Using F_3 we have $|0,0,0,0\rangle, |f'_5,0,\delta_1\rangle, |f_5,g_3,\delta_2\rangle, |f_5,g'_3,0,i'_2\rangle \in \mathcal{T}_{31}.$ Further if $h_3 = h'_4$ then $|\delta_3,h_3,0\rangle \in \mathcal{T}_{31}.$ Hence $|\mathcal{T}_{31}| = 4$ or 5.

(iv) Using F_4 we have $|0,0,0,0\rangle, |f'_5,0,h'_3,0\rangle, |\epsilon_1,h_3,i_3\rangle, |\epsilon_2,h_3,i'_3\rangle \in \mathcal{T}_{41}.$ Hence $|\mathcal{T}_{41}| = 4.$

(v) Using F_5 we have $|0,0,0,0\rangle, |\zeta_1,h'_3,i'_2\rangle, |\zeta_2,0,i_2\rangle, |0,g'_3,h_3,i'_2\rangle \in \mathcal{T}_{51}$. If $f_5 = f'_6$ then $|f'_5,g_3,\zeta_3\rangle, |f_5,g'_3,\zeta_4\rangle \in \mathcal{T}_{61}$. Hence $|\mathcal{T}_{51}| = 4$ or 6.

(vi) Using $F_6(i_2 = i_3)$ we have $|0, 0, 0, 0\rangle, |f'_5, g'_3, h'_2, 0\rangle, |f'_5, g_3, 0, i'_2\rangle, |f_5, 0, \eta_1\rangle, |\eta_2, h'_2, i_2\rangle, |\eta_3, h_2, i'_2\rangle \in \mathcal{T}_{61}.$ Hence $|\mathcal{T}_{61}| = 6.$

Appendix C: The construction of six four-qubit UOMs $F_1, F_2, ..., F_6$ (to be shortened greatly)

We introduce a simple fact from [13, Lemma 2]. It will be used in the proof of Lemma 13.

Lemma 9 (i) If $S \subseteq (\mathbb{C}^2)^{\otimes n}$ is a UPB, then for all $|v\rangle \in S$ and all integers $1 \leq j \leq n$ there is another product

vector $|w\rangle \in S$ such that $|v\rangle$ and $|w\rangle$ are orthogonal on the *j*-th subsystem or *j*-th qubit.

(ii) The number of distinct vectors of any qubit in a UPB is an even integer.

The following result from [27, Lemma 5] will be used in the proof of Lemma 14.

Lemma 10 Let $X = [x_{i,j}] \in \mathcal{O}(m, n)$ be a UOM, and $\mu(x)$ the multiplicity of element x. If $p_j = \sum \mu(x)\mu(x')$, where the summation is over all pairs $\{x, x'\}$ in column j of X, then $\sum p_j \ge m(m-1)/2$.

We refer to the positive integer p_j as the *o-number* of column j of X. It represents the number of all orthogonal pairs in column j of X. We refer readers to [27] for more details on UOMs.

We present a special partition of a positive integer into smaller positive integers. It will be also used in the proof of Lemma 14.

Lemma 11 Suppose p is the sum of 2n positive integers $a_1, a_2, ..., a_{2n}$. Then the maximum of $a_1a_2 + a_3a_4 + ... + a_{2n-1}a_{2n}$ is $\lceil \frac{p-2n+2}{2} \rceil \cdot \lfloor \frac{p-2n+2}{2} \rfloor + n-1$. It is achievable if and only if up to the permutation of subscripts, we have $a_1 = \lceil \frac{p-2n+2}{2} \rceil$, $a_2 = \lfloor \frac{p-2n+2}{2} \rfloor$ and $a_i = 1$ for i > 2.

Proof. Let $N = a_1a_2 + a_3a_4 + \dots + a_{2n-1}a_{2n}$. We fix the values of $a_1, a_3, \dots, a_{2n-1}$ and a_6, a_8, \dots, a_{2n} , and make $a_1 > a_3 > \dots > a_{2n-1}$ by renaming the subscripts. Then we have $a_2 + a_4 = p - a_1 - a_3 - a_5 - a_6 - \dots - a_{2n}$. Since $a_1 > a_3$ is given, so $a_4 = 1$ and $a_2 = p - a_1 - a_3 - a_5 - a_5$ $a_6 - \dots - a_{2n} - 1$ come to N greater. When free a_2, a_4 and a_6 , using the argument same to freeing a_2 and a_4 , N is greater if $a_4 = a_6 = 1$ and $a_2 = p - a_1 - a_3 - a_5 - a_7 - a_7 - a_8 - a_7 - a_8 - a_$ $a_8 - \dots - a_{2n} - 2$ is satisfied. Deducing the rest by this method, one obtain that N reaches the maximum when we fix $a_1, a_3, \dots, a_{2n-1}$ and take $a_4 = a_6 = \dots = a_{2n} = 1$, $a_2 = p - a_1 - a_3 - \dots - a_{2n-1} - (n-1)$. Using the similar argument to fixing $a_1, a_3, \dots, a_{2n-1}$, one can show that N reaches the maximum when we fix a_2, a_4, \dots, a_{2n} and take $a_3 = a_5 = \dots = a_{2n-1} = 1, a_1 = p - a_2 - a_4 - \dots - a_{2n} - a_{2n-1} = 0$ (n-1). It is to say that, we have $a_3 = a_4 = ... = a_{2n} = 1$ and $a_1 + a_2 = p - (2n - 2)$. So N reaches the maximum when $a_1 = \lceil \frac{p-2n+2}{2} \rceil$, $a_2 = \lfloor \frac{p-2n+2}{2} \rfloor$ and $a_i = 1$ for $i = 3, 4, \dots, 2n$. \square

From now on we will study 4-qubit UPBs of size 8, and show how to find the UPBs $\mathcal{F}_1, ..., \mathcal{F}_6$. First of all, we present the following observation by counting the data in [13].

Lemma 12 Let $\mathcal{T}_{A:B:C:D} = \{|f_1, g_1, h_1, i_1\rangle, |f_2, g_2, h_2, i_2\rangle, ..., |f_8, g_8, h_8, i_8\rangle\}$ be a UPB of size 8. If one of the following three conditions holds, then $\mathcal{T}_{AB:CD}$ is not a UPB in the coarse graining $\mathbb{C}^4 \otimes \mathbb{C}^4$.

(i) $\mathcal{T}_{A:B:C:D}$ has a qubit having at least four identical vectors.

(ii) There are three subscripts j_1, j_2, j_3 such that $|f_{j_1}\rangle = |f_{j_2}\rangle = |f_{j_3}\rangle$ and $|g_{j_1}\rangle = |g_{j_2}\rangle = |g_{j_3}\rangle$.

(iii) There are five distinct subscripts $j_1, j_2, ..., j_5$ such that $|f_{j_1}\rangle = |f_{j_2}\rangle = |f_{j_3}\rangle$ and $|g_{j_4}\rangle = |g_{j_5}\rangle$.

Take $\mathcal{T}_{AB:CD} = \{ |p_1, q_1\rangle, |p_2, q_2\rangle, ..., |p_8, q_8\rangle \},\$ Proof. where $|p_j\rangle = |f_j, g_j\rangle$ and $|q_j\rangle = |h_j, i_j\rangle$ for j = 1, 2, ..., 8.

(i) By renaming the subscripts and permuting the qubits we may assume that $|f_1\rangle = |f_2\rangle = |f_3\rangle = |f_4\rangle$. Also $|g_1\rangle$, $|g_2\rangle$, $|g_3\rangle$, $|g_4\rangle$ is linearly dependent. So $|p_1\rangle$, $|p_2\rangle, |p_3\rangle, |p_4\rangle$ are linearly dependent. Then the space spanned by $|p_1\rangle$, $|p_2\rangle$, $|p_3\rangle$, $|p_4\rangle$ has dimension at most two. Therefore, there exists a $|p\rangle \in \mathbb{C}^4$ orthogonal to $|p_1\rangle$, $|p_2\rangle, |p_3\rangle, |p_4\rangle$ and $|p_5\rangle$. Moreover, there is a $|q\rangle \in \mathbb{C}^4$ orthogonal to $|q_6\rangle$, $|q_7\rangle$ and $|q_8\rangle$. So $|p,q\rangle$ is orthogonal to $\mathcal{T}_{AB:CD}$. By definition $\mathcal{T}_{AB:CD}$ is not a UPB in $\mathbb{C}^4 \otimes \mathbb{C}^4$.

(ii) We have $|p_{j_1}\rangle = |p_{j_2}\rangle = |p_{j_3}\rangle$. Therefore, the space spanned by $|p_{j_1}\rangle$, $|p_{j_2}\rangle$, $|p_{j_3}\rangle$, $|p_{j_4}\rangle$ and $|p_{j_5}\rangle$ has dimension at most three. Besides the space spanned by $|q_{i_6}\rangle$, $|q_{j_7}\rangle$ and $|q_{j_8}\rangle$ also has dimension at most three. Then, there exist $|p\rangle$, $|q\rangle \in \mathbb{C}^4$ such that $|p,q\rangle$ is orthogonal to $\mathcal{T}_{AB:CD}$.

(iii) Since the space spanned by $|q_7\rangle$, $|q_8\rangle$ and $|q_9\rangle$ has dimension at most three, there is a $|q\rangle \in \mathbb{C}^4$ orthogonal to $|q_7\rangle$, $|q_8\rangle$ and $|q_9\rangle$. Then $|f'_{i_1}, g'_{i_2}, q\rangle$ is orthogonal to $\mathcal{T}_{AB:CD}$.

Lemma 13 Let $\mathcal{T}_{A:B:C:D} = \{ |f_1, g_1, h_1, i_1 \rangle, \}$ $|f_2, g_2, h_2, i_2\rangle, \dots, |f_8, g_8, h_8, i_8\rangle\}$ be a UPB of size 8. If there are $|f_1\rangle = |f_2\rangle = |f_3\rangle$ and $|g_2\rangle = |g_3\rangle = |g_4\rangle$, then the remaining vectors $|f_4\rangle$, $|f_5\rangle$,..., $|f_8\rangle$ are pairwise linearly independent or $\mathcal{T}_{AB:CD}$ is no longer a UPB in $\mathcal{H}_{AB}: \mathcal{H}_{CD}$. Similarly, the remaining vectors $|g_1\rangle$, $|g_5\rangle$, $|g_6\rangle, ..., |g_8\rangle$ are pairwise linearly independent or $\mathcal{T}_{AB:CD}$ is no longer a UPB in \mathcal{H}_{AB} : \mathcal{H}_{CD} .

First of all, we prove that either the vec-Proof. tors $|f_4\rangle$, $|f_5\rangle$,..., $|f_8\rangle$ are pairwise linearly independent or $\mathcal{T}_{AB:CD}$ is no longer a UPB in \mathcal{H}_{AB} : \mathcal{H}_{CD} . If the set $\{f_4, f_5, ..., f_8\}$ has two identical elements, then the same element has multipicity two, three, four or five. When the element has multiplicity three, four or five, $\mathcal{T}_{AB:CD}$ is no longer a UPB from lemma 12 (iii). Also a qubit has an even number of distinct elements from lemma 9 (ii). So when the set $\{f_4, f_5, ..., f_8\}$ has an element of multiplicity two, it must contain two different elements of both multiplicity two. It is a contradiction with lemma 12 (iii). Now we have proved it.

Using the similar argument, one may show that the vectors $|g_1\rangle$, $|g_5\rangle$, $|g_6\rangle$,..., $|g_8\rangle$ are pairwise linearly independent or $\mathcal{T}_{AB:CD}$ is no longer a UPB in $\mathcal{H}_{AB}: \mathcal{H}_{CD}$.

Lemma 14 Let $\mathcal{T}_{A:B:C:D} = \{ |f_1, g_1, h_1, i_1 \rangle, \}$ $|f_2, g_2, h_2, i_2\rangle, \dots, |f_8, g_8, h_8, i_8\rangle$ be a 4-qubit UPB of size 8. If the first and second qubit respectively have three identical vectors, then $\mathcal{T}_{A:B:C:D}$ is not a UPB of size 8 in $\mathcal{H}_{AB} \otimes \mathcal{H}_{CD}$.

Suppose $|a_1\rangle$, $|a_2\rangle$, $|a_3\rangle$ are three identical Proof. vectors among $|f_1\rangle$, $|f_2\rangle$,..., $|f_8\rangle$ and $|b_1\rangle$, $|b_2\rangle$, $|b_3\rangle$ are three identical vectors among $|g_1\rangle$, $|g_2\rangle$,..., $|g_8\rangle$. If $|a_1\rangle$, $|a_2\rangle, |a_3\rangle, |b_1\rangle, |b_2\rangle, |b_3\rangle$ are in three, five or six distinct product vectors of $\mathcal{T}_{A:B:C:D}$, then $\mathcal{T}_{AB:CD}$ is not a UPB from Lemma 12 (ii) and (iii).

We only need to investigate the case that $|a_1\rangle$, $|a_2\rangle$, $|a_3\rangle, |b_1\rangle, |b_2\rangle, |b_3\rangle$ are in four distinct product vectors. Denote by U the UOM over the UPB $\mathcal{T}_{A:B:C:D}$. Moreover, p_i are o-number of column j of the U for j = 1, 2, 3, 4. Then U has $p_1 = 5$ and $p_2 = 5$ in condition of Lemma 13. Also $p_1 + p_2 + p_3 + p_4 \ge 8(8-1)/2$ from Lemma 10 (vi). So $p_3 + p_4 \ge 18$. We have p_3, p_4 $\leq (\frac{8-2n+2}{2})^2 = n^2 - 9n + 24$ from Lemma 11 and p = 8in condition of Lemma 11. Then the possible value of *n* is 1, 2, 3, 4. If n = 1, then $a_1 = a_2 = \frac{8-2\times 1+2}{2} = 4$ in Lemma 11. That is, $\mathcal{T}_{A:B:C:D}$ has a qubit having four identical vectors. Then $\mathcal{T}_{A:B:C:D}$ is not a UPB from Lemma 12 (i). If n = 3 or 4, then $p_3, p_4 \leq 6$ or 4. It makes a contradiction with $p_3 + p_4 \ge 18$. If n = 2, then $p_3, p_4 \leq 10$. So the case n = 2 is the only case that satisfies the condition $p_3 + p_4 \ge 18$. To satisfy $p_3 + p_4 \ge 18$, one of them must be $3 \times 3 + 1 \times 1$ and the other one could be either of $3 \times 3 + 1 \times 1$, $3 \times 2 + 2 \times 1$, $2 \times 2 + 2 \times 2$. There is no harm in supposing $p_3 = 3 \times 3 + 1 \times 1$. Then $p_4 = 3 \times 3 + 1 \times 1$, $3 \times 2 + 2 \times 1$ or $2 \times 2 + 2 \times 2$. We can obtain $h_{i_1} = h_{i_2} = h_{i_3}$ and $i_{i_4} = i_{i_5}$ for five distinct descripts $i_1, i_2, i_3, i_4, i_5 \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. So $\mathcal{T}_{A:B:C:D}$ is not a UPB from Lemma 12 (iii). П

Lemma 15 Let $\mathcal{T}_{A:B:C:D} = \{ |f_1, g_1, h_1, i_1 \rangle, \}$ $|f_2, g_2, h_2, i_2\rangle, ..., |f_8, g_8, h_8, i_8\rangle$ be a 4-qubit UPB of size 8. Then $\mathcal{T}_{AB:CD}$ is not a UPB of size 8 in $\mathcal{H}_{AB} \otimes \mathcal{H}_{CD}$ when one of the following three conditions is satisfied.

(i) There are three subscripts j_1, j_2, j_3 such that $|f_{j_1}\rangle =$ $|f_{j_2}\rangle = |f_{j_3}\rangle$ and $|h_{j_1}\rangle = |h_{j_2}\rangle = |h_{j_3}\rangle$.

(ii) There are three subscripts j_1, j_2, j_3 such that

 $\begin{array}{l} |f_{j_1}\rangle = |f_{j_2}\rangle = |f_{j_3}\rangle, \ |g_{j_1}\rangle = |g_{j_2}\rangle \ and \ |h_{j_1}\rangle = |h_{j_2}\rangle. \\ (iii) \ There \ are \ three \ subscripts \ j_1, j_2, j_3 \ such \ that \end{array}$ $|f_{j_1}\rangle = |f_{j_2}\rangle, |g_{j_1}\rangle = |g_{j_2}\rangle \text{ and } |h_{j_1}\rangle = |h_{j_2}\rangle = |h_{j_3}\rangle.$

Proof. (i) We prove the assertion by contradiction. Suppose $\mathcal{T}_{AB:CD}$ is a UPB of size 8 in $\mathcal{H}_{AB} \otimes \mathcal{H}_{CD}$. Up to the equivalence, we can assume $j_1 = 1$, $j_2 = 2$ and $j_3 = 3$, and $f_1 = f_2 = f_3 = h_1 = h_2 = h_3 = 0$. We express the UOM of $\mathcal{T}_{A:B:C:D}$ as

$$U_{1} = \begin{bmatrix} 0 & g_{1} & 0 & i_{1} \\ 0 & g_{2} & 0 & i_{2} \\ 0 & g_{3} & 0 & i_{3} \\ f_{4} & g_{4} & h_{4} & i_{4} \\ f_{5} & g_{5} & h_{5} & i_{5} \\ f_{6} & g_{6} & h_{6} & i_{6} \\ f_{7} & g_{7} & h_{7} & i_{7} \\ f_{8} & g_{8} & h_{8} & i_{8} \end{bmatrix}$$
(C1)

Since the first three rows of U_1 correspond to three orthogonal product vectors, we obtain that $|g_1, i_1\rangle, |g_2, i_2\rangle, |g_3, i_3\rangle$ are orthogonal. Up to equivalence we may assume that $g_1 = g_2 = 0$ and $g_3 = 1$. Since $\mathcal{T}_{AB:CD}$ is a UPB of size 8 in $\mathcal{H}_{AB} \otimes \mathcal{H}_{CD}$, Lemma 12 (iii) shows that $g_4, g_5, ..., g_8$ are distinct. In addition, we have $g_4, g_5, ..., g_8 \neq 0$ from Lemma 14. So one of them must be 1 by Lemma 9 (ii). So U_1 is equivalent to

$$\begin{bmatrix} 0 & 0 & 0 & i_{1} \\ 0 & 0 & 0 & i'_{1} \\ 0 & 1 & 0 & i_{3} \\ f_{4} & 1 & h_{4} & i_{4} \\ f_{5} & g_{5} & h_{5} & i_{5} \\ f_{6} & g'_{5} & h_{6} & i_{6} \\ f_{7} & g_{7} & h_{7} & i_{7} \\ f_{8} & g'_{7} & h_{8} & i_{8} \end{bmatrix}$$
(C2)

We have $g_5, g_7 \neq 0, 1$ from the discussion in the paragraph above (C2). Since the first and second row vectors are orthogonal to the last four row vectors of U_2 , we obtain that $|0,0\rangle \in \mathcal{H}_A \otimes \mathcal{H}_C$ is orthogonal to $|f_j, h_j\rangle$ for j = 5, 6, 7, 8. Since $\mathcal{T}_{AB:CD}$ is a UPB of size 8, Lemma 12 (iii) shows that f_5, f_6, f_7, f_8 contain exactly two 1's, and so do h_5, h_6, h_7, h_8 . So the matrix in (C2) is equivalent to

$$U_{11} = \begin{bmatrix} 0 & 0 & 0 & i_1 \\ 0 & 0 & 0 & i'_1 \\ 0 & 1 & 0 & i_3 \\ f_4 & 1 & h_4 & i_4 \\ 1 & g_5 & h_5 & i_5 \\ 1 & g'_5 & h_6 & i_6 \\ f_7 & g_7 & 1 & i_7 \\ f_8 & g'_7 & 1 & i_8 \end{bmatrix} \quad \text{or} \quad U_{12} = \begin{bmatrix} 0 & 0 & 0 & i_1 \\ 0 & 0 & 0 & i'_1 \\ 0 & 1 & 0 & i_3 \\ f_4 & 1 & h_4 & i_4 \\ 1 & g_5 & h_5 & i_5 \\ f_6 & g'_5 & 1 & i_6 \\ 1 & g_7 & h_7 & i_7 \\ f_8 & g'_7 & 1 & i_8 \end{bmatrix} (C3)$$

where $g_5 \neq g_7, g'_7$. For U_{11} , Lemma 12 shows that $h_5, f_7, f_8 \neq 0$. Since row 5 is orthogonal to row 7 and 8, we have $i'_5 = i_7 = i_8$. So column 3 and 4 of U_{11} shows a contradiction with Lemma 12 (iii) and the fact that $\mathcal{T}_{AB;CD}$ is a UPB of size 8.

On the other hand for U_{12} , similar to the above argument for U_{11} one can show that $i_8 = i'_5$, $i_7 = i'_6$, $f_8 = f'_6$ and $h_7 = h'_5$. Then row 4 of U_{12} is not orthogonal to all four bottom row vectors of U_{12} . It is a contradiction with the fact that U_{12} is a UOM. We have proven that $\mathcal{T}_{AB:CD}$ is not a UPB of size 8 in $\mathcal{H}_{AB} \otimes \mathcal{H}_{CD}$.

(ii) We prove the assertion by contradiction. Suppose $\mathcal{T}_{AB:CD}$ is a UPB of size 8 in $\mathcal{H}_{AB} \otimes \mathcal{H}_{CD}$. Up to the equivalence, we can assume $j_1 = 1$, $j_2 = 2$ and $j_3 = 3$ and $f_1 = f_2 = f_3 = g_1 = g_2 = h_1 = h_2 = 0$. We express

the UOM of $\mathcal{T}_{A:B:C:D}$ as

$$U_{1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & g_{3} & h_{3} & i_{3} \\ f_{4} & g_{4} & h_{4} & i_{4} \\ f_{5} & g_{5} & h_{5} & i_{5} \\ f_{6} & g_{6} & h_{6} & i_{6} \\ f_{7} & g_{7} & h_{7} & i_{7} \\ f_{8} & g_{8} & h_{8} & i_{8} \end{bmatrix}$$
(C4)

We claim that there are only two cases U_{11} and U_{12} in (C5), where f_5 may be f_7 or f'_7 and $f_5, f_6, f_7 \neq 0, 1$ is satisfied.

$$U_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & g_3 & h_3 & i_3 \\ 1 & g_4 & h_4 & i_4 \\ f_5 & g_5 & h_5 & i_5 \\ f'_5 & g_6 & h_6 & i_6 \\ f_7 & g_7 & h_7 & i_7 \\ f'_7 & g_8 & h_8 & i_8 \end{bmatrix}, \quad U_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & g_3 & h_3 & i_3 \\ 1 & g_4 & h_4 & i_4 \\ 1 & g_5 & h_5 & i_5 \\ f_6 & g_6 & h_6 & i_6 \\ f_6 & g_7 & h_7 & i_7 \\ f'_6 & g_8 & h_8 & i_8 \end{bmatrix}.$$
(C5)

We can obtain that k of f_4, f_5, f_6, f_7, f_8 of U_1 equal to 1 from Lemma9(i), where $1 \le k \le 5$ and k is a positive integer. Moreover, we have $f_4, f_5, f_6, f_7, f_8 \ne 0$ and $1 \le k \le 3$ from Lemma 12(i). However, if k = 3, then column 1, 2 of U_1 make a contradiction with the fact that U_1 is a UOM and Lemma 12(iii). Then we have k = 1, 2. Namely, up to equivalence we obtain two cases, $f_4 = 1, f_5, f_6, f_7, f_8 \ne 1$ and $f_4 = f_5 = 1, f_6, f_7, f_8 \ne 1$. For $f_4 = 1, f_5, f_6, f_7, f_8 \ne 1$, at most two of f_5, f_6, f_7, f_8 are the same from 12(iii). Moreover, from Lemma 9(i) we can obtain $f_6 = f'_5, f_8 = f'_7$ up to equivalent, where f_5 may be f_7 or f'_7 . So we have proved the claim in the line above (C5).

For U_{11} in (C5), we claim that there are two cases, U_{111} and U_{112} . We can obtain $f_5, f'_5, f_7, f'_7 \neq 1$ from $f_5, f_7 \neq 0, 1$ in the line above (C5). Since row 1, 2 are othogonal to row 3, 5, 6, 7 and 8 of U_{11} , we obtain that $|0,0\rangle \in \mathcal{H}_B \otimes \mathcal{H}_C$ is orthogonal to g_j, h_j for j = 3, 5, 6, 7, 8. First, one can show at most two of g_3, g_5, g_6, g_7, g_8 are 1's. Otherwise, row 1 and 2 of U_{11} is a contradiction with Lemma 12(iii) and the fact that U_{11} is a UOM by the assumption U_1 in (C4) is a UOM. Second, one can show at most three of h_3, h_5, h_6, h_7, h_8 are 1's from Lemma 12(i). Then we have shown that two of g_3, g_5, g_6, g_7, g_8 are 1's and three of h_3, h_5, h_6, h_7, h_8 are 1's in U_{11} . If $g_3 = 1$, up to equivalence we can assume $g_5 = 1$. We directly obtain $h_6 = h_7 = h_8 = 1$. Then U_{11} becomes U_{111} . On the other hand for $g_3 \neq 1$, that is $h_3 = 1$, up to equivalence we can assume $g_5 = g_6 = 1$. We directly obtain $h_7 = h_8 = 1$. Then U_{11} becomes U_{112} . Now we have proved the claim at the beginning of this paragraph.

There is $|1, 0, 0, i'_4\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$ orthogonal to all row vectors of U_{111} and U_{112} . It is a contradiction with the fact U_{111} and U_{112} are UOMs of size 8 in $\mathcal{H}_{CD} \otimes \mathcal{H}_{CD}$ by the assumption U_1 in (C4) is a UOM.

$$U_{111} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & h_3 & i_3 \\ 1 & g_4 & h_4 & i_4 \\ f_5 & 1 & h_5 & i_5 \\ f'_5 & g_6 & 1 & i_6 \\ f_7 & g_7 & 1 & i_7 \\ f'_7 & g_8 & 1 & i_8 \end{bmatrix}, \quad U_{112} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & g_3 & 1 & i_3 \\ 1 & g_4 & h_4 & i_4 \\ f_5 & 1 & h_5 & i_5 \\ f'_5 & 1 & h_6 & i_6 \\ f_7 & g_7 & 1 & i_7 \\ f'_7 & g_8 & 1 & i_8 \end{bmatrix}$$
(C6)

For U_{12} in (C5), we claim that there are four cases U_{121} , U_{122} , U_{123} and U_{124} in (C7). Since row 1, 2 are othogonal to row 3, 6, 7 and 8 of U_{11} , we obtain that $|0,0\rangle \in \mathcal{H}_B \otimes \mathcal{H}_C$ is orthogonal to $|g_j,h_j\rangle$ for j = 3, 6, 7, 8. First, one can show at most two of g_3, g_6, g_7, g_8 are 1's. Otherwise, row 1 and 2 of U_{11} is a contradiction with

$$U_{121} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & h_3 & i_3 \\ 1 & g_4 & h_4 & i_4 \\ 1 & g_5 & h_5 & i_5 \\ f_6 & 1 & h_6 & i_6 \\ f_6 & g_7 & 1 & i_7 \\ f_6' & g_8 & 1 & i_8 \end{bmatrix}, \quad U_{122} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & h_3 & i_3 \\ 1 & g_4 & h_4 & i_4 \\ 1 & g_5 & h_5 & i_5 \\ f_6 & g_6 & 1 & i_6 \\ f_6 & g_7 & 1 & i_7 \\ f_6' & 1 & h_8 & i_8 \end{bmatrix},$$

In the following we show that neither of U_{121} , U_{122} , U_{123} , U_{124} is a UOM in $\mathcal{H}_{AB} \otimes \mathcal{H}_{CD}$. This will prove the claim of (ii).

For U_{121} in (C7), we have $f_6, f'_6 \neq 1$ from the line above (C5). We obtain $g_7, g_8 \neq 0$ from Lemma 14 and $h_3 \neq 0$ from Lemma 15 (i). Since row 3 is orthogonal to row 7, 8, we can obtain $i_7 = i_8 = i'_3$. Then U_{121} becomes U_{1211} in (C8). So there exists $|0, 0, 1, i_3\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes$ $\mathcal{H}_C \otimes \mathcal{H}_D$ orthogonal to all row vectors of U_{1211} . It shows a contradiction with the definition of UOM and the fact that U_{1211} is a UOM by the assumption U_1 in (C4) is a UOM.

For U_{122} in (C7), we have $f_6 \neq 1$ from the line above (C5). We obtain $g_6, g_7 \neq 0$ from Lemma 14 and $h_3 \neq 0$ from Lemma 15 (i). Since row 3 is orthogonal to row 6, 7, we can obtain $i_6 = i_7 = i'_3$. Then U_{122} becomes U_{1221} in (C8). So there exists $|0, 0, 1, i_3\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes$ $\mathcal{H}_C \otimes \mathcal{H}_D$ orthogonal to all row vectors of U_{1221} . It shows a contradiction with the definition of UPB and the fact Lemma 12(iii) and the fact that U_{12} is a UOM by the assumption that U_1 in (C4) is a UOM. Second, one can show at most three of h_3, h_6, h_7, h_8 are 1's from Lemma 12(i). If three of h_3, h_6, h_7, h_8 are 1's, then there exists $|0, 1, 0, i'_3\rangle$ orthogonal to U_{12} for $g_3 = 1$ and there exists $|0,1,0,i_j\rangle$ orthogonal to U_{12} for $h_3 = 1$ and $g_j = 1$ for j = 6, or 7, or 8. It is a contradiction with the definition of UPB and the fact U_{12} is a UOM by the assumption U_1 in (C4) is a UOM. Then we have shown that two of g_3, g_6, g_7, g_8 are 1's and two of h_3, h_6, h_7, h_8 are 1's in U_{12} . For $g_3 = 1$, we have two cases, $g_6 = 1$ and $g_8 = 1$. In case one can directly obtain $h_7 = h_8 = 1$. Then U_{12} becomes U_{121} . In case two one can directly obtain $h_6 = h_7 = 1$. Then U_{12} becomes U_{122} . On the other hand for $g_3 \neq 1$, that is $h_3 = 1$, we also have two cases, $g_6 = g_7 = 1$ and $g_6 = g_8 = 1$. In case one can directly obtain $h_8 = 1$. Then U_{12} becomes U_{123} . In case two one can directly obtain $h_7 = 1$. Then U_{12} becomes U_{124} . Now we have proved the claim at the beginning of this paragraph.

$$U_{123} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & g_3 & 1 & i_3 \\ 1 & g_4 & h_4 & i_4 \\ 1 & g_5 & h_5 & i_5 \\ f_6 & 1 & h_6 & i_6 \\ f_6 & 1 & h_7 & i_7 \\ f_6' & g_8 & 1 & i_8 \end{bmatrix}, \quad U_{124} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & g_3 & 1 & i_3 \\ 1 & g_4 & h_4 & i_4 \\ 1 & g_5 & h_5 & i_5 \\ f_6 & 1 & h_6 & i_6 \\ f_6 & g_7 & 1 & i_7 \\ f_6' & 1 & h_8 & i_8 \end{bmatrix}.$$
(C7)

that U_{1221} is a UOM.

$$U_{1211} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & h_3 & i_3 \\ 1 & g_4 & h_4 & i_4 \\ 1 & g_5 & h_5 & i_5 \\ f_6 & 1 & h_6 & i_6 \\ f_6 & g_7 & 1 & i'_3 \\ f'_6 & g_8 & 1 & i'_3 \end{bmatrix}, \quad U_{1221} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & h_3 & i_3 \\ 1 & g_4 & h_4 & i_4 \\ 1 & g_5 & h_5 & i_5 \\ f_6 & g_6 & 1 & i'_3 \\ f_6 & g_7 & 1 & i'_3 \\ f'_6 & 1 & h_8 & i_8 \end{bmatrix}$$
(C8)

For U_{123} in (C7), we have $f_6 \neq 0, 1$ from the line above (C5). We obtain $g_3, g_4, g_5 \neq 0$ from Lemma 14. In the following we show $h_4, h_5, h_7 \neq 1$. First, one can obtain $h_4 \neq 1$. Otherwise, we have $|1, 0, 0, i'_5\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes$ \mathcal{H}_D is orthogonal to all row vectors of U_{123} . Second, one can obtain $h_5 \neq 1$. Otherwise, we have $|1, 0, 0, i'_4\rangle \in$ $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$ is orthogonal to all row vectors of U_{123} . Lastly, one can obtain $h_7 \neq 1$. Otherwise, we have $|0, 1, 0, i'_6\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$ is orthogonal to all row vectors of U_{123} . Then we have proved

$$f_6, f'_6 \neq 0, 1, \ g_3, g_4, g_5 \neq 0, 1 \ h_4, h_5, h_7 \neq 1.$$
 (C9)

We claim that U_{123} in (C7) has two cases U_{1231} and U_{1232} in (C10). Since row 3 is orthogonal to row 6, 7, we can obtain $|1, i_3\rangle$ is orthogonal to $|h_6, i_6\rangle$ and $|h_7, i_7\rangle$ from $f_6 \neq 1, g_3 \neq 0$ by (C9). One can show that h_6, h_7 are not equal to 0 at the same time. Otherwise, column 3 of U_{123} shows a contradiction with Lemma 12 (i) and the fact U_{123} is a UOM by the assumption U_1 in (C4) is a UOM. Then we have $h_6 = 0, h_7 \neq 0$ or $h_6 \neq 0, h_7 = 0$ or $h_6, h_7 \neq 0$. Since $f_6 = f_7, g_6 = g_7$ and i_6, i_7 is undetermined, one can obtain the first two cases $h_6 = 0, h_7 \neq 0$ and

 $h_6 \neq 0, h_7 = 0$ are equivalent. Up to equivalence, we have two cases $h_6 = 0, h_7 \neq 0$ or $h_6, h_7 \neq 0$ for U_{123} . For $h_6 = 0, h_7 \neq 0$ in U_{123} , since $|1, i_3\rangle$ is orthogonal to $|h_7, i_7\rangle$ from the second line in this paragraph, we obtain $i_7 = i'_3$. Since $h_7 \neq 1$ by (C9) and row 5 is orthogonal to row 6, we have $i_6 = i'_7 = i_3$. Since $f_6 \neq 0, g_4, g_5 \neq$ $0, h_4, h_5 \neq 1$ by (C9) and row 6 is orthogonal to row 4, 5, we have $i_4 = i_5 = i'_6 = i'_3$. Then U_{123} becomes U_{1231} in (C10). For $h_6, h_7 \neq 0$ in U_{123} , since $|1, i_3\rangle$ is orthogonal to $|h_6, i_6\rangle$ and $|h_7, i_7\rangle$ from the second line in this paragraph, we can obtain $i_6 = i_7 = i'_3$. Then U_{123} becomes U_{1232} in (C10). We have proved the claim at the beginning of this paragraph.

$$U_{1231} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & g_3 & 1 & i_3 \\ 1 & g_4 & h_4 & i'_3 \\ 1 & g_5 & h_5 & i'_3 \\ f_6 & 1 & 0 & i_3 \\ f_6 & 1 & h_7 \neq 0 & i'_3 \\ f'_6 & g_8 & 1 & i_8 \end{bmatrix}, \quad U_{1232} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & g_3 & 1 & i_3 \\ 1 & g_4 & h_4 & i_4 \\ 1 & g_5 & h_5 & i_5 \\ f_6 & 1 & h_6 \neq 0 & i'_3 \\ f_6 & 1 & h_7 \neq 0 & i'_3 \\ f'_6 & g_8 & 1 & i_8 \end{bmatrix}.$$
(C10)

For U_{1231} in (C10), there exists $|1, 0, 0, i_3\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$ is orthogonal to all row vectors of U_{1231} . It shows a contradiction with the definition of UOM and the fact U_{1231} is a UOM in $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$ by the assumption U_1 in (C4) is a UOM. That is, U_{1231} is not a UOM in $\mathcal{H}_{AB} \otimes \mathcal{H}_{CD}$.

For U_{1232} in (C10), there exists $|0, 1, 0, i_3\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$ is orthogonal to all row vectors of U_{1232} . It shows a contradiction with the definition of UOM and the fact U_{1232} is a UOM in $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$ by the assumption U_1 in (C4) is a UOM. That is, U_{1232} is not a UOM in $\mathcal{H}_{AB} \otimes \mathcal{H}_{CD}$.

Therefore, U_{123} in (C10) is not a UOM in $\mathcal{H}_{AB} \otimes \mathcal{H}_{CD}$. For U_{124} in (C7), we have $f_6, f'_6 \neq 0, 1$ from the line above (C5). We obtain $g_3, g_4, g_5 \neq 0$ from Lemma 14. In the following we show that $h_4, h_5 \neq 1$. First, one can obtain $h_4 \neq 1$. Otherwise, we have $|1,0,0,i'_5\rangle \in$ $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$ is orthogonal to all row vectors of U_{124} . Second, one can obtain $h_5 \neq 1$. Otherwise, we have $|1,0,0,i'_4\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$ is orthogonal to all row vectors of U_{124} . Then we have proved

$$f_6, f'_6 \neq 0, 1, \ g_3, g_4, g_5 \neq 0, \ h_4, h_5 \neq 1.$$
 (C11)

We claim that U_{124} in (C7) has three cases U_{1241} , U_{1242} and U_{1243} in (C12). Since row 3 is orthogonal to row 6, 8, we can obtain $|1, i_3\rangle$ is orthogonal to $|h_6, i_6\rangle$ and $|h_8, i_8\rangle$ from $f_6, f'_6 \neq 1, g_3 \neq 0$ by (C11). One can show that h_6, h_8 are not equal to 0 at the same time. Otherwise, column 3 of U_{124} shows a contradiction with Lemma 12 (i) and the fact U_{124} is a UOM by the assumption U_1 in (C4) is a UOM. Then we have three cases $h_6 = 0, h_8 \neq 0$ or $h_6 \neq 0, h_8 = 0$ or $h_6, h_8 \neq 0$ for U_{124} . For $h_6 =$ $0, h_8 \neq 0$ in U_{124} , since $f_6 \neq 0, g_4, g_5 \neq 0, h_4, h_5 \neq 1$ by (C11) and row 6 is orthogonal to row 4, 5, we have $i_4 = i_5 = i'_6$. Then U_{124} becomes U_{1241} in (C12). For $h_6 \neq 0, h_8 = 0$ in U_{124} , since $f'_6 \neq 0, g_4, g_5 \neq 0, h_4, h_5 \neq 1$ by (C11) and row 8 is orthogonal to row 4, 5, we have $i_4 = i_5 = i'_8$. Then U_{124} becomes U_{1242} in (C12). For $h_6, h_8 \neq 0$ in U_{124} , since $|1, i_3\rangle$ is orthogonal to $|h_6, i_6\rangle$ and $|h_7, i_7\rangle$ from the second line in this paragraph, we can obtain $i_6 = i_8 = i'_3$. Then U_{124} becomes U_{1243} in (C12). We have proved the claim at the beginning of this paragraph.

$$U_{1241} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & g_3 & 1 & i_3 \\ 1 & g_4 & h_4 & i_6' \\ 1 & g_5 & h_5 & i_6' \\ f_6 & 1 & 0 & i_6 \\ f_6 & g_7 & 1 & i_7 \\ f_6' & 1 & h_8 \neq 0 & i_8 \end{bmatrix}, \quad U_{1242} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & g_3 & 1 & i_3 \\ 1 & g_4 & h_4 & i_8' \\ 1 & g_5 & h_5 & i_8' \\ f_6 & 1 & h_6 \neq 0 & i_6 \\ f_6 & g_7 & 1 & i_7 \\ f_6' & 1 & 0 & i_8 \end{bmatrix}, \quad U_{1243} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & g_3 & 1 & i_3 \\ 1 & g_4 & h_4 & i_4 \\ 1 & g_5 & h_5 & i_5 \\ f_6 & 1 & h_6 \neq 0 & i_3' \\ f_6 & g_7 & 1 & i_7 \\ f_6' & 1 & 0 & i_8 \end{bmatrix}.$$
(C12)

For U_{1241} in (C12), there exists $|1, 0, 0, i_6\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$ is orthogonal to all row vectors of U_{1241} . It shows a contradiction with the definition of UOM and the fact U_{1241} is a UOM in $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$ by the assumption U_1 in (C4) is a UOM. That is, U_{1241} is not a UOM in $\mathcal{H}_{AB} \otimes \mathcal{H}_{CD}$.

For U_{1242} in (C12), there exists $|1, 0, 0, i_8\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$ is orthogonal to all row vectors of U_{1242} . It shows a contradiction with the definition of UOM and the fact U_{1242} is a UOM in $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$ by the assumption U_1 in (C4) is a UOM. That is, U_{1242} is not a UOM in $\mathcal{H}_{AB} \otimes \mathcal{H}_{CD}$.

For U_{1243} in (C12), there exists $|0, 1, 0, i_3\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$ is orthogonal to all row vectors of U_{1243} . It shows a contradiction with the definition of UOM and the fact U_{1243} is a UOM in $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$ by the assumption U_1 in (C4) is a UOM. That is, U_{1243} is not a UOM in $\mathcal{H}_{AB} \otimes \mathcal{H}_{CD}$.

So U_{124} in (C10) is not a UOM in $\mathcal{H}_{AB} \otimes \mathcal{H}_{CD}$.

We haved proved the claim below (C7).

Therefore, we have proved that $\mathcal{T}_{AB:CD}$ is not a UPB of size 8 in $\mathcal{H}_{AB} \otimes \mathcal{H}_{CD}$.

(iii) We prove the assertion by contradiction. Suppose $\mathcal{T}_{AB:CD}$ is a UPB of size 8 in $\mathcal{H}_{AB} \otimes \mathcal{H}_{CD}$. Up to the equivalence, we can assume $j_1 = 1$, $j_2 = 2$ and $j_3 = 3$ and $f_1 = f_2 = g_1 = g_2 = h_1 = h_2 = h_3 = 0$. We express the UOM of $\mathcal{T}_{A:B:C:D}$ as

$$U_{1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ f_{3} & g_{3} & 0 & i_{3} \\ f_{4} & g_{4} & h_{4} & i_{4} \\ f_{5} & g_{5} & h_{5} & i_{5} \\ f_{6} & g_{6} & h_{6} & i_{6} \\ f_{7} & g_{7} & h_{7} & i_{7} \\ f_{8} & g_{8} & h_{8} & i_{8} \end{bmatrix}.$$
 (C13)

We claim that there are only two cases U_{11} and U_{12} ,

where c_1 , c_2 , c_3 , c_4 , c_5 are not 0's or 1's.

$$U_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ f_3 & g_3 & 0 & i_3 \\ f_4 & g_4 & 1 & i_4 \\ f_5 & g_5 & c_1 & i_5 \\ f_6 & g_6 & c_2 & i_6 \\ f_7 & g_7 & c_3 & i_7 \\ f_8 & g_8 & c_4 & i_8 \end{bmatrix}, \quad U_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ f_3 & g_3 & 0 & i_3 \\ f_4 & g_4 & 1 & i_4 \\ f_5 & g_5 & 1 & i_5 \\ f_6 & g_6 & c_5 & i_6 \\ f_7 & g_7 & c_5 & i_7 \\ f_8 & g_8 & c_5 & i_8 \end{bmatrix}.$$
(C14)

In fact, if one of c_1 , c_2 , c_3 , c_4 , c_5 is 0 or 1 then $\mathcal{T}_{AB:CD}$ is not a UPB from Lemma 12 (i) or $U_{11} = U_{12}$. If two of c_1 , c_2 , c_3 , c_4 are 1's, we can assume $c_1 = c_2 = 1$. Then the space spanned by $|h_4, i_4\rangle$, $|h_5, i_5\rangle$, $|h_6, i_6\rangle$, $|h_7, i_7\rangle$ has dimension at most three. Also the space spanned by $|f_1, g_1\rangle$, $|f_2, g_2\rangle$, $|f_3, g_3\rangle$, $|f_8, g_8\rangle$ has dimension at most three. So there exist ϕ , $\psi \in \mathbb{C}^4$ such that ϕ and ψ is respectively orthogonal to $|f_1, g_1\rangle$, $|f_2, g_2\rangle$, $|f_3, g_3\rangle$, $|f_8, g_8\rangle$ and $|h_4, i_4\rangle$, $|h_5, i_5\rangle$, $|h_6, i_6\rangle$, $|h_7, i_7\rangle$. It makes a contradiction with the assumption that $\mathcal{T}_{AB:CD}$ is a UPB. Therefore, the form U_{11} and U_{12} are all possible cases. We have proved the claim above (C14).

For U_{11} , since the first two rows are orthogonal to row 3, 5, 6, 7, 8, we obtain that $|0, 0\rangle \in \mathcal{H}_{AB}$ is orthogonal to $|f_j, g_j\rangle$ for j = 3, 5, 6, 7, 8. Namely, three of f_3, f_5, f_6, f_7, f_8 or g_3, g_5, g_6, g_7, g_8 are 1's. It's a contradiction with Lemma 12 (iii) and the fact that $\mathcal{T}_{AB:CD}$ is a UPB of size 8.

For U_{12} , we have $f_3, g_3, f_4, g_4, f_5, g_5, f_6, g_6, f_7, g_7, f_8, g_8 \neq 0$ from Lemma 15(ii). And the first rows are orthogonal to each of the last six rows of U_{12} . So $|0,0\rangle \in \mathcal{H}_{AB}$ is orthogonal to each of $|f_3, g_3\rangle$, $|f_6, g_6\rangle$, $|f_7, g_7\rangle$, $|f_8, g_8\rangle$. Also the first two rows both have at most two identical product vectors from Lemma 12 (iii). So we have the

$$U_{121} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & g_3 & 0 & i_3 \\ f_4 & g_4 & 1 & i_4 \\ f_5 & g_5 & 1 & i_5 \\ 1 & g_6 & a_5 & i_6 \\ f_7 & 1 & a_5 & i_7 \\ f_8 & 1 & a'_5 & i_8 \end{bmatrix}, \quad U_{122} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & g_3 & 0 & i_3 \\ f_4 & g_4 & 1 & i_4 \\ f_5 & g_5 & 1 & i_5 \\ f_6 & 1 & a_5 & i_6 \\ f_7 & 1 & a_5 & i_7 \\ 1 & g_8 & a'_5 & i_8 \end{bmatrix}$$
(C15)

For U_{121} , row 3 is orthogonal to row 7 and 8, so $|i_3\rangle$ is orthogonal to $|i_7\rangle$ and $|i_8\rangle$. That is, $i_7 = i_8 = i'_3$. Using the similar argument, we can obtain $i_6 = i_7 = i'_3$ in U_{122} because row 3 is orthogonal to row 6 and 7. So U_{121} and U_{122} are respectively equivalent to U_{1211} and U_{1221} .

$$U_{1211} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & g_3 & 0 & i_3 \\ f_4 & g_4 & 1 & i_4 \\ f_5 & g_5 & 1 & i_5 \\ 1 & g_6 & a_5 & i_6 \\ f_7 & 1 & a_5 & i'_3 \\ f_8 & 1 & a'_5 & i'_3 \end{bmatrix}, \quad U_{1221} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & g_3 & 0 & i_3 \\ f_4 & g_4 & 1 & i_4 \\ f_5 & g_5 & 1 & i_5 \\ f_6 & 1 & a_5 & i'_3 \\ f_7 & 1 & a_5 & i'_3 \\ 1 & g_8 & a'_5 & i_8 \end{bmatrix}$$
C16)

So column 3 and 4 of U_{1211} and U_{1221} shows a contradiction with Lemma 12 (iii) and the fact that $\mathcal{T}_{AB:CD}$ is a UPB of size 8.

The following proposition can be proven similarly to Lemma 12, 13, 14, 15.

- Noga Alon and László Lovász. Unextendible product bases. Journal of Combinatorial Theory, Series A, 95(1):169–179, 2001.
- [2] D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal. Unextendible Product Bases, Uncompletable Product Bases and Bound Entanglement. *Communications in Mathematical Physics*, 238:379–410, 2003.
- [3] Jianxin Chen and Nathaniel Johnston. The minimum size of unextendible product bases in the bipartite case (and some multipartite cases). *Communications in Mathematical Physics*, 333(1):351–365, 2013.
- [4] J. Tura, R. Augusiak, P. Hyllus, M. Ku, J. Samsonowicz, and M. Lewenstein. Four-qubit entangled symmetric states with positive partial transpositions. *Physical Review A*, 85(6):060302, 2012.
- [5] Jianxin Chen, Lin Chen, and Bei Zeng. Unextendible product basis for fermionic systems. *Journal of Mathematical Physics*, 55(8), 2014.
- [6] R. Augusiak, T. Fritz, Ma. Kotowski, Mi. Kotowski, M. Lewenstein, and A. Acn. Tight bell inequalities with no quantum violation from qubit unextendible product bases. *Physical Review A*, 85(4):4233–4237, 2012.
- [7] Maciej Demianowicz and Remigiusz Augusiak. From

Proposition 16 Let $\mathcal{T}_{A:B:C:D} = \{|f_1, g_1, h_1, i_1\rangle, |f_2, g_2, h_2, i_2\rangle, ..., |f_8, g_8, h_8, i_8\rangle\}$ be a 4-qubit UPB of size 8 in $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$.

(i) If there are two subscripts j_1, j_2 such that $|f_{j_1}\rangle = |f_{j_2}\rangle$, $|g_{j_1}\rangle = |g_{j_2}\rangle$ and $|h_{j_1}\rangle = |h_{j_2}\rangle$, then $\mathcal{T}_{A:B:C:D}$ is no longer a UPB in $\mathcal{H}_{AB} \otimes \mathcal{H}_{CD}$.

(ii) If there are two subscripts j_1, j_2 such that $|f_{j_1}\rangle = |f_{j_2}\rangle$, $|g_{j_1}\rangle = |g_{j_2}\rangle$, $|h_{j_1}\rangle = |h'_{j_2}\rangle$ and $|i_{j_1}\rangle = |i'_{j_2}\rangle$, then $\mathcal{T}_{A:B:C:D}$ is no longer a UPB in $\mathcal{H}_{AB} \otimes \mathcal{H}_{CD}$.

(iii) If there are two subscripts j_1, j_2 such that $|f_{j_1}\rangle = |f_{j_2}\rangle$, $|g_{j_1}\rangle = |g'_{j_2}\rangle$, $|h_{j_1}\rangle = |h_{j_2}\rangle$ and $|i_{j_1}\rangle = |i'_{j_2}\rangle$, then F_1 in Appendix A happen to be all UOMs satisfying this condition.

(iv) If there are two subscripts j_1, j_2 such that $|f_{j_1}\rangle = |f_{j_2}\rangle$ and $|g_{j_1}\rangle = |g_{j_2}\rangle$, then $\mathcal{T}_{A:B:C:D}$ is no longer UPB in $\mathcal{H}_{AB} \otimes \mathcal{H}_{CD}$.

(v) If there are two subscripts j_1, j_2 such that $|f_{j_1}\rangle = |f_{j_2}\rangle$ and $|h_{j_1}\rangle = |h_{j_2}\rangle$, then F_2, F_3, F_4, F_5 in Appendix A happen to be all UOMs satisfying this condition.

(vi) If there are five distinct subscripts j_1, j_2, j_3, j_4, j_5 such that $|f_{j_1}\rangle = |f_{j_2}\rangle = |f_{j_3}\rangle$ and $|h_{j_3}\rangle = |h_{j_4}\rangle = |h_{j_5}\rangle$, then F_2, F_3, F_4, F_5 in Appendix A happen to be all UOMs satisfying this condition.

(vii) If there are three distinct subscripts j_1, j_2, j_3 such that $|f_{j_1}\rangle = |f_{j_2}\rangle = |f_{j_3}\rangle$, then F_2, F_3, F_4, F_5 in Appendix A happen to be all UOMs satisfying this condition.

(viii) If there are three distinct subscripts j_1, j_2 such that $|f_{j_1}\rangle = |f_{j_2}\rangle$, then F_2, F_3, F_4, F_5, F_6 in Appendix A happen to be all UOMs satisfying this condition.

unextendible product bases to genuinely entangled subspaces. *Phys. Rev. A*, 98:012313, Jul 2018.

- [8] L. Dicarlo, M. D. Reed, L. Sun, B. R. Johnson, J. M. Chow, J. M. Gambetta, L. Frunzio, S. M. Girvin, M. H. Devoret, and R. J. Schoelkopf. Preparation and measurement of three-qubit entanglement in a superconducting circuit. *Nature*, 467(7315):574–8, 2010.
- [9] M. D. Reed, L Dicarlo, S. E. Nigg, L. Sun, L Frunzio, S. M. Girvin, and R. J. Schoelkopf. Realization of threequbit quantum error correction with superconducting circuits. *Nature*, 482(7385):382–5, 2012.
- [10] Lin Chen and Dragomir Z. Djokovic. Nonexistence of nqubit unextendible product bases of size 2ⁿ-5. Quantum Information Processing, 17(2):24, 2018.
- [11] Nathaniel Johnston. The structure of qubit unextendible product bases. Journal of Physics A Mathematical and Theoretical, 47(47), 2014.
- [12] Lin Chen and Dragomir Z Djokovic. Orthogonal product bases of four qubits. *Journal of Physics A: Mathematical* and Theoretical, 50(39):395301, 2017.
- [13] Nathaniel Johnston. The minimum size of qubit unextendible product bases. arXiv preprint arXiv:1302.1604, 2013.

- [14] C. H. Bennett, D. P. Divincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal. Unextendible Product Bases and Bound Entanglement. *Physical Review Letters*, 82:5385–5388, June 1999.
- [15] Hans J. Briegel and Robert Raussendorf. Persistent entanglement in arrays of interacting particles. *Phys. Rev. Lett.*, 86:910–913, Jan 2001.
- [16] Géza Tóth, Christian Knapp, Otfried Gühne, and Hans J. Briegel. Optimal spin squeezing inequalities detect bound entanglement in spin models. *Phys. Rev. Lett.*, 99:250405, Dec 2007.
- [17] Lin Chen and Shmuel Friedland. The tensor rank of tensor product of two three-qubit w states is eight. *Linear Algebra and Its Applications*, 543:1–16, 2018.
- [18] Leonid Gurvits. Classical deterministic complexity of edmonds' problem and quantum entanglement. 2003.
- [19] F Monteiro, Vivoli V Caprara, T Guerreiro, A Martin, J. D. Bancal, H Zbinden, R. T. Thew, and N Sangouard. Revealing genuine optical-path entanglement. *Physical Review Letters*, 114(17), 2015.
- [20] Y Yeo and W. K. Chua. Teleportation and dense coding with genuine multipartite entanglement. *Physical Review Letters*, 96(6):060502, 2006.
- [21] Marcus Huber and Ritabrata Sengupta. Witnessing genuine multipartite entanglement with positive maps. *Physical Review Letters*, 113(10):100501, 2014.
- [22] Adn Cabello, Alessandro Rossi, Giuseppe Vallone, Francesco De Martini, and Paolo Mataloni. Proposed bell experiment with genuine energy-time entanglement. *Physical Review Letters*, 102(4):040401, 2009.
- [23] T. Kraft, C. Ritz, N. Brunner, M. Huber, and O. Ghne.

Characterizing genuine multilevel entanglement. *Physical Review Letters*, 120(6):060502, 2018.

- [24] Gza Tth and Otfried Ghne. Detecting genuine multipartite entanglement with two local measurements. *Phys.rev.lett*, 94(6):060501, 2005.
- [25] M Huber, F Mintert, A Gabriel, and B. C. Hiesmayr. Detection of high-dimensional genuine multipartite entanglement of mixed states. *Physical Review Letters*, 104(21):210501, 2010.
- [26] Lin Chen, Kai Wang, Yi Shen, Yize Sun, and Lijun Zhao. Constructing 2 × 2 × 4 and 4 × 4 unextendible product bases and positive-partial-transpose entangled states, 2018. arXiv:1810.08932v1.
- [27] Lin Chen and Dragomir Z. Djokovic. Multiqubit upb: The method of formally orthogonal matrices. *Journal of Physics A Mathematical and Theoretical*, 51(26), 2018.
- [28] Stijn De Baerdemacker, Alexis De Vos, Lin Chen, and Li Yu. The birkhoff theorem for unitary matrices of arbitrary dimensions. *Linear Algebra and Its Applications*, 514:151–164, 2017.
- [29] S. B Bravyi. Unextendible product bases and locally unconvertible bound entangled states. *Quantum Informa*tion Processing, 3(6):309–329, 2004.
- [30] Lin Chen and Dragomir Z Dokovic. Separability problem for multipartite states of rank at most 4. Journal of Physics A Mathematical and Theoretical, 46(46):1103– 1114, 2013.
- [31] The non-orthogonal UPB is a set of product vectors that are not orthogonal to any product vector at the same time