

On classical capacity of Weyl channels

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Abstract

The additivity of minimal output entropy is proved for the Weyl channel obtained by the deformation of a q-c Weyl channel. The classical capacity of channel is calculated.

Keywords: quantum Weyl channel, classical capacity of a channel

1 Introduction

The quantum coding theorem proved independently by A.S. Holevo [1] and B. Schumacher, M.D. Westmoreland [2] posed the task of calculating the Holevo upper bound $\overline{C}(\Phi^{\otimes N})$ for a tensor product of N copies of quantum channel Φ because a classical capacity of Φ is given by the formula

$$C(\Phi) = \lim_{N \rightarrow +\infty} \frac{\overline{C}(\Phi^{\otimes N})}{N}.$$

The additivity conjecture asks whether the equality

$$\overline{C}(\Phi \otimes \Omega) = \overline{C}(\Phi) + \overline{C}(\Omega) \tag{1}$$

holds true for the fixed channel Φ and an arbitrary channel Ω . If the additivity property (1) takes place for Φ the classical capacity can be calculated as follows

$$C(\Phi) = \overline{C}(\Phi). \tag{2}$$

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The same level of interest has the additivity conjecture in the weak form asking whether

$$\overline{C}(\Phi^{\otimes N}) = N\overline{C}(\Phi)$$

takes place for a fixed channel Φ . The validity of this statement also leads to (2). The additivity conjecture for \overline{C} is closely related to the additivity conjecture for the minimal output entropy of a channel and the multiplicativity conjectures for trace norms of a channel [3]. At the moment, the additivity is proved for many significant cases [4–8] including the solution to the famous problem of Gaussian optimizers [9, 10]. On the other hand, there are channels for which the additivity conjecture doesn't hold true [11]. Recently the method of majorization was introduced to estimate the Holevo upper bound for Weyl channels [12]. In the present paper we prove the additivity conjecture for one subclass of Weyl channels that are "deformations" of q-c channels of [1]. Our method is based upon [12].

Throughout this paper we denote $\mathfrak{S}(H)$ the set of positive unit-trace operators (quantum states) in a Hilbert space H , I_H is the identity operator in H and $S(\rho) = -\text{Tr}(\rho \log \rho)$ is the von Neumann entropy of $\rho \in \mathfrak{S}(H)$. Quantum channel $\Phi : \mathfrak{S}(H) \rightarrow \mathfrak{S}(K)$ is a completely positive trace preserving map between the algebras of all bounded operators $B(H)$ and $B(K)$ in Hilbert spaces H and K respectively. Given two $\rho, \sigma \in \mathfrak{S}(H)$ for which $\text{supp} \rho \subset \text{supp} \sigma$ the quantum relative entropy is $S(\rho \parallel \sigma) = \text{Tr}(\rho \log \rho) - \text{Tr}(\rho \log \sigma)$. The property of non-increasing the relative entropy with respect to the action of a quantum channel Φ states [13]

$$S(\Phi(\rho) \parallel \Phi(\sigma)) \leq S(\rho \parallel \sigma)$$

for $\rho, \sigma \in \mathfrak{S}(H)$.

The Holevo upper bound for a quantum channel Φ is determined by the formula

$$\overline{C}(\Phi) = \sup_{\pi_j, \rho_j \in \mathfrak{S}(H)} (S(\sum_j \pi_j \Phi(\rho_j)) - \sum_j \pi_j S(\Phi(\rho_j))),$$

where the supremum is taken over all probability distributions (π_j) on the ensemble of states $\rho_j \in \mathfrak{S}(H)$.

2 Weyl channels

Here we use the techniques introduced in [14, 15] and developed in [16–19]. Fix an orthonormal basis $(e_j, j \in \mathbb{Z}_n)$ in a Hilbert space H with dimension $\dim H = n$, and consider two unitary operators in H defined by the formula

$$Ue_j = e^{\frac{2\pi i}{n}j}e_j, \quad Ve_j = e_{j+1}, \quad j \in \mathbb{Z}_n. \quad (3)$$

Formula (3) determines unitaries $W_{jk} = U^j V^k$ called Weyl operators satisfying the property

$$\sum_{j,k \in \mathbb{Z}_n} W_{jk} \rho W_{jk}^* = n I_H, \quad \rho \in \mathfrak{S}(H). \quad (4)$$

Quantum channels of the form

$$\Phi(\rho) = \sum_{j,k \in \mathbb{Z}_n} \pi_{jk} W_{jk} \rho W_{jk}^*, \quad \rho \in \mathfrak{S}(H), \quad (5)$$

where (π_{jk}) is a probability distribution, are said to be Weyl channels. Given a unitary representation λ of \mathbb{Z}_n in H and a probability distribution $(p_k, k \in \mathbb{Z}_n)$ a Weyl channel of the form

$$\Psi_\lambda(\rho) = \sum_{k \in \mathbb{Z}_n} p_k \lambda(k) \rho \lambda(k)^*, \quad \rho \in \mathfrak{S}(H),$$

is said to be a phase damping channel.

Let us fix a phase damping channel of the form

$$\Psi(\rho) = \sum_{k \in \mathbb{Z}_n} p_k V^k \rho V^{k*}, \quad \rho \in \mathfrak{S}(H),$$

where $p = (p_k, k \in \mathbb{Z}_n)$ is a probability distribution. Consider the quantum channel

$$\Phi(\rho) = \frac{1}{n} \sum_{j \in \mathbb{Z}_n} U^j \Psi(\rho) U^{j*} = \frac{1}{n} \sum_{j,k \in \mathbb{Z}_n} p_k U^j V^k \rho V^{k*} U^{j*}, \quad \rho \in \mathfrak{S}(H), \quad (6)$$

Formula (6) gives a general form of the Weyl channel invariant with respect to the action of the group $(U^j, j \in \mathbb{Z}_n)$ in the sense

$$U^j \Phi(\rho) U^{j*} = \Phi(\rho), \quad \rho \in \mathfrak{S}(H), \quad j \in \mathbb{Z}_n. \quad (7)$$

It follows from (7) that

$$\mathbb{E} \circ \Phi = \Phi, \quad (8)$$

where the expectation \mathbb{E} to the algebra of fixed elements with respect to the action of $(U^j, j \in \mathbb{Z}_n)$ is given by

$$\mathbb{E}(\rho) = \frac{1}{n} \sum_{j \in \mathbb{Z}_n} U^j \rho U^{j*}, \quad \rho \in \mathfrak{S}(H).$$

Put

$$\Xi_k(\rho) = \frac{1}{n} \sum_{j \in \mathbb{Z}_n} U^j V^k \rho V^{k*} U^{j*}, \quad \rho \in \mathfrak{S}(H), \quad j \in \mathbb{Z}_n,$$

then (6) can be represented as

$$\Phi(\rho) = \sum_{k \in \mathbb{Z}_n} p_k \Xi_k(\rho).$$

The property (8) shows that Φ is a q-c channel and the additivity of \overline{C} was shown in [1]. We place the following statement here to calculate the exact value of a classical capacity.

Proposition 1. *Given a quantum channel $\Omega : \mathfrak{S}(K) \rightarrow \mathfrak{S}(K)$ and a pure state $|\xi\rangle \langle \xi| \in \mathfrak{S}(H \otimes K)$*

$$\inf_{\rho \in \mathfrak{S}(H \otimes K)} S(\Phi \otimes \Omega(|\xi\rangle \langle \xi|)) \geq - \sum_{k \in \mathbb{Z}_n} p_k \log p_k + S(\Omega(\text{Tr}_H(|\xi\rangle \langle \xi|))).$$

Proof.

Let us define a c-q channel $\Upsilon : \mathfrak{S}(H) \rightarrow \mathfrak{S}(H \otimes K)$ by the formula

$$\Upsilon(\rho) = \sum_{k \in \mathbb{Z}_n} \langle e_k, \rho e_k \rangle (\Xi_k \otimes \Omega)(|\xi\rangle \langle \xi|), \quad \rho \in \mathfrak{S}(H).$$

Put

$$\rho = \sum_{k \in \mathbb{Z}_n} p_k |e_k\rangle \langle e_k|, \quad \sigma = \frac{1}{n} I_H. \quad (9)$$

Applying the property of non-increasing the quantum relative entropy with respect to the action of quantum channel we obtain

$$S(\Upsilon(\rho) \parallel \Upsilon(\sigma)) \leq S(\rho \parallel \sigma). \quad (10)$$

It follows from (4) that

$$\sum_{k \in \mathbb{Z}_n} \Xi_k(\rho) = I_H, \quad \rho \in \mathfrak{S}(H).$$

Hence

$$\sum_{k \in \mathbb{Z}_n} (\Xi_k \otimes \Omega)(|\xi\rangle \langle \xi|) = I_H \otimes \Omega(\text{Tr}_H(|\xi\rangle \langle \xi|))$$

and

$$\Upsilon(\sigma) = \frac{1}{n} I_H \otimes \Omega(\text{Tr}_H(|\xi\rangle \langle \xi|)). \quad (11)$$

Substituting (9)–(11) to (10) we get

$$-S((\Phi \otimes \Omega)(|\xi\rangle \langle \xi|)) - Tr \left((\Phi \otimes \Omega)(|\xi\rangle \langle \xi|) \log \left(\frac{1}{n} I_H \otimes \Omega(Tr_H(|\xi\rangle \langle \xi|)) \right) \right) \leq \sum_{k \in \mathbb{Z}_n} p_k \log p_k - Tr(\rho \log \sigma).$$

Taking into account that

$$Tr \left((\Phi \otimes \Omega)(|\xi\rangle \langle \xi|) \log \left(\frac{1}{n} I_H \otimes \Omega(Tr_H(|\xi\rangle \langle \xi|)) \right) \right) = -\log n - S(\Omega(Tr_H(|\xi\rangle \langle \xi|)))$$

and

$$Tr(\rho \log \sigma) = -\log n$$

we obtain the result.

□

Corollary 1. *Given a quantum channel $\Omega : \mathfrak{S}(K) \rightarrow \mathfrak{S}(K)$ and the q -c Weyl channel (6) the following equality holds*

$$\inf_{\rho \in \mathfrak{S}(H \otimes K)} S((\Phi \otimes \Omega)(\rho)) = \inf_{\rho \in \mathfrak{S}(H)} S(\Phi(\rho)) + \inf_{\rho \in \mathfrak{S}(K)} S(\Omega(\rho)).$$

Proof.

Notice that

$$S(\Phi(|e_j\rangle \langle e_j|)) = - \sum_{k \in \mathbb{Z}_n} p_k \log p_k \geq \inf_{\rho \in \mathfrak{S}(H)} S(\Phi(\rho))$$

for any $j \in \mathbb{Z}_n$. It follows from Proposition 1 that

$$\inf_{\rho \in \mathfrak{S}(H \otimes K)} S((\Phi \otimes \Omega)(\rho)) \geq \inf_{\rho \in \mathfrak{S}(H)} S(\Phi(\rho)) + \inf_{\rho \in \mathfrak{S}(K)} S(\Omega(\rho)). \quad (12)$$

On the other hand, the right side in (12) can not be less than the left hand side. Hence,

$$\inf_{\rho \in \mathfrak{S}(H)} S(\Phi(\rho)) = - \sum_{k \in \mathbb{Z}_n} p_k \log p_k$$

and we have the equality in (12).

□

Corollary 2. *The classical capacity of the q -c Weyl channel (6) is given by the formula*

$$C(\Phi) = \log(n) + \sum_{k \in \mathbb{Z}_n} p_k \log p_k.$$

Proof.

The statement can be derived from the fact that

$$\overline{C}(\Phi^{\otimes N}) = N \log n - \inf_{\rho \in \mathfrak{S}(H^{\otimes N})} S(\Phi^{\otimes N}(\rho))$$

for covariant channels [20]. It follows from Corollary 1 that

$$\inf_{\rho \in \mathfrak{S}(H^{\otimes N})} S(\Phi^{\otimes N}(\rho)) = N \inf_{\rho \in \mathfrak{S}(H)} S(\Phi(\rho)).$$

In the proof of Corollary 1 we have shown that

$$\inf_{\rho \in \mathfrak{S}(H)} S(\Phi(\rho)) = - \sum_{k \in \mathbb{Z}_n} p_k \log p_k. \quad (13)$$

□

3 Majorization

Let \mathfrak{J} be the index set and $|\mathfrak{J}| = d < +\infty$. Given a probability distribution $\lambda = (\lambda_J, J \in \mathfrak{J})$ we denote $\lambda^\downarrow = (\lambda_j^\downarrow, 1 \leq j \leq d)$ the probability distribution obtained by sorting λ in the decreasing order,

$$\lambda_1^\downarrow \geq \lambda_2^\downarrow \geq \cdots \geq \lambda_d^\downarrow.$$

Consider two probability distribution $\lambda = (\lambda_J, J \in \mathfrak{J})$ and $\mu = (\mu_J, J \in \mathfrak{J})$. We shall say that λ majorizes μ and write

$$\mu \prec \lambda$$

iff

$$\sum_{j=1}^k \mu_j^\downarrow \leq \sum_{j=1}^k \lambda_j^\downarrow, \quad 1 \leq k \leq d.$$

Let H_d be a Hilbert space with $\dim H_d = d$. Denote $B(H_d)$ the algebra of all bounded operators in H_d . The following statement can be derived from [12] (see Theorem 2).

Proposition 2. *Let $0 \leq X_J \leq I$, $J \in \mathfrak{J}$, $|\mathfrak{J}| = d^2$, be a set of positive operators in $B(H_d)$ such that*

$$\sum_{J \in \mathfrak{J}} X_J = dI_{H_d}.$$

Then, given a probability distribution $\pi = (\pi_J, J \in \mathfrak{J})$ the eigenvalues $\lambda = (\lambda_j)_{j=1}^d$ of the positive operator

$$A = \sum_{J \in \mathfrak{J}} \pi_J X_J$$

sorted in the decreasing order $\lambda \equiv \lambda^\downarrow$ satisfy the relation

$$\lambda \prec p,$$

where

$$p_j = \sum_{m=1+(j-1)d}^{d+(j-1)d} \pi_m^\downarrow, \quad 1 \leq j \leq d.$$

Proof.

Let $(e_j)_{j=1}^d$ be the unit eigenvectors corresponding to the eigenvalues $(\lambda_j)_{j=1}^d$. Then,

$$\sum_{j=1}^k \lambda_j = \sum_{j=1}^k \langle e_j, A e_j \rangle = \sum_{j=1}^k \sum_{J \in \mathfrak{J}} \pi_J \langle e_j, X_J e_j \rangle \leq \sum_{j=1}^k p_j, \quad 1 \leq k \leq d.$$

□

Corollary 3. *The eigenvalues λ of the positive operator A in Proposition 2 possess the property*

$$-\sum_{j=1}^d \lambda_j \log \lambda_j \geq -\sum_{j=1}^d p_j \log p_j.$$

Proof.

Since λ majorizes μ due to Proposition 2, we get the result [21].

□

4 Deformation of q-c Weyl channels

Let us come back to Weyl channels (5).

Definition. *Suppose that a probability distribution $(\pi_{jk}, j, k \in \mathbb{Z}_n)$ satisfies the relation*

$$\pi_{00} \geq \pi_{10} \geq \cdots \geq \pi_{n-10} \geq \pi_{01} \geq \cdots \pi_{n-11} \geq \pi_{02} \geq \cdots \geq \pi_{n-1n-1}. \quad (14)$$

Put

$$p_k = \sum_{j \in \mathbb{Z}_n} \pi_{jk}, \quad k \in \mathbb{Z}_n. \quad (15)$$

Then (5) is said to be the Weyl channel obtained by the deformation of q -c channel (6).

Theorem. The Weyl channel Φ obtained by the deformation of q -c channel satisfies the property

$$\inf_{\rho \in \mathfrak{S}(H^{\otimes N})} S(\Phi^{\otimes N}(\rho)) = -N \sum_{k=1}^n p_j \log p_j.$$

Proof.

Denote \mathfrak{J} the index set $(\mathbb{Z}_n \times \mathbb{Z}_n)^{\times N}$ consisting of collections $(j_1, k_1), \dots, (j_N, k_N)$, $j_s, k_s \in \mathbb{Z}_n$. Let us consider the probability distribution $\Pi = (\Pi_J, J \in \mathfrak{J})$ and a set of positive operators $(X_J, J \in \mathfrak{J})$ defined by the formula

$$\Pi_J = \prod_{s=1}^N \pi_{j_s k_s},$$

$$X_J = \left(\bigotimes_{s=1}^N W_{j_s k_s} \right) \rho \left(\bigotimes_{s=1}^N W_{j_s k_s}^* \right), \quad J \in \mathfrak{J},$$

where ρ is a fixed state in $\mathfrak{S}(H^{\otimes N})$. Then, the conditions of Proposition 2 is satisfied for (Π_J) , (X_J) and $d = n^N$. Applying Corollary 3 we obtain

$$S(\Phi(\rho)) \geq -N \sum_{j=1}^n p_j \log p_j. \quad (16)$$

The equality in (16) is achieved for any

$$\rho = |e\rangle \langle e|,$$

where

$$e = \bigotimes_{s=1}^N e_{j_s}, \quad j_s \in \mathbb{Z}_n.$$

□

Corollary 4. The classical capacity of the Weyl channel obtained by the deformation of (6) is given by the formula

$$C(\Phi) = \log(n) + \sum_{k \in \mathbb{Z}_n} p_k \log p_k.$$

Proof.

The statement can be derived from the fact that

$$\overline{C}(\Phi^{\otimes N}) = N \log n - \inf_{\rho \in \mathfrak{S}(H^{\otimes N})} S(\Phi^{\otimes N}(\rho))$$

for covariant channels [20]. It follows from Theorem that

$$\inf_{\rho \in \mathfrak{S}(H^{\otimes N})} S(\Phi^{\otimes N}(\rho)) = N \inf_{\rho \in \mathfrak{S}(H)} S(\Phi(\rho)) = -N \sum_{j=1}^N p_j \log p_j.$$

□

4.1 Example: qutrits

Because the qubit case $\dim H = 2$ is completely parsed [4] a simplest example of the introduced techniques can be given for qutrits, $\dim H = 3$. Let us define two unitary operators U and V satisfying (3)

$$Ue_0 = e_0, \quad Ue_1 = e^{i\frac{2\pi}{3}}e_1, \quad Ue_2 = e^{i\frac{4\pi}{3}}e_2,$$

$$Ve_0 = e_1, \quad Ve_1 = e_2, \quad Ve_2 = e_0.$$

Then, consider the expectation (8)

$$\mathbb{E}(x) = \frac{1}{3} \sum_{j=0}^2 U^j x U^{j*}, \quad x \in B(H).$$

Taking a probability distribution $\{p_0, p_1, p_2\}$ we can define a qc Weyl channel by the formula

$$\Phi_{qc}(\rho) = \mathbb{E} \circ \sum_{k=0}^2 p_k V^k \rho V^{k*}, \quad \rho \in \mathfrak{S}(H). \quad (17)$$

It follows from Corollary 1 and Corollary 2 that

$$\inf_{\rho \in \mathfrak{S}(H \otimes K)} S((\Phi_{qc} \otimes \Omega)(\rho)) = \inf_{\rho \in \mathfrak{S}(H)} S(\Phi_{qc}(\rho)) + \inf_{\rho \in \mathfrak{S}(K)} S(\Omega(\rho))$$

for any quantum channel $\Omega : \mathfrak{S}(K) \rightarrow \mathfrak{S}(K)$ and the classical capacity is equal to

$$C(\Phi_{qc}) = n + \sum_{k=0}^2 p_k \log p_k.$$

Suppose that $p_0 \geq p_1 \geq p_2$ and one can pick up positive numbers π_{jk} , $0 \leq j, k \leq 2$, satisfying the relations

$$\pi_{00} \geq \pi_{10} \geq \pi_{20} \geq \pi_{01} \geq \pi_{11} \geq \pi_{21} \geq \pi_{02} \geq \pi_{12} \geq \pi_{22},$$

$$p_k = \pi_{0k} + \pi_{1k} + \pi_{2k}, \quad 0 \leq k \leq 2.$$

Then,

$$\Phi(\rho) = \sum_{j,k=0}^2 \pi_{jk} U^j V^k \rho V^{k*} U^{j*}, \quad \rho \in \mathfrak{S}(H),$$

is the Weyl channel obtained by the deformation of (17). Applying Corollary 4 we obtain for a classical capacity

$$C(\Phi) = \log(3) + p_0 \log p_0 + p_1 \log p_1 + p_2 \log p_2.$$

As a concrete example one can take

$$p_0 = \frac{1}{2}, \quad p_1 = \frac{1}{3}, \quad p_2 = \frac{1}{6}.$$

In the case, one of possible deformations is given by

$$\begin{aligned} \pi_{00} &= \frac{1}{4}, \quad \pi_{10} = \frac{1}{8}, \quad \pi_{20} = \frac{1}{8}, \\ \pi_{01} &= \frac{1}{8}, \quad \pi_{11} = \frac{1}{8}, \quad \pi_{21} = \frac{1}{12}, \\ \pi_{02} &= \frac{1}{12}, \quad \pi_{12} = \frac{1}{24}, \quad \pi_{22} = \frac{1}{24}. \end{aligned}$$

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