Necessary conditions for classifying **m**-separability of multipartite entanglements

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Abstract

We study the norms of the Bloch vectors for arbitrary n-partite quantum states. A tight upper bound of the norms is derived for n-partite systems with different individual dimensions. These upper bounds are used to deal with the separability problems. Necessary conditions are presented for **m**-separable states in n-partite quantum systems. Based on the upper bounds, classification of multipartite entanglement is illustrated with detailed examples.

Keywords Bloch vectors · Norm · Upper bounds · Separability

1 Introduction

Quantum entanglement is a remarkable resource in the theory of quantum information, with numerous applications in quantum information processing, secure communication and channel protocols [1–3]. A multipartite quantum state that is not separable with respect to any bipartition is said to be genuinely multipartite entangled [4–6]. Genuinely multipartite entangled states have significant advantages in quantum tasks compared with biseparable ones [7].

The notion of genuine multipartite entanglement (GME) was introduced in [7]. Let $H_i^{d_i}, i = 1, \cdots n$, denote d_i-dimensional Hilbert spaces. An n-partite state $\rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes \cdots \otimes H_n^{d_n}$ can be expressed as $\rho = \sum\limits_i p_i |\psi_i\rangle\langle\psi_i|$, where $\sum\limits_i p_i = 1, 0 < p_i \leq 1, |\psi_i\rangle \in H_1^{d_1} \otimes H_2^{d_2} \otimes \cdots \otimes H_n^{d_n}$ are normalized pure states. ρ is biseparable if $|\psi_i\rangle$ $(i = 1, \cdots, n)$ can be expressed as one of the forms: $|\psi_i\rangle = |\psi_i^{j_1\cdots j_{k-1}}\rangle \otimes |\psi_i^{j_k\cdots j_n}\rangle$, where $|\psi_i^{j_1\cdots j_{k-1}}\rangle$ and $|\psi_i^{j_k\cdots j_n}\rangle$ denote pure states in $H_i^{j_1} \otimes \cdots \otimes H_i^{j_{k-1}}$ and $H_i^{j_k} \otimes \cdots \otimes H_i^{j_n}$, respectively, $j_1 \neq \cdots \neq j_n \in \{1, \cdots, n\}$, $k = 2, ..., n-1$. Otherwise, ρ is called genuine multipartite entangled. Correspondingly, we say that the state $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ is **m**-separable if all the $|\psi_i\rangle$ are tensor products of m vectors in the subspaces of $H_1^{d_1} \otimes H_2^{d_2} \otimes \cdots \otimes H_n^{d_n}$.

Any quantum state has Bloch representation in multipartite high-dimensional quantum systems. By using the norms of the Bloch vectors, the density operators in lower dimensions were discussed in [8,9]. For bipartite and multipartite quantum states, separable conditions have been presented in [10–13]. The norms of the Bloch vectors for any qudit quantum states with subsystems less than or equal to four have been investigated in [14]. Then in [15], Tănăsescu et al. generalized the result of [14] for four-partite quantum systems, which provided an upper bound on the entanglement measure given by the Bloch vector norm and a necessary algebraic condition for separability of a general multi-partite quantum system under any arbitrary partition function. Two multipartite entanglement measures for n-qubit and n-qudit pure states are given in [16,17]. In [18], the sum of relative isotropic strengths of any three-qudit state over d-dimensional Hilbert space cannot exceed $d-1$ have been discussed, and the trade-off relations and monogamy-like relations of the sum of spin correlation strengths for pure three- and four-partite systems are derived. Some sufficient or necessary conditions of GME were presented in [19–21]. To the detection of GME, the common criterion is the entanglement witnesses [5,22–24]. In [6], the norms of the Bloch vectors give rise to a general framework to detect different classes of GME for arbitrary dimensional quantum systems. Recently, based on the norms of the correlation tensors, Zhao et al. [25] have been studied the separability criteria by matrix method and necessary conditions of separability for multipartite systems are given under arbitrary partition.

In this paper, we study the Bloch representations of quantum states with arbitrary number of subsystems. In Section 2, we present tight upper bounds for the norms of Bloch vectors in n-qudit quantum states. These upper bounds are then used to derive tight upper bounds for entanglement measures in [16,17]. The upper bounds of the norms of the Bloch vectors are useful to study the separability. In Section 3, we investigate different subclasses of bi-separable states in n-partite systems. Necessary conditions for **m**-separability and complete classification of n-partite quantum systems are presented.

2 Upper bounds of the norms of Bloch vectors

Let λ_i, $i = 1, \cdots, d^2-1$, denote the generators of the special unitary group $SU(d)$, which satisfy $\lambda_i^\dagger = \lambda_i$, $Tr(\lambda_i) = 0$, $Tr(\lambda_i\lambda_j) = 2\delta_{ij}$. The following theorem gives the general result for n-partite quantum states.

Theorem 1. *Let* $\rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes \cdots \otimes H_n^{d_n}$ $(n \geq 3, 2 \leq d_1 \leq d_2 \leq \cdots \leq d_n, d_n \leq d_1 \cdots d_{n-1})$

be an n-partite quantum state. We have

$$\left\|\mathbf{T}^{(12\cdots n)}\right\|^2 \leq 2^n \left\{1 - \frac{\sum\limits_{1\leq i_1<\cdots<i_{n-1}\leq n} d_{i_1}\cdots d_{i_{n-1}} - \sum\limits_{i=1}^{n} d_i + (n-2)d_n}{(n-2)d_1\cdots d_{n-1}d_n^2}\right\}. \tag{1}$$

Proof. ρ has the Bloch representation:

$$\begin{aligned}\rho =&\frac{1}{d_1\cdots d_n} I_{d_1}\otimes\cdots\otimes I_{d_n} + \frac{1}{2}\Bigg(\frac{1}{d_2\cdots d_n}\sum_{i_1=1}^{d_1^2-1} t_{i_1}^{(1)}\lambda_{i_1}\otimes I_{d_2}\otimes\cdots\otimes I_{d_n}+\cdots \\ &+\frac{1}{d_1\cdots d_{n-1}}\sum_{i_n=1}^{d_n^2-1} t_{i_n}^{(n)} I_{d_1}\otimes I_{d_2}\otimes\cdots\otimes\lambda_{i_n}\Bigg)+\cdots \\ &+\frac{1}{2^n}\sum_{k=1}^{n}\sum_{i_k=1}^{d_k^2-1} t_{i_1\cdots i_n}^{(1\cdots n)}\lambda_{i_1}\otimes\lambda_{i_2}\otimes\cdots\otimes\lambda_{i_n},\end{aligned} \tag{2}$$

where I_{d_i} denotes the $d_i\times d_i$ identity matrix , $i=1,\cdots,n$, $t_{i_1}^{(1)} = Tr(\rho\lambda_{i_1}\otimes I_{d_2}\otimes\cdots\otimes I_{d_n}),\cdots,t_{i_{j_1}\cdots i_{j_k}}^{(j_1\cdots j_k)} = Tr(\rho\lambda_{i_{j_1}}\otimes\cdots\otimes\lambda_{i_{j_k}}\otimes I_{d_{j_{k+1}}}\otimes\cdots\otimes I_{d_{j_n}}),\cdots,t_{i_1\cdots i_n}^{(1\cdots n)} = Tr(\rho\lambda_{i_1}\otimes\lambda_{i_2}\otimes\cdots\otimes\lambda_{i_n})$ and $\mathbf{T}^{(1)},\cdots,\mathbf{T}^{(j_1\cdots j_k)},\cdots,\mathbf{T}^{(1\cdots n)}$ are the vectors (tensors) with the elements $t_{i_1}^{(1)},\cdots,t_{i_{j_1}\cdots i_{j_k}}^{(j_1\cdots j_k)},\cdots,t_{i_1\cdots i_n}^{(1\cdots n)}$ $(1\leq j_1<\cdots<j_k\leq n, i_s=1,\cdots,d_s^2-1, s=1,\cdots,n)$, respectively.

Set

$$\begin{aligned}\left\|\mathbf{T}^{(1)}\right\|^2 &= \sum_{i_1=1}^{d_1^2-1}\left(t_{i_1}^{(1)}\right)^2, \\ &\cdots, \\ \left\|\mathbf{T}^{(j_1\cdots j_k)}\right\|^2 &= \sum_{s=j_1}^{j_k}\sum_{i_s=1}^{d_s^2-1}\left(t_{i_{j_1}\cdots i_{j_k}}^{(j_1\cdots j_k)}\right)^2, \\ &\cdots, \\ \left\|\mathbf{T}^{(1\cdots n)}\right\|^2 &= \sum_{s=1}^{n}\sum_{i_s=1}^{d_s^2-1}\left(t_{i_1\cdots i_n}^{(1\cdots n)}\right)^2,\end{aligned}$$

$$\begin{aligned}x_1 &= \tfrac{1}{d_2\cdots d_n}\left\|\mathbf{T}^{(1)}\right\|^2+\cdots+\tfrac{1}{d_1\cdots d_{n-1}}\left\|\mathbf{T}^{(n)}\right\|^2, \\ x_2 &= \tfrac{1}{d_3\cdots d_n}\left\|\mathbf{T}^{(12)}\right\|^2+\cdots+\tfrac{1}{d_1\cdots d_{n-2}}\left\|\mathbf{T}^{(n-1,n)}\right\|^2, \\ &\cdots, \\ x_n &= \left\|\mathbf{T}^{(12\cdots n)}\right\|^2.\end{aligned}$$

For a pure state $\rho=|\psi\rangle\langle\psi|$, one has $Tr(\rho^2)=1$, namely,

$$Tr(\rho^2) = \frac{1}{d_1\cdots d_n}+\frac{1}{2}x_1+\frac{1}{2^2}x_2+\cdots+\frac{1}{2^n}x_n = 1. \tag{3}$$

In the following we denote $\rho_{j_1},\rho_{j_2\cdots j_n}$ the reduced density matrix for the subsystem $H_{j_1}^{d_{j_1}}$ and $H_{j_2}^{d_{j_2}}\otimes\cdots\otimes H_{j_n}^{d_{j_n}}$, $j_1\neq\cdots\neq j_n\in\{1,2,\cdots,n\}$. One computes that,

$$\rho_{j_1} = \frac{1}{d_{j_1}} I_{d_{j_1}} + \frac{1}{2} \sum_{i_{j_1}=1}^{d_{j_1}^2-1} t_{i_{j_1}}^{(j_1)} \lambda_{i_{j_1}},$$

$$\begin{aligned}
\rho_{j_2\cdots j_n} =& \frac{1}{d_{j_2}\cdots d_{j_n}} I_{d_{j_2}} \otimes \cdots \otimes I_{d_{j_n}} + \\
& \frac{1}{2}\Bigg(\frac{1}{d_{j_3}\cdots d_{j_n}} \sum_{i_{j_3}=1}^{d_{j_3}^2-1} t_{i_{j_3}}^{(j_3)} \lambda_{i_{j_3}} \otimes \cdots \otimes I_{d_{j_n}} + \cdots \\
& + \frac{1}{d_{j_2}\cdots d_{j_{n-1}}} \sum_{i_{j_n}=1}^{d_{j_n}^2-1} t_{i_{j_n}}^{(j_n)} I_{d_{j_2}} \otimes \cdots \otimes \lambda_{i_{j_n}} \Bigg) + \cdots \\
& + \frac{1}{2^{n-1}} \sum_{s=2}^{n} \sum_{i_{j_s}=1}^{d_{j_s}^2-1} t_{i_{j_2}\cdots i_{j_n}}^{(j_2\cdots j_n)} \lambda_{i_{j_2}} \otimes \cdots \otimes \lambda_{i_{j_n}}.
\end{aligned}$$

For a pure state $\rho = |\psi\rangle\langle\psi|$, we have

$$Tr(\rho_{j_1}^2) = Tr(\rho_{j_2\cdots j_n}^2), \tag{4}$$

which holds for any $j_1 \neq \cdots \neq j_n \in \{1, 2, \cdots, n\}$. We obtain

$$\sum_{j_1=1}^{n} \frac{1}{d_{j_1}} Tr(\rho_{j_1}^2) = \sum_{j_1=1}^{n} \frac{1}{d_{j_1}} Tr(\rho_{j_2\cdots j_n}^2). \tag{5}$$

Hence we get

$$\begin{aligned}
\sum_{i=1}^{n} \frac{1}{d_i^2} + \frac{1}{2} \sum_{i=1}^{n} \frac{1}{d_i} \left\| \mathbf{T}^{(i)} \right\|^2 =& \frac{n}{d_1\cdots d_n} + \frac{n-1}{2} x_1 + \\
& \frac{n-2}{2^2} x_2 + \cdots + \frac{1}{2^{n-1}} x_{n-1}.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{1}{2^2} x_2 =& \frac{1}{n-2} \left(\sum_{i=1}^{n} \frac{1}{d_i^2} - \frac{n}{d_1\cdots d_n} \right) + \\
& \frac{1}{2(n-2)} \left(\sum_{i=1}^{n} \frac{1}{d_i} \left\| \mathbf{T}^{(i)} \right\|^2 - (n-1) x_1 \right) \\
& - \cdots - \frac{1}{2^{n-1}(n-2)} x_{n-1}.
\end{aligned} \tag{6}$$

Substituting (6) into (3), we get

$$\begin{aligned}\frac{1}{2^n}x_n =& \left[1-\frac{1}{n-2}\left(\sum_{i=1}^{n}\frac{1}{d_i^2}-\frac{2}{d_1\cdots d_n}\right)\right]- \\ & \frac{1}{2(n-2)}\left(\sum_{i=1}^{n}\frac{1}{d_i}\left\|\mathbf{T}^{(i)}\right\|^2-x_1\right)-\cdots-\frac{n-3}{2^{n-1}(n-2)}x_{n-1} \\ \leq & 1-\frac{\sum\limits_{1\leq i_1<\cdots<i_{n-1}\leq n} d_{i_1}\cdots d_{i_{n-1}}-\sum\limits_{i=1}^{n}d_i+(n-2)d_n}{(n-2)d_1\cdots d_{n-1}d_n^2}\end{aligned}$$

where the inequality holds for

$$\sum_{i=1}^{n}\frac{1}{d_i^2}-\frac{2}{d_1\cdots d_n}\geq\frac{\sum\limits_{1\leq i_1<\cdots<i_{n-1}\leq n} d_{i_1}\cdots d_{i_{n-1}}-\sum\limits_{i=1}^{n}d_i+(n-2)d_n}{d_1\cdots d_{n-1}d_n^2} \tag{7}$$

$$\sum_{i=1}^{n}\frac{1}{d_i}\left\|\mathbf{T}^{(i)}\right\|^2-x_1=\sum_{i=1}^{n}\left(\frac{1}{d_i}-\frac{1}{d_1\cdots d_{i-1}d_{i+1}\cdots d_n}\right)\left\|\mathbf{T}^{(i)}\right\|^2\geq 0, \tag{8}$$

the inequation (8) holds for $d_i\leq d_1\cdots d_{i-1}d_{i+1}\cdots d_n, i=1,\cdots,n$. And $x_3,\cdots,x_{n-1}\geq 0$. In fact, the inequation (7) holds if and only if

$$\sum_{i=1}^{n-1}(d_n-d_i)\left(\frac{1}{d_i^2}-\frac{1}{\prod\limits_{i=1}^{n}d_i}\right)\geq 0. \tag{9}$$

Since $d_n\geq d_i$, and $d_i^2\leq\prod\limits_{i=1}^{n}d_i, i=1,\cdots,n-1$, so the inequation (9) holds, which is equivalent to hold for inequation (7). Hence we get

$$x_n\leq 2^n\left\{1-\frac{\sum\limits_{1\leq i_1<\cdots<i_{n-1}\leq n} d_{i_1}\cdots d_{i_{n-1}}-\sum\limits_{i=1}^{n}d_i+(n-2)d_n}{(n-2)d_1\cdots d_{n-1}d_n^2}\right\},\quad n\geq 3.$$

Then we consider a mixed state ρ with ensemble representation $\rho=\sum\limits_i p_i|\psi_i\rangle\langle\psi_i|$, where $\sum\limits_i p_i=1, 0<p_i\leq 1$, by the convexity of the Frobenius norm one derives

$$\begin{aligned}\left\|\mathbf{T}^{(12\cdots n)}(\rho)\right\|^2 =& \left\|\sum_i p_i\mathbf{T}^{(12\cdots n)}(|\psi_i\rangle\langle\psi_i|)\right\|^2 \\ \leq & \sum_i p_i\left\|\mathbf{T}^{(12\cdots n)}(|\psi_i\rangle\langle\psi_i|)\right\|^2 \\ \leq & 2^n\left\{1-\frac{\sum\limits_{1\leq i_1<\cdots<i_{n-1}\leq n} d_{i_1}\cdots d_{i_{n-1}}-\sum\limits_{i=1}^{n}d_i+(n-2)d_n}{(n-2)d_1\cdots d_{n-1}d_n^2}\right\},\quad n\geq 3,\end{aligned}$$

which ends the proof. ■

Remark 1: Theorem 1 is a generalization of Proposition 1 and Proposition 2 given in [25]. When $n = 3, 2 \leq d_1 \leq d_2 \leq d_3, d_3 \leq d_1 d_2$, we obtain that

$$\left\|\mathbf{T}^{(123)}\right\|^2 \leq 8\left(1 - \frac{d_1 d_2 + d_1 d_3 + d_2 d_3 - d_1 - d_2}{d_1 d_2 d_3^2}\right)$$

which coincide with the upper bound in [25]. When $n = 4, 2 \leq d_1 \leq d_2 \leq d_3 \leq d_4, d_4 \leq d_1 d_2 d_3$, we obtain that

$$\left\|\mathbf{T}^{(1234)}\right\|^2 \leq 16\left(1 - \frac{d_1 d_2 d_3 + d_1 d_2 d_4 + d_1 d_3 d_4 + d_2 d_3 d_4 - d_1 - d_2 - d_3 + d_4}{d_1 d_2 d_3 d_4^2}\right)$$

which also coincide with the upper bound in [25].

As a special case, consider $d_1 = \cdots = d_n = d$ in Theorem 1. We have

Corollary 1. *Let $\rho \in H_1^d \otimes H_2^d \otimes \cdots \otimes H_n^d$ $(n \geq 3, d \geq 2)$ be an n-qudit quantum state. We have*

$$\left\|\mathbf{T}^{(12\cdots n)}\right\|^2 \leq \frac{2^n\left[(n-2)d^n - nd^{n-2} + 2\right]}{(n-2)d^n}. \tag{10}$$

Remark 2: The upper bound of Corollary 1 is the same as in [15] and [25]. And Corollary 1 is the generalization of the results of [14]. When $n = 3, 4$, the results of Corollary 1 reduce to the ones in [14] and [15]. And when $n = 3$, the upper bound of Corollary 1 is $\frac{8(d-1)^2(d+2)}{d^3}$, which tighter than the upper bound of Corollary 2.2 given in [18].

The Bloch vectors are used to define a valid entanglement measure in [16,17] as follows. For an n-qudit pure state, the entanglement measure is defined as:

$$E_{\mathbf{T}}(|\psi\rangle) = \left(\frac{d}{2}\right)^{\frac{n}{2}} \left\|\mathbf{T}^{(1\cdots n)}\right\| - \left(\frac{d(d-1)}{2}\right)^{\frac{n}{2}}, \tag{11}$$

where $\mathbf{T}^{(1\cdots n)}$ is defined as a vector with elements $t_{i_1 \cdots i_n}^{(1\cdots n)} = Tr(\rho \lambda_{i_1} \otimes \lambda_{i_2} \otimes \cdots \otimes \lambda_{i_n})$. Our results can give rise to an upper bound of the entanglement:

Corollary 2. *For any n-qudit pure state $\rho \in H_1^d \otimes H_2^d \otimes \cdots \otimes H_n^d$ $(n \geq 3, d \geq 2)$, the entanglement measure has the upper bounds:*

$$E_{\mathbf{T}}(|\psi\rangle) \leq \left(\frac{d}{2}\right)^{\frac{n}{2}} \left[\sqrt{d^n - \frac{n}{n-2} d^{n-2} + \frac{2}{n-2}} - (d-1)^{\frac{n}{2}}\right], \tag{12}$$

which coincide with the upper bound in [15].

3 The Necessary conditions for m-separable states

Now we study separability problems of n-partite quantum systems based on the upper bounds of the norms of Bloch vectors. Let $\rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes \cdots \otimes H_n^{d_n}$, $n \geq 3$, $2 \leq d_1 \leq d_2 \leq$

$\cdots \leq d_n$. If ρ can be written as $\rho = \sum\limits_i p_i|\psi_i\rangle\langle\psi_i|$, where $\sum\limits_i p_i = 1$, $0 < p_i \leq 1$, $|\psi_i\rangle$ is one of the following sets: $\{|\phi_{j_1}\rangle \otimes |\phi_{j_2\cdots j_n}\rangle\}, \cdots, \{|\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle\}, \{|\phi_{j_1j_2}\rangle \otimes |\phi_{j_3\cdots j_n}\rangle\}, \cdots,$ where $j_1 \neq \cdots \neq j_n \in \{1, \cdots n\}$. Then ρ is called $(1, n-1)$ separable, $\cdots$, $\underbrace{(1, \cdots, 1)}_{n}$ separable, $(2, n-2)$ separable, $\cdots$, respectively.

Lemma 1. *Let $\rho_j \in H_j^{d_j}$ $(j = 1, \cdots, n, d_j \geq 2)$ be the reduced density operator of ρ. We have*

$$\left\|\mathbf{T}^{(j)}\right\|^2 \leq \frac{2(d_j - 1)}{d_j}. \tag{13}$$

Proof. ρ_j has the Bloch representation:

$$\rho_j = \frac{1}{d_j} I_{d_j} + \frac{1}{2} \sum_{i_j=1}^{d_j^2-1} t_{i_j}^{(j)} \lambda_{i_j}, \tag{14}$$

where $t_{i_j}^{(j)} = Tr(\rho_j \lambda_{i_j})$, $\mathbf{T}^{(j)}$ is a vector with entries $t_{i_j}^{(j)}$, $j = 1, \cdots, n$, $i_j = 1, \cdots, d_j^2 - 1$. Since $Tr(\rho_j^2) \leq 1$, i.e. $\frac{1}{d_j} + \frac{1}{2}\left\|\mathbf{T}^{(j)}\right\|^2 \leq 1$, one obtains (13). ■

Lemma 2. *Let $\rho_{jk} \in H_j^{d_j} \otimes H_k^{d_k}$ $(1 \leq j < k \leq n, 2 \leq d_j \leq d_k)$ be the reduced density operator of ρ. We have*

$$\left\|\mathbf{T}^{(jk)}\right\|^2 \leq \frac{2^2(d_j^2 - 1)}{d_j^2}. \tag{15}$$

Proof. ρ_{jk} has the Bloch representation:

$$\begin{aligned}\rho_{jk} =& \frac{1}{d_j d_k} I_{d_j} \otimes I_{d_k} + \frac{1}{2d_k} \sum_{i_j=1}^{d_j^2-1} t_{i_j}^{(j)} \lambda_{i_j} \otimes I_{d_k} + \\ & \frac{1}{2d_j} \sum_{i_k=1}^{d_k^2-1} t_{i_k}^{(k)} I_{d_j} \otimes \lambda_{i_k} + \frac{1}{4} \sum_{i_j=1}^{d_j^2-1} \sum_{i_k=1}^{d_k^2-1} t_{i_j i_k}^{(jk)} \lambda_{i_j} \otimes \lambda_{i_k},\end{aligned} \tag{16}$$

where $t_{i_j}^{(j)} = Tr(\rho_{jk} \lambda_{i_j} \otimes I_{d_k})$, $t_{i_k}^{(k)} = Tr(\rho_{jk} I_{d_j} \otimes \lambda_{i_k})$, $t_{i_j i_k}^{(jk)} = Tr(\rho_{jk} \lambda_{i_j} \otimes \lambda_{i_k})$. $\mathbf{T}^{(j)}$, $\mathbf{T}^{(k)}$ and $\mathbf{T}^{(jk)}$ are vectors with entries $t_{i_j}^{(j)}$, $t_{i_k}^{(k)}$ and $t_{i_j i_k}^{(jk)}$ $(1 \leq j < k \leq n, i_j = 1, \cdots, d_j^2 - 1, i_k = 1, \cdots, d_k^2 - 1)$. Set

$$\left\|\mathbf{T}^{(j)}\right\|^2 = \sum_{i_j=1}^{d_j^2-1} \left(t_{i_j}^{(j)}\right)^2,$$

$$\left\|\mathbf{T}^{(k)}\right\|^2 = \sum_{i_k=1}^{d_k^2-1} \left(t_{i_k}^{(k)}\right)^2,$$

$$\left\|\mathbf{T}^{(jk)}\right\|^2 = \sum_{i_j=1}^{d_j^2-1} \sum_{i_k=1}^{d_k^2-1} \left(t_{i_j i_k}^{(jk)}\right)^2.$$

For a pure state $\rho_{jk} = |\psi\rangle\langle\psi|$, one has $Tr(\rho_{jk}^2) = 1$, namely,

$$Tr(\rho_{jk}^2) = \frac{1}{d_j d_k} + \frac{1}{2d_k}\left\|\mathbf{T}^{(j)}\right\|^2 + \frac{1}{2d_j}\left\|\mathbf{T}^{(k)}\right\|^2 + \frac{1}{4}\left\|\mathbf{T}^{(jk)}\right\|^2 = 1. \tag{17}$$

Let ρ_j and ρ_k be the reduced density matrices with respect to the subsystems $1 \leq j < k \leq n$. Since for a pure state ρ_{jk}, $Tr(\rho_j^2) = Tr(\rho_k^2)$, i.e. $\frac{1}{d_j} + \frac{1}{2}\left\|\mathbf{T}^{(j)}\right\|^2 = \frac{1}{d_k} + \frac{1}{2}\left\|\mathbf{T}^{(k)}\right\|^2$. Therefore, we get

$$
\begin{aligned}
\left\|\mathbf{T}^{(jk)}\right\|^2 =& \frac{2^2(d_j^2-1)}{d_j^2} - \frac{2(d_j+d_k)}{d_j d_k}\left\|\mathbf{T}^{(j)}\right\|^2 \\
\leq& \frac{2^2(d_j^2-1)}{d_j^2}.
\end{aligned}
\tag{18}
$$

Now we consider a mixed state ρ_{jk} with ensemble representation $\rho_{jk} = \sum\limits_i p_i|\psi_i\rangle\langle\psi_i|$, where $\sum\limits_i p_i = 1$, $0 < p_i \leq 1$, by the convexity of the Frobenius norm one derives

$$
\begin{aligned}
\left\|\mathbf{T}^{(jk)}(\rho_{jk})\right\|^2 =& \left\|\sum_i p_i \mathbf{T}^{(jk)}(|\psi_i\rangle\langle\psi_i|)\right\|^2 \\
\leq& \sum_i p_i \left\|\mathbf{T}^{(jk)}(|\psi_i\rangle\langle\psi_i|)\right\|^2 \\
\leq& \frac{2^2(d_j^2-1)}{d_j^2},
\end{aligned}
$$

which ends the proof. ■

For the n-partite quantum state $\rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes \cdots \otimes H_n^{d_n}, n \geq 3, 2 \leq d_1 \leq \cdots \leq d_n$, consider the $\mathbf{m}$-partition of n-qudit quantum state ρ, we denote $\mathbf{m} = (k_1, \cdots, k_m)$, where $\sum\limits_{s=1}^{m} k_s = n, 1 \leq k_1 \leq \cdots \leq k_m \leq n-1$ and $n_j = \sum\limits_{s=1}^{j} k_s, 1 \leq j \leq m, 1 \leq n_j \leq n$. By Theorem 1, $d_{n_j} \leq d_{n_{j-1}+1}\ d_{n_{j-1}+2} \cdots d_{n_j - 1}$ if $3 \leq k_j \leq n-1, 1 \leq j \leq m$. Moreover, denote the following:

$$
\begin{aligned}
a_1 &= \tfrac{2(d_1-1)}{d_1}, \\
&\cdots, \\
a_{n_p} &= \tfrac{2\left(d_{n_p}-1\right)}{d_{n_p}}, \\
\\
a_{n_b} &= \tfrac{2^2\left(d_{n_b-1}^2-1\right)}{d_{n_b-1}^2}, \\
&\cdots, \\
a_{n_{b+(q-1)}} &= \tfrac{2^2\left(d_{n_{b+(q-1)}-1}^2-1\right)}{d_{n_{b+(q-1)}-1}^2} \\
&\cdots, \\
a_{n_c} &= 2^t\left\{1 - \tfrac{\sum d_{i_{n_c-(t-1)}}\cdots d_{i_{n_c-1}} - \sum\limits_{i=0}^{t-1} d_{n_c+i}\ +(n-2)d_{n_c}}{(t-2)d_{n_c-(t-1)}\cdots d_{n_c-1}d_{n_c}^2}\right\}, \\
&\cdots, \\
a_{n_{c+(s-1)}} &= 2^t\left\{1 - \tfrac{\sum d_{i_{n_{c+(s-1)}-(t-1)}}\cdots d_{i_{n_{c+(s-1)}-1}} - \sum\limits_{i=0}^{t-1} d_{n_c+(s-1)+i} + (n-2)d_{n_{c+(s-1)}}}{(t-2)d_{n_{c+(s-1)}-(t-1)}\cdots d_{n_{c+(s-1)}-1}d_{n_{c+(s-1)}}^2}\right\}, \quad t \geq 3.
\end{aligned}
\tag{19}
$$

where $p+q+\cdots+s=m,\ p+2q+\cdots+ts=n, 1\leq p,b,q,\cdots,c,s\leq m$.

Then from Lemmas 1, 2 and Theorem 1, we have

$$\begin{gathered}
\left\|\mathbf{T}^{(1)}\right\|^2\leq a_1,\\
\cdots,\\
\left\|\mathbf{T}^{(n_p)}\right\|^2\leq a_{n_p},\\
\\
\left\|\mathbf{T}^{(n_b-1,n_b)}\right\|^2\leq a_{n_b},\\
\cdots,\\
\left\|\mathbf{T}^{\left(n_{b+(q-1)}-1,n_{b+(q-1)}\right)}\right\|^2\leq a_{n_{b+(q-1)}},\\
\\
\left\|\mathbf{T}^{(n_c-(t-1),\cdots,n_c)}\right\|^2\leq a_{n_c},\\
\cdots,\\
\left\|\mathbf{T}^{\left(n_{c+(s-1)}-(t-1),\cdots,n_{c+(s-1)}\right)}\right\|^2\leq a_{n_{c+(s-1)}},\quad t\geq 3.
\end{gathered}$$

where $\mathbf{T}^{(u)}$ is a vector with the entries $t^{(u)}_{i_u}$ $(u=1,\cdots,n_p,\ i_u=1,\cdots,d_u^2-1)$. $\mathbf{T}^{(xy)}$ is a vector with entries $t^{(xy)}_{i_x i_y}$ $(x=n_b-1,\cdots,n_{b+(q-1)}-1, y=n_b,\cdots,n_{b+(q-1)}, i_x=1,\cdots,d_x^2-1, i_y=1,\cdots,d_y^2-1)$. $\cdots$. $\mathbf{T}^{(x_1,\cdots,x_t)}$ is a vector with entries $t^{(x_1\cdots x_t)}_{i_{x_1}\cdots i_{x_t}}$ $(x_1=n_c-(t-1),\cdots,n_{c+(s-1)}-(t-1),\cdots,x_t=n_c,\cdots,n_{c+(s-1)}, i_{x_1}=1,\cdots,d_{x_1}^2-1,\cdots,i_{x_t}=1,\cdots,d_{x_t}^2-1)$.

The following theorem gives the necessary conditions of $\mathbf{m}$-separability.

Theorem 2. *Let $\rho\in H_1^{d_1}\otimes H_2^{d_2}\otimes\cdots\otimes H_n^{d_n}$ $(n\geq 3, 2\leq d_1\leq d_2\leq\cdots\leq d_n)$ be an n-partite quantum state. If ρ is $\mathbf{m}$-separable we have*

$$\left\|\mathbf{T}^{(12\cdots n)}\right\|^2\leq\prod_{f=1}^{n_p}a_f\prod_{g=n_b}^{n_{b+(q-1)}}a_g\cdots\prod_{h=n_c}^{n_{c+(s-1)}}a_h,\tag{20}$$

where $\mathbf{m}=(k_1,\cdots,k_m)$, $\sum\limits_{s=1}^{m}k_s=n, 1\leq k_1\leq\cdots\leq k_m\leq n-1$ and $n_j=\sum\limits_{s=1}^{j}k_s, 1\leq j\leq m, 1\leq n_j\leq n$, $d_{n_j}\leq d_{n_{j-1}+1}\ d_{n_{j-1}+2}\cdots d_{n_j-1}$ if $3\leq k_j\leq n-1, 1\leq j\leq m$. a_f, a_g, a_h $(f=1,\cdots,n_p, g=n_b,\cdots,n_{b+(q-1)}, h=n_c,\cdots,n_{c+(s-1)})$ are given in (19).

Proof. If $\rho=|\psi\rangle\langle\psi|$ is an $\mathbf{m}$-separable pure state, where $|\psi\rangle\in H_1^{d_1}\otimes H_2^{d_2}\otimes\cdots\otimes H_n^{d_n}$. Without lose of generality, assume that

$$\begin{aligned}
|\psi\rangle=&|\phi_1\rangle\otimes\cdots\otimes|\phi_{n_p}\rangle\otimes|\phi_{n_b-1,n_b}\rangle\otimes\cdots\otimes\left|\phi_{n_{b+(q-1)}-1,n_{b+(q-1)}}\right\rangle\otimes\cdots\\
&\otimes\left|\phi_{n_c-(t-1),\cdots,n_c}\right\rangle\otimes\cdots\otimes\left|\phi_{n_{c+(s-1)}-(t-1),\cdots,n_{c+(s-1)}}\right\rangle.
\end{aligned}$$

We have

$$
\begin{aligned}
t_{i_1\cdots i_n}^{(1\cdots n)} =& Tr\left(|\psi\rangle\langle\psi|\lambda_{i_1}\otimes\cdots\otimes\lambda_{i_n}\right)\\
=& Tr\left(|\phi_1\rangle\langle\phi_1|\lambda_{i_1}\right)\cdots Tr\left(|\phi_{n_p}\rangle\langle\phi_{n_p}|\lambda_{i_{n_p}}\right)\cdot\\
& Tr\left(\left|\phi_{n_b-1,n_b}\right\rangle\left\langle\phi_{n_b-1,n_b}\right|\lambda_{i_{n_b-1}}\otimes\lambda_{i_{n_b}}\right)\cdots\\
& Tr\Bigg(\left|\phi_{n_{b+(q-1)}-1,n_{b+(q-1)}}\right\rangle\left\langle\phi_{n_{b+(q-1)}-1,n_{b+(q-1)}}\right|\\
& \lambda_{i_{n_{b+(q-1)}-1,n_{b+(q-1)}}}\otimes\lambda_{i_{n_{b+(q-1)}-1,n_{b+(q-1)}}}\Bigg)\cdots\\
& Tr\Bigg(\left|\phi_{n_c-(t-1),\cdots,n_c}\right\rangle\left\langle\phi_{n_c-(t-1),\cdots,n_c}\right|\\
& \lambda_{i_{n_c-(t-1)}}\otimes\cdots\otimes\lambda_{i_{n_c}}\Bigg)\cdots\\
& Tr\Bigg(\left|\phi_{n_{c+(s-1)}-(t-1),\cdots,n_{c+(s-1)}}\right\rangle\left\langle\phi_{n_{c+(s-1)}-(t-1),\cdots,n_{c+(s-1)}}\right|\\
& \lambda_{i_{n_{c+(s-1)}-(t-1)}}\otimes\cdots\otimes\lambda_{i_{n_{c+(s-1)}}}\Bigg)\\
=& t_{i_1}^{(1)}\cdots t_{i_{n_p}}^{(n_p)}\ t_{i_{n_b-1}i_{n_b}}^{(n_b-1,n_b)}\cdots t_{i_{n_{b+(q-1)}-1}i_{n_{b+(q-1)}}}^{\left(n_{b+(q-1)}-1,n_{b+(q-1)}\right)}\cdots\\
& t_{i_{n_c-(t-1)}\cdots i_{n_c}}^{(n_c-(t-1),\cdots,n_c)}\cdots t_{i_{n_{c+(s-1)}-(t-1)}\cdots i_{n_{c+(s-1)}}}^{\left(n_{c+(s-1)}-(t-1),\cdots,n_{c+(s-1)}\right)}.
\end{aligned}
\tag{21}
$$

Thus

$$
\begin{aligned}
\left\|\mathbf{T}^{(12\cdots n)}\right\|^2 =& \left\|\mathbf{T}^{(1)}\right\|^2\cdots\left\|\mathbf{T}^{(n_p)}\right\|^2\\
& \left\|\mathbf{T}^{(n_b-1,n_b)}\right\|^2\cdots\left\|\mathbf{T}^{(n_{b+(q-1)}-1,n_{b+(q-1)})}\right\|^2\cdots\\
& \left\|\mathbf{T}^{(n_c-(t-1),\cdots,n_c)}\right\|^2\cdots\left\|\mathbf{T}^{(n_{c+(s-1)}-(t-1),\cdots,n_{c+(s-1)})}\right\|^2\\
\leq& \prod_{f=1}^{n_p}a_f\prod_{g=n_b}^{n_{b+(q-1)}}a_g\cdots\prod_{h=n_c}^{n_{c+(s-1)}}a_h.
\end{aligned}
\tag{22}
$$

Then for any mixed state $\rho=\sum_k p_k|\psi_k\rangle\langle\psi_k|\in H_1^{d_1}\otimes H_2^{d_2}\otimes\cdots\otimes H_n^{d_n}$, where $\sum_k p_k=1$, $0<p_k\leq 1$,by the convexity of the Frobenius norm one derives

$$
\begin{aligned}
\left\|\mathbf{T}^{(12\cdots n)}(\rho)\right\|^2 =& \left\|\sum_k p_k\mathbf{T}^{(12\cdots n)}(|\psi_k\rangle\langle\psi_k|)\right\|^2\\
\leq& \sum_k p_k\left\|\mathbf{T}^{(12\cdots n)}(|\psi_k\rangle\langle\psi_k|)\right\|^2\\
\leq& \prod_{f=1}^{n_p}a_f\prod_{g=n_b}^{n_{b+(q-1)}}a_g\cdots\prod_{h=n_c}^{n_{c+(s-1)}}a_h.
\end{aligned}
\tag{23}
$$

■

Remark 3: The upper bounds of Theorem 2 is a generalization of Theorem 2 given in [15] and Theorem 7 given in [25], respectively. Set $d_1 = \cdots = d_n = d$, and $a_1 = \cdots = a_{n_p}$, $a_{n_b} = \cdots = a_{n_{b+(q-1)}} = a_2, \cdots, a_{n_c} = \cdots = a_{n_{c+(s-1)}} = a_t$. Then (20) gives rise to

$$\left\|\mathbf{T}^{(12\cdots n)}\right\|^2 \leq a_1^p a_2^q \cdots a_t^s. \tag{24}$$

which coincide with Theorem 2 in [15] and Theorem 7 in [25].

Remark 4: Let $\rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes H_3^{d_3} \otimes H_4^{d_4}$ be a four-partite quantum state. One has

$$\left\|\mathbf{T}^{(1234)}\right\|^2 \leq \begin{cases} \frac{2^4(d_1-1)}{d_1}\left\{1-\frac{d_2d_3+d_2d_4+d_3d_4-2d_4}{d_2d_3d_4^2}\right\}, & if\ \rho\ is\ (1,3)\ separable; \\ \frac{2^4\left(d_1^2-1\right)\left(d_3^2-1\right)}{d_1^2d_3^2}, & if\ \rho\ is\ (2,2)\ separable; \\ \frac{2^4(d_1-1)(d_2-1)\left(d_3^2-1\right)}{d_1d_2d_3^2}, & if\ \rho\ is\ (1,1,2)\ separable; \\ \frac{2^4\prod\limits_{i=1}^{4}(d_i-1)}{\prod\limits_{i=1}^{4}d_i}, & if\ \rho\ is\ (1,1,1,1)\ separable. \end{cases} \tag{25}$$

The following two examples show that the upper bounds in Theorem 2 are nontrivial and are tight.

Example 1: Consider the quantum state $\rho \in H_1^2 \otimes H_2^2 \otimes H_3^2 \otimes H_4^2 \otimes H_5^2$,

$$\rho = x(|\psi\rangle\langle\psi| + |\varphi\rangle\langle\varphi|) + \frac{1-2x}{32}I_{32}, \tag{26}$$

where $|\psi\rangle = \frac{1}{\sqrt{2}}(|00000\rangle + |11111\rangle)$, $|\varphi\rangle = \frac{1}{2}(|00001\rangle + |00010\rangle + |00100\rangle + |01000\rangle)$. Since $\left\|\mathbf{T}^{(12345)}\right\|^2 = \sum\limits_{i_1,\cdots,i_5=1}^{3}\left(t_{i_1\cdots i_5}^{(1\cdots 5)}\right)^2$, where $t_{i_1\cdots i_5}^{(1\cdots 5)} = Tr(\rho\lambda_{i_1}\otimes\cdots\otimes\lambda_{i_5})$ are the entries of $\mathbf{T}^{(12345)}$, we have $\left\|\mathbf{T}^{(12345)}\right\|^2 = 20x^2$. Thus for $\frac{3\sqrt{5}}{10} < x \leq \frac{\sqrt{15}}{5}$, ρ is not $(1,4)$ or $(1,2,2)$ separable. For $\frac{\sqrt{15}}{5} < x \leq 1$, ρ is not $(2,3)$ separable. For $\frac{\sqrt{5}}{5} < x \leq \frac{3\sqrt{5}}{10}$, ρ is not $(1,1,3)$ separable. For $\frac{\sqrt{15}}{10} < x \leq \frac{\sqrt{5}}{5}$, ρ is not $(1,1,1,2)$ separable. For $\frac{\sqrt{5}}{10} < x \leq \frac{\sqrt{15}}{10}$, ρ is not $(1,1,1,1,1)$ separable.

Example 2: Consider the quantum state $\rho \in H_1^2 \otimes H_2^3 \otimes H_3^4 \otimes H_4^5$,

$$\rho = x|\psi\rangle\langle\psi| + \frac{1-x}{120}I_{120}, \tag{27}$$

where $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1|0\rangle_2|0\rangle_3|4\rangle_4 + |1\rangle_1|0\rangle_2|0\rangle_3|0\rangle_4)$, $|0\rangle_1 := [1,0]^T$, $|1\rangle_1 := [0,1]^T$, $|0\rangle_2 := [1,0,0]^T$, $|0\rangle_3 := [1,0,0,0]^T$, $|0\rangle_4 := [1,0,0,0,0]^T$, $|4\rangle_4 := [0,0,0,0,1]^T$ (T is the transpose). Since $\left\|\mathbf{T}^{(1234)}\right\|^2 = \sum\limits_{k=1}^{4}\sum\limits_{i_k=1}^{d_k^2-1}\left(t_{i_1\cdots i_4}^{(1\cdots 4)}\right)^2$, where $t_{i_1\cdots i_4}^{(1\cdots 4)} = Tr(\rho\lambda_{i_1}\otimes\cdots\otimes\lambda_{i_4})$ are the entries of $\mathbf{T}^{(1234)}$, we can compute that $\left\|\mathbf{T}^{(1234)}\right\|^2 = 6x^2$. Thus for $\frac{\sqrt{263}}{15} < x \leq \frac{\sqrt{30}}{4}$, ρ is not $(1,3)$

separable. For $\frac{\sqrt{30}}{4} < x \leq 1$, ρ is not $(2,2)$ separable. For $\frac{\sqrt{30}}{6} < x \leq \frac{\sqrt{263}}{15}$, ρ is not $(1,1,2)$ separable. For $\frac{2\sqrt{30}}{15} < x \leq \frac{\sqrt{30}}{6}$, ρ is not $(1,1,1,1)$ separable.

From the above results, we are able to classify the entanglement of n-partite quantum states by using the norms of the Bloch vector $\left\|\mathbf{T}^{(12\cdots n)}\right\|^2$. The upper bounds of $\left\|\mathbf{T}^{(12\cdots n)}\right\|^2$ can be used to identify the $\mathbf{m}$-separable n-partite quantum states, which include the fully separable states and the genuine multipartite entangled states as special classes.

4 Conclusion

Classification and detection of quantum entanglement are basic and fundamental problems in theory of quantum entanglement. We have investigated the norms of the Bloch vectors for arbitrary n-partite quantum systems. Tight upper bounds of the norms have been derived, and used to derive tight upper bounds for entanglement measure defined by the norms of Bloch vectors. The upper bounds have a close relationship to the separability. Necessary conditions have been presented for $\mathbf{m}$-separable quantum states. With these upper bounds a complete classification of n-partite quantum states has been obtained. Our results may highlight further studies on the quantum entanglement.

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