A Formulation of Rényi Entropy on C*-Algebras

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Abstract

The entropy of probability distribution defined by Shannon has several extensions. Rényi entropy is one of the general extensions of Shannon entropy and is widely used in engineering, physics, and so on. On the other hand, the quantum analogue of Shannon entropy is von Neumann entropy. Furthermore, the formulation of this entropy was extended to on C^* -algebras by Ohya (S-mixing entropy). In this paper, we formulate Renyi entropy on C^* -algebras based on S-mixing entropy and prove several inequalities for the uncertainties of states in various reference systems.

Keywords: Quantum Information Theory; Quantum Entropy; *S*-mixing entropy; Rényi Entropy; Quantum Statistical Mechanics; Operator Algebras.

Contents

1	Introduction	2
2	Preliminaries	2
	2.1 Rényi Entropy	. 2
	2.2 Decomposition Theory	. 4
	2.3 S -Mixing Entropy	. 6

3	Rényi Entropy on C^* -Algebras			
	3.1	Density Case	9	
	3.2	General Case	10	

1 Introduction

Shannon introduced the entropy as the information amount of information systems represented by probability spaces [13]. Rényi defined a general extension of Shannon entropy on probability spaces which is called Rényi entropy [11]. Rényi entropy is more general than Shannon entropy in the sense of a positive number α , and it corresponds to Shannon entropy when $\alpha \to 1$. This entropy is useful and widely used in physics, engineering, and so on [3], [4].

On the other hand, von Neumann entropy measures the complexity (or the information amount) of a quantum system [15]. In 1984, Ohya formulated the general extension of von Neumann entropy which is called S-mixing entropy on C^* -algebras [6],[7], [8], [16]. S-mixing entropy depends on choosing subset (reference system) of the set of all states on the C^* algebra. Thanks to the property, one can measures the uncertainty of the state depending on reference systems. Mukhamedov and Watanabe formulated an extension of S-mixing entropy by taking the set of all quantum channels as the reference system. Moreover, they showed that the entropy can apply to detect entangled states and calculated the complexities of qubit and phase-damping channels [5].

In this paper, we formulate Rényi entropy on C^* -algebras based on S-mixing entropy and show that the introduced entropy corresponds to S-mixing entropy when $\alpha \to 1$. Furthermore, we prove that our Rényi entropy is a general extension of quantum Rényi entropy [9], [14] if $\alpha > 1$. Moreover, by using our Rényi entropy, we investigate the uncertainties of states measured from various reference systems.

We organize the paper as follows: In Section 2, we recall the notations and some properties of the Rényi entropy on probability spaces. Furthermore, we review the decomposition theory of states on C^* -algebras and the definition of S-mixing entropy. In Section 3, we formulate Rényi entropy on C^* -algebras based on the definition of S-mixing entropy and show several properties of it. Furthermore, by using the introduced entropy, we prove the equalities or inequalities of the complexities of states measured from different reference systems.

2 Preliminaries

In this section, we review the definitions of Rényi entropy and \mathcal{S} -mixing entropy, and those several properties.

2.1 Rényi Entropy

In this chapter, log denotes the logarithm of base 2.

Definition 1 Let $\{p_1, p_2, \dots, p_n\}$ be the probability distribution of a random variable X. The Rényi entropy is defined by

$$S_{\alpha}(X) := \frac{1}{1-\alpha} \log \sum_{k=1}^{n} p_k^{\alpha} \quad , \quad \alpha \in [0, +\infty) \setminus \{1\}.$$

$$\tag{1}$$

This entropy corresponds to the Shannon entropy when $\alpha \to 1$. Namely, the following theorem holds.

Theorem 1 Under the above assumptions,

$$\lim_{\alpha \to 1} S_{\alpha}(X) = -\sum_{k=1}^{n} p_k \log p_k \tag{2}$$

is satisfied.

Furthermore, Rényi entropy has the additivity.

Theorem 2 If X and Y are independent random variables,

$$S_{\alpha}(X,Y) = S_{\alpha}(X) + S_{\alpha}(Y). \tag{3}$$

Moreover, since

$$\frac{\partial}{\partial \alpha} S_{\alpha} \le 0,$$

one can see that this entropy is a decreasing function with respect to the parameter α .

Rényi entropy has important roles for the coding theory. For instance, the following theorem exists for the entropy [2], [9].

Let \mathcal{X} be a finite alphabet set and X be a rondam variable of \mathcal{X} . Let C be a source code, that is, a map from \mathcal{X} to the set of finite-length strings of symbols of a binary alphabet. Then C(x) denotes the codeword of $x \in \mathcal{X}$ and l(x) denotes the length of C(x). Now we define the cost of the coding:

$$L_{\beta}(C) := \frac{1}{\beta} \log \sum_{x} p(x) 2^{\beta l(x)}$$

where p(x) is the pbability of x and $\beta > -1$.

Theorem 3 Let $\alpha = 1/(1+\beta)$. For a uniquely decodable code, the following inequality holds:

$$L_{\beta}(C) \ge S_{\alpha}(X). \tag{4}$$

Furthermore, there exists a uniquely decodable code C satisfying

$$L_{\beta}(C) \le S_{\alpha}(X) + 1. \tag{5}$$

2.2 Decomposition Theory

A quantum state can be decomposed into simpler components. In this section, we recall the mathematical theory on the decompositions of states [1], [14] that we need as follows.

Let $(\mathcal{A}, \mathfrak{S}, \theta(G))$ be a C^* -dynamical system, that is, \mathcal{A} is a C^* -algebra, \mathfrak{S} is the set of all states φ on \mathcal{A} , and $\theta(G)$ is the set of all *-automorphisms on \mathcal{A} associated with a group G. The triplet $(\mathcal{A}, \mathfrak{S}, \theta(G))$ describes the dynamics of a quantum system [14].

Moreover, let $I(\theta)$ be the set of all θ -invariant states (i.e. $\varphi \circ \theta_g = \varphi$, $\forall g \in G$), and $K_{\beta}(\theta)$ ($G = \mathbb{R}$) be the set of all states satisfying KMS condition with respect to θ_t ($t \in \mathbb{R}$).

Definition 2 The decomposition from an θ -invariant state into extremal θ -invariant states is called ergodic decomposition.

Since $I(\theta)$ and $K_{\beta}(\theta)$ are weak*-compact and convex subset of \mathfrak{S} , we deal with the case where spaces have such conditions.

Let S be a compact and convex subspace of a locally convex Hausdorff space. Moreover, let exS be the set of all extreme points of S. According to the Krein-Mil'man theorem [10], $exS \neq \phi$ and the weak*-closure of convex hull of exS equals to S, i.e. $\overline{co}^{w^*}exS = S$.

Definition 3 The decomposition from S into exS is called extremal decomposition.

Let $M(\mathcal{S})$ be the set of all normal Borel measures on \mathcal{S} . Furthermore, define

$$M_1(\mathcal{S}) := \{ \mu \in M(\mathcal{S}), \ \mu(\mathcal{S}) = 1 \}.$$
(6)

Definition 4 For any $\mu \in M(\mathcal{S})$,

$$b(\mu) := \int_{\mathcal{S}} \omega d\mu(\omega) \tag{7}$$

is called the barycenter of μ .

Moreover, let $C_{\mathbb{R}}(\mathcal{S})$ be the set of all real continuous functions on \mathcal{S} and

 $K(\mathcal{S}) := \{ f \in C_{\mathbb{R}}(\mathcal{S}) ; f \text{ are convex functions} \}.$

For two measures $\mu, \nu \in M(\mathcal{S})$, define " \prec " as follows :

$$\mu \prec \nu \iff \mu(f) \le \nu(f), \quad \forall f \in K(\mathcal{S}).$$

Then \prec gives an ordering on $M(\mathcal{S})$. Let us denote $M^m(\mathcal{S})$ as the set of all maximal elements with respect to the ordering.

Furthermore, we recall the following theorems.

Theorem 4 If S is a metricable compact convex set ;

- 1. exS is a G_{δ} set.
- 2. $\mu \in M_1^m(\mathcal{S})$ iff $\mu(\mathrm{ex}\mathcal{S}) = 1$.
- 3. For any $\varphi \in S$, there exist $\mu \in M_1^m(S)$ such that $\varphi = b(\mu)$.

Theorem 5 If S is a compact convex set ;

- 1. Any $\mu \in M_1^m(\mathcal{S})$ has $\exp \mathcal{S}$ as their pseudo-support (i.e. for any Bair sets Q such that $\exp \mathcal{S} \subset Q \subset \mathcal{S}, \ \mu(Q) = 1$).
- 2. For any $\varphi \in S$, there exist μ which satisfy (1) such that $\varphi = b(\mu)$.

Moreover, we have the following theorem for uniqueness of maximal measure μ .

Let \mathcal{X} be a locally convex Hausdorff space, \mathcal{S} be a compact convex subset of \mathcal{X} , and \mathcal{K} be a convex cone whose vortex is 0. Furthermore, let \mathcal{S} be the base of \mathcal{K} , i.e.

$$\mathcal{K} = \{ \lambda \omega \; ; \; \lambda \ge 0, \; \omega \in \mathcal{S} \}.$$

Then \mathcal{K} is the convex cone generated by $\{1\} \times \mathcal{S}$. Defining

$$\omega_1 \ge \omega_2 \iff \omega_1 - \omega_2 \in \mathcal{K}$$

then \geq gives an ordering on \mathcal{K} .

Definition 5 If \mathcal{K} is the lattice with respect to the above \geq , \mathcal{S} is called Choquet simplex.

Theorem 6 If S is compact convex, the following are equivalent:

- 1. S is a Choquet simplex.
- 2. For any $\varphi \in S$, there exists a unique maximal probability measure μ .

Let $M_{\varphi}(\mathcal{S})$ be the set of all μ which is its barycenter equals to the state φ on the C^* -algebra, i.e.

$$M_{\varphi}(\mathcal{S}) := \{ \mu \in M_1(\mathcal{S}), \ b(\mu) = \varphi \}.$$
(8)

For φ satisfying (8), one obtains the integral representation of φ :

$$\varphi = \int_{\mathcal{S}} \omega d\mu(\omega). \tag{9}$$

It is called the *barycentric decomposition* of φ . According to Theorem 6, this dcomposition is not unique unless \mathcal{S} is a Choquet simplex.

Furthermore, we review the orthogonality of states. Let $\{\mathcal{H}_{\varphi}, \pi_{\varphi}, x_{\varphi}\}$ be the GNS representation defined by φ . For $\varphi_1, \varphi_2 \in \mathfrak{S}$, set $\varphi := \varphi_1 + \varphi_2 \in \mathcal{A}_+^*$. Then the following are euivalent:

- 1. Let $\psi \in \mathcal{A}_+^*$. If $\psi \leq \varphi_1$ and $\psi \leq \varphi_2$, $\psi = 0$.
- 2. There exists a projection $E \in \pi_{\varphi}(\mathcal{A})'$ such that

$$\begin{aligned} \varphi_1(A) &= \langle x_{\varphi}, E\pi_{\varphi}(A)x_{\varphi} \rangle, \\ \varphi_2(A) &= \langle x_{\varphi}, (I-E)\pi_{\varphi}(A)x_{\varphi} \rangle. \end{aligned}$$

3.
$$\mathcal{H}_{\varphi} = \mathcal{H}_{\varphi_1} \oplus \mathcal{H}_{\varphi_2}, \, \pi_{\varphi} = \pi_{\varphi_1} \oplus \pi_{\varphi_2}, \, x_{\varphi} = x_{\varphi_1} \oplus x_{\varphi_2}.$$

Definition 6 The states φ_1 , φ_2 satisfying the above conditions are called mutually orthogonal and denoted by $\varphi_1 \perp \varphi_2$.

Definition 7 For any Borel sets $Q \subset \mathfrak{S}$ (i.e. $Q \in \mathcal{B}(\mathfrak{S})$), $\mu \in M(\mathfrak{S})$ satisfying

$$\left(\int_{Q}\omega d\mu\right)\perp\left(\int_{\mathfrak{S}\backslash Q}\omega d\mu\right)$$

is called orthogonal measure on \mathfrak{S} .

We define $\mathcal{O}_{\varphi}(\mathfrak{S})$ as the set of all orthogonal probability measures whose barycenters are φ .

2.3 *S*-Mixing Entropy

If $\mu \in M_{\varphi}(\mathcal{S})$ has countable supports, that is, (9) can be written as

$$\varphi = \sum \lambda_k \varphi_k \tag{10}$$

where $\lambda_k > 0$; $\sum \lambda_k = 1$ and $\{\varphi_k\} \subset exS$, we denote the set of all such measures as $D_{\varphi}(S)$. **Definition 8** Under the above assumptions, the entropy of $\varphi \in S$ is given by

$$S^{\mathcal{S}}(\varphi) := \begin{cases} \inf\{-\sum \lambda_k \log \lambda_k; \ \mu = \{\lambda_k\} \in D_{\varphi}(\mathcal{S})\} \\ +\infty \qquad (\mu \notin D_{\varphi}(\mathcal{S})) \end{cases}$$
(11)

The above entropy is called *S*-mixing entropy. Since one can regard that the complexity of the system is $+\infty$ if φ has uncountable states, Ohya defined $S^{\mathcal{S}}(\varphi) := +\infty$ ($\mu \notin D_{\varphi}(\mathcal{S})$).

 $S^{\mathcal{S}}(\varphi)$ depends on the set \mathcal{S} chosen, thus it represents the amount of complexity of the state measured from the reference system \mathcal{S} . That is, this entropy takes measuring the uncertainty of states from various reference systems into account.

Furthermore, if φ is faithful normal and $S = \mathfrak{S}$, this entropy corresponds to von Neumann entropy [6], [14].

By the way, since one can regard that the complexities of real physical systems are finite, we denote the subset of ${\cal S}$ as

 $\mathcal{S}_r := \{ \varphi \in \mathcal{S} ; S^{\mathcal{S}}(\varphi) < \infty \}.$

Since $S = \overline{co}^{w^*} ex S$, the following proposition holds.

Proposition 1

$$\bar{S}_r^{w^*} = \mathcal{S}.\tag{12}$$

3 Rényi Entropy on C*-Algebras

In this section, we define Rényi entropy on C^* -algebras based on \mathcal{S} -mixing entropy and show that the introduced entropy includes \mathcal{S} -mixing entropy and quantum Rényi entropy as the special cases. Furthermore, by using our Rényi entropy, we investigate the uncertainty of states in different reference systems.

Definition 9 Under the same assumptions and notations with Definition 8, we define:

$$S_{\alpha}^{\mathcal{S}}(\varphi) := \inf \left\{ (1-\alpha)^{-1} \log \sum_{k} \lambda_{k}^{\alpha} \right\} \quad ; \quad \alpha \in [0, +\infty) \setminus \{1\}$$
(13)

where the infimum is taken over all $\mu = \{\lambda_k\} \in D_{\varphi}(\mathcal{S})$. Moreover, if $\mu \notin D_{\varphi}(\mathcal{S})$, $S_{\alpha}^{\mathcal{S}}(\varphi) := \infty$.

We call (13) S-mixing Rényi entropy. From the analogue of classical case, one can see the following theorem:

Theorem 7 $S^{\mathcal{S}}_{\alpha}(\varphi)$ is monotone decreasing with respect to the parameter α .

Furthermore, in analogy with the classical case, we have the following theorem.

Theorem 8 For any $\varphi \in \mathcal{S}$,

$$\lim_{\alpha \to 1} S_{\alpha}^{\mathcal{S}}(\varphi) = S^{\mathcal{S}}(\varphi) \tag{14}$$

holds.

Proof According to the classical case, for $\mu \in D_{\varphi}(\mathcal{S})$,

$$\lim_{\alpha \to 1} (1 - \alpha)^{-1} \log \sum_{k} \lambda_k^{\alpha} = -\sum_{k} \lambda_k \log \lambda_k$$
(15)

holds. We shall denote $\tilde{S}^{\mathcal{S}}_{\alpha}(\varphi) := (1-\alpha)^{-1} \log \sum_{k} \lambda_{k}^{\alpha}$, $\tilde{S}^{\mathcal{S}}(\varphi) := -\sum_{k} \lambda_{k} \log \lambda_{k}$. Then we have

$$0 \leq \inf_{\{\lambda_k\}} \tilde{S}^{\mathcal{S}}(\varphi) - \inf_{\{\lambda'_k\}} \tilde{S}^{\mathcal{S}}_{\alpha}(\varphi) = \sup(-\tilde{S}^{\mathcal{S}}_{\alpha}(\varphi)) - \sup(-\tilde{S}^{\mathcal{S}}(\varphi))$$
$$\leq \sup(\tilde{S}^{\mathcal{S}}(\varphi) - \tilde{S}^{\mathcal{S}}_{\alpha}(\varphi)) \quad , \quad \forall \alpha > 1.$$
(16)

$$0 \le \inf_{\{\lambda'_k\}} \tilde{S}^{\mathcal{S}}_{\alpha}(\varphi) - \inf_{\{\lambda_k\}} \tilde{S}^{\mathcal{S}}(\varphi) \le \sup(\tilde{S}^{\mathcal{S}}_{\alpha}(\varphi) - \tilde{S}^{\mathcal{S}}(\varphi)) \quad , \quad 0 \le \forall \alpha < 1.$$
(17)

Due to (15), the right hand sides of (16) and (17) go to 0 when $\alpha \rightarrow 1$. Therefore we obtain the theorem.

Now we prove that our S-mixing Rényi entropy includes the density case [9], [14]. Let $\mathbf{T}(\mathcal{H})$ be the set of all trace class operators on a Hilbert space \mathcal{H} , and $\mathbf{T}(\mathcal{H})_{+,1} := \{A \in \mathbf{T}(\mathcal{H}) ; \text{Tr}A = 1\}.$

Definition 10 For any $\rho \in \mathbf{T}(\mathcal{H})_{+,1}$ and any $\alpha \in [0, +\infty) \setminus \{1\}$, the quantum Rényi entropy is defined by

$$S_{\alpha}(\rho) := (1 - \alpha)^{-1} \log \operatorname{Tr} \rho^{\alpha}.$$
(18)

Lemma 1 Let $\rho = \sum_{n} \lambda_n \rho_n$ be the decomposition into pure states (i.e. dim $(\operatorname{ran}\rho_n) = 1$). For any $\alpha > 1$,

$$S_{\alpha}(\rho) \le (1-\alpha)^{-1} \log \sum_{n} \lambda_n^{\alpha}$$
(19)

holds. If $\rho_n \perp \rho_m$ $(n \neq m)$, one obtains the equality.

Proof Let $\rho = \sum_k p_k E_k$ be the Schatten decomposition [12] of ρ . Then for any $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} p_k \ge \sum_{k=1}^{n} \lambda_n$$

is satisfied [14]. Therefore we have $\sum_{k=1}^{n} p_k^{\alpha} \geq \sum_{k=1}^{n} \lambda_n^{\alpha}$ ($\forall \alpha \in [0, +\infty) \setminus \{1\}$). Moreover, according to the monotonicity of log,

$$(1-\alpha)^{-1}\log\sum_{k=1}^{n}p_{k}^{\alpha}\leq(1-\alpha)^{-1}\log\sum_{k=1}^{n}\lambda_{k}^{\alpha}\quad,\quad\forall\alpha>1.$$

Since $0 \leq \sum_{k=1}^{n} p_k^{\alpha} < 1$ (resp. $0 \leq \sum_{k=1}^{n} \lambda_n^{\alpha} < 1$), there exists the limit : $\lim_{n \to \infty} \log \sum_{k=1}^{n} p_k^{\alpha}$

(resp. $\lim_{n \to \infty} \log \sum_{k=1}^n \lambda_k^{\alpha}$). Thus we have

$$(1-\alpha)^{-1}\log\sum_{k=1}^{\infty}p_k^{\alpha} \le (1-\alpha)^{-1}\log\sum_{k=1}^{\infty}\lambda_k^{\alpha} \quad , \quad \forall \alpha > 1$$

This gives the inequality (19).

Moreover, if $\rho_n \perp \rho_m$ $(n \neq m)$, $\rho = \sum_n \lambda_n \rho_n$ becomes the Schatten decomposition of ρ . Thus $\lambda_n = p_n$. Therefore

$$S_{\alpha}(\rho) = (1 - \alpha)^{-1} \log \sum_{n} \lambda_n^{\alpha}.$$

Using this lemma, we prove the following theorem.

Theorem 9 Let \mathcal{A} be a C^{*}-algebra. If a state φ can be written as $\varphi(A) = \text{Tr}\rho A \ (\forall A \in \mathcal{A}),$

$$S^{\mathfrak{S}}_{\alpha}(\varphi) = S_{\alpha}(\rho) \quad , \quad \forall \alpha > 1,$$

$$\tag{20}$$

where \mathfrak{S} is the set of all states on \mathcal{A} .

Proof Let $\rho = \sum_k \lambda_k \rho_k$ be the decomposition into pure states ρ_k (i.e. $\rho_k^2 = \rho_k$, $\forall k$). Denoting

$$\varphi_k(A) = \mathrm{Tr}\rho_k A \ (\forall A \in \mathcal{A}),$$

then $\varphi = \sum \lambda_k \varphi_k$ is the extremal decomposition. Furthermore, if $\varphi \in ex\mathfrak{S}$, ρ is a pure state (i.e. $\rho = \rho^2$). Therefore according to Lemma 1,

$$S_{\alpha}(\varphi) = \inf\{(1-\alpha)^{-1}\log\sum\lambda_{k}^{\alpha}\} = S_{\alpha}(\rho)$$

holds.

Therefore, if $\alpha > 1$, *S*-mixing Rényi entropy includes the quantum Rényi entropy as the special case. If $0 \le \alpha < 1$, the following inequality holds.

Theorem 10 Under the above settings, for any $0 \le \alpha < 1$,

$$S^{\mathfrak{S}}_{\alpha}(\varphi) \le S_{\alpha}(\rho). \tag{21}$$

Proof If $0 \le \alpha < 1$, there holds

$$(1-\alpha)^{-1}\log\sum_n \lambda_n^{\alpha} \le (1-\alpha)^{-1}\log\sum_n p_n^{\alpha}.$$

This result induces the inequality (21).

3.1 Density Case

Since S-mixing Rényi entropy depends on S, we can consider the complexity of the state measured from the reference system S. In this chapter, we study the complexities of density operators by taking different reference systems.

Let $\mathbf{C}(\mathcal{H})$ be the set of all compact operators on \mathcal{H} . Then $\mathcal{A} := \mathbf{C}(\mathcal{H}) + \mathbb{C}I$ is a C^* -algebra. Now let $\theta(\mathbb{R})$ be the set of all 1-parameter strongly continuous automorphisms on \mathcal{A} and let

$$\theta_t(\cdot) := U_t \cdot U_{-t} \quad , \quad \theta_t \in \theta(\mathbb{R})$$

where U_t is a unitary operator on \mathcal{A} . Furthermore, when $\mathcal{S} = \mathfrak{S}$, we simply denote $S^{\mathfrak{S}}_{\alpha}(\varphi)$ by $S_{\alpha}(\varphi)$.

Theorem 11 If φ is faithful and θ -invariant, and if eigenvalues of ρ are non-degenerate,

$$S_{\alpha}^{I(\theta)}(\varphi) = S_{\alpha}(\varphi) \tag{22}$$

holds.

Proof Since $\varphi \in I(\theta)$, for any $t \in \mathbb{R}$ and unitaries U_t , $[U_t, \rho] = 0$ holds. Moreover, if φ is faithful, $\rho > 0$ is satisfied. Furthermore, since the eigenvalues of ρ are non-degenerate, we can put $\rho = |x_k\rangle \langle x_k|$ where x_k are any eigenvectors of ρ . Therefore, for any $t \in \mathbb{R}$ and any k,

$$[U_t, \rho_k] = 0$$

holds. Hence $\rho_k \in I(\theta)$. Thus we obtain the following inequality:

$$S_{\alpha}(\varphi) \ge S_{\alpha}^{I(\theta)}(\varphi).$$

Next, we show the opposite inequality. Let $\varphi = \sum \lambda_k \varphi_k$ be the ergodic decomposition (i.e. $\varphi_k \in \text{ex}I(\theta)$), and ρ_k be a density adjusted φ_k . Then ρ_k is a pure state. Therefore $\varphi_k \in \text{ex}\mathfrak{S}$. Hence

$$S_{\alpha}(\varphi) \leq S_{\alpha}^{I(\theta)}(\varphi).$$

Theorem 12 If $\varphi \in K_{\beta}(\theta)$, $S_{\alpha}^{K(\theta)} = 0$.

Proof Let H be a Hamiltonian of a physical system, and $\beta := 1/kT$ (k; the Boltzmann constant, T; the temperature). Denote

$$\rho = \frac{e^{-\beta H}}{\operatorname{Tr} e^{-\beta H}} \quad , \quad e^{-\beta H} \in \mathbf{T}(\mathcal{H})$$

and

$$\varphi(A) := \mathrm{Tr}\rho A \quad , \quad A \in \mathcal{A}.$$

Then φ is a unique KMS state for β and $\theta_t(A) := u_t A u_{-t}$ ($u_t := \exp(itH)$). Therefore, if $\varphi \in K_{\beta}(\theta)$, from uniqueness of a state,

$$S_{\alpha}^{K(\theta)}(\varphi) = 0.$$

3.2 General Case

In this section, we study the complexities of general states by taking different \mathcal{S} .

Theorem 13 For any KMS states $\varphi \in K_{\beta}(\theta)$, the following inequalities hold:

1.
$$S_{\alpha}^{I(\theta)}(\varphi) \ge S_{\alpha}^{K(\theta)}(\varphi)$$

2. $S_{\alpha}(\varphi) \ge S_{\alpha}^{K(\theta)}(\varphi)$.

Proof 1. The decomposition from $\varphi \in K_{\beta}(\theta)$ into $\operatorname{ex} K_{\beta}(\theta)$ is unique [1]. We put the decomposition $\varphi = \sum \lambda_n \varphi_n$. Then $\varphi_n \perp \varphi_m$ $(n \neq m)$ holds. On the other hand, since $\operatorname{ex} K_{\beta}(\theta) \subset I(\theta)$, φ_n can be decomposed into the elements of $\operatorname{ex} I(\theta)$, that is, ergodic states. Let $\varphi_n = \sum \mu_k^{(n)} \psi_k$ ($\psi_k \in \operatorname{ex} I(\theta)$) be the ergodic decomposition. Because of the uniqueness of

the decomposition into φ_n , we can regard $(1-\alpha)^{-1} \log \sum_n (\lambda_n)^{\alpha}$ as the constant. Furthermore, $0 \leq \sum_k (\mu_k^{(n)})^{\alpha} < 1$ holds. Therefore we have

$$(1-\alpha)^{-1}\log\sum_{k,n}(\lambda_n\mu_k^{(n)})^{\alpha} = (1-\alpha)^{-1}\log\sum_n(\lambda_n)^{\alpha}\sum_k(\mu_k^{(n)})^{\alpha}$$
$$= \frac{1}{\alpha-1}\left\{-\log\sum_n\lambda_n^{\alpha} + \left(-\log\sum_{n,k}(\mu_k^{(n)})^{\alpha}\right)\right\}$$
$$\geq (1-\alpha)^{-1}\log\sum_n\lambda_n^{\alpha} = S_{\alpha}^{K(\theta)}(\varphi).$$

By taking the infimum over all $\{\mu_k^{(n)}\}$, we obtain $S_{\alpha}^{I(\theta)}(\varphi) \geq S_{\alpha}^{K(\theta)}(\varphi)$.

2. Since $\exp K_{\beta}(\theta) \subset \mathfrak{S}$, we obtain the inequality in the same way as 1.

Moreover, in order to investigate the inequality between $S^{I(\theta)}_{\alpha}(\varphi)$ and $S_{\alpha}(\varphi)$, we need *G*-commutativity of $(\mathcal{A}, \theta(G))$. Thus, we recall the definition. Let $(\mathcal{H}_{\varphi}, \pi_{\varphi}, x_{\varphi})$ be the *GNS*-representation defined by φ and $\{u_g^{\varphi} ; g \in G\}$ be the strongly continuous unitary group on \mathcal{H}_{φ} .

Definition 11 Let E_{φ} be a projection from \mathcal{H}_{φ} to the set of u_g^{φ} -invariant vectors. If $E_{\varphi}\pi_{\varphi}(\mathcal{A})''E_{\varphi}$ is a commutative von Neumann algebra, $(\mathcal{A}, \theta(G))$ is called G-commutative for φ .

Furthermore, we mention the following theorem.

Theorem 14 For $\varphi \in I(\theta)$, the following are satisfied:

- 1. There exists $\mu \in \mathcal{O}_{\varphi}(I(\theta))$ whose pseudo-support is $exI(\theta)$.
- 2. If $(\mathcal{A}, \theta(G))$ is G-commutative, $I(\theta)$ is a Choquet simplex. Therefore, then the above μ is a unique maximal measure.

Now we prove the following inequalities.

Theorem 15 If $(\mathcal{A}, \theta(\mathbb{R}))$ is G-commutative for φ ,

$$S_{\alpha}(\varphi) \ge S_{\alpha}^{I(\theta)}(\varphi) \ge S_{\alpha}^{K(\theta)}(\varphi).$$
(23)

Proof According to Theorem 14, the ergodic decomposition of φ is unique. Hence the first inequality is satisfied. The second one holds from Theorem 13.

References

- O. Bratteli, D. W. Robinson : Operator Algebras and Quantum Statistical Mechanics I, Springer, New York (1981).
- [2] L. L. Campbell : A coding theorem and Rényi entropy, Information and Control, 8, 429-523 (1965).
- [3] M. S. Hughes, J. N. Marsh, J. M. Arbeit, R. G. Neumann, R. W. Fuhrhop, K. D. Wallace, L. Thomas, J. Smith, K. Agyem, G. M. Lanza, S. A. Wickline, and J. E. McCarthy : Application of Renyi entropy for ultrasonic molecular imaging, J. Acoust. Soc. Am., 125 (5), 3141-3145 (2007).
- [4] Y. Kusaki, T. Takayanagi : Renyi entropy for local quenches in 2D CFT from numerical conformal blocks, J. High Energy Physics, 01 (2018) 115.
- [5] F. Mukhamedov, N. Watanabe : On S-mixing entropy of quantum channels, Quantum Inf. Process, 17, 148-168 (2018).
- [6] M. Ohya : Entropy transmission in C*-dynamical systems, J. Math. Anal. Appl. 100, 222-235 (1984).
- [7] M. Ohya, D. Petz : Quantum Entropy and its Use, Springer, Berlin (1993).
- [8] M. Ohya, N. Watanabe : Foundation of Quantum Communication Theory, Makino Pub. Co., Tokyo (1998).
- [9] D. Petz: Quantum Information Theory and Quantum Statistics, Springer, Berlin (2008).
- [10] R. R. Phelps : Lecture on Choquet's Theorem, Van Nostrand (1966).
- [11] A. Rényi : On the foundations of information theory, Rev. Int. Stat. Inst. 33, pp. 1-14 (1965).
- [12] R. Schatten : Norm Ideals of Completely Continuous Operators. Springer, Berlin (1970).
- [13] C. E. Shannon : Mathematical theory of communication, Bell Systems Tech. J. 27, pp. 379-423 and 623-656 (1948).
- [14] H. Umegaki, M. Ohya : Quantum Entropies, Kyoritsu Pub., Tokyo (1984).
- [15] J. von Neumann : Die Mathematischen Grundlagen der Quantenmechanik, Springer, Berlin (1932).
- [16] N. Watanabe : Note on entropies of quantum dynamical systems, Found. Phys., 41, 549-563 (2011).