# $H_2$ -reducible matrices in six-dimensional mutually unbiased bases

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# Abstract

Finding four six-dimensional mutually unbiased bases (MUBs) containing the identity matrix is a long-standing open problem in quantum information. We show that if they exist, then the  $H_2$ -reducible matrix in the four MUBs has exactly nine  $2 \times 2$  Hadamard submatrices. We apply our result to exclude from the four MUBs some known CHMs, such as symmetric  $H_2$ -reducible matrix, the Hermitian matrix, Dita family, Bjorck's circulant matrix, and Szollosi family. Our results represent the latest progress on the existence of six-dimensional MUBs.

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#### Contents

I.	Introduction	2
II.	Preliminaries	3
III.	The $H_2$ -reducible matrix in an MUB trio	6
IV.	application	9
v.	Conclusions	12
	Acknowledgements	12
	References	13

#### I. INTRODUCTION

In quantum physics, mutually unbiased bases (MUBs) present a basic notion of describing physical observables [1]. MUBs have been extensively useful in quantum tomography, discrete Wigner functions [2, 3] and King problem [4]. In particular, MUBs minimize the uncertainty of estimating density matrices and may conceal security in quantum key distribution protocols. It's been proven that the complete set of d-dimensional MUBs has d+1 MUBs. The main problem on MUBs is to prove whether MUBs in the d-dimensional Hilbert space  $\mathbb{C}^d$  is complete for any integer d. It has been proven true when d is prime power. However it is widely conjectured that four MUBs may not exist in  $\mathbb{C}^6$ . Much effort has been devoted to the problem in the past decades [5–25].

In this paper we investigate this conjecture in terms of the so-called  $H_2$ -reducible matrix [26, 27]. The latter is a  $6 \times 6$  complex Hadamard matrix (CHM) containing a  $2 \times 2$  Hadamard submatrices, which is proportional to a  $2 \times 2$  unitary matrix. The  $H_2$ -reducible matrix represents a large subset of CHMs covering many known affine CHMs say the Fourier matrix and non-affine CHMs such as Hermitean family. So the  $H_2$ -reducible matrix plays an important

role whose existence in an MUB trio is worth being studied. In [28], we have investigated the four-MUB conjecture in terms of the  $H_2$ -reducible matrix and concluded that the  $H_2$ -reducible matrix belonging to an MUB trio has exactly nine or eighteen  $2 \times 2$  Hadamard submatrices. In this paper,we concentrate on the problem further. We review preliminary results on CHMs and  $H_2$ -reducible matrices in Lemma 1. Next we review preliminary results on MUB trio in Lemma 3 and 4. In Theorem 7, we show that if an  $H_2$ -reducible matrix belongs to an MUB trio, then the matrix has exactly nine  $2 \times 2$  Hadamard submatrices. This is the main result of this paper supported by the preliminary Lemma 2 and 5. We furthermore apply our result to exclude some known CHMs as members of MUB trio. They include the affine CHMs say the Fourier matrix, Dita family, Bjorck's circulant matrix, and non-affine CHMs such as Hermitean and Szollosi family. Our results present the latest progress on the existence of four six-dimensional MUBs. They are also related to other topics in quantum information, e.g., unitary matrices, tensor rank and unextendible product basis [29–31].

The rest of this paper is structured as follows. In Sec. II we construct the notion of CHMs, equivalence and complex equivalence of matrices, as well as the parametrization of  $H_2$ -reducible matrices. In Sec. III we introduce the main result of this paper. In Sec. IV we apply our result to exclude some known CHMs as members of MUB trio. We conclude in Sec. V.

## II. PRELIMINARIES

In this section we introduce the notations and facts used in this paper. We refer to the  $n \times n$  complex Hadamard matrix (CHM)  $H_n = [u_{ij}]_{i,j=1,\dots,n}$  as a matrix with orthogonal row vectors and entries of modulus one. That is,  $H_n^{\dagger}H_n = nI_n$  and  $|u_{ij}| = 1$ . To find out the connection between different CHMs, we define the equivalence and complex equivalence. We refer to the monomial unitary matrix as a unitary matrix each of whose row and columns has exactly one nonzero entry, and it has modulus one. Two  $n \times n$  matrices U and V are complex equivalent when U = PVQ where P,Q are both monomial unitary matrices. If P,Q are both permutation matrices then we say that U,V are equivalent. Evidently if U,V are equivalent then they are complex equivalent, and the converse fails. The number of real entries of a CHM may be changed under complex equivalence, while it is unchanged under equivalence. For example, it is straightforward to show that any  $n \times n$  CHM is complex

equivalent to a CHM containing at least 2n + 1 entry one. They are in the first column and row of the CHM.

In quantum physics, a pure state is described by a unit vector in linear algebra. Two states in  $\mathbb{C}^d$  are MU when their inner product is of modulus  $\frac{1}{\sqrt{d}}$ . Two MUBs are orthonormal basis are MU when their elements are pairwise MU. For convenience we refer to a unitary matrix as an MUB consisting of the column vectors of the unitary matrix. For d = 6, it has been a long-standing open problem whether four MUBs  $I_6, V, W, X$  exist. If it exists then we refer to the three unitary matrices V, W, X as an MUB trio.

In the following we review Theorem 11 of the paper [26]. We shall use it in the proof of Lemma 5 and Theorem 7. The result parameterizes every  $H_2$ -reducible matrix.

**Lemma 1** (i) The  $H_2$ -reducible CHM is complex equivalent to the CHM H in [26, Theorem 11], namely

$$H = \begin{bmatrix} F_2 & Z_1 & Z_2 \\ Z_3 & \frac{1}{2} Z_3 A Z_1 & \frac{1}{2} Z_3 B Z_2 \\ Z_4 & \frac{1}{2} Z_4 B Z_1 & \frac{1}{2} Z_4 A Z_2 \end{bmatrix}$$

$$= \begin{bmatrix} I_2 & 0 & 0 \\ 0 & Z_3 & 0 \\ 0 & 0 & Z_4 \end{bmatrix} \cdot \begin{bmatrix} F_2 & I_2 & I_2 \\ I_2 & \frac{1}{2} A & \frac{1}{2} B \\ I_2 & \frac{1}{2} B & \frac{1}{2} A \end{bmatrix} \cdot \begin{bmatrix} I_2 & 0 & 0 \\ 0 & Z_1 & 0 \\ 0 & 0 & Z_2 \end{bmatrix}, \qquad (1)$$

where

$$F_{2} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad Z_{1} = \begin{bmatrix} 1 & 1 \\ z_{1} & -z_{1} \end{bmatrix}, \quad Z_{2} = \begin{bmatrix} 1 & 1 \\ z_{2} & -z_{2} \end{bmatrix},$$

$$Z_{3} = \begin{bmatrix} 1 & z_{3} \\ 1 & -z_{3} \end{bmatrix}, \quad Z_{4} = \begin{bmatrix} 1 & z_{4} \\ 1 & -z_{4} \end{bmatrix},$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^{*} & -A_{11}^{**} \end{bmatrix}, \quad B = \begin{bmatrix} -1 - A_{11} & -1 - A_{12} \\ -1 - A_{12}^{**} & 1 + A_{11}^{**} \end{bmatrix},$$

$$A_{11} = -\frac{1}{2} + i \frac{\sqrt{3}}{2} (\cos \theta + e^{-i\phi} \sin \theta),$$

$$A_{12} = -\frac{1}{2} + i \frac{\sqrt{3}}{2} (\cos \theta + e^{-i\phi} \sin \theta),$$

$$B_{11} = -\frac{1}{2} - i \frac{\sqrt{3}}{2} (\cos \theta + e^{-i\phi} \sin \theta),$$

$$B_{12} = -\frac{1}{2} - i \frac{\sqrt{3}}{2} (\cos \theta + e^{i\phi} \sin \theta),$$

$$\theta, \phi \in [0, 2\pi), \quad |z_{j}| = 1,$$

$$z_{3}^{2} = \mathcal{M}_{A}(z_{1}^{2}) = \mathcal{M}_{B}(z_{2}^{2}),$$

$$z_{4}^{2} = \mathcal{M}_{A}(z_{2}^{2}) = \mathcal{M}_{B}(z_{1}^{2}),$$

$$\mathcal{M}_{A}(z) = \frac{A_{12}^{2}z - A_{11}^{2}}{(A_{11}^{2})^{*}z - (A_{12}^{2})^{*}},$$

$$\mathcal{M}_{B}(z) = \frac{B_{12}^{2}z - B_{11}^{2}}{(B_{11}^{2})^{*}z - (B_{12}^{2})^{*}}.$$
(2)

(ii) Suppose 
$$M = \begin{bmatrix} F_2 & Z_1 & Z_2 \\ Z_3 & a & b \\ Z_4 & c & d \end{bmatrix}$$
 is an  $H_2$ -reducible matrix where  $F_2, Z_1, Z_2, Z_3$  and  $Z_4$  are

given in (2). Then  $\bar{M}$  is the same as the matrix H in (1) satisfying (2). In particular, a, b, c, d are  $2 \times 2$  Hadamard submatrices described in (2).

We review Theorem 12 in the recent paper [28] on  $H_2$ -reducible matrices and MUBs. This is the main result of [28]. We shall use it in the proof of Theorem 7 as the main result of this paper.

**Lemma 2** If an  $H_2$ -reducible matrix belongs to an MUB trio, then the matrix has exactly nine or eighteen  $2 \times 2$  Hadamard submatrices.

Next we review a fact from [23, Lemma 11]. It gives the necessary condition by which a  $6 \times 6$  CHM is a member of some MUB trio. This is used in the proof of Lemma 5.

**Lemma 3** The CHM in an MUB trio contains no a real  $2 \times 3$  or  $3 \times 2$  submatrix up to complex equivalence.

Finally we review a fact on complex numbers used in the proof of Theorem 7.

**Lemma 4** (i) Suppose a + b + c = 0 with complex numbers a, b, c of modulus one. Then  $(a, b, c) \propto (1, \omega, \omega^2)$  or  $(1, \omega^2, \omega)$  with  $\omega = e^{\frac{2\pi i}{3}}$ .

(ii) Suppose a+b+c+d=0 with complex numbers a,b,c,d of modulus one. Then a=-b,-c or -d.

## III. THE $H_2$ -REDUCIBLE MATRIX IN AN MUB TRIO

In this section we show that the  $H_2$ -reducible matrix in an MUB trio has exactly nine  $2 \times 2$  Hadamard submatrices. This is presented in Theorem 7. For this purpose we construct a preliminary lemma.

**Lemma 5** Suppose M is a  $6 \times 6$  CHM containing more than nine  $2 \times 2$  Hadamard submatrices, and M belongs to an MUB trio. Then up to complex equivalence we may assume that M is the matrix in (1) with the entry (3,3) of M being -1.

**Proof.** Evidently M is an  $H_2$ -reducible matrix. It follows from Lemma 1 (i) that there exist two  $6 \times 6$  monomial unitary matrices P, Q such that

$$H := [h_{ij}] := H(\theta, \phi, z_1, z_2, z_3, z_4) = PMQ$$

$$= \begin{bmatrix} F_2 & Z_1 & Z_2 \\ Z_3 & \frac{1}{2}Z_3AZ_1 & \frac{1}{2}Z_3BZ_2 \\ Z_4 & \frac{1}{2}Z_4BZ_1 & \frac{1}{2}Z_4AZ_2 \end{bmatrix}$$

$$= \begin{bmatrix} I_2 & 0 & 0 \\ 0 & Z_3 & 0 \\ 0 & 0 & Z_4 \end{bmatrix} \cdot \begin{bmatrix} F_2 & I_2 & I_2 \\ I_2 & \frac{1}{2}A & \frac{1}{2}B \\ I_2 & \frac{1}{2}B & \frac{1}{2}A \end{bmatrix} \cdot \begin{bmatrix} I_2 & 0 & 0 \\ 0 & Z_1 & 0 \\ 0 & 0 & Z_2 \end{bmatrix},$$
(3)

where  $F_2, Z_1, ..., Z_4, A, B$  containing the parameters  $\theta, \phi, z_1, z_2, z_3, z_4$  with  $\theta, \phi \in [0, \pi)$ , and  $|z_j| = 1$  are given in (2). Since H and M are complex equivalent, H still has more than

nine  $2 \times 2$  Hadamard submatrices. Using (3), we obtain that  $h_{ij} = -1$  for some (i, j) such that  $i, j \neq 1$  and  $(i, j) \neq (2, 2)$ . If one of  $z_1, ..., z_4$  is -1 then H has a  $2 \times 4$  or  $4 \times 2$  real submatrix. It is a contradiction with Lemma 3, because H belongs to an MUB trio. Hence  $h_{ij} = -1$  with  $(i, j) \in \{3, 4, 5, 6\} \times \{3, 4, 5, 6\}$ .

We present claim one as follows. If we assume that H in (3) with  $h_{33}, h_{35}, h_{53}$  or  $h_{55} = -1$  does not belong to any MUB trio, then neither does H with  $h_{ij} = -1$  and  $(i,j) \in \{3,4,5,6\} \times \{3,4,5,6\}$ . To prove the claim, we consider  $H(\theta,\phi,z_1,z_2,z_3,z_4)$  with  $h_{43} = -1$ . Let the permutation matrix  $R = I_2 \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus I_2$ . Then (3) implies that  $RH(\theta,\phi,z_1,z_2,z_3,z_4) = [h'_{ij}]$  with  $h'_{33} = -1$ . Using the assumption, we obtain that  $H(\theta,\phi,z_1,z_2,-z_3,z_4) = [h'_{ij}]$  does not belong to any MUB trio. Neither does  $H(\theta,\phi,z_1,z_2,z_3,z_4)$  with  $h_{43} = -1$ . One can similarly show that  $H(\theta,\phi,z_1,z_2,z_3,z_4)$  with  $h_{34},h_{44},...$  or  $h_{66} = -1$ . We have proven claim one.

We present claim two as follows. If we assume that H in (3) with  $h_{33} = -1$  does not belong to any MUB trio, then neither does H with  $h_{35}, h_{53}, h_{55} = -1$ . To prove the claim, we consider  $H(\theta, \phi, z_1, z_2, z_3, z_4)$  with  $h_{55} = -1$ . Let the permutation matrix  $R_1 = I_2 \oplus$ 

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$
 Then (3) implies that  $R_1H(\theta, \phi, z_1, z_2, z_3, z_4)R_1 = H(\theta, \phi, z_2, z_1, z_4, z_3) = [h_{ij}'']$ 

with  $h_{33}'' = -1$ . Using the assumption, we obtain that  $[h_{ij}'']$  does not belong to any MUB trio. We have proven claim two. One can similarly show that H with  $h_{35} = -1$  belongs to an MUB trio if and only if so does H with  $h_{53} = -1$ . To prove the claim, if suffices to show that  $H(\theta, \phi, z_1, z_2, z_3, z_4)$  with  $h_{53} = -1$  does not belong to any MUB trio. Indeed, (3) implies that  $R_1H(\theta, \phi, z_1, z_2, z_3, z_4) = H(\theta + \pi, \phi, z_1, z_2, z_4, z_3) = [h_{ij}''']$  with  $h_{33}''' = -1$ . Using the assumption, we obtain that  $[h_{ij}''']$  does not belong to any MUB trio. We have proven claim two.

By combining claim one and two, we have proven the assertion.

From the proof of Lemma 5, one can similarly obtain the following observation.

Corollary 6 Suppose M is an  $H_2$ -reducible matrix in (1). If M with  $h_{33} = x$  does not belong to any MUB trio, then neither does M with  $h_{ij} = x$  and (i, j) is one of (3, 4), (3, 5), ..., (6, 6).

Now we are in a position to prove the main result of this section.

**Theorem 7** If an  $H_2$ -reducible matrix belongs to an MUB trio, then the matrix has exactly nine  $2 \times 2$  Hadamard submatrices.

In other words, the member of MUB trio has no CHM containing the  $3 \times 3$  submatrix  $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & * \\ 1 & * & -1 \end{bmatrix}$  up to complex equivalence.

**Proof.** Suppose  $M = [m_{ij}]$  is an  $H_2$ -reducible matrix belonging to an MUB trio. It follows from Lemma 2 that M has exactly nine or eighteen  $2 \times 2$  Hadamard submatrices. We shall exclude the latter by contradiction, and the assertion follows.

Assume that M has exactly eighteen  $2 \times 2$  Hadamard submatrices. Using Lemma 5, we may assume that M is the matrix in (1) with  $m_{33} = -1$ . Applying Lemma 4 (ii) to row 1, 3 of M and column 5, 6 of M, we obtain that one of  $\begin{bmatrix} m_{12} & m_{15} \\ m_{32} & m_{35} \end{bmatrix}$  and  $\begin{bmatrix} m_{12} & m_{16} \\ m_{32} & m_{36} \end{bmatrix}$  is an Hadamard submatrix. Let the permutation matrix  $R_1 = I_2 \oplus I_2 \oplus I_2 \oplus I_3 \oplus I_4 \oplus I_5 \oplus I_5$ . Using (1), one can show that  $MR_1$  is still an  $H_2$ -reducible matrix containing exactly eighteen  $2 \times 2$  Hadamard submatrices in an MUB trio. For convenience we still name  $MR_1$  as  $M = [m_{ij}]$ . So  $\begin{bmatrix} m_{12} & m_{15} \\ m_{32} & m_{35} \end{bmatrix}$  is a  $2 \times 2$  Hadamard submatrix. Similarly by studying column 1, 3, we may assume that  $\begin{bmatrix} m_{21} & m_{23} \\ m_{51} & m_{53} \end{bmatrix}$  is a  $2 \times 2$  Hadamard submatrix. Using Lemma 1, we can determine the eighteen  $2 \times 2$  Hadamard submatrices in M. We obtain four equations  $m_{33} = -1$ ,  $m_{35} = -z_3$ ,  $m_{53} = -z_1$ , and  $m_{55} = z_2 z_4$ . Using (1), one can derive the expressions of  $M = [m_{ij}]$ . By solving the four equations, we obtain  $m_{66} = -1$ . It means that M has more than eighteen  $2 \times 2$  Hadamard submatrices. It is a contradiction with the assumption that M has exactly eighteen  $2 \times 2$  Hadamard Hadamard submatrices. So we have excluded the option that M has exactly eighteen  $2 \times 2$  Hadamard submatrices. We have proven the assertion.

In the next section we introduce the application of Theorem 7.

#### IV. APPLICATION

In this section, we will exclude some known CHMs as members of MUB trio by using Theorem 7. First of all we introduce Theorem 8 and Theorem 9, which exculde symmetric  $H_2$ -reducible matrix and the Hermitian matrix respectively from MUB trio.

**Theorem 8** The CHM in any MUB trio is not a symmetric  $H_2$ -reducible matrix.

**Proof.** Suppose that H is a symmetric  $H_2$ -reducible matrix in an MUB trio. The first row of H is  $(h_1, h_2, h_3, h_4, h_5, h_6)$ . Let  $D = \operatorname{diag}(h_1^{-\frac{1}{2}}, h_1^{-\frac{1}{2}}h_2^{-1}, h_1^{-\frac{1}{2}}h_3^{-1}, h_1^{-\frac{1}{2}}h_4^{-1}, h_1^{-\frac{1}{2}}h_5^{-1}, h_1^{-\frac{1}{2}}h_6^{-1})$ . Then D is a monomial unitary matrix, and H' = DHD is a symmetric  $H_2$ -reducible matrix whose first row and column consist of ones. According to the Corollary 3 of [26], H' has at least one element equaling to -1. If the element -1 does not belong to the diagonal of H', then H' contains at least two elements equaling to -1 by the symmetry of H'. Hence H' does not belong to any MUB trio by Theorem 7. So the element -1 belongs to the diagonal of H'. From the proof of Lemma 5, we know that there is a permutation matrix P s.t. H'' = PH'P, and the element of the third row and the third column of H'' is -1, meanwhile H'' is a symmetric  $H_2$ -reducible matrix which has the form in (1). We assume that

$$H'' = \begin{bmatrix} F_2 & Z_1 & Z_2 \\ Z_3 & \frac{1}{2}Z_3AZ_1 & \frac{1}{2}Z_3BZ_2 \\ Z_4 & \frac{1}{2}Z_4BZ_1 & \frac{1}{2}Z_4AZ_2 \end{bmatrix}$$

where

$$F_{2} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad Z_{1} = \begin{bmatrix} 1 & 1 \\ z_{1} & -z_{1} \end{bmatrix}, \quad Z_{2} = \begin{bmatrix} 1 & 1 \\ z_{2} & -z_{2} \end{bmatrix},$$

$$Z_{3} = \begin{bmatrix} 1 & z_{3} \\ 1 & -z_{3} \end{bmatrix}, \quad Z_{4} = \begin{bmatrix} 1 & z_{4} \\ 1 & -z_{4} \end{bmatrix},$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^{*} & -A_{11}^{**} \end{bmatrix}, \quad B = \begin{bmatrix} -1 - A_{11} & -1 - A_{12} \\ -1 - A_{12}^{**} & 1 + A_{11}^{**} \end{bmatrix},$$

$$A_{11} = -\frac{1}{2} + i \frac{\sqrt{3}}{2} (\cos \theta + e^{-i\phi} \sin \theta),$$

$$A_{12} = -\frac{1}{2} + i \frac{\sqrt{3}}{2} (\cos \theta + e^{-i\phi} \sin \theta),$$

$$B_{11} = -\frac{1}{2} - i \frac{\sqrt{3}}{2} (\cos \theta + e^{-i\phi} \sin \theta),$$

$$B_{12} = -\frac{1}{2} - i \frac{\sqrt{3}}{2} (\cos \theta + e^{i\phi} \sin \theta),$$

$$\theta, \phi \in [0, 2\pi), \quad |z_{j}| = 1,$$

$$z_{3}^{2} = \mathcal{M}_{A}(z_{1}^{2}) = \mathcal{M}_{B}(z_{2}^{2}),$$

$$z_{4}^{2} = \mathcal{M}_{A}(z_{2}^{2}) = \mathcal{M}_{B}(z_{1}^{2}),$$

$$\mathcal{M}_{A}(z) = \frac{A_{12}^{2}z - A_{11}^{2}}{(A_{11}^{2})^{*}z - (A_{12}^{2})^{*}},$$

$$\mathcal{M}_{B}(z) = \frac{B_{12}^{2}z - B_{11}^{2}}{(B_{11}^{2})^{*}z - (B_{12}^{2})^{*}}.$$

$$(4)$$

Suppose that  $h''_{mn}(1 \le m, n \le 6)$  is the element of the m'th row and the n' column of H''. Because of the symmetry of H'', we have  $h''_{mn} = h''_{nm}$ ,  $z_1 = z_3$  and  $z_2 = z_4$ . By  $h''_{34} = h''_{43}$ , we have

$$\frac{1}{2}(A_{11} + A_{12}z_1 - \overline{A_{12}}z_3 + \overline{A_{11}}z_1z_3) = \frac{1}{2}(A_{11} - A_{12}z_1 + z_3(\overline{A_{12}} + \overline{A_{11}}z_1)). \tag{5}$$

Namely

$$(-\cos\theta + e^{i\phi}\sin\theta)z_1 = 0. (6)$$

Obviously  $z_1 \neq 0$ . So  $-\cos \theta + e^{i\phi} \sin \theta = 0$ , namely  $\tan \theta = e^{-i\phi}$ . Since  $\tan \theta$  is real, then  $e^{-i\phi} = 1$  or  $e^{-i\phi} = -1$ . Now we can verify that  $h''_{45} = h''_{54} = -1$  by using (4). Hence H'' dose not belong to any MUB trio by Theorem 7. To sum up we have H dose not belong to any MUB trio. So we complete this proof.

**Theorem 9** The CHM in any MUB trio is not an Hermitian matrix.

Suppose that  $H_6$  is an Hermitian matrix in an MUB trio. From the paper [32] by Kyle Beauchamp and Remus Nicoara we know that  $H_6$  is equivalent to  $H(\theta)$ , where:

$$H(\theta) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \frac{1}{x} & -y & -\frac{1}{x} & y \\ 1 & x & -1 & t & -t & -x \\ 1 & -\frac{1}{y} & \frac{1}{t} & -1 & \frac{1}{y} & -\frac{1}{t} \\ 1 & -x & -\frac{1}{t} & y & 1 & \frac{1}{z} \\ 1 & \frac{1}{y} & -\frac{1}{x} & -t & z & 1 \end{bmatrix},$$

$$(7)$$

and  $\theta \in [-\pi, -arcos(\frac{-1+\sqrt{3}}{2})] \cup [arcos(\frac{-1+\sqrt{3}}{2}), \pi]$ , the parameters x, y, z, t are given by:

$$y = e^{i\theta}, z = \frac{1 + 2y - y^2}{y(-1 + 2y + y^2)}$$
(8)

$$x = \frac{1 + 2y + y^2 - \sqrt{2}\sqrt{1 + 2y + 2y^3 + y^4}}{1 + 2y - y^2} \tag{9}$$

$$x = \frac{1 + 2y + y^2 - \sqrt{2}\sqrt{1 + 2y + 2y^3 + y^4}}{1 + 2y - y^2}$$

$$t = \frac{1 + 2y + y^2 - \sqrt{2}\sqrt{1 + 2y + 2y^3 + y^4}}{-1 + 2y + y^2}.$$
(9)

One can verify that  $H(\theta)$  is an  $H_2$ -reducible matrix of more than eighteen  $2 \times 2$  Hadamard submatrices. So  $H(\theta)$  does not belong to any MUB trio by Theorem 7, of cause  $H_6$  does not belong to any MUB trio. Hence we complete this proof. 

Next we shall investigate some affine and non-affine CHMs. First the two CHMs constructed on p256 of [26], and the Dita family [16, Eq. (5.45)] which are affine CHMs, all have more than nine  $2 \times 2$  Hadamard submatrices. So they are both excluded by Theorem 7. One can similarly exclude the Fourier family and its transpose as known affine families. Next a special CHM is Bjorck's circulant matrix [16, Eq. (5.46)],

$$C_{6} = \begin{bmatrix} 1 & id & -d & -i & -d^{*} & id^{*} \\ id^{*} & 1 & id & -d & -i & -d^{*} \\ -d^{*} & id^{*} & 1 & id & -d & -i \\ -i & -d^{*} & id^{*} & 1 & id & -d \\ -d & -i & -d^{*} & id^{*} & 1 & id \\ id & -d & -i & -d^{*} & id^{*} & 1 \end{bmatrix},$$

$$(11)$$

where  $d = \frac{1-\sqrt{3}}{2} + i\sqrt{\frac{\sqrt{3}}{2}}$ . One can show that  $C_6$  has more than nine  $2 \times 2$  Hadamard submatrices. It is known that every circulant Hadamard matrix is equivalent to either the  $6 \times 6$  Fourier matrix or  $C_6$ . So every circulant Hadamard matrix does not belong to any MUB trio.

Third Theorem 7 excludes some non-affine CHMs too, such as the Szollosi family [4, Eq. (C.12)]

$$X(a,b) = H(x,y,u,v) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & x^{2}y & xy^{2} & \frac{xy}{uv} & uxy & vxy \\ 1 & \frac{x}{y} & x^{2}y & \frac{x}{u} & \frac{x}{v} & uvx \\ 1 & uvx & uxy & -1 & -uxy & -uvx \\ 1 & \frac{x}{u} & vxy & -\frac{x}{u} & -1 & -vxy \\ 1 & \frac{x}{v} & \frac{xy}{uv} & -\frac{xy}{u} & -\frac{x}{v} & -1 \end{bmatrix},$$
(12)

where entries x, y and u, v are solutions of the equations  $f_{\alpha} = 0$  and  $f_{-\alpha} = 0$ , respectively such that  $f_{\alpha}(z) = z^3 - \alpha z^2 + \alpha^* z - 1$  and  $\alpha = a + bi$  restricted by  $D(\alpha) \leq 0$  and  $D(-\alpha) \leq 0$  with  $D(\alpha) = |\alpha|^4 + 18 |\alpha|^2 - 8Re[\alpha^3] - 27$ . One can show that both Hermitean and Szollosi families are  $H_2$ -reducible matrices of more than nine  $2 \times 2$  Hadamard submatrices. So they are not members of any MUB trio in terms of Theorem 7.

#### V. CONCLUSIONS

We have shown that if four six-dimensional MUBs containing the identity matrix exist, then the  $H_2$ -reducible matrix in the four MUBs has exactly nine  $2\times2$  Hadamard submatrices. We have applied our result to exclude some known affine and non-affine CHMs as members of MUB trio, such as symmetric  $H_2$ -reducible matrix, the Hermitian matrix, Dita family, Bjorck's circulant matrix, and Szollosi family. The next step is to exclude every  $H_2$ -reducible matrix as a member of MUB trio.

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