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The Witten Index for 1D Supersymmetric Quantum Walks with Anisotropic Coins

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Abstract Chirally symmetric discrete-time quantum walks possess supersymmetry, and their Witten indices can be naturally defined. The Witten index gives a lower bound for the number of topologically protected bound states. The purpose of this paper is to give a complete classification of the Witten index associated with a one-dimensional split-step quantum walk. It turns out that the Witten index of this model exhibits striking similarity to the one associated with a Dirac particle in supersymmetric quantum mechanics.

Keywords Quantum walks, Supersymmetry, Witten index, Split-step quantum walks

1 Introduction

Discrete-time quantum walks are versatile platforms realising topological phenomena [1,2,3,4,5,6,7,8]. Kitagawa et al. [9] proposed a split-step quantum walk with chiral symmetry and experimentally observed topologically protected bound states [10] (see [11] for a comprehensive review). For such bound states, Fuda et al. [12, 13] proved the robustness against compact perturbations and the spatial exponential decay property with mathematical rigour. Barkhofen et al. [14] implemented a chiral symmetric discrete-time quantum walk with supersymmetry. Recently, the first author of the present paper [15] proved that all the chiral symmetric quantum walks possess supersymmetry and that a discrete-time quantum walk has chiral symmetry if and only if the product of two unitary involutions represents its evolution operator.

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From these facts, we know that a discrete-time quantum walk can possess supersymmetry even if it does not have apparent chiral symmetry. Indeed, all homogeneous one-dimensional two-state quantum walks [16,27], multi-dimensional quantum walks [12], various types of quantum walks on graphs [17,18,19,20,21,22,23,24,25], and several quantum-walk based algorithms [26,28] have evolution operators that can be represented by the product of two unitary involutions, and therefore they exhibit supersymmetry. See [15] for more details, and [29] for many examples of inhomogeneous one-dimensional quantum walks [30,31,32,33,34] whose evolutions are written by the product of two unitary evolutions.

As shown in [15], a supersymmetric quantum walk (SUSYQW) assigns the Witten index, which provides a lower bound for the number of topologically protected bound sates. In this paper, we classify the Witten index for the split-step quantum walk entirely.

1.1 Witten index for SUSYQWs

To give a precise definition of the Witten index for SUSYQWs introduced in [15], we briefly review here the supersymmetric structure of chiral symmetric quantum walks. We say that a unitary operator U on a Hilbert space $\mathcal H$ has chiral symmetry if there exists a unitary involution Γ on $\mathcal H$ (i.e., $\Gamma^{-1} = \Gamma^* = \Gamma$) such that $\Gamma U\Gamma = U^{-1}$. U has chiral symmetry if and only if it can be represented as a product of two unitary involutions Γ and $C := \Gamma U$.

Suppose that U is the evolution of a chiral symmetric quantum walk. Namely, there are two unitary involutions Γ and C such that $U = \Gamma C$. We call $Q := [\Gamma, C]/2i$ a supercharge and $H = Q^2$ the superhamiltonian, where [X,Y] := XY - YX is the commutator. A direct calculation proves that Γ and H commute and hence H can be decomposed into $H_+ \oplus H_-$ with respect to the decomposition $\ker(\Gamma - 1) \oplus \ker(\Gamma + 1)$. We now define a topological index $\inf(\Gamma, C)$ so that it coincides with the Witten index $\inf(\Gamma, C)$ of H with respective to Γ , i.e.,

$$\operatorname{ind}(\Gamma, C) = \Delta_{\Gamma}(H) := \dim \ker H_{+} - \dim \ker H_{-}.$$
 (1)

In this sense, we call a pair (Γ, C) of two unitary involutions a SUSYQW with the evolution $U = \Gamma C$ and call the index $\operatorname{ind}(\Gamma, C)$ the Witten index for the SUSYQW. We say that a SUSYQW (Γ, C) is Fredholm if H is Fredholm. As shown in [15], the Fredholmness of (Γ, C) depends only on U (or equivalently H), and it is independent of the choice of (Γ, C) . However, the Witten index $\operatorname{ind}(\Gamma, C)$ depends on the choice of (Γ, C) . If (Γ, C) is Fredholm, then the index $\operatorname{ind}(\Gamma, C)$ is robust against compact perturbations (see §I).

1.2 Main result

In this paper, we study a split-step quantum walk [13, 12, 35], which unifies Kitagawa's split-step quantum walk [11] and a usual one-dimensional quantum walk [16, 27, 36].

Let $\mathcal{H}:=\ell^2(\mathbb{Z},\mathbb{C}^2)$ be the state space of the split-step quantum walk. Idetifying \mathcal{H} with $\ell^2(\mathbb{Z})\oplus\ell^2(\mathbb{Z})$, we define a shift operator Γ on \mathcal{H} as

$$\Gamma = \begin{pmatrix} p & qL \\ q^*L^* - p \end{pmatrix},\tag{2}$$

where L is the left shift operator on $\ell^2(\mathbb{Z})$. We suppose that $(p,q) \in \mathbb{R} \times \mathbb{C}$ satisfies $p^2 + |q|^2 = 1$, which ensures that Γ is a unitary involution. We define a coin operator on \mathcal{H} as

$$C = \begin{pmatrix} a_1 & b^* \\ b & a_2 \end{pmatrix},\tag{3}$$

where a_1, a_2 and b are the multiplication operator on $\ell^2(\mathbb{Z})$ by functions $a_j : \mathbb{Z} \to \mathbb{R}$ (j = 1, 2) and $b : \mathbb{Z} \to \mathbb{C}$. We assume $a_j(x)^2 + |b(x)|^2 = 1$ $(x \in \mathbb{Z}, j = 1, 2)$, which garantees that C is a unitary involution. The evolution operator of the split-step quantum walk is defined as the product of Γ and C, i.e., $U = \Gamma C$. Since Γ and C are unitary involutions, the evolution U has chiral symmetry, and (Γ, C) defines a SUSYQW as explained in Subsection 1.1.

As shown in [15], the modulus of $\operatorname{ind}(\Gamma, C)$ provides the lower bound for the number of topological bound states. Therefore, if $\operatorname{ind}(\Gamma, C)$ is nonzero, the corresponding quantum walk with the evolution $U = \Gamma C$ has a topological bound state. Motivated by this fact, we give a complete classification of the Witten index for the split-step quantum walk (Γ, C) defined by (2) and (3).

To state our main result, we suppose that the coin C is anisotropic [37,38], i.e., $C(x) := \begin{pmatrix} a_1(x) \ b^*(x) \\ b(x) \ a_2(x) \end{pmatrix}$ has limits as $x \to \pm \infty$. We denote these limits by $C(L) = \lim_{x \to -\infty} C(x)$ and $C(R) = \lim_{x \to +\infty} C(x)$. Clearly, the limit coins are unitary and hermitian. We say that a unitary and hermitian matrix is trivial if it equals +1 or -1. If the limit coins $C(\sharp)$ (\sharp = L, R) are nontrivial, they can be assumed to be a unitary involution of the form

$$C(\sharp) = \begin{pmatrix} a(\sharp) & b^*(\sharp) \\ b(\sharp) & -a(\sharp) \end{pmatrix}$$

without loss of generality (see Section 2.2 for more details). We are now in a position to state our main result.

Theorem A. Let Γ and C be defined by (2) and (3). Suppose that C is anisotropic and the limit coins $C(\sharp)$ (\sharp = L, R) are nontrivial. Then

$$(\Gamma, C)$$
 is Fredholm if and only if $|p| \neq |a(\sharp)|$ for each $\sharp = L, R$. (A1)

In this case, we have

$$\operatorname{ind}(\Gamma, C) = \begin{cases} +\operatorname{sgn} p, & |a(R)| < |p| < |a(L)|, \\ -\operatorname{sgn} p, & |a(L)| < |p| < |a(R)|, \\ 0, & otherwise. \end{cases}$$
(A2)

The case where at least one of the two limits C(L) and C(R) is a trivial unitary involution is excluded here, since the pair (Γ, C) with this property automatically fails to be Fredholm (Lemma 11). Theorem A provides a necessary and sufficient condition for the Fredholmness of 1D split-step SUSYQWs endowed with anisotropic coins, together with complete classification of the associated Witten index.

Note also that the model takes its simplest form when p = 0, but the associated Witten index is 0 in this case by the formula (A2). It is therefore important to consider non-zero p as well as the trivial case p = 0.

There are close links between quantum walks and Dirac particles. In a continuous limit, quantum walks converge to Dirac particles [39,40] (see [41] for a mathematically rigorous and general proof). Klein's paradox and Zitterbewegung in quantum walks were found in [42,43,44]. Theorem A inspires a new relation between quantum walks and Dirac particles in comparison with the rusult of Bole et al. [45]: for the Dirac operator $Q = -i\sigma_2 d/dx + \sigma_1 \phi(x)$ on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ with an anisotropic scalar potential $\phi(x)$ satisfying $\lim_{x\to\pm\infty}\phi(x)=\phi_\pm\in\mathbb{R}$, the Witten index equals ± 1 if $\pm\phi_-<0<\pm\phi_+$ and it equals 0 otherwise.

1.3 Organisation and strategy of the paper

The present paper is organised as follows. In §2 we go through some preliminary results including the precise definition of the one-dimensional split-step SUSYQW (Γ, C) . It is shown in §3 that the Witten index of (Γ, C) is given by the Fredholm index of a certain well-defined operator Q_{ϵ_+} on $\ell^2(\mathbb{Z})$ (see Theorem 6 for details);

$$\operatorname{ind}(\Gamma, C) = \operatorname{ind} Q_{\epsilon_{+}} = \dim \ker Q_{\epsilon_{+}} - \dim \ker Q_{\epsilon_{+}}^{*}. \tag{4}$$

We show that the operator Q_{ϵ_+} is of the form $Q_{\epsilon_+} = \alpha L + \alpha' L^* + \beta$, where α, α', β are \mathbb{C} -valued sequences indexed by \mathbb{Z} and L, L^* are the left and right shift operators on $\ell^2(\mathbb{Z})$ respectively. In §4 we separately compute the two dimensions on the right hand side (4). With the explicit form of Q_{ϵ_+} mentioned above in mind, we shall end up solving second-order linear difference equations of the form

$$\alpha(x)\Psi(x+1) + \alpha'(x)\Psi(x-1) + \beta(x)\Psi(x) = 0, \qquad \Psi = (\Psi(x))_{x \in \mathbb{Z}} \in \ell^2(\mathbb{Z}), \quad (5)$$

which is known to have two linearly independent algebraic solutions. Here, we need not only to algebraically solve Equation (5), but also to ensure the solutions Ψ to be square summable. This is precisely why the difference on the right-hand side of (4) can still be non-zero.

In §5 we prove Theorem A by making use of the index formula (4). The present paper concludes with §6, the main focus of which is a possible generalisation of the Witten index associated with SUSYQWs which fail to be Fredholm. Finally, §I contains a brief summary of the several invariance principles of the Witten index, each of which plays a supplementary role in this paper.

2 Preliminaries

The primary focus on the present paper is discrete-time quantum walks, and so we shall henceforth assume that all (linear) operators in this paper are everywhere-defined bounded operators.

2.1 A brief overview of supersymmetry

Here, we give a brief overview of supersymmetry by going through some preliminary results in a somewhat rapid manner. What follows can be found in any standard textbook on the subject (see, for example, [46, §5] or [47, §7.13]), and so proofs are omitted. An abstract operator Γ on a Hilbert space $\mathcal H$ is called an *involution*, if $\Gamma^2=1$. Note that if an operator possesses any two of the properties "involutory", "unitary" and "self-adjoint", then it possesses the third. We shall make use of the following finite-dimensional example throughout this paper;

Example 1 (2×2 case). A 2×2 matrix C is a unitary involution if and only if it is of the following form:

$$C = \begin{pmatrix} a_1 & b^* \\ b & a_2 \end{pmatrix}, \tag{6}$$

where the triple $(a_1, a_2, b) \in \mathbb{R} \times \mathbb{R} \times \mathbb{C}$ satisfies

$$b(a_1 + a_2) = 0$$
 and $a_j^2 + |b|^2 = 1$, $j = 1, 2$.

In particular, C = -1 or C = +1, which will be referred to as *trivial unitary involutions*, satisfies all of the above equalities. It is then easy to observe that a 2×2 matrix C is a non-trivial unitary involution if and only if it is of the following form:

$$C = \begin{pmatrix} a & b^* \\ b & -a \end{pmatrix} \text{ and } a^2 + |b|^2 = 1.$$
 (7)

A self-adjoint operator Q on \mathcal{H} is called a *supercharge* with respect to a unitary involution Γ , if it satisfies the anti-commutation relation $Q\Gamma + \Gamma Q = 0$, where the left hand side is commonly denoted by the symbol $\{Q, \Gamma\}$. With a canonical decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ by $\mathcal{H}_\pm := \ker(\Gamma \mp 1)$ in mind, a supercharge Q and the *superhamiltonian* $H := Q^2$ admit the following block-operator representations respectively;

$$Q = \begin{bmatrix} 0 & Q_{-} \\ Q_{+} & 0 \end{bmatrix}, \text{ where } Q_{\pm} : \mathcal{H}_{\pm} \to \mathcal{H}_{\mp} \text{ satisfy } Q_{\pm}^{*} = Q_{\mp}, \tag{8}$$

$$H = \begin{bmatrix} H_{+} & 0 \\ 0 & H_{-} \end{bmatrix}, \text{ where } H_{\pm} : \mathcal{H}_{\pm} \to \mathcal{H}_{\pm} \text{ satisfy } H_{\pm}^{*} = H_{\pm}.$$
 (9)

The superhamiltonian H simultaneously represents two non-negative hamiltonians H_+ , H_- whose spectra are identical except possibly for 0. The **Witten index** of the superhamiltonian H with respect to Γ is given by

$$\Delta_{\Gamma}(H) := \dim \ker H_{+} - \dim \ker H_{-}, \tag{10}$$

which measure the difference in the number of zero-energy ground states of H_+ , H_- , whenever the right-hand side is well-defined. Recall that Q_+ is a Fredholm operator

if and only if both dim ker Q_{\pm} are finite-dimensional and the range of Q_{+} is closed. In this case, the *Fredholm index* of Q_{+} is defined by ind $Q_{+} := \dim \ker Q_{+} - \dim \ker Q_{-}$. The following two results about a general bounded operator $A : \mathcal{H} \to K$ are useful:

$$\ker A^* A = \ker A,\tag{11}$$

$$\inf \sigma(A^*A) \setminus \{0\} > 0$$
 if and only if A has a closed range, (12)

where the proof of (12) can be found, for example, in [47, Lemma 7.27]). With (11) in mind, the Witten index has a precise interpretation as $\Delta_{\Gamma}(H) = \operatorname{ind} Q_{+}$, provided that Q_{+} is a Fredholm operator.

Following [15] we introduce the supercharge associated with a supersymmetric quantum walk (SUSYQW);

Definition 2. We call a pair (Γ, C) of two unitary involutions on a Hilbert space \mathcal{H} a SUSYQW with the evolution operator $U = \Gamma C$.

For a SUSYQW (Γ, C) , $Q := [\Gamma, C]/2i$ is a supercharge with respect to Γ , i.e., $\{Q, \Gamma\} = 0$. We define the Witten inex of the SUSYQW as

$$\operatorname{ind}(\Gamma, C) = \Delta_{\Gamma}(H),$$

where the right-hand side is defined by (10) with the superhamiltonian $H = Q^2$ for the supercharge Q of the SUSYQW.

Definition 3 (Fredholmness). A SUSYQW (Γ, C) is said to be *Fredholm*, if Q_+ as in (8) is a Fredholm operator.

2.2 Definition of the model

Given $X = \mathbb{C}$ or $X = \mathbb{C}^2$, we shall consider the Hilbert space of square-summable X-valued sequences:

$$\ell^2(\mathbb{Z},X):=\{\Psi:\mathbb{Z}\to X\mid \sum_{x\in\mathbb{Z}}\|\Psi(x)\|_X^2<\infty\},$$

where $\|\cdot\|_X$ is the standard norm defined on X. We shall agree to write elements of \mathbb{C}^2 as 2×1 column vectors. With this convention in mind, an element Ψ of $\ell^2(\mathbb{Z},\mathbb{C}^2)$ is written by $\Psi=(\Psi_1,\Psi_2)^T$, where $\Psi_1,\Psi_2\in\ell^2(\mathbb{Z},\mathbb{C}^2)$. On the Hilbert space $\ell^2(\mathbb{Z}):=\ell^2(\mathbb{Z},\mathbb{C})$, the left-shift operator L and the right-shift operator L^* are given respectively by

$$(L\Psi)(x) = \Psi(x+1)$$
 and $(L^*\Psi)(x) = \Psi(x-1), \qquad x \in \mathbb{Z}$

Evidently, we have $LL^* = L^*L = 1$. Let $\mathcal{H} = \ell^2(\mathbb{Z}, \mathbb{C}^2)$ be the state space of a quantum walker throughout the present paper. With the canonical identification $\mathcal{H} = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ in mind, we are now in a position to introduce the precise definition of the model we shall consider throughout this paper;

Definition 4. A *(one-dimensional) split-step SUSYQW* is a pair (Γ, C) of two unitary involutions on \mathcal{H} that are of the following forms;

$$\Gamma = \begin{pmatrix} p & qL \\ q^*L^* & -p \end{pmatrix}_{\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})} \text{ and } C = \begin{pmatrix} a_1 & b^* \\ b & a_2 \end{pmatrix}_{\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})}, \tag{13}$$

where the pair $(p, q) \in \mathbb{R} \times (\mathbb{C} \setminus \{0\})$ and the triple (a_1, a_2, b) of \mathbb{C} -valued sequences satisfy all of the following conditions:

$$\theta = \operatorname{Arg} q,\tag{14}$$

$$p^2 + |q|^2 = 1, (15)$$

$$a_j(x)^2 + |b(x)|^2 = 1, j = 1, 2,$$
 (16)

$$b(x)(a_1(x) + a_2(x)) = 0, (17)$$

where we assume that both (16) and (17) hold true for each $x \in \mathbb{Z}$. Note that the sequences $a_j = (a_j(x))_{x \in \mathbb{Z}}$ and $b = (b(x))_{x \in \mathbb{Z}}$ here are canonically identified with their associated multiplication operators on $\ell^2(\mathbb{Z})$.

Definition 5 (anisotropic coins). Let $L = -\infty$, and let $R = +\infty$. Let (Γ, C) be a split-step SUSYQW. The coin operator C is called an *anisotropic coin*, if it admits the following two-sided limits:

$$\lim_{x \to \sharp} C(x) = C(\sharp) = \begin{pmatrix} a_1(\sharp) & b(\sharp)^* \\ b(\sharp) & a_2(\sharp) \end{pmatrix}_{\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})}, \qquad \sharp = L, R, \tag{18}$$

where we assume that (17) and (16) both hold true for each x = L, R. Note that if $C(\sharp)$ is a non-trivial unitary involution, then we shall assume without loss of generality (see Example 1 for details) that

$$a(\sharp) := a_1(\sharp) = -a_2(\sharp).$$
 (19)

As in Definition 5 we shall always let $L = -\infty$ and $R = +\infty$ throughout this paper. This commonly used convention is, for example, in accordance with [37].

3 Diagonalisation

3.1 The main result

The ultimate purpose of the current section is to prove the following index formula for the Witten index;

Theorem 6. Let (Γ, C) be a split-step SUSYQW, where C may or may not be anisotropic. Then there exists a unitary operator ϵ on \mathcal{H} such that the supercharge $2iQ := [\Gamma, C]$ admits off-diagonalisation of the following form with respect to the orthogonal decomposition $\mathcal{H} = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$:

$$\epsilon^* Q \epsilon = \begin{pmatrix} 0 & Q_{\epsilon_-} \\ Q_{\epsilon_+} & 0 \end{pmatrix}_{\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})}, \tag{20}$$

where the three operators $\epsilon, Q_{\epsilon_+}, Q_{\epsilon_-}$ are given respectively by

$$\epsilon = \frac{1}{\sqrt{2}} \left(\frac{\sqrt{1+p}}{\sqrt{1-p}e^{-i\theta}L^*} \frac{-\sqrt{1-p}}{\sqrt{1+p}e^{-i\theta}L^*} \right)_{\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})},\tag{21}$$

$$-2iQ_{\epsilon_{+}} = (1 \pm p)e^{i\theta}Lb - (1 \mp p)e^{-i\theta}b^{*}L^{*} \pm |q|(a_{2}(\cdot + 1) - a_{1}). \tag{22}$$

Furthermore, the split-step quantum walk (Γ, C) is Fredholm if and only if Q_{ϵ_+} is a Fredholm operator. In this case, we have

$$\operatorname{ind}(\Gamma, C) = \operatorname{ind} Q_{\epsilon_{+}} = \dim \ker Q_{\epsilon_{+}} - \dim \ker Q_{\epsilon_{-}}. \tag{23}$$

Remark 7. A direct computation shows that the supercharge Q itself is not representable as an off-diagonal matrix with respect to the $\ell^2(\mathbb{Z})$ -decomposition $\mathcal{H} = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$, unlike the standard representation (8) which makes use of the canonical decomposition $\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_-$. To avoid confusion, we shall henceforth adhere to the convention that the round parentheses are used in the former representations, whereas the square parentheses are used in the latter representations.

3.2 The significance of diagonalisation

The main result of the current section, Theorem 6, might look rather technical at first glance, but as we shall see shortly the basic idea behind the proof is nothing but simple diagonalisation of the shift operator as in the following lemma;

Lemma 8. Let (Γ, C) be a split-step SUSYQW. The operator ϵ given by (21) is a unitary operator which diagonalises the shift operator Γ as follows:

$$\epsilon^* \Gamma \epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{24}$$

Proof. It is left as an easy exercise for the reader to verify that ϵ is unitary, and that the following two equalities hold true:

$$2\epsilon^* \begin{pmatrix} X & 0 \\ 0 & X' \end{pmatrix} \epsilon = \begin{pmatrix} (1+p)X + (1-p)LX'L^* & -|q|(X-LX'L^*) \\ -|q|(X-LX'L^*) & (1-p)X + (1+p)LX'L^* \end{pmatrix}, \tag{25}$$

$$2\epsilon^* \begin{pmatrix} X & 0 \\ 0 & X' \end{pmatrix} \epsilon = \begin{pmatrix} (1+p)X + (1-p)LX'L^* & -|q|(X-LX'L^*) \\ -|q|(X-LX'L^*) & (1-p)X + (1+p)LX'L^* \end{pmatrix}, \qquad (25)$$

$$2\epsilon^* \begin{pmatrix} 0 & Y' \\ Y & 0 \end{pmatrix} \epsilon = \begin{pmatrix} qLY + q^*Y'L^* & \frac{-(1-p)qLY + (1+p)q^*Y'L^*}{|q|} \\ \frac{(1+p)qLY - (1-p)q^*Y'L^*}{|q|} & -qLY - q^*Y'L^* \end{pmatrix}. \qquad (26)$$

With these two equalities in mind, we obtain (24) as follows:

$$\begin{split} 2\epsilon^* \Gamma \epsilon &= 2\epsilon^* \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix} \epsilon + 2\epsilon^* \begin{pmatrix} 0 & qL \\ q^*L^* & 0 \end{pmatrix} \epsilon \\ &= \begin{pmatrix} 2p^2 & -2p|q| \\ -2p|q| & -2p^2 \end{pmatrix} + \begin{pmatrix} 2|q|^2 & 2p|q| \\ 2p|q| & -2|q|^2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}. \end{split}$$

Remark 9. The diagonalisation of the form (24) is not unique. Indeed, as the experienced reader might immediately notice, one can introduce the discrete Fourier transform \mathscr{F} following [48] and consider following unitary transform;

$$\mathscr{F}\Gamma\mathscr{F}^{-1} = \begin{pmatrix} p & qe^{i(\cdot)} \\ q^*e^{-i(\cdot)} & -p \end{pmatrix},\tag{27}$$

where the right-hand side is diagonalisable in infinitely many different ways. Since the transform (27) is reversible, we can then obtain diagonalisation of the form (24). The unitary operator ϵ given explicitly by (21) is constructed in this precise manner.

In what follows, we shall make use of the unitary invariance of the Witten index as in Theorem 31. Let us fix an arbitrary unitary operator ϵ which gives the diagonalisation (24). We can then consider a new unitarily equivalent SUSYQW given by $(\Gamma_{\epsilon}, C_{\epsilon}) := (\epsilon^* \Gamma \epsilon, \epsilon^* C \epsilon)$. Since the new shift operator Γ_{ϵ} is given by (24), we see immediately that the two subspaces $\mathcal{H}_{\epsilon_+} := \ker(\Gamma_{\epsilon} \mp 1)$ are given respectively by

$$\mathcal{H}_{\epsilon_+} = \ell^2(\mathbb{Z}) \oplus \{0\} \text{ and } \mathcal{H}_{\epsilon_-} = \{0\} \oplus \ell^2(\mathbb{Z}).$$

Since the two subspaces $\mathcal{H}_{\epsilon_{\pm}}$ can be canonically identified with $\ell^2(\mathbb{Z})$, the following abstract version of Theorem 6 holds true;

Lemma 10. Let (Γ, C) be a split-step SUSYQW, and let ϵ be any unitary operator which gives diagonalisation (24). Then the new supercharge $Q_{\epsilon} := \epsilon^* Q \epsilon$ admits the following off-diagonal block matrix representation with respect to the decomposition $\mathcal{H} = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$:

$$Q_{\epsilon} = \begin{pmatrix} 0 & Q_{\epsilon_{-}} \\ Q_{\epsilon_{+}} & 0 \end{pmatrix}_{\ell^{2}(\mathbb{Z}) \oplus \ell^{2}(\mathbb{Z})}.$$
 (28)

Furthermore, (Γ, C) is a Fredholm operator if and only if $Q_{\epsilon_+}: \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is a Fredholm operator. In this case, we have

$$\operatorname{ind}(\Gamma, C) = \operatorname{ind}Q_{\epsilon_{+}} = \dim \ker Q_{\epsilon_{+}} - \dim \ker Q_{\epsilon_{-}}.$$

Proof. As in §2.1 the new supercharge $Q_{\epsilon} = \epsilon^* Q \epsilon$ admits

$$Q_{\epsilon} = \begin{bmatrix} 0 & Q'_{\epsilon_{-}} \\ Q'_{\epsilon_{+}} & 0 \end{bmatrix}_{\mathcal{H}_{\epsilon_{+}} \oplus \mathcal{H}_{\epsilon_{-}}}.$$

Observe first that $\ell^2(\mathbb{Z})$ can be canonically identified with \mathcal{H}_{\pm} by the unitary operators $\gamma_{\pm}:\ell^2(\mathbb{Z})\to\mathcal{H}_{\pm}$ defined respectively by the following formulas:

$$\gamma_{+}(\Psi) := \begin{pmatrix} \Psi \\ 0 \end{pmatrix} \text{ and } \gamma_{-}(\Psi) := \begin{pmatrix} 0 \\ \Psi \end{pmatrix}, \qquad \Psi \in \ell^{2}(\mathbb{Z}).$$

If we let $Q_{\epsilon_\pm} = \gamma_\mp Q'_{\epsilon_\pm} \gamma_\pm$, then $\gamma_\mp^* Q_{\epsilon_\pm} = Q'_{\epsilon_\pm} \gamma_\pm$. More explicitly,

$$Q'_{\epsilon_+}\begin{pmatrix} \Psi \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ Q_{\epsilon_+} \Psi \end{pmatrix} \text{ and } Q'_{\epsilon_-}\begin{pmatrix} 0 \\ \Psi \end{pmatrix} = \begin{pmatrix} Q_{\epsilon_-} \Psi \\ 0 \end{pmatrix}, \qquad \Psi \in \ell^2(\mathbb{Z}).$$

With these two equalities in mind, we obtain

$$Q_{\epsilon}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = Q_{\epsilon}\begin{pmatrix} \Psi_1 \\ 0 \end{pmatrix} + Q_{\epsilon}\begin{pmatrix} 0 \\ \Psi_2 \end{pmatrix} = Q'_{\epsilon_+}\begin{pmatrix} \Psi_1 \\ 0 \end{pmatrix} + Q'_{\epsilon_-}\begin{pmatrix} 0 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} Q_{\epsilon_-}\Psi_2 \\ Q_{\epsilon_+}\Psi_1 \end{pmatrix}, \qquad \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \in \mathcal{H}.$$

Therefore, (28) holds true. Here, the following easy computation shows that the operators $Q'_{\epsilon_{\tau}}Q'_{\epsilon_{\tau}}$ and $Q_{\epsilon_{\pi}}Q_{\epsilon_{\tau}}$ are unitarily equivalent:

$$\gamma_+^* Q_{\epsilon_-}' Q_{\epsilon_+}' \gamma_\pm = (\gamma_+^* Q_{\epsilon_-}' \gamma_\mp) (\gamma_\mp^* Q_{\epsilon_+}' \gamma_\pm) = Q_{\epsilon_\mp} Q_{\epsilon_+}.$$

With this fact in mind, we obtain the following two equalities:

$$\dim \ker Q'_{\epsilon_{\mp}}Q'_{\epsilon_{\pm}} = \dim \ker Q_{\epsilon_{\mp}}Q_{\epsilon_{\pm}},$$

$$\sigma(Q'_{\epsilon_{\mp}}Q'_{\epsilon_{\pm}}) \setminus \{0\} = \inf \sigma(Q_{\epsilon_{\mp}}Q_{\epsilon_{\pm}}) \setminus \{0\}.$$

That is, $(\Gamma_{\epsilon}, C_{\epsilon})$ is Fredholm if and only if $Q_{\epsilon_{+}}$ is a Fredholm operator and

$$\operatorname{ind}\left(\varGamma_{\epsilon}, C_{\epsilon}\right) = \operatorname{ind} Q'_{\epsilon_{+}} = \dim \ker Q_{\epsilon_{+}} - \dim \ker Q_{\epsilon_{-}}.$$

The claim now follows from Theorem 31.

3.3 Proof of Theorem 6

By virtue of Lemma 10 we may choose to work with any unitary ϵ which gives diagonalisation (24) in order to compute the Witten index. In particular, as in Theorem 6, we shall henceforth work with the one given explicitly by (21) in this paper. In order to prove Theorem 6, it remains to show that (22) holds true:

Proof of Equality (22). Let ϵ be the unitary operator given by (21). We shall first find the matrix representation of the time evolution $U_{\epsilon} := \epsilon^* U \epsilon$. We obtain

$$U_{\epsilon} = \epsilon^* U \epsilon = (\epsilon^* \Gamma \epsilon)(\epsilon^* C \epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \epsilon^* \begin{pmatrix} a_1 & b^* \\ b & a_2 \end{pmatrix} \epsilon =: \frac{1}{2} \begin{pmatrix} U'_+ & U_- \\ U_+ & U'_- \end{pmatrix}, \tag{29}$$

where (25)-(26) allow us to prove:

$$2\epsilon^* \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \epsilon = \begin{pmatrix} (1+p)a_1 + (1-p)a_2(\cdot +1) & -|q|(a_1-a_2(\cdot +1)) \\ -|q|(a_1-a_2(\cdot +1)) & (1-p)a_1 + (1+p)a_2(\cdot +1) \end{pmatrix},$$

$$2\epsilon^* \begin{pmatrix} 0 & b^* \\ b & 0 \end{pmatrix} \epsilon = \begin{pmatrix} qLb + q^*b^*L^* & \frac{-(1-p)qLb + (1+p)q^*b^*L^*}{|q|} \\ \frac{(1+p)qLb - (1-p)q^*b^*L^*}{|q|} & -qLb - q^*b^*L^* \end{pmatrix},$$

where $La_2 = a_2(\cdot + 1)L$ is used in the first equality. Note that (29) becomes

$$2\epsilon^* \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \epsilon + 2\epsilon^* \begin{pmatrix} 0 & b^* \\ b & 0 \end{pmatrix} \epsilon = 2\epsilon^* \begin{pmatrix} a_1 & b^* \\ b & a_2 \end{pmatrix} \epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} U'_+ & U_- \\ U_+ & U'_- \end{pmatrix} = \begin{pmatrix} U'_+ & U_- \\ -U_+ & -U'_- \end{pmatrix}.$$

It can then be shown that the following equalities hold true:

$$U'_{\pm} = qLb + q^*b^*L^* \pm (1 \pm p)a_1 \pm (1 \mp p)a_2(\cdot + 1), \tag{30}$$

$$U_{\pm} = -(1 \pm p)e^{i\theta}Lb + (1 \mp p)e^{-i\theta}b^*L^* \pm |q|(a_1 - a_2(\cdot + 1)) = 2iQ_{\epsilon_{\pm}}.$$
 (31)

We are now in a position to prove (22). We get

$$2iQ_{\epsilon} = U_{\epsilon} - U_{\epsilon}^* = \frac{1}{2} \begin{pmatrix} U_{+} \ U_{-}' \\ U_{+}' \ U_{-} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} U_{+}^* \ (U_{+}')^* \\ (U_{-}')^* \ U_{-}^* \end{pmatrix} = \begin{pmatrix} 0 & \frac{U_{-}' - (U_{+}')^*}{2} \\ \frac{U_{+}' - (U_{-}')^*}{2} & 0 \end{pmatrix},$$

where the last equality follows from the fact that U_{\pm} given by (30) are both self-adjoint. On the other hand, U_{\pm} given by (31) admit $(U'_{\mp})^* = -U'_{\pm}$, and so

$$2iQ_{\epsilon} = \begin{pmatrix} 0 & \frac{U'_{-}(U'_{+})^{*}}{2} \\ \frac{U'_{+}-(U'_{-})^{*}}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & U'_{-} \\ U'_{+} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2iQ_{\epsilon_{-}} \\ 2iQ_{\epsilon_{+}} & 0 \end{pmatrix}.$$

The claim follows.

3.4 Coin operators with trivial limits

We shall conclude the current section with one simple corollary of Theorem 6. Recall that in Theorem A the case where at least one of the two limits C(L) and C(R) is a trivial unitary involution is excluded. The following result explains why.

Lemma 11. If (Γ, C) is a split-step SUSYQW endowed with an anisotropic coin C with the property that at least one of the two limits C(L) and C(R) is trivial, then $\dim \ker Q_{\epsilon_{\pm}} = \infty$. That is, (Γ, C) automatically fails to be Fredholm in this case.

Proof. We may assume without loss of generality that C(L) is trivial, and that C(x) = C(L) for each $x \le 0$ due to the topological invariance (84). If $\Psi_{\pm} \in \ker Q_{\epsilon_{\pm}}$, then it follows from (22) that

$$0 - 0 \pm |q|(a_2(x+1) - a_1(x))\Psi_+(x) = 0, \qquad x \le -1,$$

where $a_2(x+1)-a_1(x)=a_2(L)-a_1(L)=0$. Thus, for each $x\leq -1$ the vectors $\Psi_\pm(x)\in \mathbb{C}^2$ can be freely chosen regardless of the other required conditions $(Q_{\epsilon_\pm}\Psi_\pm)(x)=0$ for each $x\geq 0$. This implies $\dim\ker Q_{\epsilon_\pm}=\infty$.

4 Classification of dim ker $Q_{\epsilon_{\pm}}$

4.1 The main result

In order to state the main theorem of the current section, we introduce the following definition;

Definition 12. Let (Γ, C) be a split-step SUSYQW with an anisotropic coin C. We shall consider the following mutually exclusive cases:

$$b(L) = 0 \text{ and } b(R) = 0, \tag{I}$$

$$b(L) = 0$$
 and $b(R) \neq 0$, (II)

$$b(L) \neq 0$$
 and $b(R) = 0$, (II')

$$b(L) \neq 0$$
 and $b(R) \neq 0$. (III)

We say that the coin operator C is of *Type I*, if the two unitary involutions C(L) and C(R) are both non-trivial and if (I) holds true. Type II, II', III coins are defined likewise. That is, we shall always assume that (19) holds true for each $\sharp = L$, R, whenever the four types of the isotropic coin thus defined.

With this definition in mind, the ultimate aim of the current section is to prove the following classification result:

Theorem 13. Let (Γ, C) be a split-step SUSYQW, and let C be an anisotropic coin of the following specific form;

$$C(x) = \begin{cases} C(R), & x \ge 1, \\ C(L), & x \le 0, \end{cases}$$
 (32)

where $C(\sharp)$ is assumed to be non-trivial for each $\sharp = L$, R. Let $d_{\pm} = \dim \ker Q_{\epsilon_{\pm}}$.

1. If C is of Type I, then d_{\pm} are uniquely determined by the pair (a(L),a(R)):

$$d_{\pm} = \begin{cases} 1, & a(L)a(R) < 0, \\ 0, & a(L)a(R) > 0. \end{cases}$$
 (33)

2. If C is of Type II, then d_{\pm} are uniquely determined by the triple (p, a(L), a(R)):

$$d_{\pm} = \begin{cases} 1, & \mp p + a(L)a(R) < 0, \\ 0, & \mp p + a(L)a(R) \ge 0. \end{cases}$$
 (34)

3. If C is of Type II', then d_{\pm} are uniquely determined by the triple (p, a(L), a(R)):

$$d_{\pm} = \begin{cases} 1, & \pm p + a(L)a(R) < 0, \\ 0, & \pm p + a(L)a(R) \ge 0, \end{cases}$$
 (35)

4. If C is of Type III, then d_{\pm} are uniquely determined by the triple (p, a(L), a(R)):

$$d_{\pm} = \begin{cases} 1, & a(R) < \pm p < a(L), \\ 1, & a(L) < \mp p < a(R), \\ 0, & otherwise. \end{cases}$$
 (36)

Remark 14. The following comments about Theorem 13 are worth mentioning:

- 1. Note that the ultimate purpose of the present paper is not the computation of each individual d_{\pm} , but rather the difference $d_{+}-d_{-}$. Since the latter quantity is invariant under compact perturbations, we may impose (32) without loss of generality.
- 2. If p = 0, then $d_+ = d_-$ regardless of the coin type:

$$d_{\pm} = \begin{cases} 1, & a(\mathbf{L})a(\mathbf{R}) < 0, \\ 0, & a(\mathbf{L})a(\mathbf{R}) \ge 0. \end{cases}$$

That is, the Witten index of (Γ, C) is always zero in this case.

4.2 Preliminaries

4.2.1 Notation

We shall always adhere to the notation introduced here throughout the remaining part of the current section. Let (Γ, C) be a split-step SUSYQW endowed with an anisotropic coin C, and let $C(\sharp)$ be non-trivial for each $\sharp = L$, R. Recall that $Q_{\epsilon_{\pm}}$ introduced in Theorem 6 are operators of the following forms:

$$Q_{\epsilon_{+}} = (1 \pm p)e^{i\theta}b(\cdot + 1)L - (1 \mp p)e^{-i\theta}b^{*}L^{*} \pm |q|(a_{2}(\cdot + 1) - a_{1}),$$

where the unnecessary constant -2i is removed for notational simplicity. We have

$$Q_{\epsilon_{\pm}} = \alpha_{\pm}(\cdot + 1)L - \alpha_{\mp}^* L^* \pm \beta, \tag{37}$$

$$\alpha_{\pm} := (1 \pm p)e^{i\theta}b,\tag{38}$$

$$\beta := |q|(a_2(\cdot + 1) - a_1),\tag{39}$$

where the two-sided limits of the last two sequences will be denoted respectively by

$$\alpha_{\pm}(\sharp) := \lim_{x \to \sharp} \alpha_{\pm}(x) = (1 \pm p)e^{i\theta}b(\sharp),$$
 $\sharp = L, R,$

$$\beta(\sharp) := \lim_{x \to \sharp} \beta(x) = |q|(a_2(\sharp) - a_1(\sharp)) = -2|q|a(\sharp), \qquad \sharp = L, R,$$

where the last equality follows from (19). We shall also make use of the simplification assumption (32) throughout this subsection, so that

$$\alpha_{\pm}(x) = \begin{cases} \alpha_{\pm}(R), & x \ge 1, \\ \alpha_{\pm}(L), & x \le 0, \end{cases}$$
(40)

$$\beta(x) = \begin{cases} \beta(R), & x \ge 1, \\ -|q|(a(L) + a(R)), & x = 0, \\ \beta(L), & x \le -1. \end{cases}$$
 (41)

4.2.2 A sketch for the proof of Theorem 13

The main theorem of the current section, Theorem 13, does require a lengthy argument as we need to separately consider the four types of the coin operator. However, the basic idea behind the proof is in fact elementary. Note first that the equation $(Q_{\epsilon_{\pm}}\Psi)(x)=0$ is equivalent to

$$\alpha_{+}(x+1)\Psi(x+1) - \alpha_{\mp}(x)^{*}\Psi(x-1) \pm \beta(x)\Psi(x) = 0. \tag{42}$$

This equation, known as the second-order linear difference equation, can then be put into the following first-order matrix equation;

$$\begin{pmatrix} \Psi(x+1) \\ \Psi(x) \end{pmatrix} = \begin{pmatrix} \frac{\mp \beta(x)}{\alpha_{\pm}(x+1)} & \frac{\alpha_{\mp}(x)^*}{\alpha_{\pm}(x+1)} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Psi(x) \\ \Psi(x-1) \end{pmatrix}, \tag{43}$$

whenever $\alpha_{\pm}(x+1) \neq 0$. This idea of transforming a difference equation to the associated matrix equation of less order is well-known (see, for example, [49]), and this is precisely the approach we are going to take. Note that we need not only to algebraically solve Equation (42), but also to ensure the solutions to be square summable. The following coefficient matrix shall be used throughout this section;

$$A_{\pm}(x) := \begin{pmatrix} \frac{\mp \beta(x)}{\alpha_{\pm}(x+1)} & \frac{\alpha_{\mp}(x)^{*}}{\alpha_{\pm}(x+1)} \\ 1 & 0 \end{pmatrix}, \tag{44}$$

$$A_{\pm}(\sharp) := \lim_{x \to \sharp} A_{\pm}(x) = \begin{pmatrix} \frac{\mp \beta(\sharp)}{\alpha_{\pm}(\sharp)} & \frac{\alpha_{\mp}(\sharp)^*}{\alpha_{\pm}(\sharp)} \\ 1 & 0 \end{pmatrix}, \qquad \sharp = L, R, \tag{45}$$

where $A_{\pm}(L)$ are well-defined if C is of either Type II or III, and $A_{\pm}(R)$ are well-defined if C is of either Type II' or III.

4.2.3 Abstract matrix difference equations

We are interested in solving a *first-order linear matrix difference equation* which is an equation of the following form:

$$\Phi(x+1) = A_0 \Phi(x), \qquad x \in \mathbb{N}, \tag{46}$$

where $\mathbb{N} = \{1, 2, ...\}$ and A_0 is a fixed invertible 2×2 matrix. An easy inductive argument shows that (46) is equivalent to the following equation:

$$\Phi(x+1) = \underbrace{A_0 \dots A_0}_{x \text{ times}} \Phi(1) = A_0^x \Phi(1), \qquad x \ge 0.$$
 (47)

We call any \mathbb{C}^2 -valued sequence Φ satisfying (46) an *algebraic solution* with in mind that it may fail to be square summable. It is easy to see from (47) that any algebraic solution Φ is uniquely determined by the initial value $\Phi(1)$. Given a \mathbb{C}^2 -valued sequence Φ , we have that Φ is an algebraic solution to (46) and $\sum_{x \in \mathbb{N}} \|\Phi(x)\|^2 < \infty$ if and only if $\Phi \in \ker(L \oplus L - \bigoplus_{x \in \mathbb{Z}} A_0)$. The following well-known result is included merely for the sake of completeness (See, for example, the proof of [49, Theorem 2.15] which makes use of the discrete analogue of the Wronskian);

Lemma 15. Let A_0 be a fixed invertible 2×2 matrix, and let Φ, Φ' be two algebraic solutions to the difference equation (46). Then Φ, Φ' are linearly independent if and only if $\Phi(x_0), \Phi'(x_0)$ are linearly independent for any $x_0 \in \mathbb{N}$.

Proof. If $\Phi(x_0)$, $\Phi'(x_0)$ are linearly independent for each $x_0 \in \mathbb{Z}$, then Φ , Φ' are obviously linearly independent. To prove the converse, suppose that Φ , Φ' are linearly independent, and that $c\Phi(x_0) + c'\Phi'(x_0) = 0$ for some fixed $x_0 \in \mathbb{N}$ and some $c, c' \in \mathbb{C}$. Since Φ , Φ' are both solutions to the difference equation (46), we have $c\Phi(x) + c'\Phi'(x) = 0$ for each $x \in \mathbb{N}$, and so the linear independence of Φ , Φ' gives c = c' = 0. It follows that $\Phi(x_0)$, $\Phi'(x_0)$ are linearly independent for each $x_0 \in \mathbb{N}$. \Box

To put it another way, Lemma 15 states that two algebraic solutions Φ , Φ' to (46) are either identically linearly independent or identically linearly dependent. It is then easy to observe that $\dim \ker(L \oplus L - \bigoplus_{x \in \mathbb{N}} A_0) \leq 2$. Here, the equality may not hold, since an algebraic solution to (47) may fail to be square summable. To check the square summability of solutions, the following lemma is useful:

Lemma 16. Let A_0 be a fixed invertible 2×2 matrix with two distinct eigenvalues z_1, z_2 , so that A_0 admits diagonalisation of the following form for some invertible matrix P:

$$A_0 = P \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} P^{-1}.$$

Suppose that Φ is an algebraic solution to the difference equation (46), and that

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} := P^{-1}\Phi(1).$$
 (48)

Then we have $\Phi \in \ker(L \oplus L - \bigoplus_{x \in \mathbb{N}} A_0)$ if and only if the following sum is finite;

$$\sum_{x \in \mathbb{N}} \left(|k_1|^2 |z_1|^{2x} + |k_2|^2 |z_2|^{2x} \right) < \infty.$$

Proof. It follows from (47) that

$$\Phi(x+1) = A_0^x \Phi(1) = P\begin{pmatrix} z_1^x & 0 \\ 0 & z_2^x \end{pmatrix} P^{-1} \Phi(1) = P\begin{pmatrix} k_1 z_1^x \\ k_2 z_2^x \end{pmatrix}, \qquad x \ge 0.$$

Then there exist constants C_1 , $C_2 > 0$, such that

$$C_1 \left\| \begin{pmatrix} k_1 z_1^x \\ k_2 z_2^x \end{pmatrix} \right\|^2 \le \|\Phi(x+1)\|^2 \le C_2 \left\| \begin{pmatrix} k_1 z_1^x \\ k_2 z_2^x \end{pmatrix} \right\|^2, \qquad x \ge 0,$$

where

$$\left\| \begin{pmatrix} k_1 z_1^x \\ k_2 z_2^x \end{pmatrix} \right\|^2 = |k_1|^2 |z_1|^{2x} + |k_2|^2 |z_2|^{2x}.$$

The claim follows.

In fact, we shall end up solving Equation (46) with a constraint on the initial condition, and so we introduce the following notation:

Lemma 17. Given an invertible 2×2 matrix A_0 and two complex numbers $a_0, b_0 \in \mathbb{C}$, we introduce the following subspace of $\ell^2(\mathbb{N}, \mathbb{C}^2)$:

$$S(A_0, a_0, b_0) := \left\{ \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \in \ker \left(L \oplus L - \bigoplus_{x \in \mathbb{N}} A_0 \right) \mid a_0 \Phi_1(1) + b_0 \Phi_2(1) = 0 \right\}. \tag{49}$$

If $a_0, b_0 \in \mathbb{C}$ are both non-zero, then dim $S(A_0, a_0, b_0) \leq 1$.

Proof. If $\Phi, \Phi' \in S(A_0, a_0, b_0)$, then $\Phi(1), \Phi'(1)$ are linearly dependent vectors in \mathbb{C}^2 :

$$a_0\Phi(1)+b_0\Phi'(1)=a_0\begin{pmatrix}\Phi_1(1)\\\Phi_2(1)\end{pmatrix}+b_0\begin{pmatrix}\Phi'_1(1)\\\Phi'_2(1)\end{pmatrix}=\begin{pmatrix}a_0\Phi_1(1)+b_0\Phi_2(1)\\a_0\Phi'_1(1)+b_0\Phi'_2(1)\end{pmatrix}=0.$$

Thus, Φ , Φ' are also linearly dependent by Lemma 15. The claim follows.

4.2.4 Concrete matrix difference equations

With (49) in mind, we are now in a position to state the explicit forms of the matrix difference equations we need to solve;

Lemma 18. If the type of the coin C is one of II,II',III, then we have the following associated linear isomorphisms respectively:

$$\ker Q_{\epsilon_{\pm}} \ni (\Psi(x))_{x \in \mathbb{Z}} \longmapsto \left(\begin{pmatrix} \Psi(x) \\ \Psi(x-1) \end{pmatrix} \right)_{x \in \mathbb{N}} \in \mathcal{S}(A_{\pm}(R), \alpha_{\pm}(R), \pm \beta(0)), \tag{50}$$

$$\ker Q_{\epsilon_{\pm}} \ni (\Psi(x))_{x \in \mathbb{Z}} \longmapsto \left(\begin{pmatrix} \Psi(-(x-1)) \\ \Psi(-x) \end{pmatrix} \right)_{x \in \mathbb{N}} \in \mathcal{S}(A_{\pm}(L)^{-1}, \mp \beta(0), \alpha_{\mp}(L)^{*}), \quad (51)$$

$$\ker Q_{\epsilon_{\pm}} \ni \Psi \longmapsto \left(\begin{pmatrix} \Psi(x) \\ \Psi(x-1) \end{pmatrix} \right)_{x \in \mathbb{Z}} \in \ker \left(L \oplus L - \bigoplus_{x \in \mathbb{Z}} A_{\pm}(x) \right). \tag{52}$$

Proof. If *C* is of Type II, then $\alpha_{\pm}(L) = 0$ and $\beta(L) \neq 0$. Thus $\Psi \in \ker Q_{\epsilon_{\pm}}$ if and only if (42) holds true for each $x \geq 1$ together with the following two conditions:

$$\alpha_{+}(R)\Psi(1) \pm \beta(0)\Psi(0) = 0$$
 and $\Psi(x) = 0$, $x \le -1$.

As in §4.2.2, Equation (42) is equivalent to the following:

$$L \oplus L \begin{pmatrix} \Psi(x) \\ \Psi(x-1) \end{pmatrix} = \begin{pmatrix} \frac{\mp \beta(x)}{\alpha_{\pm}(x+1)} & \frac{\alpha_{\mp}(x)^*}{\alpha_{\pm}(x+1)} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Psi(x) \\ \Psi(x-1) \end{pmatrix} = A_{\pm}(R) \begin{pmatrix} \Psi(x) \\ \Psi(x-1) \end{pmatrix}, \qquad x \ge 1.$$

and so (50) is a well-defined operator. It remains to show that (50) is surjective, since the injectivity is obvious. It is easy to verify that any vector in $S(A_{\pm}(R), \alpha_{\pm}(R), \pm \beta(0))$ must be of the form $(L\Psi_0, \Psi_0)$ for some $\Psi_0 \in \ell^2(\mathbb{N})$ satisfying

$$L\oplus L\begin{pmatrix} \Psi_0(x+1)\\ \Psi_0(x) \end{pmatrix} = A_\pm(\mathsf{R})\begin{pmatrix} \Psi_0(x+1)\\ \Psi_0(x) \end{pmatrix}, \qquad x\geq 1,$$

We define $\Psi \in \ell^2(\mathbb{Z}, \mathbb{C}^2)$ by

$$\Psi(x-1) := \begin{cases} 0, & x \le 0, \\ \Psi_0(x), & x \ge 1. \end{cases}$$

Then it is easy to show that $\Psi \in \ker Q_{\epsilon_{\pm}}$, and that it gets mapped to $(L\Psi_0, \Psi_0)$ under (50). Therefore, (50) is a well-defined linear isomorphism.

Similarly, if C is of Type II', then $\alpha_{\pm}(R) = 0$ and $\beta(R) \neq 0$. Thus $\Psi \in \ker Q_{\epsilon_{\pm}}$ if and only if (42) holds true for each $x \leq -1$ together with the following two conditions:

$$\mp \beta(0)\Psi(0) + \alpha_{\mp}(L)^*\Psi(-1) = 0$$
 and $\Psi(x) = 0$, $x \ge +1$.

As before (42) is equivalent to the following:

$$\begin{pmatrix} \Psi(x+1) \\ \Psi(x) \end{pmatrix} = A_{\pm}(L) \begin{pmatrix} \Psi(x) \\ \Psi(x-1) \end{pmatrix}, \qquad x \leq -1.$$

If we introduce the change of variable $x \leftrightarrow -x$, then the above equation becomes

$$L \oplus L \begin{pmatrix} \Psi(-(x-1)) \\ \Psi(-x) \end{pmatrix} = \begin{pmatrix} \Psi(-x) \\ \Psi(-x-1) \end{pmatrix} = A_{\pm}(L)^{-1} \begin{pmatrix} \Psi(-(x-1)) \\ \Psi(-x) \end{pmatrix}, \qquad x \ge 1.$$

and so (51) is a well-defined operator. It remains to show that (51) is surjective, since the injectivity is obvious. It is easy to verify that any vector in $S(A_{\pm}(L)^{-1}, \mp \beta(0), \alpha_{\mp}(L)^*)$ must be of the form $(\Psi_0, L\Psi_0)$ for some $\Psi_0 \in \ell^2(\mathbb{N})$ with

$$\begin{pmatrix} \Psi_0(x) \\ \Psi_0(x+1) \end{pmatrix} = A_{\pm}(L) \begin{pmatrix} \Psi_0(x+1) \\ \Psi_0(x+2) \end{pmatrix}, \qquad x \ge 1.$$

We define $\Psi \in \ell^2(\mathbb{Z}, \mathbb{C}^2)$ by

$$\Psi(-(x-1)) := \begin{cases} 0, & x \le 0, \\ \Psi_0(x), & x \ge 1. \end{cases}$$

Then it is easy to show that $\Psi \in \ker Q_{\epsilon_{\pm}}$, and that it gets mapped to $(\Psi_0, L\Psi_0)$ under (51). Therefore, (51) is a well-defined linear isomorphism. The fact that (52) is a linear isomorphism if C is of Type III is left as an easy exercise.

As in Lemma 16, diagonalisation of the coefficient matrices $A_{\pm}(\sharp)$ is important.

Lemma 19. If $A_{\pm}(\sharp)$ are well-defined for $\sharp = L, R$, then the two matrices $A_{\pm}(\sharp)$ have two non-zero distinct eigenvalues $z_{\pm,1}(\sharp), z_{\pm,2}(\sharp)$ of the following forms:

$$z_{\pm,j}(\sharp) = \frac{q^*}{1 \pm p} \left(\frac{(-1)^j \pm a(\sharp)}{b(\sharp)} \right), \qquad j = 1, 2.$$
 (53)

Moreover, the two matrices $A_{\pm}(\sharp)$ admit diagonalisation of the following form;

$$A_{\pm}(\sharp) = P_{\pm}(\sharp) \begin{pmatrix} z_{\pm,1}(\sharp) & 0\\ 0 & z_{\pm,2}(\sharp) \end{pmatrix} P_{\pm}(\sharp)^{-1}, \tag{54}$$

$$P_{\pm}(\sharp) := \begin{pmatrix} z_{\pm,1}(\sharp) & z_{\pm,2}(\sharp) \\ 1 & 1 \end{pmatrix}. \tag{55}$$

Proof. It is left as an easy exercise to show that a 2×2 matrix of the form

$$\begin{pmatrix} s & t \\ 1 & 0 \end{pmatrix}$$

has two eigenvalues $2z_j = s + (-1)^j \sqrt{s^2 + 4t}$, where j = 1, 2, together with the following eigenvalue equations:

$$\begin{pmatrix} s & t \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_j \\ 1 \end{pmatrix} = z_j \begin{pmatrix} z_j \\ 1 \end{pmatrix}, \qquad j = 1, 2.$$

With this result in mind, the matrices $A_{\pm}(\sharp)$ given by (45) are as follows;

$$2z_{\pm,j}(\sharp) = \frac{\mp \beta(\sharp) + (-1)^j \sqrt{\beta(\sharp)^2 + 4\alpha_{\pm}(\sharp)\alpha_{\mp}(\sharp)^*}}{\alpha_{\pm}(\sharp)}, \qquad j = 1, 2,$$

where

$$\beta(\sharp)^2 + 4\alpha_+(\sharp)\alpha_{\pm}(\sharp)^* = 4|q|^2a(\sharp)^2 + 4(1-p^2)|b(\sharp)|^2 = 4|q|^2 > 0.$$

It follows that

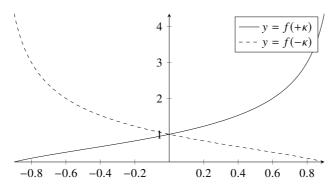
$$z_{\pm,j}(\sharp) = \frac{\mp \beta(\sharp) + (-1)^j \sqrt{\beta(\sharp)^2 + 4\alpha_\pm(\sharp)\alpha_\mp(\sharp)^*}}{2\alpha_\pm(\sharp)} = \frac{q^*}{1 \pm p} \left(\frac{(-1)^j \pm a(\sharp)}{b(\sharp)}\right).$$

The claim follows.

Definition 20. We define the increasing function $f:[-1,1] \to [0,+\infty]$ by

$$f(\kappa) := \begin{cases} \sqrt{\frac{1+\kappa}{1-\kappa}}, & \kappa \neq 1 \\ +\infty, & \kappa = 1. \end{cases}$$

The following figure shows the graphs of $y = f(\pm \kappa)$:



We shall also make use of the following obvious identities:

$$f(\kappa)^{-1} = f(-\kappa),\tag{56}$$

$$f(\kappa)f(\kappa') = f\left(\frac{\kappa + \kappa'}{1 + \kappa \kappa'}\right),$$
 (57)

$$f(\kappa)f(\kappa') < 1$$
 if and only if $\kappa + \kappa' < 0$, (58)

where $\kappa, \kappa' \in (-1, 1)$.

Corollary 21. With the notation introduced in Lemma 19 in mind, we have

$$|z_{\pm,j}(\sharp)| = \begin{cases} f(\mp p)f(\mp a(\sharp)), & j = 1, \\ f(\mp p)f(\pm a(\sharp)), & j = 2. \end{cases}$$
 (59)

Proof. Since $qq^* = (1-p)(1+p)$ and $b(\sharp)b(\sharp)^* = ((-1)^j - a(\sharp))((-1)^j + a(\sharp))$,

$$\frac{q^*}{1 \pm p} = \frac{1 \mp p}{q} \text{ and } \frac{b(\sharp)^*}{(-1)^j \pm a(\sharp)} = \frac{(-1)^j \mp a(\sharp)}{b(\sharp)}.$$
 (60)

where

$$\left| \frac{q^*}{1 \pm p} \right| = \frac{|q|}{|1 \pm p|} = \frac{\sqrt{(1+p)(1-p)}}{\sqrt{(1 \pm p)^2}} = \sqrt{\frac{1 \mp p}{1 \pm p}} = f(\mp p).$$

On the other hand,

$$\left| \frac{(-1)^j \pm a(\sharp)}{b(\sharp)} \right| = \frac{|(-1)^j \pm a(\sharp)|}{|b(\sharp)|} = \frac{\sqrt{((-1)^j \pm a(\sharp))^2}}{\sqrt{(1+a(\sharp))(1-a(\sharp))}} = \sqrt{\frac{((-1)^j \pm a(\sharp))^2}{(1+a(\sharp))(1-a(\sharp))}}.$$

We get

$$\left| \frac{(-1)^1 \pm a(\sharp)}{b(\sharp)} \right| = \sqrt{\frac{(-1 \pm a(\sharp))^2}{(1 + a(\sharp))(1 - a(\sharp))}} = \sqrt{\frac{(1 \mp a(\sharp))^2}{(1 + a(\sharp))(1 - a(\sharp))}} = \sqrt{\frac{1 \mp a(\sharp)}{1 \pm a(\sharp)}},$$

$$\left| \frac{(-1)^2 \pm a(\sharp)}{b(\sharp)} \right| = \sqrt{\frac{(1 \pm a(\sharp))^2}{(1 + a(\sharp))(1 - a(\sharp))}} = \sqrt{\frac{1 \pm a(\sharp)}{1 \mp a(\sharp)}}.$$

The claim follows.

4.3 Proof of Theorem 13

It remains to prove (33)-(36).

4.3.1 Type I coin

Proof of Equality (33). If C is an anisotropic coin of Type I, then $\beta(\sharp) \neq 0$ for each $\sharp = L$, R. It follows that $\Psi \in \ker Q_{\epsilon_{\pm}}$ if and only if $\Psi(x) = 0$ whenever $x \neq 0$. We get

$$\beta(0) = -|q|(a(\mathbf{R}) + a(\mathbf{L})) = \begin{cases} 0, & a(\mathbf{L})a(\mathbf{R}) < 0, \\ \text{non-zero,} & a(\mathbf{L})a(\mathbf{R}) > 0. \end{cases}$$

The claim follows.

4.3.2 Type II coin

If C is an anisotropic coin of Type II, then we shall make use of the isomorphism (50);

$$d_{\pm} = \dim Q_{\epsilon_{\pm}} = \dim \mathbb{S}(A_{\pm}(\mathbf{R}), \alpha_{\pm}(\mathbf{R}), \pm \beta(0)) \leq 1.$$

We shall compute d_{\pm} by making use of Lemma 16. As in Lemma 19, the matrices $A_{\pm}(R)$ admit diagonalisation of the following form:

$$A_{\pm}(\mathbf{R}) = P_{\pm}(\mathbf{R}) \begin{pmatrix} z_{\pm,1}(\mathbf{R}) & 0 \\ 0 & z_{\pm,2}(\mathbf{R}) \end{pmatrix} P_{\pm}(\mathbf{R})^{-1},$$

$$P_{\pm}(\mathbf{R}) = \begin{pmatrix} z_{\pm,1}(\mathbf{R}) & z_{\pm,2}(\mathbf{R}) \\ 1 & 1 \end{pmatrix}.$$

Given \mathbb{C} -valued sequences $\Phi_{\pm} = (\Phi_{\pm}(x))_{x \in \mathbb{N}}$, we have $\Phi_{\pm} \in \mathcal{S}(A_{\pm}(R), \alpha_{\pm}(R), \pm \beta(0))$ if and only if Φ_{\pm} are square-summable and the following equalities hold true:

$$\Phi_{\pm}(x+1) = A_{\pm}(R)\Phi_{\pm}(x), \qquad x \in \mathbb{N}, \tag{61}$$

$$\Phi_{\pm}(1) = m_{\pm} \begin{pmatrix} \frac{\mp \beta(0)}{\alpha_{\pm}(R)} \\ 1 \end{pmatrix}, \qquad \exists m_{\pm} \in \mathbb{C}.$$
 (62)

Lemma 22. If the sequences Φ_{\pm} satisfy (61) and (62), then

$$\begin{pmatrix} k_{\pm,1} \\ k_{\pm,2} \end{pmatrix} := P_{\pm}(\mathbf{R})^{-1} \Phi_{\pm}(1) = \frac{-m_{\pm}}{2} \begin{pmatrix} -1 \pm a(\mathbf{L}) \\ -1 \mp a(\mathbf{L}) \end{pmatrix}. \tag{63}$$

Proof. If we let $\sharp = R$, then

$$\begin{pmatrix} k_{\pm,1} \\ k_{\pm,2} \end{pmatrix} = \begin{pmatrix} z_{\pm,1}(\sharp) & z_{\pm,2}(\sharp) \\ 1 & 1 \end{pmatrix}^{-1} \Phi_{\pm}(1) = \frac{1}{\det P_{+}(\sharp)} \begin{pmatrix} 1 & -z_{\pm,2}(\sharp) \\ -1 & z_{\pm,1}(\sharp) \end{pmatrix} \Phi_{\pm}(1),$$
 (64)

where Lemma 19 implies

$$\det P_{\pm}(\sharp) = z_{\pm,1}(\sharp) - z_{\pm,2}(\sharp) = \frac{q^*}{1 \pm p} \left(\frac{-1 \pm a(\sharp)}{b(\sharp)} \right) - \frac{q^*}{1 \pm p} \left(\frac{+1 \pm a(\sharp)}{b(\sharp)} \right) = \frac{-2q^*}{b(\sharp)(1 \pm p)}.$$

With this equality in mind we obtain

$$\frac{\det P_{\pm}(\mathbf{R})}{m_{\pm}} \begin{pmatrix} k_{\pm,1} \\ k_{\pm,2} \end{pmatrix} = \begin{pmatrix} 1 & -z_{\pm,2}(\mathbf{R}) \\ -1 & z_{\pm,1}(\mathbf{R}) \end{pmatrix} \begin{pmatrix} \frac{\mp \beta(0)}{\alpha_{\pm}(\mathbf{R})} \\ 1 \end{pmatrix} = \begin{pmatrix} -\left(\frac{\pm \beta(0)}{\alpha_{\pm}(\mathbf{R})} + z_{\pm,2}(\mathbf{R})\right) \\ \frac{\pm \beta(0)}{\alpha_{\pm}(\mathbf{R})} + z_{\pm,1}(\mathbf{R}) \end{pmatrix},$$

where

$$\frac{\pm \beta(0)}{\alpha_{\pm}(R)} + z_{\pm,j}(R) = \frac{q^*(\mp a(L) \mp a(R))}{(1 \pm p)b(R)} + \frac{q^*((-1)^j \pm a(R))}{(1 \pm p)b(R)} = \frac{q^*((-1)^j \mp a(L))}{(1 \pm p)b(R)}$$

Therefore

$$\frac{\det P_{\pm}(\mathbf{R})}{m_{\pm}} \begin{pmatrix} k_{\pm,1} \\ k_{\pm,2} \end{pmatrix} = \frac{q^*}{(1 \pm p)b(\mathbf{R})} \begin{pmatrix} -1 \pm a(\mathbf{L}) \\ -1 \mp a(\mathbf{L}) \end{pmatrix} = \frac{-\det P_{\pm}(\mathbf{R})}{2} \begin{pmatrix} -1 \pm a(\mathbf{L}) \\ -1 \mp a(\mathbf{L}) \end{pmatrix}.$$

Proof of Equality (34). We shall first assume a(L) = 1. If the sequences Φ_{\pm} satisfy (61) and (62), then

$$\begin{pmatrix} k_{+,1} \\ k_{+,2} \end{pmatrix} = \begin{pmatrix} 0 \\ m_+ \end{pmatrix} \text{ and } \begin{pmatrix} k_{-,1} \\ k_{-,2} \end{pmatrix} = \begin{pmatrix} m_- \\ 0 \end{pmatrix}.$$

Thus Φ_+ is square summable (resp. Φ_- is square summable) if and only if

$$|m_{+}|^{2} \sum_{x=0}^{\infty} |z_{+,2}(\mathbf{R})|^{2x} < \infty$$
 $\left(\text{resp. } |m_{-}|^{2} \sum_{x=0}^{\infty} |z_{-,1}(\mathbf{R})|^{2x} < \infty\right),$

where (59) gives

$$|z_{+,2}(R)| = f(-p)f(+a(R))$$
 and $|z_{-,1}(R)| = f(+p)f(+a(R))$.

Therefore, we obtain

$$d_{\pm} = \begin{cases} 1, & \mp p + a(R) < 0, \\ 0, & \mp p + a(R) \ge 0. \end{cases}$$

An analogous argument gives that if a(L) = -1, then

$$d_{\pm} = \begin{cases} 1, & \mp p - a(\mathbf{R}) < 0, \\ 0, & \mp p - a(\mathbf{R}) \ge 0. \end{cases}$$

Thus, (34) is proved.

4.3.3 Type II' coin

This case is nothing but a repetition of the previous argument, but we include the proof for completeness. If C is an anisotropic coin of Type II', then we shall make use of the isomorphism (51);

$$d_{\pm} = \dim Q_{\epsilon_{\pm}} = \dim \mathbb{S}(A_{\pm}(L)^{-1}, \mp \beta(0), \alpha_{\mp}(L)^{*}) \le 1.$$

We shall compute d_{\pm} by making use of Lemma 16. As in Lemma 19, the matrices $A_{\pm}(L)^{-1}$ admit diagonalisation of the following form:

$$\begin{split} A_{\pm}(\mathbf{L})^{-1} &= P_{\pm}(\mathbf{L}) \begin{pmatrix} z_{\pm,1}(\mathbf{L})^{-1} & 0 \\ 0 & z_{\pm,2}(\mathbf{L})^{-1} \end{pmatrix} P_{\pm}(\mathbf{L})^{-1}, \\ P_{\pm}(\mathbf{L}) &= \begin{pmatrix} z_{\pm,1}(\mathbf{L}) & z_{\pm,2}(\mathbf{L}) \\ 1 & 1 \end{pmatrix}. \end{split}$$

Given \mathbb{C} -valued sequences $\Phi_{\pm} = (\Phi_{\pm}(x))_{x \in \mathbb{N}}$, we get $\Phi_{\pm} \in \mathcal{S}(A_{\pm}(L)^{-1}, \mp \beta(0), \alpha_{\mp}(L)^*)$ if and only if Φ_{\pm} are square-summable and the following algebraic conditions hold:

$$\Phi_{\pm}(x+1) = A_{\pm}(L)^{-1}\Phi_{\pm}(x), \qquad x \in \mathbb{N}, \tag{65}$$

$$\Phi_{\pm}(1) = m_{\pm} \begin{pmatrix} 1\\ \frac{\pm \beta(0)}{\alpha_{\tau}(1)^*} \end{pmatrix}, \qquad \exists m_{\pm} \in \mathbb{C}, \tag{66}$$

Lemma 23. If the sequences Φ_{\pm} satisfy (65) and (66), then

$$\begin{pmatrix} k_{\pm,1} \\ k_{\pm,2} \end{pmatrix} := P_{\pm}(\mathbf{L})^{-1} \Phi_{\pm}(1) = \frac{m_{\pm}}{\det P_{\pm}(\mathbf{L})} \begin{pmatrix} \frac{1 \pm a(\mathbf{R})}{+1 \mp a(\mathbf{L})} \\ \frac{1 \mp a(\mathbf{R})}{-1 \mp a(\mathbf{L})} \end{pmatrix}$$

Proof. It follows from (64) that

$$\det P_{\pm}(\mathbf{L}) \begin{pmatrix} k_{\pm,1} \\ k_{\pm,2} \end{pmatrix} = m_{\pm} \begin{pmatrix} 1 & -z_{\pm,2}(\mathbf{L}) \\ -1 & z_{\pm,1}(\mathbf{L}) \end{pmatrix} \begin{pmatrix} 1 \\ \frac{\pm \beta(0)}{\alpha_{\mp}(\mathbf{L})^*} \end{pmatrix} = m_{\pm} \begin{pmatrix} -\left(-1 + z_{\pm,2}(\mathbf{L}) \frac{\pm \beta(0)}{\alpha_{\mp}(\mathbf{L})^*} \right) \\ -1 + z_{\pm,1}(\mathbf{L}) \frac{\pm \beta(0)}{\alpha_{\mp}(\mathbf{L})^*} \end{pmatrix}$$

With (60) in mind, (53) becomes

$$z_{\pm,j} = \frac{(1 \mp p)}{q} \frac{b(L)^*}{(-1)^j \mp a(L)}, \qquad j = 1, 2.$$

We get

$$z_{\pm,j}(\mathbf{L})\frac{\pm\beta(0)}{\alpha_{\mp}(\mathbf{L})^{*}} = \frac{(1\mp p)}{q}\frac{b(\mathbf{L})^{*}}{(-1)^{j}\mp a(\mathbf{L})}\frac{|q|(\mp a(\mathbf{L})\mp a(\mathbf{R}))}{(1\mp p)e^{-i\theta}b(\mathbf{L})^{*}} = \frac{(-1)^{j+1}\mp a(\mathbf{R})}{(-1)^{j}\mp a(\mathbf{L})} + 1.$$

Thus we obtain

$$\begin{pmatrix} k_{\pm,1} \\ k_{\pm,2} \end{pmatrix} = \frac{m_{\pm}}{\det P_{\pm}(L)} \begin{pmatrix} \frac{1 \pm a(R)}{+1 \mp a(L)} \\ \frac{1 \mp a(R)}{-1 \mp a(L)} \end{pmatrix}.$$

Proof of Equality (35). We shall first assume a(R) = 1. If the sequences Φ_{\pm} satisfy (65) and (66), then as before Φ_{\pm} is square summable (resp. Φ_{-} is square summable) if and only if

$$m_+^2 \sum_{x=0}^{\infty} |z_{+,1}(L)|^{-2x} < \infty$$
 $\left(\text{resp. } m_-^2 \sum_{x=0}^{\infty} |z_{-,2}(L)|^{-2x} < \infty\right),$

where (59) together with (56) gives

$$|z_{+,1}(L)|^{-1} = f(+p)f(+a(L))$$
 and $|z_{-,2}(L)|^{-1} = f(-p)f(+a(L))$.

Therefore, we obtain

$$d_{\pm} = \begin{cases} 1, & \pm p + a(L) < 0, \\ 0, & \pm p + a(L) \ge 0. \end{cases}$$

An analogous argument gives that if a(R) = -1, then

$$d_{\pm} = \begin{cases} 1, & \pm p - a(L) < 0, \\ 0, & \pm p - a(L) \ge 0. \end{cases}$$

The claim follows.

4.3.4 Type III coin

Let C be of Type III. This case turns out to be the hardest case. Here, we shall make use of the isomorphism (52):

$$d_{\pm} = \dim \ker Q_{\epsilon_{\pm}} = \dim \ker \left(L \oplus L - \bigoplus_{x \in \mathcal{T}} A_{\pm}(x) \right).$$

Lemma 24. Given arbitrary \mathbb{C}^2 -valued sequences $\Phi_{\pm} = (\Phi_{\pm}(x))_{x \in \mathbb{Z}}$, we define two sequences $\Phi_{\pm,L}$, $\Phi_{\pm,R}$ by

$$\Phi_{\pm,L}(x) := \Phi_{\pm}(-x+1) \text{ and } \Phi_{\pm,R}(x) := \Phi_{\pm}(x), \qquad x \in \mathbb{N}$$

Then $\Phi_{\pm} \in \ker (L \oplus L - \bigoplus_{x \in \mathbb{Z}} A_{\pm}(x))$ if and only if the following three conditions are simultaneously satisfied:

$$\Phi_{\pm,R} \in \ker \left(L \oplus L - \bigoplus_{x \in \mathbb{N}}^{\infty} A_{\pm}(R) \right),$$
(67)

$$\Phi_{\pm,L} \in \ker \left(L \oplus L - \bigoplus_{x \in \mathbb{N}}^{\infty} A_{\pm}(L)^{-1} \right),$$
(68)

$$\Phi_{\pm,R}(1) = A_{\pm}(0)\Phi_{\pm,L}(1). \tag{69}$$

Proof. Evidently, we have $\Phi_{\pm} \in \ker (L \oplus L - \bigoplus_{x \in \mathbb{Z}} A_{\pm}(x))$ if and only if the following three conditions are simultaneously satisfied:

$$\begin{split} & \Phi_{\pm}(+x+1) = A_{\pm}(\mathbf{R}) \Phi_{\pm}(+x), & x \in \mathbb{N}, \\ & \Phi_{\pm}(-x+1) = A_{\pm}(\mathbf{L}) \Phi_{\pm}(-x), & x \in \mathbb{N}, \\ & \Phi_{\pm}(0+1) = A_{\pm}(0) \Phi_{\pm}(0), \end{split}$$

where the last condition is obviously (69) and the first two conditions are equivalent to the following two equations respectively:

$$\begin{split} (L \oplus L) \Phi_{\pm,\mathrm{R}}(x) &= A_{\pm}(\mathrm{R}) \Phi_{\pm,\mathrm{R}}(x), \\ (L \oplus L) \Phi_{\pm,\mathrm{L}}(x) &= A_{\pm}(\mathrm{L})^{-1} \Phi_{\pm,\mathrm{L}}(x). \end{split}$$

The claim follows.

Lemma 25. We have

$$P_{\pm}(\mathbf{R})^{-1}A_{\pm}(0)P_{\pm}(\mathbf{L}) = \begin{pmatrix} z_{\pm,1}(\mathbf{L}) & 0\\ 0 & z_{\pm,2}(\mathbf{L}) \end{pmatrix}.$$
 (70)

Proof. As in (60), we can let

$$r_{\pm} = \frac{q^*}{1 \pm p} = \frac{1 \mp p}{q}.$$

We have

$$\begin{split} P_{\pm}(\sharp) &= \begin{pmatrix} z_{\pm,1}(\sharp) & z_{\pm,2}(\sharp) \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} r_{\pm} \left(\frac{-1 \pm a(\sharp)}{b(\sharp)} \right) & r_{\pm} \left(\frac{+1 \pm a(\sharp)}{b(\sharp)} \right) \\ 1 & 1 \end{pmatrix}, \\ A_{\pm}(0) &= \begin{pmatrix} \frac{\mp \beta(0)}{\alpha_{\pm}(\mathbf{R})} & \frac{\alpha_{\mp}(\mathbf{L})^*}{\alpha_{\pm}(\mathbf{R})} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\pm |q|(a(\mathbf{R}) + a(\mathbf{L}))}{(1 \pm p)e^{i\theta}b(\mathbf{R})} & \frac{(1 \mp p)e^{-i\theta}b(\mathbf{L})^*}{(1 \pm p)e^{i\theta}b(\mathbf{R})} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\pm r_{\pm}(a(\mathbf{R}) + a(\mathbf{L}))}{b(\mathbf{R})} & \frac{r_{\pm}^2b(\mathbf{L})^*}{b(\mathbf{R})} \\ 1 & 0 \end{pmatrix}. \end{split}$$

On one hand,

$$\begin{split} P_{\pm}(\mathbf{R}) \begin{pmatrix} z_{\pm,1}(\mathbf{L}) & 0 \\ 0 & z_{\pm,2}(\mathbf{L}) \end{pmatrix} &= \begin{pmatrix} z_{\pm,1}(\mathbf{R}) & z_{\pm,2}(\mathbf{R}) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z_{\pm,1}(\mathbf{L}) & 0 \\ 0 & z_{\pm,2}(\mathbf{L}) \end{pmatrix} \\ &= \begin{pmatrix} z_{\pm,1}(\mathbf{L}) z_{\pm,1}(\mathbf{R}) & z_{\pm,2}(\mathbf{L}) z_{\pm,2}(\mathbf{R}) \\ z_{\pm,1}(\mathbf{L}) & z_{\pm,2}(\mathbf{L}) \end{pmatrix}. \end{split}$$

On the other hand,

$$\begin{split} A_{\pm}(0)P_{\pm}(\mathbf{L}) &= \begin{pmatrix} \frac{\pm r_{\pm}(a(\mathbf{R}) + a(\mathbf{L}))}{b(\mathbf{R})} & \frac{r_{\pm}^2b(\mathbf{L})^*}{b(\mathbf{R})} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r_{\pm} \left(\frac{-1 \pm a(\mathbf{L})}{b(\mathbf{L})} \right) & r_{\pm} \left(\frac{\pm 1 \pm a(\mathbf{L})}{b(\mathbf{L})} \right) \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\pm r_{\pm}^2(-1 \pm a(\mathbf{L}))(a(\mathbf{R}) + a(\mathbf{L}))}{b(\mathbf{L})b(\mathbf{R})} & + \frac{r_{\pm}^2|b(\mathbf{L})|^2}{b(\mathbf{L})b(\mathbf{R})} & \frac{\pm r_{\pm}^2(+1 \pm a(\mathbf{L}))(a(\mathbf{R}) + a(\mathbf{L}))}{b(\mathbf{L})b(\mathbf{R})} & + \frac{r_{\pm}^2|b(\mathbf{L})|^2}{b(\mathbf{L})b(\mathbf{R})} \\ z_{\pm,1}(\mathbf{L}) & z_{\pm,2}(\mathbf{L}) \end{pmatrix}, \end{split}$$

where for each j = 1, 2, we have

$$\begin{split} & \frac{\pm r_{\pm}^2((-1)^j \pm a(\mathbf{L}))(a(\mathbf{R}) + a(\mathbf{L}))}{b(\mathbf{L})b(\mathbf{R})} + \frac{r_{\pm}^2|b(\mathbf{L})|^2}{b(\mathbf{L})b(\mathbf{R})} \\ & = \frac{r_{\pm}^2}{b(\mathbf{L})b(\mathbf{R})} \left(\pm (-1)^j a(\mathbf{R}) \pm (-1)^j a(\mathbf{L}) + a(\mathbf{L})a(\mathbf{R}) + a(\mathbf{L})^2 + |b(\mathbf{L})|^2 \right) \\ & = \frac{r_{\pm}^2}{b(\mathbf{L})b(\mathbf{R})} \left(\pm (-1)^j a(\mathbf{R}) \pm (-1)^j a(\mathbf{L}) + a(\mathbf{L})a(\mathbf{R}) + 1 \right) \\ & = \frac{r_{\pm}^2}{b(\mathbf{L})b(\mathbf{R})} ((-1)^j \pm a(\mathbf{L}))((-1)^j \pm a(\mathbf{R})) \\ & = z_{\pm,j}(\mathbf{L})z_{\pm,j}(\mathbf{R}). \end{split}$$

We obtain (70) as follows;

$$A_{\pm}(0)P_{\pm}(\mathbf{L}) = \begin{pmatrix} z_{\pm,1}(\mathbf{L})z_{\pm,1}(\mathbf{R}) & z_{\pm,2}(\mathbf{L})z_{\pm,2}(\mathbf{R}) \\ z_{\pm,1}(\mathbf{L}) & z_{\pm,2}(\mathbf{L}) \end{pmatrix} = P_{\pm}(\mathbf{R}) \begin{pmatrix} z_{\pm,1}(\mathbf{L}) & 0 \\ 0 & z_{\pm,2}(\mathbf{L}) \end{pmatrix}.$$

Corollary 26. Suppose that \mathbb{C}^2 -valued sequences Φ_{\pm} satisfy algebraic equations $\Phi_{\pm}(x+1) = A_{\pm}(x)\Phi_{\pm}(x)$ for each $x \in \mathbb{Z}$, and that

$$\begin{pmatrix} k_{\pm,1} \\ k_{\pm,2} \end{pmatrix} := P_{\pm}(L)^{-1} \Phi_{\pm}(0).$$

Then $\Phi_{\pm} \in \ker (L \oplus L - \bigoplus_{x \in \mathbb{Z}} A_{\pm}(x))$ if and only if the following sum is finite for each j = 1, 2:

$$|k_{\pm,j}|^2 \sum_{x \in \mathbb{N}} \left(|z_{\pm,j}(\mathbf{L})|^{-2x} + |z_{\pm,j}(\mathbf{R})|^{+2x} \right) < \infty.$$
 (71)

Moreover,

$$|z_{+,1}(L)|^{-1} < 1$$
 and $|z_{+,1}(R)| < 1$ if and only if $-p \in (a(L), a(R)),$ (72)

$$|z_{+2}(L)|^{-1} < 1$$
 and $|z_{+2}(R)| < 1$ if and only if $+p \in (a(R), a(L)),$ (73)

$$|z_{-1}(L)|^{-1} < 1$$
 and $|z_{-1}(R)| < 1$ if and only if $-p \in (a(R), a(L)),$ (74)

$$|z_{-2}(L)|^{-1} < 1$$
 and $|z_{-2}(R)| < 1$ if and only if $+p \in (a(L), a(R))$. (75)

Furthermore, we have $d_{\pm} \leq 1$.

Note that $k_{\pm,1}$ and $k_{\pm,2}$ cannot be simultaneously both non-zero.

Proof. Recall that $\Phi \in \ker (L \oplus L - \bigoplus_{x \in \mathbb{Z}} A_{\pm}(x))$ if and only if (67)-(69) hold true. With the notation introduced in Lemma 24 in mind, we have

$$\begin{split} \begin{pmatrix} k_{\pm,1} \\ k_{\pm,2} \end{pmatrix} &= P_{\pm}(\mathbf{L})^{-1} \Phi_{\pm}(0) = P_{\pm}(\mathbf{L})^{-1} \Phi_{\pm,\mathbf{L}}(1), \\ \begin{pmatrix} k'_{\pm,1} \\ k'_{\pm,2} \end{pmatrix} &:= P_{\pm}(\mathbf{R})^{-1} \Phi_{\mathbf{R}}(1) = P_{\pm}(\mathbf{R})^{-1} A_{\pm}(0) \Phi_{\pm,\mathbf{L}}(1) = \begin{pmatrix} z_{\pm,1}(\mathbf{L}) k_{\pm,1} \\ z_{\pm,2}(\mathbf{L}) k_{\pm,2} \end{pmatrix}, \end{split}$$

where the second last equality follows from (69) and the last equality follows from (70). It follows from Lemma 16 that $\Phi \in \ker (L \oplus L - \bigoplus_{x \in \mathbb{Z}} A_{\pm}(x))$ if and only if

$$\begin{split} |k_{\pm,j}|^2 \sum_{x \in \mathbb{N}} |z_{\pm,j}(\mathbf{L})|^{-2x} &< \infty, & j = 1, 2, \\ |k_{\pm,j}|^2 |z_{\pm,j}(\mathbf{L})|^2 \sum_{x \in \mathbb{N}} |z_{\pm,j}(\mathbf{R})|^{+2x} &< \infty, & j = 1, 2, \end{split}$$

and so we get the criterion (71). It follows from (59) that

$$|z_{\pm,1}(L)|^{-1} = f(\pm p)f(\pm a(L)),$$
 $|z_{\pm,1}(R)| = f(\mp p)f(\mp a(R)),$ $|z_{\pm,2}(L)|^{-1} = f(\pm p)f(\mp a(L)),$ $|z_{\pm,2}(R)| = f(\mp p)f(\pm a(R)),$

where

$$|z_{\pm,1}(L)|^{-1} < 1$$
 and $|z_{\pm,1}(R)| < 1$ if and only if $\pm a(L) < \mp p < \pm a(R)$, $|z_{\pm,2}(L)|^{-1} < 1$ and $|z_{\pm,2}(R)| < 1$ if and only if $\pm a(L) < \mp p < \pm a(R)$.

Thus (72)-(75) hold true. Finally, we may assume without loss of generality that a(L) < a(R), so that (73)-(74) both fail to hold. That is, $\Phi_{\pm} \in \ker \left(L \oplus L - \bigoplus_{x \in \mathbb{Z}} A_{\pm}(x)\right)$ are always of the following forms:

$$\begin{pmatrix} k_{+,1} \\ 0 \end{pmatrix} = P_{\pm}(L)^{-1}\Phi_{+}(0) \text{ and } \begin{pmatrix} 0 \\ k_{-,2} \end{pmatrix} = P_{\pm}(L)^{-1}\Phi_{-}(0),$$

and so $d_{\pm} \neq 2$. This is because the conclusion of Lemma 15 still holds true, if the indexing set \mathbb{N} is replaced by \mathbb{Z} .

Proof of Equality (36). The claim immediately follows from (72)-(75). \Box

5 Proof of the main theorem

We are finally in a position to prove Theorem A, the main theorem of the present paper. This will be done in two separate steps: proof of the Fredholmness characterisation as in (A1) and proof of the index formula (A2).

5.1 The Fredholmness

In order to prove the Fredholmness characterisation (A1), let us first discuss the following simple characterisation of the closedness of the range of Q_{ϵ_n} ;

Lemma 27. With the notation introduced in Lemma 10 in mind, the operator Q_{ϵ_+} has a closed range if and only if the time-evolution U has spectral gaps 1 at ± 1 .

Proof. It is a well-known fact that Q_{ϵ_+} has a closed range if and only if $\sigma(Q_{\epsilon_+}^*Q_{\epsilon_+}) \setminus \{0\} > 0$ (see, for example, [47, Lemma 7.27]). Since $\sigma(H_{\epsilon}) = \sigma(H)$ and $H = (\operatorname{Im} U)^2$, the claim follows from the spectral theorem.

To put it another way, Lemma 27 states that the closedness of the operator Q_{ϵ_+} is nothing but a spectral property of the time-evolution $U = \Gamma C$. The following result is therefore useful;

Theorem 28. Let (Γ, C) be a split-step SUSYQW with an anisotropic coin C, and let

$$U_{\sharp} = \Gamma \bigoplus_{x \in \mathbb{Z}} C(\sharp), \qquad \sharp = L, R.$$

Then the essential spectrum of the time-evolution $U = \Gamma C$ is given by

$$\sigma_{\rm ess}(U) = \sigma(U_{\rm L}) \cup \sigma(U_{\rm R}).$$
 (76)

More explicitly, we have $\sigma(U_{\sharp}) = \{z \in \mathbb{T} \mid \text{Re } z \in I_{\sharp}\}$ for each $\sharp = L, R$, where

$$I_{\sharp} := \begin{cases} \{\pm 1\}, & \text{if } C(\sharp) \text{ is trivial,} \\ [pa(\sharp) - |qb(\sharp)|, pa(\sharp) + |qb(\sharp)|], & \text{otherwise.} \end{cases}$$

$$(77)$$

Proof of Theorem 28. This result is standard, and so we will only give a brief sketch of the proof. Firstly, the well-known equality (76) can be proved by either using Weyl's criterion for the essential spectrum or an elegant C^* -algebraic approach (see, for example, [37, Theorem 2.2]). Secondly, the fact that each $\sigma(U_{\sharp})$ is characterised by (77) is an easy consequence of the standard approach which makes use of the discrete Fourier transform.

We are now in a position to show that (A1) is also a characterisation of the closedness of the range of Q_{ϵ_+} ;

Theorem 29. Let (Γ, C) be a split-step SUSYQW endowed with an isotropic coin C. With the notation introduced in Theorem 6 in mind, the operator Q_{ϵ_+} has a closed range if and only if $|p| \neq |a(\sharp)|$ whenever $C(\sharp)$ is non-diagonal, where $\sharp = L, R$.

Proof. It immediately follows from Theorem 28 that for each $\sharp = L$, R, the set $\sigma(U_{\sharp})$ is a discrete subset of \mathbb{T} if and only if $C(\sharp)$ is a diagonal matrix (i.e. $b(\sharp) = 0$). With Lemma 27 in mind, we have that the time-evolution U has spectral gaps at ± 1 if and

 $^{^{1}~}$ That is to say, ± 1 are not accumulation points of the spectrum of U.

only if the set I_{\sharp} does not contain both -1 and +1, whenever the limit $C(\sharp)$ is not a diagonal matrix. For such \sharp , we introduce the following parametrisation:

$$\Theta := \arcsin(p) \text{ and } \Theta_{\sharp} := \arcsin(a(\sharp)).$$

With this parametrisation in mind, we obtain

$$p = \sin \Theta,$$
 $q = e^{i \operatorname{Arg} q} \cos \Theta,$ $a(\sharp) = \sin \Theta_{\sharp},$ $b(\sharp) = e^{i \operatorname{Arg} b(\sharp)} \cos \Theta_{\sharp}.$

The addition formula for the cosine gives:

$$pa(\sharp) \pm |qb(\sharp)| = \sin\Theta\sin\Theta_{\sharp} \pm \cos\Theta\cos\Theta_{\sharp} = \pm\cos(\Theta \mp \Theta_{\sharp}),$$

so that the set I_{\sharp} becomes the following closed interval:

$$I_{\sharp} = [-\cos(\Theta + \Theta_{\sharp}), +\cos(\Theta - \Theta_{\sharp})], \tag{78}$$

where $-\pi < \Theta \pm \Theta_{\sharp} < \pi$. Thus, we have $\pm 1 \in I_{\sharp}$ if and only if $\Theta \mp \Theta_{\sharp} = 0$. That is, I_{\sharp} does not contain both -1 and +1 if and only if $|p| \neq |a(\sharp)|$. The claim follows. \square

Proof of the characterisation (A1). Let (Γ, C) be a split-step SUSYQW endowed with an anisotropic coin C, and let $C(\sharp)$ be a non-trivial unitary involution for each $\sharp = L$, R. Recall the Fredholmness is invariant under compact perturbations (see (84) for details). Therefore, we may assume without loss of generality that C is of the form (32), so that Theorem 13 implies $d_{\pm} \leq 1$. That is, Theorem 29 implies that (Γ, C) is Fredholm if and only if $|p| \neq |a(\sharp)|$ for each $\sharp = L$, R. Note that if $C(\sharp)$ is diagonal, then $|p| \neq |a(\sharp)| = 1$ obviously holds true. The claim follows.

5.2 Proof of the index formula (A2)

From here on, we shall assume that (Γ, C) is a Fredholm SUSYQW and prove (A2) by considering the four coin types separately:

5.2.1 Type I coin

Proof of Equality (A2). If C is of Type I, then (A2) becomes

$$\operatorname{ind}(\Gamma, C) = 0, (79)$$

since $|a(\sharp)| = 1$ for each $\sharp = L, R$ by assumption. In fact, (79) immediately follows (33). The claim follows.

5.2.2 Type II coin

Proof of Equality (A2). If C is of Type II, then (A2) becomes

$$\operatorname{ind}(\Gamma, C) = \begin{cases} +\operatorname{sgn} p, & |a(R)| < |p|, \\ 0, & \text{otherwise,} \end{cases}$$
(80)

since |a(L)| = 1 by assumption. It follows from Theorem 13 that $d_+ \neq d_-$ if and only if one of the following two conditions holds true:

$$(d_{\pm}, d_{\mp}) = (1, 0)$$
 if and only if $\mp p \le a(L)a(R) < \pm p$,

where the last condition is equivalent to $|a(R)| < \pm p$, since $|p| \neq |a(R)|$. Thus (80) holds true.

5.2.3 Type II' coin

Proof of Equality (A2). If C is of Type II', then (A2) becomes

$$\operatorname{ind}(\Gamma, C) = \begin{cases} -\operatorname{sgn} p, & |a(L)| < |p|, \\ 0, & \text{otherwise,} \end{cases}$$
 (81)

since |a(R)| = 1 by assumption. It follows from Theorem 13 that $d_+ \neq d_-$ if and only if one of the following two conditions holds true:

$$(d_{\pm}, d_{\mp}) = (1, 0)$$
 if and only if $\pm p \le a(L)a(R) < \mp p$,

where the last condition is equivalent to $|a(R)| < \mp p$, since $|p| \neq |a(R)|$.

5.2.4 Type III coin

Proof of Equality (A2). Let us assume that C is of Type III coin operator. Note first that if |a(L)| = |a(R)|, then ind $(\Gamma, C) = 0$. Thus, we shall assume $|a(L)| \neq |a(R)|$ from here on. By the invariance principle (85), we shall assume without loss of generality that a(R) < a(L) throughout, and so

ind
$$(\Gamma, C) = \pm 1$$
 if and only if $\pm p \in (a(R), a(L))$ and $\mp p \notin (a(R), a(L))$.

We show first that (A2) holds true, under the assumption |a(R)| < |a(L)| first. That is, we need to check

$$\operatorname{ind}(\Gamma, C) = \begin{cases} +\operatorname{sgn} p, & \text{if } p \neq 0 \text{ and } |a(R)| < |p| < |a(L)|, \\ 0, & \text{otherwise} \end{cases}$$

Since |a(R)| < |a(L)| and a(R) < a(L), we must always have a(L) = |a(L)|. If $a(R) \ge 0$, then

ind
$$(\Gamma, C) = \pm 1$$
 if and only if $a(R) < \pm p < a(L)$.

On the other hand, if a(R) < 0, then

ind
$$(\Gamma, C) = \pm 1$$
 if and only if $\pm p \in (|a(R)|, |a(L)|)$.

It remains to prove that (A2) holds true, under the other assumption |a(L)| < |a(R)|. That is, we need to check

$$\operatorname{ind}\left(\Gamma,C\right) = \begin{cases} -\operatorname{sgn} p, & \text{if } p \neq 0 \text{ and } |a(\mathsf{L})| < |p| < |a(\mathsf{R})|, \\ 0, & \text{otherwise.} \end{cases}$$

As before invariance principle (85) allows us to assume without loss of generality that a(R) < a(L). Since |a(L)| < |a(R)| and a(R) < a(L), we must always have a(R) = -|a(R)|. If a(L) < 0, then

ind
$$(\Gamma, C) = \pm 1$$
 if and only if $|a(L)| < \mp p < |a(R)|$.

On the other hand, if $a(L) \ge 0$, then

ind
$$(\Gamma, C) = \pm 1$$
 if and only if $|a(L)| < \mp p < |a(R)|$.

6 Concluding Remarks

A somewhat natural question arises. Can we still define the Witten index, if a given SUSYQW fails to be Fredholm? The answer to this question turns out to be yes, and we shall give a brief account of how research towards this direction can be undertaken. In fact, the standard theory of supersymmetry is already capable of dealing with the Witten index which cannot be interpreted as the Fredholm index by making use of a certain trace formula. See, for example, [45] or [50]. These papers provide a theoretical foundation in what follows.

Let (Γ, C) be a one-dimensional split-step SUSYQW, and let ϵ be any unitary operator which gives diagonalisation of the shift operator as in (24). We can then consider the unitarily equivalent SUSYQW $(\epsilon^* \Gamma \epsilon, \epsilon^* C \epsilon)$ together with the new associated supercharge $Q_{\epsilon} := \epsilon^* \Gamma \epsilon$ and superhamiltonian $H_{\epsilon} := \epsilon^* H \epsilon = Q_{\epsilon}^2$ admitting:

$$Q_{\epsilon} = \begin{pmatrix} 0 & Q_{\epsilon_{-}} \\ Q_{\epsilon_{+}} & 0 \end{pmatrix}, \qquad H_{\epsilon} = \begin{pmatrix} H_{\epsilon_{+}} & 0 \\ 0 & H_{\epsilon_{-}} \end{pmatrix},$$

where the first equality follows from (28). We say that the triple (Γ, C, ϵ) is *trace-compatible*, if $H_{\epsilon_+} - H_{\epsilon_-}$ is a trace-class operator on $\ell^2(\mathbb{Z})$. We can then define the *Witten index* of the triple (Γ, C, ϵ) by

$$\operatorname{ind}(\Gamma, C, \epsilon) := \lim_{t \to \infty} \operatorname{tr}(e^{-tH_{\epsilon_+}} - e^{-tH_{\epsilon_-}}), \tag{82}$$

whenever the limit exists. It is not known to the authors whether or not Formula (82) depends on ϵ . As in [45], if the SUSYQW (Γ , C) turns out to be Fredholm, then the above limit exists, and we get

$$\operatorname{ind}(\Gamma, C, \epsilon) = \operatorname{ind}(\Gamma, C),$$

where the left hand side does not depend on ϵ in this Fredholm case. That is, in principle, we should be able to recover (A2) by simply evaluating the trace-formula (82). Research towards this direction is work in progress, and this will be part of the PhD dissertation of the second author. The present paper concludes with the following simple example;

Example 30. Let (Γ, C) be a one-dimensional split-step SUSYQW whose coin operator C has the property that b(x) = 0 for each $x \in \mathbb{Z}$. With the notation introduced in Theorem 6 in mind, we obtain

$$-2iQ_{\epsilon_+} = 0 - 0 \pm |q|(a_2(\cdot + 1) - a_1) =: \pm \beta.$$

Then the superhamiltonian becomes

$$H_{\epsilon} = Q_{\epsilon}^2 = \begin{pmatrix} 0 & -\frac{i\beta}{2} \\ +\frac{i\beta}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\frac{i\beta}{2} \\ +\frac{i\beta}{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{\beta^2}{4} & 0 \\ 0 & \frac{\beta^2}{4} \end{pmatrix}.$$

This implies $H_+ = H_-$, so that (Γ, C, ϵ) is trace-compatible and ind $(\Gamma, C, \epsilon) = 0$.

I Supplementary Material

An (abstract) *supersymmetric quantum walk* (SUSYQW) is a pair (Γ, C) of two unitary involutions on a Hilbert space \mathcal{H} .

The Witten index for SUSYQWs turns out to enjoy the following two invariance principles, each of which will a significant role in this paper.

Theorem 31 (Invariance of the Witten index). *The following two assertions hold true:*

1. Unitary Invariance. Let (Γ, C) and (Γ', C') be two SUSYQWs that are unitarily equivalent in the sense $(\Gamma', C') = (\epsilon^* \Gamma \epsilon, \epsilon^* C \epsilon)$ for some unitary operator ϵ on \mathcal{H} . Then (Γ, C) is a Fredholm SUSYQW if and only if so is $(\epsilon^* \Gamma \epsilon, \epsilon^* C \epsilon)$. In this case,

$$\operatorname{ind}(\Gamma, C) = \operatorname{ind}(\epsilon^* \Gamma \epsilon, \epsilon^* C \epsilon). \tag{83}$$

 Topological Invariance. Let (Γ, C) and (Γ, C') be two SUSYQWs sharing the same shift operator Γ, and let C – C' be a compact operator. Then (Γ, C) is a Fredholm SUSYQW if and only if so is (Γ, C'). In this case.

$$\operatorname{ind}(\Gamma, C) = \operatorname{ind}(\Gamma, C'). \tag{84}$$

Remark 32. The invariance principle (83) can be used to classify SUSYQWs in the following precise sense. If the Witten indices associated with two given SUSYQWs (Γ, C) and (Γ', C') do not agree to each other, then they cannot be unitarily equivalent. This is, of course, analogous to the manner in which we use the homotopy/homology groups to prove that certain topological spaces are not homotopy equivalent.

Here is yet another important principle of the Witten index:

Theorem 33 ([15, Corollary 3.7]). *If one of* (Γ, C) , $(-\Gamma, C)$, $(\Gamma, -C)$ *is a Fredholm SUSYQW, then so are the rest. In this case we have the following formulas:*

$$\operatorname{ind}(\Gamma, -C) = \operatorname{ind}(\Gamma, C), \tag{85}$$

$$\operatorname{ind}(-\Gamma, C) = -\operatorname{ind}(\Gamma, C). \tag{86}$$

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