Quantum geometric information flows and relativistic generalizations of G. Perelman thermodynamics for nonholonomic Einstein systems with black holes and stationary solitonic hierarchies

Iuliana Bubuianu *

Radio Iaşi, 44 Lascăr Catargi street, Iaşi, 700107, Romania

Sergiu I. Vacaru †

Physics Department, California State University at Fresno, Fresno, CA 93740, USA; and Dep. Theoretical Physics and Computer Modelling, 101 Storozhynetska street; Yuriy Fedkovych Chernivtsi National University, Chernivtsi, 58029, Ukraine

Elşen Veli Veliev ‡

Department of Physics, Kocaeli University, 41380, Izmit, Turkey

Jan 19, 2022 (published in QINP)

Abstract

We investigate classical and quantum geometric information flow theories (GIFs and QGIFs) when the geometric flow evolution and field equations for nonholonomic Einstein systems, NES, are derived from Perelman-Lyapunov type entropic type functionals. The term NES encodes models when the fundamental physical equations are subjected to nonholonomic (equivalently, non-integrable, anholonomic) constraints. There are used canonical geometric variables that allow a general decoupling and integration of systems of nonlinear partial differential equations describing GIFs and QGIFs and Ricci soliton type configurations. Our approach is different from the constructions elaborated for special classes of solutions characterized by area-hypersurface entropy, related holographic, and dual gauge-gravity models involving generalizations of the Bekenstein-Hawking entropy. We formulate the theory of QGIFs which in certain quasi-classical limits encodes GIFs and models with flow evolution of NES. There are computed respectively the von Neumann, relative and conditional entropy; mutual information, entanglement, and Rényi entropy. We construct explicit examples of generic off-diagonal exact and parametric solutions describing stationary solitonic gravitational hierarchies and deformations of black hole configurations. Finally, we show how Perelman's thermodynamic values and extensions to QGIF models can be computed for various new classes of exact solutions which can not be described following the Bekenstein-Hawking approach.

Keywords: Quantum geometric information flows; relativistic geometric flows; Perelman W-entropy; modified gravity; solitonic hierarchies; nonholonomic Ricci solitons; entanglement and Rényi entropy.

PACS2010: 02.40.-k, 02.90.+p, 03.67.-a, 04.50.-h, 04.50.Kd, 04.90.+e, 05.90.+m
MSC2010: 53C44, 53C50, 53C80, 81P45, 94A17, 83C15, 83C55, 83C99, 83D99, 35Q75, 37J60, 37D35

* email: iulia.bubu@gmail.com

† **corresponding author**; emails: sergiu.vacaru@gmail.com ;
Address for post correspondence in 2021-2022 as a visitor senior researcher at YF CNU Ukraine is: 37 Yu. Gagarin street, ap. 3, Chernivtsi, Ukraine, 58008

‡ email: elsen@kocaeli.edu.tr and elsenveli@hotmail.com

Contents

1 Introduction

This work provides an exposition of the subject of modelling modified gravity theories, MGTs, and general relativity, GR, in the framework of classical and quantum geometric information flow (respectively, GIF and QGIF) theories. Although a great amount of research has been devoted recently to quantum information theory, gravity and entanglement (for reviews of results and some important directions of research, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]), almost all constructions and applications related to gravity theories involve area–hypersurface, holography and dual gauge-gravity models for various types of entropic and thermodynamic type MGTs. Such approaches are elaborated with respective generalizations of the concepts and formulas for the Bekenstein–Hawking entropy and black hole, BH, thermodynamics [14, 15, 16, 17].

In a series of papers [18, 19, 20, 21, 24, 25, 26], we deal with the following question: *Does a more general concept of gravitational entropy can be elaborated in such a form that would allow us to characterize more general classes of generic off-diagonal solutions (not only for BHs and cosmological solutions with horizons and holographic configurations) depending on all spacetime/ phase space coordinates in various types of MGTs, GR and Finsler like generalizations ?* The answer was found positive for relativistic and various type generalizations of the geometric flow theorye and by constructing in explicit form and providing corresponding physical interpretations of a series of new classes of exact and parametric solutions in GR and MGTs (supersymmetric/ noncommutative string and brane gravity models, entropic gravity...) and nonholonomc geometric flow models. The main idea exploited and developed in our works was to characterize and derive such gravity theories and their solutions using the concepts of F- and W-functional due to G. Perelman [27] used in order to prove the Thurston-Poincaré conjecture in the theory of Ricci flows of Riemannian metrics. In this article, we do not address issues on modifications of the theory of Ricci flows from the viewpoint of formulating (and possible proofs) of a generalized geometric conjectures for pseudo-Riemannian (or non-commutative/supersymmetric etc.) metrics or/and non-Riemannian connections, which is a very difficult mathematical task requesting studies on thousands of pages with rigorous topological and geometric analysis methods as in monographs [28, 29, 30, 31] and related applications in physics [32, 18, 19, 20, 21]. We develop corresponding geometric methods and find new classes of exact solutions when the W-functional (it is also called the W-entropy because it is like a "minus-entropy") allows us to elaborate on associated geometric and quantum thermodynamic and information models which are more general than those involving the Bekenstein–Hawking entropy and allow to characterize general classes of solutions in geometric flow and MGTs.

The G. Perelman functionals and related geometric evolution and thermodynamic models can not be applied directly for deriving and investigating realistic classical and quantum gravity and mater field theories. To ensure the relativistic invariance and compatibility with GR and/or other MGTs is necessary to consider certain types of nonholonomic deformations of the F- and W-functionals. Here we note that in literature on mathematics, mechanics and physics the term "anholonomic", i.e. with non-integrable distributions/ structures is used equivalently with "nonholonomic", see [22, 23, 24, 25, 26] and references therein for reviews and applications of such methods in modern gravity theories. Respective geometric flow theories can be elaborated with different types of evolution parameters[1]. Nevertheless, Ricci soliton configurations (determined by self-similar geometric flows) can be defined for all types of such models which are described equivalently by certain types of (modified) Einstein equations. Such equations can be solved in very general forms using geometric and analytic methods elaborated in our works, see details and reviews in [24, 25, 26, 33, 34, 35, 20, 21].

Even it is not clear at present if relativistic/ noncommutative / supersymmetric analogues of the Poincaré hypothesis can be formulated and proven, we can always construct various classes of exact and parametric solutions for systems of nonlinear partial differential equations, PDEs, describing a diversity of evolution/ dynamical/ diffusion/ kinetic / thermodynamic / information classical and quantum systems. Such physically important nonlinear PDEs can be decoupled and integrated in general forms using the so-called anholonomic frame deformation method, AFDM (reviews of results and applications to modern gravity and geometric flow theories can be found in [33, 34, 35, 24, 26] and references therein). To generate exact generic off-diagonal solutions is important to write down the equations in so-called canonical nonholonomic geometric variables (in our works, we use "hats" on symbols in order to emphasize that respective geometric objects are constructed in canonical forms adapted to a respective nonlinear connection structure, see definition in next section).

In this article, we perform all classical and quantum models constructions using canonical geometric data for conventional nonholonomic Einstein systems, NES, which can be under (quantum) geometric flow evolution on a temperature like parameter, or described by certain self-similar Ricci type soliton configurations (with a fixed value of such a parameter). Here we note that only certain classes of black hole and cosmological solutions in MGTs and GR (for very special cases with respective conditions on higher symmetry, hypersurface horizons and asymptotic/ initial behaviour) are characterized by Bekenstein–Hawing type entropies. In another turn, generalized G. Perelman entropic/ thermodynamic values can be defined and computed in all cases at least for any finite spacetime region with well-defined causality and regularizing singularities conditions. It is possible to write equivalently such geometric (flow) and statistical thermodynamic like values in canonical nonholonomic variables, or in certain analogous Finsler-Lagrange-Hamilton, or almost Kähler, variables. This is very important for elaborating analogous, entropic, deformation quantization and brane/ gauge like gravity models (see, for instance, [36, 37, 18, 24, 25, 26]). To apply such methods to GR and standard particle physics we have to impose at the end (when corresponding classes of exact solutions were found in exact form) certain additional nonholonomic constraints in order to extract Levi-Civita (torsionless), LC, configurations. If we work from the very beginning only with the LC-connection, it is not possible to decouple/ integrate in general forms nonlinear classical and quantum fundamental evolution/ dynamical equations. This is a general property of the systems of nonlinear PDEs on curved spacetimes.

The formalism developed in this and partner works [38, 39, 40] allows us to address issues related to various types of quasi-classical and quantum equations (nonlinear Schrödinger, Liouville, and Dirac ones, and noncommutative quantum deformations of the Einstein equations). We are able to deal with the basic elements fundamental concepts of quantum information theory such as entanglement and multipartite states, teleportation, and quantum interference for NES under geometric flow evolution. There are three main goals in this article: 1) To prove a general decoupling and integrability of stationary (on spacetime coordinates) of geometric flow equations with a temperature like evolution parameter. 2) To provide an introduction to the theory of GIFs and QGIFs of NES and show how generalized Perelman entropies and associated quantum mechanical, statistical and thermodynamic geometric theories are used for elaborating such theories. 3) To consider possible applications of the AFDM and study properties of stationary solutions (in particular, of BHs and their solitonic off-diagonal deformations) under nonlinear geometric flows, interactions and for solitonic gravitational and matter field configurations when the W-entropy can be defined and computed but the concept of Bekenstein-Hawking entropy is not applicable, or not enough, for characterizing such classes of exact solutions and QGIFs. In this paper, we use canonical nonholonomic variables which allow us to provide analogous thermodynamic and GIF and QGIF descriptions of any solution found following the AFDM, or other type methods, for MGTs and GR, and/or can be derived as some emergent/ entropic gravity models.

[1] for a temperature like one, which is used in this work, or as a time like parameter treated as a "complex temperature"; such theories have, for instance, very different topological and global classical and quantum properties but, for additional assumptions, preserve much similarities which are exploited for local and perturbative models, for instance, in quantum statistics, particle and condensed matter physics etc.

Here we consider also this important issue: One of the basic concepts in quantum information theory is that of qubit. Given a Hilbert space associated to a physical system is possible to realize a qubit as any two-dimensional subspace of that Hilbert space. Such a physical realization is not, in general, localized in a physical space. In [38, 39], we study the concept of quibit for QGIFs when certain analogous Hamilton and/or thermodynamic models are determined by W-entropy. For such theories, we can formulate physical realizations that are well-localized in certain effective phase spaces and associate a qubit as a two-dimensional quantum state attached to a point in a base space. In a curved spacetime context, we can represent a qubit as a sequence of two-dimensional quantum states evolving along a spacetime trajectory and/or with sets of world lines of qubits covering a spacetime region. Other type ambiguities are related to the fact that there are no finite-dimensional (in particular, two dimensional ones) faithful unitary representations of the Lorentz group. Naively, it would appear that it is impossible to elaborate a mathematical formalism which would describe localized qubits in a unique both relativistic and unitary form. The problem (and an explicit method how to solve it using WKB approximations) is analyzed in [41]. Here we note that for QGIF theories there are used analogous statistical thermodynamic models and respective quantum generalizations of GIF entropies. Such constructions are well-defined for geometric flows in spacetimes and phase spaces with conventional 3+1 splitting of dimensions [19] and there are rigorous proofs for noncommutative geometric models (in the A. Connes approach and for theories with almost Kähler structures and/or Siberg-Witten transforms) and deformation quantization [36, 37, 18].

Let us explain the structure of our work: This paper aims to be self-contained and that why we include in section 2 (and some footnotes in other sections) a necessary background material on the geometry of nonholonomic Lorentz manifolds, their relativistic geometric flow evolution equations and nonholonomic Ricci solitons described by modified Einstein equations. The geometric constructions are performed in canonical nonholonomic variables which allow a straightforward decoupling and integration of geometric flow and gravitational field equations. We define the nonholonomic canonical version of G. Perelman F-functional and W-functional (equivalently, W-entropy) and show how associated thermodynamic models can be elaborated.

Section 3 is devoted to the theory of (classical) geometric information flows, GIFs, and quantum information geometric flows, QGIFs, and respective entanglement of nonholonomic Einstein systems, NES. Using statistical distribution functions determined by the W-entropy and thermodynamic values constructed for GIFs, we define and show how to compute the Shanon, conditional and relative entropy of such systems. Quantum mechanical models, QMs, are elaborated for the canonical density matrix and von Neumann entropy as respective quantum analogues of the statistical density matrix for NES GIFs. We investigate properties of entanglement and gravity for QGIFs using (and showing how to compute) respective inequalities for relative entropy, mutual information and the Rényi entropy for classical and quantum geometric thermodynamics and information flows.

In section 4, we apply the AFDM and prove the existence of an important decoupling properties and explicit integrability of nonholonomic geometric flows and Ricci soliton equations for stationary (i.e. depending on certain space coordinates) NES determined by solitonic configurations of gravitational and matter fields. Such new classes of solutions are described by generic off-diagonal metrics and generalized connections depending on all spacetime coordinates and temperature like parameters via general classes of generating functions and (effective) sources of solitonic configurations of gravity and matter fields. Possible stationary parameterizations of main geometric objects and matter sources are stated in Table 1.

In section 5, we show how exact and parametric solutions can be constructed for flow evolution of stationary configurations and NES. There are reviewed and cited [19, 20, 21, 25, 33, 34, 35, 24, 26] for details, methods and examples of other type solutions. We provide Table 2 summarizing the AFDM for generating stationary solutions for geometric flows of NES and nonholonomic Ricci solitons and off-diagonal solitonic configurations. The formulas are re-defined for new classes of solutions describing for instance, black hole (BH) nonholonomic/ ellipsoid deformations of horizons by a nontrivial vacuum with a solitonic configuration, or embedding of BH in Ricci solitonic vacuum and nonvacuum backgrounds with encoding solitonic gravitational and matter field structures. In explicit form, the AFDM formalism is considered for computing small parametric and generic off–diagonal deformations of the Kerr solution under solitonic geometric flows and nontrivial solitonic gravitational and matter field configurations. We emphasize that the thermodynamic properties of such generalized classes of stationary solitonic solutions can not be described by a Bekenstein-Hawking entropy and respective classical or quantum generalizations.

G. Perelman W-entropy and main geometric thermodynamic values can be always defined and computed in explicit form for various classes of solutions in geometric flow, MGTs and GR. In section 6 we consider key steps and provide explicit examples how to compute such values for three classes of generic off-diagonal solutions stationary solitonic configurations and respective models GIF and QGIF theories. To avoid cumbersome formulas we consider special classes of normalization, generating and integration functions. The fist example is for stationary solitonic generating functions which can be of solitonic or non-solitonic character. Then, in the second example, we compute G. Perelman's thermodynamic values for BHs deformed by general (not introducing small parameters) stationary solitonic generating functions and sources. Finally, the formulas for such values are considered for small parametric stationary deformations of BH solutions and respective GIF and QGIF models.

An outlook, conclusions and discussions are presented in section 7.

2 Geometric flow equations for nonholonomic Einstein systems

In this section, we provide an introduction to the theory relativistic geometric evolution flows and nonholonomic Einstein systems, NES. The constructions are performed in canonical nonholonomic variables with the so-called "hat" connections (all correspondingly adapted to non–integrable distributions of geometric/physical objects) which will allow us to decouple and integrate physically important systems of nonlinear PDEs, see section 4. For geometric details, proofs, examples and constructions with other type variables for geometric flows and MGTs, we cite [19, 20, 21, 35, 24] and references therein.

2.1 Preliminaries: nonholonomic Lorentz manifolds with $2+2$ splitting

We consider a pseudo-Riemannian manifold $\mathbf{V}$ determined by a metric field $\mathbf{g}=\{g_{\alpha\beta}(u)\}$ with local pseudo-Euclidean signature $(+++-)$ and enabled with a conventional 2+2 splitting into horizontal (h) and vertical (v) components defined by a Whitney sum

$$\mathbf{N}:\ T\mathbf{V}=h\mathbf{V}\oplus v\mathbf{V}, \qquad (1)$$

where $T\mathbf{V}$ is the tangent bundle.[2] A N–connection structure (1) is determined locally by a corresponding set of coefficients N_i^a, when $\mathbf{N}=N_i^a(u)dx^i\otimes\partial_a$. For any h-v–splitting, we can define N–adapted local bases, $\mathbf{e}_\nu=(\mathbf{e}_i,e_a)$, and cobases, $\mathbf{e}^\mu=(e^i,\mathbf{e}^a)$, when

$$\begin{aligned}
\mathbf{e}_\nu &= (\mathbf{e}_i=\partial/\partial x^i-\ N_i^a\partial/\partial y^a,\ e_a=\partial_a=\partial/\partial y^a), &(2)\\
\mathbf{e}^\mu &= (e^i=dx^i,\mathbf{e}^a=dy^a+\ N_i^a dx^i). &(3)
\end{aligned}$$

In general, such N–adapted bases are nonholonomic because there are satisfied relations of type

$$[\mathbf{e}_\alpha,\mathbf{e}_\beta]=\mathbf{e}_\alpha\mathbf{e}_\beta-\mathbf{e}_\beta\mathbf{e}_\alpha=W_{\alpha\beta}^\gamma\mathbf{e}_\gamma, \qquad (4)$$

with nontrivial anholonomy coefficients $W_{ia}^b=\partial_a N_i^b, W_{ji}^a=\Omega_{ij}^a=\mathbf{e}_j\left(N_i^a\right)-\mathbf{e}_i(N_j^a)$. There are encoded holonomic (integrable) bases if and only if $W_{\alpha\beta}^\gamma=0$. We can elaborate on $\mathbf{V}$ and respective (co) tangent bundles a N–adapted differential and integral calculus and a corresponding variational formalism there are used N–elongated operators (2) and (3) and introduced additionally a covariant derivative D defined as a metric-affine connection. All geometric constructions and physical models can be re-defined in terms of distinguished

[2] On abstract and coordinate indices (we shall underline such indices if order to emphasize that they are for a coordinate base) we adopt such conventions: The local coordinates are labelled $u^\mu=(x^i,y^a)$, (in brief, we shall write $u=(x,y)$), where $i,j,...=1,2$ and $a,b,...=3,4$. Cumulative small Greek indices run values $\alpha,\beta,...=1,2,3,4$, where $u^4=y^4=t$ is a time like coordinate. An arbitrary local basis is denoted $e^\alpha=(e^i,e^a)$ and the corresponding dual one, co-basis, is written in the form $e_\beta=(e_j,e_b)$. There are always some nontrivial frame transforms to corresponding coordinate bases, $\partial_{\alpha'}=(\partial_{i'},\partial_{a'})$ [for instance, $\partial_{i'}=\partial/\partial x^{i'}$], and cobasis, when $e_\beta=A_\beta^{\ \beta'}(u)\partial_{\beta'}$ and $e^\alpha=A^\alpha_{\ \alpha'}(u)du^{\alpha'}$, for $du^{\alpha'}=(dx^{i'},dy^{a'})$, are considered as frame (vierbein) transforms. There are used primed, underlined indices etc. for other type frame/coordinate systems related via respective classes of nonlinear transforms. It is applied the Einstein summation rule on repeating up–low indices if the contrary will be not stated.

objects (in brief, d–objects) when the coefficients are determined with respect to N–adapted (co) frames and their tensor products.[3]

A manifold $(\mathbf{V},\mathbf{N})$ endowed with a nontrivial structure $W^{\gamma}_{\alpha\beta}$ (4) is called nonholonomic (equivalently, anholonomic) if it is defined as a union of non-integrable distributions in any point $u \in \mathbf{V}$. We call it as a **nonholonomic Lorentz manifold** if at least locally such a curved spacetime possess a causal structure like that for the special relativity theory determined by a local Minkowski metric.

For nonholonomic manifolds, there is a subclass of linear connections which are adapted to the N–connection structure and called **distinguished connections** (in brief, **d–connections**). Such a d-connection $\mathbf{D} = (h\mathbf{D}, v\mathbf{D})$ on $\mathbf{V}$ preserves under parallel transport the N–connection splitting (1). A general linear connection D is not adapted to a chosen h-v–decomposition, i.e. it is not a d–connection.[4] For instance, the Levi–Civita, LC, connection in GR is not a d–connection. To a d–connection $\mathbf{D}$, we can associate an operator of covariant derivative, $\mathbf{D}_{\mathbf{X}}\mathbf{Y}$ (for a d–vector $\mathbf{Y}$ in the direction of a d–vector $\mathbf{X}$). We can compute N–adapted coefficients for $\mathbf{D} = \{\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = (L^{i}_{jk}, L^{a}_{bk}, C^{i}_{jc}, C^{a}_{bc})\}$ defined with respect to tensor products of N–adapted frames (2) and (3).

For any d–connection $\mathbf{D}$ and d–vectors $\mathbf{X},\mathbf{Y} \in T\mathbf{V}$, the d–torsion, $\mathbf{T}$, the nonmetricity, $\mathbf{Q}$, and the d–curvature, $\mathbf{R}$, tensors (in N–adapted forms, they are d–tensors) are defined and computed in standard form,

$$\mathbf{T}(\mathbf{X},\mathbf{Y}) := \mathbf{D}_{\mathbf{X}}\mathbf{Y} - \mathbf{D}_{\mathbf{Y}}\mathbf{X} - [\mathbf{X},\mathbf{Y}],\ \mathbf{Q}(\mathbf{X}) := \mathbf{D}_{\mathbf{X}}\mathbf{g}, \mathbf{R}(\mathbf{X},\mathbf{Y} := \mathbf{D}_{\mathbf{X}}\mathbf{D}_{\mathbf{Y}} - \mathbf{D}_{\mathbf{Y}}\mathbf{D}_{\mathbf{X}} - \mathbf{D}_{[\mathbf{X},\mathbf{Y}]}.$$

Such values are defined locally by N–adapted coefficients are correspondingly labeled using h- and v–indices,

$$\begin{aligned} \mathbf{T} &= \{\mathbf{T}^{\gamma}{}_{\alpha\beta} = \left(T^{i}{}_{jk}, T^{i}{}_{ja}, T^{a}{}_{ji}, T^{a}{}_{bi}, T^{a}{}_{bc}\right)\}, \mathbf{Q} = \{\mathbf{Q}^{\gamma}{}_{\alpha\beta}\}, \\ \mathbf{R} &= \{\mathbf{R}^{\alpha}{}_{\beta\gamma\delta} = \left(R^{i}{}_{hjk}, R^{a}{}_{bjk}, R^{i}{}_{hja}, R^{c}{}_{bja}, R^{i}{}_{hba}, R^{c}{}_{bea}\right)\}, \end{aligned} \tag{5}$$

see explicit formulas in [19, 20, 21, 35, 24].

With respect to a dual local coordinate basis du^{α}, any metric tensor $\mathbf{g}$ on $(\mathbf{V},\mathbf{N})$ can be parameterized in a (general) off-diagonal form,

$$\mathbf{g} = \underline{g}_{\alpha\beta}(u) du^{\alpha} \otimes du^{\beta},\ \text{where}\ \ \underline{g}_{\alpha\beta}(u) = \left[\begin{array}{cc} g_{ij} + N^{a}_{i} N^{b}_{j} g_{ab} & N^{e}_{j} g_{ae} \\ N^{e}_{i} g_{be} & g_{ab} \end{array} \right], \tag{6}$$

with 6 independent coefficients.[5] Any metric $\mathbf{g}$ can be written equivalently as a d–tensor (d–metric)

$$\mathbf{g} = g_{\alpha}(u) \mathbf{e}^{\alpha} \otimes \mathbf{e}^{\beta} = g_{i}(x) dx^{i} \otimes dx^{i} + g_{a}(x,y) \mathbf{e}^{a} \otimes \mathbf{e}^{a}, \tag{7}$$

or, in brief, $\mathbf{g} = (h\mathbf{g}, v\mathbf{g})$. A metric $\mathbf{g}$ (6) with N–coefficients N^{e}_{j} is generic off–diagonal if the anholonomy coefficients $W^{\gamma}_{\alpha\beta}$ (4) are not zero.

For any metric field / d–metric $\mathbf{g}$, we can define two important linear connection structures following such geometric conditions:

$$\mathbf{g} \rightarrow \left\{ \begin{array}{lll} \nabla : & \nabla \mathbf{g} = 0;\ {}^{\nabla}\mathbf{T} = 0, & \text{the Levi–Civita connection;} \\ \widehat{\mathbf{D}} : & \widehat{\mathbf{D}}\ \mathbf{g} = 0;\ h\widehat{\mathbf{T}} = 0,\ v\widehat{\mathbf{T}} = 0, & \text{the canonical d–connection.} \end{array} \right. \tag{8}$$

The LC–connection ∇ can be introduced without any N–connection structure but the canonical d–connection $\widehat{\mathbf{D}}$ (in a series of our works, it is called also the "hat"-connection) depends generically on a prescribed nonholonomic h- and v-splitting. In above formulas, $h\widehat{\mathbf{T}}$ and $v\widehat{\mathbf{T}}$ are respective torsion components of $\widehat{\mathbf{D}}$ which vanish on

[3] For instance, any vector $Y(u) \in T\mathbf{V}$ can be parameterized as a d–vector, $\mathbf{Y} = \mathbf{Y}^{\alpha}\mathbf{e}_{\alpha} = \mathbf{Y}^{i}\mathbf{e}_{i} + \mathbf{Y}^{a}e_{a}$, or $\mathbf{Y} = (hY, vY)$, with $hY = \{\mathbf{Y}^{i}\}$ and $vY = \{\mathbf{Y}^{a}\}$. Similarly, we can determine and compute the coefficients of d–tensors, N–adapted differential forms, d–connections, d–spinors etc.

[4] We do not use boldface symbols for not N–adapted geometric objects.

[5] Any symmetric second rank tensor on a 4-d $\mathbf{V}$ has, in general, 10 independent coefficients but 4 components of a metric tensor can be always transformed into zero by locall coordinate transform. We can not transform arbitrary geometric data $(\mathbf{g},\mathbf{N})$ into a diagonal metric in a finite region $U \subset \mathbf{V}$ even a Minkowski metric can be assigned (using frame/coordinate transforms) in a point $u \in \mathbf{V}$.

conventional h- and v–subspaces. Nevertheless, there are nonzero torsion components, $hv\widehat{\mathbf{T}}$. Here we note that $\widehat{\mathbf{T}}$ is defined completely by the coefficients of a d-metric and N-connection (i.e. of respective off-diagonal metric) structures which is different from the well-known Cartan connection in Riemann-Cartan geometry.

It should be emphasized that all geometric constructions on a $\mathbf{V}$ can be performed equivalently with ∇ and/or $\widehat{\mathbf{D}}$ and related via a canonical distorting relation

$$\widehat{\mathbf{D}}[\mathbf{g},\mathbf{N}] = \nabla[\mathbf{g}] + \widehat{\mathbf{Z}}[\mathbf{g},\mathbf{N}]. \tag{9}$$

In this formula, both linear connections and the distorting tensor $\widehat{\mathbf{Z}}$ are uniquely determined by data $(\mathbf{g},\mathbf{N})$ as an algebraic combination of coefficients of $\widehat{\mathbf{T}}^{\gamma}{}_{\alpha\beta}$. The N–adapted coefficients for $\widehat{\mathbf{D}}$ and corresponding torsion, $\widehat{\mathbf{T}}^{\gamma}{}_{\alpha\beta}$, Ricci d–tensor, $\widehat{\mathbf{R}}\ _{\beta\gamma}$, and Einstein d–tensor, $\widehat{\mathbf{E}}\ _{\beta\gamma}$, can be computed in standard form as in the case of LC-connection ∇, or any metric-affine connection D (abstract formulas are similar, but the N-adapted coefficients are different and can be defined only for d-connections). The canonical distortion relation (9) defines respective distortion relations of the Riemannian, Ricci and Einstein tensors and respective curvature scalars which are uniquely determined by data $(\mathbf{g},\mathbf{N})$. Any (pseudo) Riemannian geometry can be equivalently formulated using $(\mathbf{g},\nabla)$ or $(\mathbf{g},\widehat{\mathbf{D}})$. In our partner works [38, 39] (see references therein), we use also generalized Finsler-Lagrange-Hamilton variables with respective canonical d-connection structures (they can be introduced even in GR but also in other type MGTs) which is important for elaborating QGIF theories of Lagrange-Hamilton nonlinear mechanical systems and analogous/ emergent gravity theories.

The canonical d–connection $\widehat{\mathbf{D}}$ has a very important role in elaborating the AFDM for constructing exact and parametric solutions in geometric flow and MGTs. It allows us to decouple the geometric flow evolution and gravitational and matter field equations with respect to N–adapted frames of reference. This is not possible if we work only with ∇ and there are very limited possibilities for "pure" Finsler connections (like the Cartan, Chern, or Berwald d-connections). Constructing certain general classes of solutions for $\widehat{\mathbf{D}}$, we can impose at the end the condition $\widehat{\mathbf{T}} = 0$ and extract LC–configurations

$$\widehat{\mathbf{D}}_{|\widehat{\mathbf{T}}=0} = \nabla, \tag{10}$$

or to re-define the solutions and respective physical models in certain Lagrange-Hamilton, or almost symplectic, variables which are more convenient for elaborating and study quantum field theories, quantum mechanical models and related quantum information theories.

2.2 Hypersurface and (non) relativistic nonholonomic geometric flows

In this work, we consider families of 4–d Lorentz nonholonomic manifolds $\mathbf{V}(\tau)$ determined by metrics $\mathbf{g}(\tau) = \mathbf{g}(\tau,u)$ of signature $(+++-)$ and N–connections $\mathbf{N}(\tau)$ parameterized by a positive parameter $\tau, 0 \leq \tau \leq \tau_0$. Any $\mathbf{V} \subset \mathbf{V}(\tau)$ can be enabled with a double nonholonomic 2+2 and 3+1 splitting, see [19, 20, 21, 35, 24] for details on the geometry of spacetimes enabled with such double distributions. The local coordinates are labeled as on nonholonomic Lorentz manifold when $u^{\alpha} = (x^i, y^a) = (x^{\grave{\imath}}, u^4 = t)$ for $i,j,k,... = 1,2;\ a,b,c,... = 3,4;$ and $\grave{\imath},\grave{j},\grave{k} = 1,2,3$, but dependencies on a temperature like parameter τ will be emphasized for geometric flow evolution models. The 3+1 splitting can be chosen in such a form that any open region $U \subset \mathbf{V}$ is covered by a family of 3-d spacelike hypersurfaces $\widehat{\Xi}_t$ parameterized by a time like parameter t.[6]

The (non-relativistic) Ricci flow evolution equations were postulated heuristically by R. Hamilton [28] (in physics, similar equations had been considered by D. Friedan [32] a few years earlier). We write such equations in the form

$$\frac{\partial g_{\grave{\imath}\grave{j}}}{\partial \xi} = -2\ R_{\grave{\imath}\grave{j}}. \tag{11}$$

[6]There are two generic different types of geometric flow theories when 1) $\tau(\chi)$ is a re-parametrization of a temperature like parameter used for labeling 4-d Lorentz spacetime configurations and 2) $\xi(t)$ is a time like parameter when 3-d spacelike configurations evolve relativistically on a "redefined" time like coordinate. In next sections, we shall study in details only theories of type 1 even a number of formulas and geometric constructions will be presented with a relativistic time/ parameters which can be used for theories of type 2.

The equations (11) describe a nonlinear diffusion process for geometric flow evolution of 3-d Riemannian metrics.[7] In modified and normalized forms (see below some more details related to nonholonomic generalizations and formulas (17) and (18)), the Hamilton equations (11) and various nonholonomic deformations can be proven following a corresponding variational calculus for Perelman's W- and F–functionals [29, 30, 31].

For self-similar configurations in fixed points, the geometric flows (11) are described by the so-called Ricci soliton equations

$$R_{\grave{i}\grave{j}} - \lambda g_{\grave{i}\grave{j}} = \nabla_{\grave{i}} v_{\grave{j}} + \nabla_{\grave{j}} v_{\grave{i}}, \tag{12}$$

for $\lambda = \pm 1, 0$ and a vector field $v_{\grave{j}}$. The formulas (12) consist a variant of Einstein equations with cosmological constant and a special source determined by $v_{\grave{j}}$ if the 3-d Riemannian metrics are transformed into pseudo-Riemannian ones.

2.2.1 Geometric evolution of d-metrics and the Laplace d-operator for 3+1 splitting

For a region $U \subset \mathbf{V}$ with 2+2 splitting defined by data $(\mathbf{N}, \mathbf{g})$, we consider an additional structure of 3-d hypersurfaces Ξ_t parameterized by time like coordinate $y^4 = t$ for coordinates $u^\alpha = (x^i, y^a) = (x^{\grave{i}}, t)$. The metric structure can be represented in a d-metric form and/or with 3+1 splitting,

$$\begin{aligned} \mathbf{g} &= \mathbf{g}_{\alpha'\beta'}(\tau,\ u) d\ \mathbf{e}^{\alpha'}(\tau) \otimes d\mathbf{e}^{\beta'}(\tau) \\ &= q_i(\tau, x^k) dx^i \otimes dx^i + \mathbf{q}_3(\tau, x^k, y^a) \mathbf{e}^3(\tau) \otimes \mathbf{e}^3(\tau) - [\ {}_{q}N(\tau, x^k, y^a)]^2 \mathbf{e}^4(\tau) \otimes \mathbf{e}^4(\tau). \end{aligned} \tag{13}$$

In (13), there are considered "shift" coefficients $\mathbf{q}_{\grave{i}} = (q_i, \mathbf{q}_3)$ related to a 3-d metric $\mathbf{q}_{ij} = diag(\mathbf{q}_{\grave{i}}) = (q_i, \mathbf{q}_3)$ on a hypersurface Ξ_t if $\mathbf{q}_3 = \mathbf{g}_3$ and $[\ {}_{q}N]^2 = -\mathbf{g}_4$, where ${}_{q}N$ is the lapse function (our notations are different from those in [42] when we use a left label q in order to avoid ambiguities with the coefficients N_i^a for the N-connection). The parameter τ can be of temperature type like in thermodynamic theories or considered as a time like one when τ is identified with $y^4 = ct$ (depending on the type of geometric evolution flow model we study).

To elaborate on relativistic geometric flows and thermodynamical models we can use a N–adapted 3+1 decomposition for the canonical d–connection, $\mathbf{D} = (\ {}_{\shortmid}\mathbf{D},\ {}^{t}D)$ and d–metric $\mathbf{g} := (\mathbf{q},\ {}_{q}N)$ of a 4–d nonholonomic Lorentz manifold $\mathbf{V}$. On closed 3-d spacelike hypersurfaces, both the geometric flow and MGTs can be formulated in two equivalent forms using the connections ${}_{\shortmid}\nabla$ and/or ${}_{\shortmid}\mathbf{D}$ when the evolution of geometric objects are determined by the evolution of the hypersurface metric $\mathbf{q}$ and an extension to $\mathbf{g}$. We introduce the canonical Laplacian d-operator, ${}_{\shortmid}\widehat{\Delta} := {}_{\shortmid}\mathbf{D}\ {}_{\shortmid}\mathbf{D}$ and consider the canonical distortion tensor ${}_{\shortmid}\mathbf{Z}$. Using distortions ${}_{\shortmid}\nabla = {}_{\shortmid}\mathbf{D} - {}_{\shortmid}\mathbf{Z}$ (which is a 3-d version of the canonical distortion relation (9)), we compute

$$ {}_{\shortmid}\widehat{\Delta} = {}_{\shortmid}\mathbf{D}_\alpha\ {}_{\shortmid}\mathbf{D}^\alpha = {}_{\shortmid}\Delta + {}_{\shortmid}^{Z}\widehat{\Delta}$$

$$\begin{aligned} \text{where}\ {}_{\shortmid}\Delta &= {}_{\shortmid}\nabla_{\grave{i}}\ {}_{\shortmid}\nabla^{\grave{i}} = {}_{\shortmid}\nabla_\alpha\ {}_{\shortmid}\nabla^\alpha, \\ {}_{\shortmid}^{Z}\widehat{\Delta} &= {}_{\shortmid}\mathbf{Z}_{\grave{i}}\ {}_{\shortmid}\mathbf{Z}^{\grave{i}} - [\ {}_{\shortmid}\mathbf{D}_{\grave{i}}(\ {}_{\shortmid}\mathbf{Z}^{\grave{i}}) + {}_{\shortmid}\mathbf{Z}_{\grave{i}}(\ {}_{\shortmid}\mathbf{D}^{\grave{i}})] = {}_{\shortmid}\mathbf{Z}_\alpha\ {}_{\shortmid}\mathbf{Z}^\alpha - [\ {}_{\shortmid}\mathbf{D}_\alpha(\ {}_{\shortmid}\mathbf{Z}^\alpha) + {}_{\shortmid}\mathbf{Z}_\alpha(\ {}_{\shortmid}\mathbf{D}^\alpha)]; \\ {}_{\shortmid}\mathbf{R}_{\grave{i}\grave{j}} &= {}_{\shortmid}R_{\grave{i}\grave{j}} - {}_{\shortmid}\mathbf{Z}ic_{\grave{i}\grave{j}},\ {}_{\shortmid}\mathbf{R}\ _{\beta\gamma} = {}_{\shortmid}R\ _{\beta\gamma} - {}_{\shortmid}\mathbf{Z}ic_{\beta\gamma}, \\ \\ {}_{\shortmid}^{s}R &= {}_{\shortmid}R - \mathbf{g}^{\beta\gamma}\ {}_{\shortmid}\mathbf{Z}ic_{\beta\gamma} = {}_{\shortmid}R - \mathbf{q}^{\grave{i}\grave{j}}\ {}_{\shortmid}\mathbf{Z}ic_{\grave{i}\grave{j}} = {}_{\shortmid}R - {}_{\shortmid}\mathbf{Z}, \\ {}_{\shortmid}\mathbf{Z} &= \mathbf{g}^{\beta\gamma}\ {}_{\shortmid}\mathbf{Z}ic_{\beta\gamma} = \mathbf{q}^{\grave{i}\grave{j}}\ {}_{\shortmid}\mathbf{Z}ic_{\grave{i}\grave{j}} = {}_{h}\widehat{Z} + {}_{v}\widehat{Z},\ {}_{h}\widehat{Z} = g^{ij}\ \mathbf{Z}ic_{ij},\ {}_{v}\widehat{Z} = h^{ab}\ \mathbf{Z}ic_{ab}; \\ R &= {}_{h}R + {}_{v}R,\ \ {}_{h}R := g^{ij}\ R_{ij},\ {}_{v}R = h^{ab}\ R_{ab}. \end{aligned} \tag{14}$$

Such values can be computed in explicit form for any class of exact solutions of modified Einstein equations when a double 2+2 and 3+1 splitting is prescribed and the LC–conditions can be imposed additionally. The 3-d distortion formulas (14) are important for defining and computing gravitational thermodynamic values on space like hypersurfaces.

[7] This can be found for small deformations of a 3–d Euclidean metric $g_{\grave{i}\grave{j}} \approx \delta_{\grave{i}\grave{j}} + h_{\grave{i}\grave{j}}$, with $\delta_{\grave{i}\grave{j}} = diag[1,1,1]$ and $h_{\grave{i}\grave{j}}| \ll 1$, when the Ricci tensor approximates the Laplace operator $\Delta = \frac{\partial^2}{(\partial u^1)^2} + \frac{\partial^2}{(\partial u^2)^2} + \frac{\partial^2}{(\partial u^3)^2}$ and we obtain a linear diffusion equation if $R_{\grave{i}\grave{j}} \sim \Delta h_{\grave{i}\grave{j}}$.

2.2.2 Nonholonomc 3–d hypersurface Perelman's functionals

On a normalized 3-d spacelike closed hypersurface ${}^{c}\widehat{\Xi} \subset \mathbf{V}$, the normalized version of R. Hamilton equations can be written in a coordinate basis,

$$\partial_{\xi} q_{\grave{i}\grave{j}} = -2\ {}_{\shortmid}R_{\grave{i}\grave{j}} + \frac{2\grave{r}}{5} q_{\grave{i}\grave{j}},\ q_{\grave{i}\grave{j}|\xi=0} = q^{[0]}_{\grave{i}\grave{j}}[x^{\grave{i}}]. \tag{15}$$

For these formulas, the left label "c" is used for "compact and closed" regions. We shall prefer to write explicitly only the dependence on parameter variable ξ (writing in brief $q_{\grave{i}\grave{j}}(\xi) = q_{\grave{i}\grave{j}}(x^{\grave{i}}, \xi)$) if that will do not result in ambiguities. The Ricci tensor ${}_{\shortmid}R_{\grave{i}\grave{j}}$ is computed for the Levi–Civita connection ${}_{\shortmid}\nabla$ of $q_{\grave{i}\grave{j}}(\xi)$ parameterized by a real variable ξ, $0 \leq \xi < \xi_0$, for a differentiable function $\xi(t)$ and can be distorted to 3-d nonholonomic canonical variables. In the standard Riemannian approach, the boundary conditions in (15) are stated for $\xi = 0$ when a normalizing factor $\grave{r} = \int_{{}^{c}\widehat{\Xi}}\ {}_{\shortmid}R\sqrt{|q_{\grave{i}\grave{j}}|}d\grave{x}^3 / \int_{{}^{c}\widehat{\Xi}} \sqrt{|q_{\grave{i}\grave{j}}|}d\grave{x}^3$ is introduced in order to preserve the volume of ${}^{c}\widehat{\Xi}$, i.e. $\int_{{}^{c}\Xi} \sqrt{|q_{\grave{i}\grave{j}}|}d\grave{x}^3$.

To generate solutions of (15) for $q_{\grave{i}\grave{j}} \subset g_{\alpha\beta}$ with $g_{\alpha\beta}$ considered as a solution of the 4-d (modified) Einstein equations we have to relate a nontrivial normalizing factor $\grave{r}$ and a respective cosmological constant. The equation (11) can be written in any nonholonomic basis using respective formulas for hypersurface geometric evolution of frame fields, $\partial_{\xi} e_{\grave{i}}^{\ \underline{\grave{i}}} = q^{\underline{\grave{i}\grave{j}}}\ {}_{\shortmid}R_{\underline{\grave{j}\grave{k}}} e_{\grave{i}}^{\ \underline{\grave{k}}}$, when $q_{\grave{i}\grave{j}}(\xi) = q_{\underline{\grave{i}\grave{j}}}(\xi) e_{\grave{i}}^{\ \underline{\grave{i}}}(\xi) e_{\grave{j}}^{\ \underline{\grave{j}}}(\xi)$ for $e_{\grave{i}}(\xi) = e_{\grave{i}}^{\ \underline{\grave{i}}}(\xi)\partial_{\underline{\grave{i}}}$ and $e^{\grave{j}}(\xi) = e^{\grave{j}}_{\ \underline{\grave{j}}}(\xi) dx^{\underline{\grave{j}}}$. As in standard Hamilton–Perelman theory, there is a unique solution for such systems of linear ordinary differential equations, ODEs, for any $\xi \in [0, \xi_0)$ and such a solution can be extended for a family of 3-d hypersurfaces.

In nonholonomic variables and for the hypersurface canonical d–connection ${}_{\shortmid}\widehat{\mathbf{D}}$, the Perelman's functionals can be written in terms of integrals on families of 3-d hypersurfaces

$$ {}_{\shortmid}\widehat{\mathcal{F}} = \int_{\widehat{\Xi}_t} e^{-f}\sqrt{|q_{\grave{i}\grave{j}}|}d\grave{x}^3 (\ {}_{\shortmid}^{s}\widehat{R} + |\ {}_{\shortmid}\widehat{\mathbf{D}} f|^2), \text{ and } {}_{\shortmid}\widehat{\mathcal{W}} = \int_{\widehat{\Xi}_t} M \sqrt{|q_{\grave{i}\grave{j}}|}d\grave{x}^3 [\xi(\ {}_{\shortmid}^{s}\widehat{R} + |\ {}_{\shortmid}^{h}\widehat{\mathbf{D}} f| + |\ {}_{\shortmid}^{v}\widehat{\mathbf{D}} f|)^2 + f - 6], \tag{16}$$

where the scaling function f satisfies $\int_{\widehat{\Xi}_t} M\sqrt{|q_{\grave{i}\grave{j}}|}d\grave{x}^3 = 1$ for $M = (4\pi\xi)^{-3} e^{-f}$. The functionals ${}_{\shortmid}\widehat{\mathcal{F}}$ and ${}_{\shortmid}\widehat{\mathcal{W}}$ transform into standard Perelman functionals on a hypersurface $\widehat{\Xi}_t$ if ${}_{\shortmid}\widehat{\mathbf{D}} \to {}_{\shortmid}\nabla$. The W–entropy ${}_{\shortmid}\widehat{\mathcal{W}}$ is a Lyapunov type non–decreasing functional and can be considered as an alternative to the Hawking-Bekenstein entropy for the case of hypersufaces for BHs and various holographic generalizations. Such entropy type functionals can be used for elaborating hypersurface thermodynamic models and computing respective statistical distribution and energy functionals (we shall consider this in next sections for 4-d configurations).

2.2.3 Nonholonomic Ricci flow equations for 3–d hypersurface metrics

We can consider another type dependencies of geometric objects in formulas (16) on a smooth parameter $\upsilon(\xi)$ for which $\partial\upsilon/\partial\xi = -1$. For simplicity, we can omit the normalization terms. Elaborating on a variational N-adapted calculus or using geometric abstract symbolic methods for global constructions on manifolds, we prove the nonholonomic geometric evolution (modified by nonholonomic distortion) Hamilton equations for any induced 3–d metric $\mathbf{q}$ and canonical d–connection ${}_{\shortmid}\widehat{\mathbf{D}}$. We obtain a flow evolution system of PDEs,

$$\begin{aligned}
\partial_{\upsilon}\mathbf{q}_{\grave{i}\grave{j}} &= -2(\ {}_{\shortmid}\widehat{\mathbf{R}}_{\grave{i}\grave{j}} + \ {}_{\shortmid}\widehat{\mathbf{Z}}ic_{\grave{i}\grave{j}}), && (17)\\
{}_{\shortmid}\widehat{\mathbf{R}}_{i\grave{a}} &= -\ {}_{\shortmid}\widehat{\mathbf{Z}}ic_{i\grave{a}}, && (18)\\
\partial_{\upsilon} f &= -(\ {}_{\shortmid}\widehat{\Delta} - \ {}_{\shortmid}^{Z}\widehat{\Delta})f + |(\ {}_{\shortmid}\widehat{\mathbf{D}} - \ {}_{\shortmid}\widehat{\mathbf{Z}})f|^2 - \ {}_{\shortmid}^{s}\widehat{R} - \ {}_{\shortmid}\widehat{\mathbf{Z}}.
\end{aligned}$$

The distortion tensors in these equations are completely determined by $\mathbf{q}_{\grave{i}\grave{j}}$, see formulas (14), when

$$\partial_{\upsilon}\ {}_{\shortmid}\widehat{\mathcal{F}}(q,\ {}_{\shortmid}\widehat{\mathbf{D}}, f) = 2\int_{{}^{c}\widehat{\Xi}_t} e^{-f}\sqrt{|q_{\grave{i}\grave{j}}|}d\grave{x}^3 [|\ {}_{\shortmid}\widehat{\mathbf{D}}_{\grave{i}\grave{j}} + \ {}_{\shortmid}\widehat{\mathbf{Z}}ic_{\grave{i}\grave{j}} + (\ {}_{\shortmid}\widehat{\mathbf{D}}_{\grave{i}} - \ {}_{\shortmid}\widehat{\mathbf{Z}}_{\grave{i}})(\ {}_{\shortmid}\widehat{\mathbf{D}}_{\grave{j}} - \ {}_{\shortmid}\widehat{\mathbf{Z}}_{\grave{j}})f|^2],$$

when the normalizing function f is subjected to the condition that $\int_{\ _{c}\widehat{\Xi}_t} e^{-f}\sqrt{|q_{\grave{i}\grave{j}}|}d\grave{x}^3$ is constant for a fixed ξ and $f(\upsilon(\xi)) = f(\xi)$.

The system of nonlinear PDEs (17) and (18) is equivalent to (15) (for self-similar conditions when $\partial_\upsilon \mathbf{q}_{\grave{i}\grave{j}} = 0$, we obtain 3-d Ricci soliton equations) up to certain re–definition of nonholonomic frames and variables. We have to consider (18) as additional constraints because in nonholonomic variables the Ricci d–tensor is (in general) nonsymmetric.

2.2.4 Geometric evolution to nonholonomic 4–d Lorentz configurations

The geometric evolution of 3-d d-metrics embedded into 4-d d-metrics $\mathbf{q}(\xi(t)) \subset \mathbf{g}(\xi(t)) := (\mathbf{q}(\xi(t)),\ _{q}N(\xi(t)))$ is described by respective generalizations of functionals (16) resulting in nonholonomic deformations of the Hamilton equations.

For foliations $\widehat{\Xi}_t$ adapted to a N-connection structures and parameterized by a spacetime coordinate t, we can introduce such 4–d functionals

$$\widehat{\mathcal{F}}(\mathbf{q},\ _{\shortmid}\widehat{\mathbf{D}}, f) = \int_{t_1}^{t_2} dt\ \ _{q}N(\xi)\ _{\shortmid}\widehat{\mathcal{F}}(\mathbf{q},\ _{\shortmid}\widehat{\mathbf{D}}, f), \text{ and } \widehat{\mathcal{W}}(\mathbf{q},\ _{\shortmid}\widehat{\mathbf{D}}, f) = \int_{t_1}^{t_2} dt\ \ _{q}N(\xi)\ \ _{\shortmid}\widehat{\mathcal{W}}(\mathbf{q},\ _{\shortmid}\widehat{\mathbf{D}}, f(\xi)).$$

In these relativistic flow formulas, the parameter ξ for 3-d Ricci flows is extended as a time like coordinate on a open region in a 4-d $\mathbf{V}$ determined by a Lorentzian d-metric $\mathbf{g}(\xi)$, For elaborating alternative models (which consists the main goals of this work), we can consider additional dependencies on a temperature parameter τ both for the 3-d and 4-d configurations. Using frame transforms on 4-d nonholonomic Lorentz manifolds, such values can be re–defined respectively in terms of data $(\mathbf{g}(\tau), \widehat{\mathbf{D}}(\tau))$. We obtain

$$\widehat{\mathcal{F}}(\tau) = \int_{t_1}^{t_2}\int_{\widehat{\Xi}_t} e^{-\widehat{f}}\sqrt{|\mathbf{g}_{\alpha\beta}|}d^4u(\ ^{s}R + |\widehat{\mathbf{D}}\widehat{f}|^2),\ \widehat{\mathcal{W}}(\tau) = \int_{t_1}^{t_2}\int_{\widehat{\Xi}_t} \widehat{M}\sqrt{|\mathbf{g}_{\alpha\beta}|}d^4u[\tau(\ ^{s}R + |\ ^{h}\widehat{\mathbf{D}}\widehat{f}| + |\ ^{v}\widehat{\mathbf{D}}\widehat{f}|)^2 + \widehat{f} - 8], \tag{19}$$

where the scaling function $\widehat{f}$ satisfies $\int_{t_1}^{t_2}\int_{\widehat{\Xi}_t}\widehat{M}\sqrt{|\mathbf{g}_{\alpha\beta}|}d^4u = 1$ for $\widehat{M} = (4\pi\tau)^{-3} e^{-\widehat{f}}$. In these formulas τ is a temperature like parameter describing relativistic evolution of geometric objects and associated thermodynamic values.

The functionals $\widehat{\mathcal{F}}(\tau)$ and $\widehat{\mathcal{W}}(\tau)$ for 4-d pseudo-Riemannian metrics are not just static thermodynamic entropies like in the 3–d Riemannian case. They determine certain relativistic thermo field models with flows of entropic values on respective temperature like parameter τ and a time like coordinate ξ. Nevertheless, they describe nonlinear general relativistic diffusion type processes if $\mathbf{q} \subset \mathbf{g}$ and $_{\shortmid}\widehat{\mathbf{D}} \subset \widehat{\mathbf{D}}$ are determined by certain lapse and shift functions as certain solutions of 4–d gravitational equations. The canonical values (19) can be geometrically and physically motivated for any exact/parametric solution in GR or MGT, when the AFDM is applicable. It is not clear if such functionals are well defined for arbitrary geometric flows of pseudo-Riemannian metrics.

The systems of nonliner PDEs (17) and (18) can be generalized to 4–d configurations when the coefficients are determined by the Ricci d–tensors and distortions contain the left label "$_{\shortmid}$ ". Using the hypersurface d-metric $\mathbf{q}_{\alpha\beta} = \mathbf{g}_{\alpha\beta} + \mathbf{n}_\alpha\mathbf{n}_\beta$, we write

$$\begin{aligned} \partial_\upsilon \mathbf{g}_{\alpha\beta} &= -2(\ _{\shortmid}\widehat{\mathbf{R}}_{\alpha\beta} + \ _{\shortmid}\widehat{\mathbf{Z}}ic_{\alpha\beta}) - \partial_\upsilon(\mathbf{n}_\alpha\mathbf{n}_\beta), \\ _{\shortmid}\widehat{\mathbf{R}}_{ia} &= -\ _{\shortmid}\widehat{\mathbf{Z}}ic_{ia}, \text{ for } _{\shortmid}\widehat{\mathbf{R}}_{\alpha\beta} \text{ with } \alpha \neq \beta. \end{aligned} \tag{20}$$

The term $\partial_\upsilon(\mathbf{n}_\alpha\mathbf{n}_\beta)$ can be computed in explicit form using formulas for the geometric relativistic evolution of N–adapted frames.[8]

[8] There are some important formulas on flow evolution on a parameter $\upsilon \in [0, \upsilon_0)$ of N–adapted frames in a 4–d nonholonomic Lorentz manifold computed as $\mathbf{e}_\alpha(\upsilon) = \ \mathbf{e}_\alpha^{\ \underline{\alpha}}(\upsilon, u)\partial_{\underline{\alpha}}$. For N-adapted frame/coordinate transforms, the frame coefficients are

$$\mathbf{e}_\alpha^{\ \underline{\alpha}}(\upsilon, u) = \left[\begin{array}{cc} e_i^{\ \underline{i}}(\upsilon, u) & -N_i^b(\upsilon, u)\ e_b^{\ \underline{a}}(\upsilon, u) \\ 0 & e_a^{\ \underline{a}}(\upsilon, u) \end{array}\right],\ \mathbf{e}^\alpha_{\ \underline{\alpha}}(\upsilon, u) = \left[\begin{array}{cc} e^i_{\ \underline{i}} = \delta^i_{\underline{i}} & e^b_{\ \underline{i}} = N_k^b(\upsilon, u)\ \ \delta^k_{\underline{i}} \\ e^i_{\ \underline{a}} = 0 & e^a_{\ \underline{a}} = \delta^a_{\underline{a}} \end{array}\right],$$

For 4-d configurations with a corresponding re–definition of the scaling function, $f \to \widehat{f}$, and using necessary type N–adapted distributions, we can construct models of geometric evolution of vacuum gravitational fields with h– and v–splitting for $\widehat{\mathbf{D}}$,

$$\begin{aligned} \partial_v \mathbf{g}_{ij} &= -2\widehat{\mathbf{R}}_{ij},\ \partial_v \mathbf{g}_{ab} = -2\widehat{\mathbf{R}}_{ab}, \\ \partial_v \widehat{f} &= -\widehat{\square}\widehat{f} + \left|\widehat{\mathbf{D}}\widehat{f}\right|^2 - {}^s\widehat{R}, \end{aligned} \quad (21)$$

Proofs are possible if we follow a similar calculus to that presented in the proof of Proposition 1.5.3 of [29] but in N–adapted form.

2.3 Nonholonomic geometric flows and Ricci solitons in canonical variables

We consider a family of Riemannian metrics, $g_{\grave{i}\grave{j}}(\tau) = g_{\grave{i}\grave{j}}(\tau, x^{\grave{k}})$ parameterized by a temperature like parameter τ and defined on a 3-d spacelike hypersurface $\Xi \subset \mathbf{V}$. The Ricci flows of such metrics are described by standard Hamilton equations

$$\frac{\partial g_{\grave{i}\grave{j}}}{\partial \tau} = -2\ R_{\grave{i}\grave{j}}[\nabla], \quad (22)$$

where the Ricci tensor $R_{\grave{i}\grave{j}}$ is determined by geometric data $(g_{\grave{i}\grave{j}}, \nabla)$. Rigorous mathematical results on such systems of PDEs and related applications for the proof of the Thurston-Poincaré conjecture can be found in [28, 27, 29, 30, 31].

The normalizing function in (25) can be chosen in such a form that for self-similar configurations with a fixed evolution parameter τ_0 such PDEs transform into modified Einstein equations (24) for NES. In h- and v–split components, such equations are written

$$\widehat{\mathbf{R}}_{ij} = \widehat{\mathbf{\Upsilon}}_{ij};\ \widehat{\mathbf{R}}_{ab} = \widehat{\mathbf{\Upsilon}}_{ab}; \widehat{\mathbf{R}}_{ia} = \widehat{\mathbf{R}}_{ai} = 0;\ \widehat{\mathbf{R}}_{ij} = \widehat{\mathbf{R}}_{ji};\ \widehat{\mathbf{R}}_{ab} = \widehat{\mathbf{R}}_{ba}.$$

In geometric flow theory, such equations are called as (nonholonomic) Ricci solitons if they are of type (12) but for the canonical d-connection and another type fixing of the evolution parameter,

$$\widehat{\mathbf{R}}_{\alpha\beta} - \lambda\ \mathbf{g}_{\alpha\beta} = \widehat{\mathbf{D}}_\alpha \mathbf{v}_\beta + \widehat{\mathbf{D}}_\beta \mathbf{v}_\alpha. \quad (23)$$

Such systems of nonlinear PDEs are determined by some geometric data $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$ and a cosmological constant λ and d-vector field $\mathbf{v}_\alpha(u)$.

The gravitational field equations for nonholonomic Einstein systems, NES, can be written in similar forms to the nonholonomic Ricci soliton equations in canonical variables,

$$\widehat{\mathbf{R}}_{\alpha\beta} = \widehat{\mathbf{\Upsilon}}_{\alpha\beta}. \quad (24)$$

Such equations transform into the standard Einstein equation in GR if there are imposed additional nonholonomic constraints, or found some smooth limits, for extracting LC–configurations, $\widehat{\mathbf{D}}_{|\widehat{\mathcal{T}}=0} = \nabla$, when $\widehat{\mathbf{T}}^\gamma{}_{\alpha\beta} = 0$ and the sources $\widehat{\mathbf{\Upsilon}}_{\alpha\beta}$ are constructed as in standard particle physics but with distortions of linear connections (9). Here we note that a nonholonomic vacuum Einstein space is characterized by a more rich geometric structure which can be with generic off-diagonal and locally anisotropic interactions, encoding nonlinear evolution scenarios etc.

In (24), there are considered effective and matter fields sources of type $\widehat{\mathbf{\Upsilon}}_{\mu\nu} = {}^e\widehat{\mathbf{\Upsilon}}_{\mu\nu} + {}^m\widehat{\mathbf{\Upsilon}}_{\mu\nu}$, where ${}^e\widehat{\mathbf{\Upsilon}}_{\mu\nu}$ are effective sources determined by distortions of the linear connections and effective Lagrangians for gravitational fields in MGTs [43, 22, 20, 21, 35, 24]. Such a source is not zero even in GR if there are nonzero

---

when the h- and v- components of a d-metric evolve as $\tilde{g}_{ij}(v) = e_i^{\ \underline{i}}(v,u)\ e_j^{\ \underline{j}}(v,u)\eta_{\underline{ij}}$ and $\tilde{g}_{ab}(v) = e_a^{\ \underline{a}}(v,u)\ e_b^{\ \underline{b}}(v,u)\eta_{\underline{ab}}$. For local parameterizations of Minkowski type with $\eta_{\underline{ij}} = diag[+,+]$ and $\eta_{\underline{ab}} = diag[+,-]$ corresponding to the chosen signature of $\tilde{\mathbf{g}}^{[0]}_{\alpha\beta}(u)$, we have the evolution equations $\frac{\partial}{\partial v}\mathbf{e}^\alpha_{\ \underline{\alpha}} = \mathbf{g}^{\alpha\beta}\ \widehat{\mathbf{R}}_{\beta\gamma}\ \ \mathbf{e}^\gamma_{\ \underline{\alpha}}$.

distortions (9) from $\widehat{\mathbf{D}}$ to ∇. The source for matter field, ${}^{m}\widehat{\mathbf{\Upsilon}}_{\mu\nu}$, can be constructed using a N–adapted variational calculus for ${}^{m}\mathcal{L}(\mathbf{g},\widehat{\mathbf{D}}, {}^{A}\varphi)$, when

$$
{}^{m}\widehat{\mathbf{\Upsilon}}_{\mu\nu} = \varkappa({}^{m}\widehat{\mathbf{T}}_{\mu\nu} - \frac{1}{2}\mathbf{g}_{\mu\nu} {}^{m}\widehat{\mathbf{T}}) \rightarrow \varkappa({}^{m}T_{\mu\nu} - \frac{1}{2}\mathbf{g}_{\mu\nu} {}^{m}T)
$$

for [coefficients of $\widehat{\mathbf{D}}$] $\rightarrow$ [coefficients of ∇] even, in general, $\widehat{\mathbf{D}} \neq \nabla$. In such formulas, we consider ${}^{m}\widehat{\mathbf{T}} = \mathbf{g}^{\mu\nu} {}^{m}\widehat{\mathbf{T}}_{\mu\nu}$ for ${}^{m}\widehat{\mathbf{T}}_{\alpha\beta} := -\frac{2}{\sqrt{|\mathbf{g}_{\mu\nu}|}}\frac{\delta(\sqrt{|\mathbf{g}_{\mu\nu}|} \ {}^{m}\mathcal{L})}{\delta \mathbf{g}^{\alpha\beta}}$. In this work, we shall consider only ${}^{m}\mathcal{L} = {}^{\phi}\mathcal{L}(\mathbf{g},\widehat{\mathbf{D}}, \phi)$ determined by a scalar field $\phi(x,u)$ and/or geometric evolution of scalar fields $\phi(\tau) = \phi(\tau, x, u)$, when ${}^{m}\widehat{\mathbf{T}}_{\alpha\beta} = {}^{\phi}\widehat{\mathbf{T}}_{\alpha\beta}$.

The geometric flow equations (22) can be generalized for 4-d Lorentzian metrics describing modified Ricci flows of NES. In N-adapted from and for the canonical d-connection, such equations are written in the form

$$
\begin{aligned}
\frac{\partial \mathbf{g}_{ij}}{\partial \tau} &= -2\left(\widehat{\mathbf{R}}_{ij} - \widehat{\mathbf{\Upsilon}}_{ij}\right); \ \frac{\partial \mathbf{g}_{ab}}{\partial \tau} = -2\left(\widehat{\mathbf{R}}_{ab} - \widehat{\mathbf{\Upsilon}}_{ab}\right); \qquad (25)\\
\widehat{\mathbf{R}}_{ia} &= \widehat{\mathbf{R}}_{ai} = 0; \ \widehat{\mathbf{R}}_{ij} = \widehat{\mathbf{R}}_{ji}; \ \widehat{\mathbf{R}}_{ab} = \widehat{\mathbf{R}}_{ba};\\
\partial_{\tau} \ \widehat{f} &= -\widehat{\square}\widehat{f} + \left| \widehat{\mathbf{D}} \ \widehat{f}\right|^{2} - \ {}_{s}\widehat{R} + \widehat{\mathbf{\Upsilon}}_{a}^{a}, \qquad (26)
\end{aligned}
$$

where $\widehat{\square}(\tau) = \widehat{\mathbf{D}}^{\alpha}(\tau) \ \widehat{\mathbf{D}}_{\alpha}(\tau)$ is used for the geometric flows of the d'Alambert operator and sources $\widehat{\mathbf{\Upsilon}}_{\alpha\beta}(\tau) = [\widehat{\mathbf{\Upsilon}}_{ij}(\tau), \widehat{\mathbf{\Upsilon}}_{ab}(\tau)]$ are constructed as for (24) but for d-metrics and respective geometric objects with τ-parameter dependence. A normalization function $\widehat{f}(\tau, u)$ has to be introduced for proofs of such systems of nonlinear PDEs from certain nonholonomically generalized F- and W-functionals (see next section and [18, 19, 20, 21]). Various classes of exact and parametric solutions describing geometric flow evolution of off-diagonal stationary and cosmological configurations can be described using the AFDM [33, 34, 35, 24, 26].

We emphasize that all systems of PDEs for geometric and physical models on $\mathbf{V}$ can be expressed in different equivalent forms using different geometric data $(\mathbf{g}, \mathbf{N}, \nabla) \Leftrightarrow (\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}}) \Leftrightarrow (\mathbf{g}, \mathbf{N}, \mathbf{D})$, involving respective distortion relations. Certain classes of nonholonomic variables (geometric data) are convenient, for instance, for elaborating canonical methods of quantization, other type ones can be useful for finding exact solutions. In this work, we shall give priority to $\widehat{\mathbf{D}}$ because such a formalism allows us to encode directly into geometric flow and information theories various classes of generic off-diagonal solutions.

2.4 G. Perelman functionals and geometric thermodynamic models for gravity

The aim of this subsection is to define nonholonomic canonical modifications of the F- and W-functionals [27] which allow us to prove the nonholonomic geometric flow equations (25). The W-entropy will be used for constructing an associated statistical thermodynamic model for geometric flows of NES.

2.4.1 F- and W-functionals for nonholonomic Einstein systems

G. Perelman entropic like functionals can be postulated using different types of nonholonomic variables with conventional 2+2 and 3+1 decomposition of dimensions or double fibration splitting [19, 20, 21].

In nonholonomic canonical variables, the relativistic versions of G. Perelman functionals are postulated

$$
\begin{aligned}
\widehat{\mathcal{F}} &= \int (4\pi\tau)^{-2} e^{-\widehat{f}}\sqrt{|\mathbf{g}|}d^4u(\ {}_{s}\widehat{R} + |\widehat{\mathbf{D}}\widehat{f}|^2) \text{ and} \qquad (27)\\
\widehat{\mathcal{W}} &= \int \widehat{\mu}\sqrt{|\mathbf{g}|}d^4u[\tau(\ {}_{s}\widehat{R} + | \ {}_{h}\widehat{\mathbf{D}}\widehat{f}| + | \ {}_{v}\widehat{\mathbf{D}}\widehat{f}|)^2 + \widehat{f} - 4], \qquad (28)
\end{aligned}
$$

where the normalizing function $\widehat{f}(\tau, u)$ is subjected to the conditions $\int \widehat{\mu}\sqrt{|\mathbf{g}|}d^4u = \int_{t_1}^{t_2}\int_{\Xi_t} \widehat{\mu}\sqrt{|\mathbf{g}|}d^4u = 1$, for a classical integration measure $\widehat{\mu} = (4\pi\tau)^{-2} e^{-\widehat{f}}$ and the Ricci scalar ${}_{s}\widehat{R}$ is taken for the Ricci d-tensor $\widehat{\mathbf{R}}_{\alpha\beta}$ of a d-connection $\widehat{\mathbf{D}}$.

Applying a N-adapted variational calculus on $\mathbf{g}_{\alpha\beta}$ (27) or (28) (see similar details in [19, 20, 21] and, for Riemannian configurations, in [29, 30, 31]), we can prove the geometric flow evolution equations (25) for NES. Here we note that the functional $\widehat{\mathcal{W}}$ (28) defines a nonholonomic canonical and relativistic generalizations of so-called W-entropy introduced in [27]. Various types of 4-d - 10-d $\mathcal{W}$–entropies and associated statistical and quantum thermodynamics values are used for elaborating models of classical and (commutative and noncommutative/ supersymmetric) quantum geometric flows and geometric information flows, see [36, 37, 18, 19, 20, 21] and our partner works [38, 39, 40].

2.4.2 Thermodynamic models for (modified) Einstein geometric flows

A geometric flow evolution of NES can be characterized by analogous thermodynamic models. Respective geometric and statistical thermodynamic values can be defined in nonholonomic canonical variables (with hats, which is different from tilde, underlined and other type variables used for GIF and QGIF theories in partner works [24, 38, 39]).[9]

We associate to (28) a respective thermodynamic generating functions defined in canonical variables

$$\widehat{\mathcal{Z}}[\mathbf{g}(\tau)] = \int (4\pi\tau)^{-2} e^{-\widehat{f}} \sqrt{|\mathbf{g}|} d^4u(-\widehat{f}+2), \text{ for } \mathbf{V}. \tag{29}$$

Such values are with functional dependence on $\mathbf{g}(\tau)$ (we shall not write this in explicit forms if it do not result in ambiguities). A density state defined as in footnote 9 is a functional $\widehat{\sigma}[\mathbf{g}(\tau)] = \widehat{\mathcal{Z}}^{-1}e^{-\beta E}$. We can consider also the geometric evolution densities $\widehat{\rho}[\ _1\mathbf{g}]$ and $\widehat{\rho}'[\ _1\mathbf{g}]$, where the left label 1 is used in order to distinguish two d-metrics $\mathbf{g}$ and $\ _1\mathbf{g}$.

Using (29) and (28) and respective 3+1 parameterizations of d-metrics (see formulas in the last footnote and (13) with a time like coordinate $y^4 = t$ and temperature like evolution parameter τ), we define and compute analogous thermodynamic values for geometric evolution flows of NES,

$$\begin{aligned}
\widehat{\mathcal{E}} &= -\tau^2 \int (4\pi\tau)^{-2} e^{-\widehat{f}} \sqrt{|q_1 q_2 \mathbf{q}_3 (\,_qN)|}\delta^4 u(\ _s\widehat{R} + |\widehat{\mathbf{D}}\widehat{f}|^2 \ - \frac{2}{\tau}), \\
\widehat{\mathcal{S}} &= -\int (4\pi\tau)^{-2} e^{-\widehat{f}} \sqrt{|q_1 q_2 \mathbf{q}_3 (\,_qN)|}\delta^4 u \left[\tau\left(\ _s\widehat{R} + |\widehat{\mathbf{D}}\widehat{f}|^2\right) + \widehat{f} - 4\right], \\
\widehat{\eta} &= -2\tau^4 \int (4\pi\tau)^{-2} e^{-\widehat{f}} \sqrt{|q_1 q_2 \mathbf{q}_3 (\,_qN)|}\delta^4 u[|\ \widehat{\mathbf{R}}_{\alpha\beta} + \widehat{\mathbf{D}}_\alpha\ \widehat{\mathbf{D}}_\beta \widehat{f} - \frac{1}{2\tau}\mathbf{g}_{\alpha\beta}|^2],
\end{aligned} \tag{30}$$

where $\delta^4 u$ contains N-elongated differentials in order to compute such integrals in N-adapted forms. Using such values, we can compute in canonical variables the respective free energy and relative entropy,

$$\widehat{\mathcal{F}}\ (\ _1\mathbf{g}) = \widehat{\mathcal{S}}(\ _1\mathbf{g}) - \beta^{-1}\widehat{\mathcal{S}}(\ _1\mathbf{g}) \text{ and } \widehat{\mathcal{S}}(\ _1\mathbf{g} \shortparallel \mathbf{g}) = \beta[\widehat{\mathcal{F}}(\ _1\mathbf{g}) - \widehat{\mathcal{F}}(\mathbf{g})], \text{ where}$$

$$\begin{aligned}
\widehat{\mathcal{E}}(\ _1\mathbf{g}) &= -\tau^2 \int (4\pi\tau)^{-2} e^{-\widehat{f}} \sqrt{|q_1 q_2 \mathbf{q}_3 (\,_qN)|}\delta^4 u[\ _s\widehat{R}(\ _1\underline{\mathbf{g}}) + |\widehat{\mathbf{D}}(\ _1\mathbf{g})\widehat{f}(\tau,u)|^2 \ - \frac{2}{\tau}], \\
\widehat{\mathcal{S}}(\ _1\mathbf{g}) &= -\int (4\pi\tau)^{-2} e^{-\widehat{f}} \sqrt{|q_1 q_2 \mathbf{q}_3 (\,_qN)|}\delta^4 u \left[\tau\left(\ _s\widehat{R}(\ _1\mathbf{g}) + |\widehat{\mathbf{D}}(\ _1\mathbf{g})\widehat{f}(\tau,u)|^2\right) + \widehat{f}(\tau,u) - 4\right].
\end{aligned} \tag{31}$$

[9] Let us remember some most important concepts and formulas from statistical thermodynamics. A partition function $Z = \int \exp(-\beta E) d\omega(E)$ is considered for a canonical ensemble at temperature $\beta^{-1} = T$ and a measure $\omega(E)$ as density of states. In standard form, there are computed such canonical values: average flow energy, $\mathcal{E} = \langle E \rangle := -\partial \log Z/\partial\beta$; flow entropy, $\mathcal{S} := \beta \langle E \rangle + \log Z$; flow fluctuation, $\eta := \langle (E - \langle E \rangle)^2 \rangle = \partial^2 \log Z/\partial\beta^2$.

A partition function (equivalently, thermodynamic generating function) Z allows us to define a conventional *state density* (for quantum models, a *density matrix*) $\sigma(\beta, E) = Z^{-1}e^{-\beta E}$. The *relative entropy* between two state densities ρ and σ is defined/computed

$$\mathcal{S}(\rho \shortparallel \sigma) := -\mathcal{S}(\rho) + \int (\beta\mathcal{E} + \log Z)\rho d\omega(E) = \beta[\mathcal{E}(\rho) - T\mathcal{S}(\rho)] + \log Z.$$

In this formula, the *average energy* is computed for the density matrix ρ, $\mathcal{E}(\rho) = \int \mathcal{E}\rho d\omega(E)$, and the formula $\log\sigma = -\beta\mathcal{E} - \log Z$ is used. We define the *free energy* using formula $\mathcal{F}(\rho) := \mathcal{E}(\rho) - T\mathcal{S}(\rho)$. If $\log Z$ is independent on ρ, we get $\mathcal{S}(\sigma \shortparallel \sigma) = 0$ and $\mathcal{S}(\rho \shortparallel \sigma) = \beta[\mathcal{F}(\rho) - \mathcal{F}(\sigma)]$. Elaborating geometric flow evolution and analogous thermodynamics systems, we consider that under evolution it is preserved the thermal equilibrium at temperature β with maps of density states $\rho \to \rho'$ keeping the same density state σ. Such systems are characterized by inequalities $\mathcal{S}(\rho \shortparallel \sigma) \geq \mathcal{S}(\rho' \shortparallel \sigma)$, and $\mathcal{F}(\rho) \geq \mathcal{F}(\rho')$.

Finally, we conclude that generating functions (29) and respective thermodynamical values (30) can be written equivalently in terms of the canonical d–connections $\widehat{\mathbf{D}}$ and/or for an another type $\mathbf{D}$ if we consider nonholonomic deformations to certain systems of nonlinear PDEs on $\mathbf{V}$.

3 Classical and quantum geometric information flows and gravity

We formulate the theory of (quantum) geometric information flows (respectively, GIFs and QGIFs) and nonholonomic Einstein systems, NES.

3.1 The geometric information flow theory of nonholonomic Einstein systems

We consider for QGIGs and NES the basic aspects of classical information theory with fundamental concepts of Shannon[10], conditional and relative entropies and applications in modern physics, see [1, 2, 44, 45, 46] and references therein. There are used the W-entropy functional (28) and the associated thermodynamical models elaborated in section 2.4.2 and partner works [38, 39].

For classical GIFs and NES, the canonical thermodynamic values are determined by data $\left[\widehat{\mathcal{W}};\widehat{\mathcal{Z}},\widehat{\mathcal{E}},\widehat{\mathcal{S}},\widehat{\eta}\right]$, see (28) and (30). We can introduce probabilities on a discrete network with random variables, for instance, $\widehat{p}_{\underline{n}} = 2^{-\ \widehat{\mathcal{E}}(b_{\underline{n}})}$. Continuous GIF models encoding geometric evolution of NES in canonical covariant variables are characterized by the thermodynamic entropy $\widehat{\mathcal{S}}[\mathbf{g}(\tau)]$ and/or the W-entropy $\widehat{\mathcal{W}}[\mathbf{g}(\tau)]$ (28) and certain constructions without statistical thermodynamics values. NES under canonical geometric evolution flows are denoted in general form as $\widehat{B} = \widehat{B}[\mathbf{g}(\tau)]$; such systems are determined by flows of corresponding canonical d-metrics on nonholonomic Lorentz spacetimes.

Now, let us consider how to construct models of information thermodynamics determined by geometric flows of NES.[11] Conventionally, we work with two such GIFs and NES, $\widehat{A} = \widehat{A}[\mathbf{g}(\tau)]$ and $\widehat{B} = \widehat{B}[\ _{1}\mathbf{g}(\tau)]$. To study conditional GIFs of gravitational systems we shall use geometric flow models on $\mathbf{V}\otimes\mathbf{V}$ when the local coordinates are $(u,\ _{1}u)$ and the normalizing functions are of type $\ _{AB}\widehat{f}(u,\ _{1}u)$. A d-metric structure on such tensor products of nonholonomic Lorentz manifolds can be parameterized in the form $\ _{AB}\mathbf{g} = \{\mathbf{g} = [q_1, q_2, \mathbf{q}_3,_{q} N],\ _{1}\mathbf{g} = [\ _{1}q_1,\ _{1}q_2,\ _{1}\mathbf{q}_3,_{1q} N]\}$. Respectively, we can define, for instance, a canonical d–connection $\ _{AB}\widehat{\mathbf{D}} =\ _{A}\widehat{\mathbf{D}} +\ _{B}\widehat{\mathbf{D}}$ and corresponding scalar curvature $\ _{sAB}\widehat{R} =\ _{s}\widehat{R} +\ _{s1}\widehat{R}$.

[10] Let B is a random variable taking certain values $b_{\underline{1}}, b_{\underline{2}}, ..., b_{\underline{k}}$, for instance, as a long message of symbols $\underline{N} \gg 1$ containing different k letters. The respective probabilities to observe such values are denoted $p_{\underline{1}}, p_{\underline{2}}, ..., p_{\underline{k}}$. The Shannon entropy is defined $S_B := -\sum_{\underline{j}=1}^{\underline{k}} p_{\underline{j}} \log p_{\underline{j}} \geq 0$ for $\sum_{\underline{j}=1}^{\underline{k}} p_{\underline{j}} = 1$, when $\underline{N}S_B$ is the number of bits of information which can be extracted from a message with $\underline{N}$ symbols. One could be correlations between letters for a more complex random process. In the "ideal gaze" limit (ignoring correlations), we approximate the entropy of a long message to be $\underline{N}S$ with the entropy S for a message consisting of only one letter. For a statistical thermodynamical model and a classical Hamiltonian H, we determin the probability of a i-th symbol b_i via formula $p_{\underline{i}} = 2^{-H(b_{\underline{i}})}$.

[11] Let us remember some basic concepts from the classical information theory. We consider a message with many letters when any letter is a random variable X taking possible values $x_{\underline{1}}, ..., x_{\underline{k}}$. A receiver get a random variable Y defined by letters $y_{\underline{1}}, ..., y_{\underline{r}}$. The goal is to compute how many bits of information a receiver will obtain form a message with N letters (with random variables $X, Y, Z, ...$). For one variable, the probability to observe $X = x_{\underline{i}}$ is denoted $P_X(x_{\underline{i}})$ for $\sum_{\underline{i}} P_X(x_{\underline{i}}) = 1$. A sender and a receiver communicate via a random process of two variables defined by a joint distribution $P_{X,Y}(x_{\underline{i}}, y_{\underline{j}})$ as the probabilities, respectively, to send is $X = x_{\underline{i}}$ and to hear is $Y = y_{\underline{j}}$. $P_Y(y_{\underline{j}}) = \sum_{\underline{i}} P_{X,Y}(x_{\underline{i}}, y_{\underline{j}})$ is the probability to receive $Y = y_{\underline{j}}$ when summation is taken over all choices that could be send. By definition, the *conditional probability* $P_{X|Y}(x_{\underline{i}}|y_{\underline{j}}) := \frac{P_{X,Y}(x_{\underline{i}}, y_{\underline{j}})}{P_Y(y_{\underline{j}})}$ is a value characterizing receiving $Y = y_{\underline{j}}$. We estimate the probability that it was sent x_i. For receiver's messages,
$P_X(x_{\underline{i}}) = \sum_{\underline{j}} P_{X,Y}(x_{\underline{i}}, y_{\underline{j}})$ or we can consider $P_X(x_{\underline{i}})$ as an independent probability density.

There are defined such important values: $S_{X|Y=y_j} := -\sum_{\underline{i}} P_{X|Y}(x_{\underline{i}}|y_{\underline{j}}) \log P_{X|Y}(x_{\underline{i}}|y_{\underline{j}})$ is the Shanon entropy of the conditional probability; $S_{XY} := -\sum_{\underline{i},\underline{j}} P_{X,Y}(x_{\underline{i}}, y_{\underline{j}}) \log P_{X,Y}(x_{\underline{i}}, y_{\underline{j}})$ is the entropy of the joint distribution;
$S_Y := -\sum_{\underline{i},\underline{j}} P_{X,Y}(x_{\underline{i}}, y_{\underline{j}}) \log P_Y(y_{\underline{j}})$ is the total received information content; $S_X := -\sum_{\underline{i},\underline{j}} P_{X,Y}(x_{\underline{i}}, y_{\underline{j}}) \log P_X(x_{\underline{i}})$ is the total sent information content. Using such formulas, the *conditional entropy* is by definition $S_{X|Y} := \sum_{\underline{j}} P_Y(y_{\underline{j}}) S_{X|Y=y_j} = S(X|Y) = S_{XY} - S_Y \geq 0$. The *mutual information* between X and Y is defined $I(X;Y) := S_X - S_{XY} + S_Y \geq 0$ which is a measure of how much we learn about X observing Y.

The canonical thermodynamic GIF and NES entropies for respective systems are $\widehat{\mathcal{S}}[\widehat{A}]$ and $\widehat{\mathcal{S}}[\widehat{B}]$ being respectively defined by $\mathbf{g}(\tau)$ and $\ _1\mathbf{g}(\tau)$ as in (30). They can be considered as analogs of S_X and S_Y used in the last footnote. As an analog of S_{XY} for GIFs, we introduce the thermodynamic generating function (as a generalization of (29))

$$_{AB}\widehat{\mathcal{Z}}[\mathbf{g}(\tau),\ _1\mathbf{g}(\tau)] = \int\ _1\int (4\pi\tau)^{-4} e^{-\ _{AB}\widehat{f}} \sqrt{|\ \mathbf{g}|}\sqrt{|\ _1\mathbf{g}|} d^4u\ d^4\ _1u(-\ _{AB}f + 8),\ \text{for } \mathbf{V}\otimes\mathbf{V}. \qquad (32)$$

This results in a GIF NES canonical thermodynamic entropy function

$$\begin{aligned} _{AB}\widehat{\mathcal{S}} = \widehat{\mathcal{S}}\ [\widehat{A},\widehat{B}] \ &= \ -\int\ _1\int (4\pi\tau)^{-4} e^{-\ _{AB}\widehat{f}} \sqrt{|q_1 q_2 \mathbf{q}_3({}_qN)|}\sqrt{|\ _1q_1\ _1q_2\ \ _1\mathbf{q}_3({}_{1q}N)|}\delta^4u\ \delta^4\ _1u \\ & \left[\tau\left(\ _s\widehat{R} +\ _{s1}\widehat{R} + |\ \widehat{\mathbf{D}}\ _{AB}\widehat{f} +\ _1\widehat{\mathbf{D}}\ _{AB}\widehat{f}|^2\right) +\ _{AB}\widehat{f} - 8\right]. \end{aligned} \qquad (33)$$

Using these values, we claim (proofs can be performed in any point of respective causal curves on Lorentz manifolds) that the formulas for the conditional entropy and mutual information are respectively generalized for GIFs of NES,

$$\widehat{\mathcal{S}}\ [\widehat{A}|\widehat{B}] :=\ _{AB}\widehat{\mathcal{S}} - \widehat{\mathcal{S}}[\widehat{B}] \geq 0 \text{ and } \widehat{\mathcal{J}}\ [\widehat{A};\widehat{B}] := \widehat{\mathcal{S}}[\widehat{A}] -\ _{AB}\widehat{\mathcal{S}} + \widehat{\mathcal{S}}[\widehat{B}] \geq 0.$$

Similar claims can be formulated for the W-entropy $\widehat{\mathcal{W}}[\mathbf{g}(\tau)]$ (28),

$$\widehat{\mathcal{W}}\ [\widehat{A}|\widehat{B}] :=\ _{AB}\widehat{\mathcal{W}} - \widehat{\mathcal{W}}[\widehat{B}] \geq 0 \text{ and } \widehat{\mathcal{J}}\ [\widehat{A};\widehat{B}] := \widehat{\mathcal{W}}[\widehat{A}] -\ _{AB}\widehat{\mathcal{W}} + \widehat{\mathcal{W}}[\widehat{B}] \geq 0.$$

These formulas can be are computed respectively for the W–entropy instead of the S-entropy used in the standard probability theory and generalizations in information theory.

We can define and calculate the relative entropy S and mutual information I between two distributions following definitions of the standard probability and information theory[12],

$$S(P_X||Q_X) := \sum_{i,j} P_{X,Y}(x_{\underline{i}}, y_{\underline{j}})[\log P_{X,Y}(x_{\underline{i}}, y_{\underline{j}}) - \log(P_X(x_{\underline{i}})P_Y(y_{\underline{j}}))] = S_X - S_{XY} + S_Y = I(X;Y);$$

$$\begin{aligned} S(P_{X,Y}||Q_{X,Y}) &:= S_X - S_{XY} + S_Y = I(X;Y); \\ S(P_{X,Y,Z}||Q_{X,Y,Z}) &:= S_{XY} - S_{XYZ} - S_{YZ} = I(X;YZ). \end{aligned}$$

Such values are subjected to important inequalities

$$\begin{aligned} & I(X;Y) := S_X + S_Y - S_{XY} \geq 0,\ \text{subadditivity of entropy ;} \\ & S(P_{X,Y}||Q_{X,Y}) \geq S(P_X||Q_X), S(P_{X,Y,Z}||Q_{X,Y,Z}) \geq S(P_{X,Y}||Q_{X,Y}),\ \text{monotonicity of relative entropy.} \end{aligned}$$

[12]For convenience, we remember some basic formulas on relative entropy and mutual information which are necessary for considerations in this paper. In this paragraph, we do not use "hats" on respective symbols because such values can be defined in general form not obligatory encoding canonical nonholonomic variables. The relative entropy is introduced for two probability distributions P_X and Q_X. Considering $X = x_{\underline{i}}$, with $\underline{i}= \{1, 2, ...s\}$, we state $p_{\underline{i}} = P_X(x_{\underline{i}})$ and $q_{\underline{i}} = Q_X(x_{\underline{i}})$ for some long messages with $\underline{N}$ letters. Our goal is to decide which distribution describes a random process more realistically. Let us define the relative entropy per observation $S(P_X||Q_X) := \sum_{\underline{i}} p_{\underline{i}}(\log p_{\underline{i}} - \log q_{\underline{i}}) \geq 1$ under the assumption that $\underline{N}S(P_X||Q_X) \gg 1$. This value is asymmetric both on P_X and Q_X and measures the difference between these two probability distributions (it is considered that P_X is for the correct answer and Q_X is taken as an initial hypothesis).

At the next step, we can consider a pair of random variables X and Y and respective two probability distributions. The fist one is taken as a possible correlated joint distribution $P_{X,Y}(x_{\underline{i}}, y_{\underline{j}})$ and $P_X(x_{\underline{i}}) := \sum_{\underline{j}} P_{X,Y}(x_{\underline{i}}, y_{\underline{j}}), P_Y(y_{\underline{j}}) := \sum_{\underline{i}} P_{X,Y}(x_{\underline{i}}, y_{\underline{j}})$. We also use a second probability distribution $Q_{X,Y}(x_{\underline{i}}, y_{\underline{j}}) = P_X(x_{\underline{i}})\ P_Y(y_{\underline{j}})$ defined in a form ignoring correlations between X and Y. In general, $Q_{X,Y}(x_{\underline{i}}, y_{\underline{j}})$ can be with correlations of type $Q_X(x_{\underline{i}}) := \sum_{\underline{j}} Q_{X,Y}(x_{\underline{i}}, y_{\underline{j}})$. We can introduce three random variables X, Y, Z described by a joint probability distribution and related values, $P_{X,Y,Z}(x_{\underline{i}}, y_{\underline{j}}, z_{\underline{k}})$ and $P_X(x_{\underline{i}}) := \sum_{\underline{j},\underline{k}} P_{X,Y,Z}(x_{\underline{i}}, y_{\underline{j}}, z_{\underline{k}}), P_{Y,Z}(y_{\underline{j}}, z_{\underline{k}}) := \sum_{\underline{i}} P_{X,Y,Z}(x_{\underline{i}}, y_{\underline{j}}, z_{\underline{k}})$. Ignoring the correlations between X and YZ, we define $Q_{X,Y,Z}(x_{\underline{i}}, y_{\underline{j}}, z_{\underline{k}}) := P_X(x_{\underline{i}})P_{Y,Z}(y_{\underline{j}}, z_{\underline{k}})$. Other type values can be defined for observation of the subsystem XY, when $P_{X,Y}(x_{\underline{i}}, y_{\underline{j}}) := \sum_{\underline{k}} P_{X,Y,Z}(x_{\underline{i}}, y_{\underline{j}}, z_{\underline{k}}), Q_{X,Y}(x_{\underline{i}}, y_{\underline{j}}) := \sum_{\underline{k}} Q_{X,Y,Z}(x_{\underline{i}}, y_{\underline{j}}, z_{\underline{k}}) = P_X(x_{\underline{i}})P_Y(y_{\underline{j}})$.

For three random variables, we can introduce also the concepts and condition of strong subadditivity

$$S_X - S_{XYZ} - S_{YZ} \geq S_X - S_{XY} + S_Y, \text{ or } S_{XY} + S_{YZ} \geq S_Y + S_{XYZ},$$

which is equivalent for the condition of monotonity of mutual information $I(X;YZ) \geq I(X;Y)$.

Above definitions and formulas for S and I can be generalized respectively for the relative entropy and mutual information of GIFs and NES. Proofs can be performed for causal lines and nonholonomic variables generated by some thermodynamic generating functions ${}_A\widehat{\mathcal{Z}} := \widehat{\mathcal{Z}}[\ \mathbf{g}(\tau)]$ and ${}_B\widehat{\mathcal{Z}} := {}_1\widehat{\mathcal{Z}}[\ {}_1\mathbf{g}(\tau)]$, see (29), as analogs of certain values $p_i = P_X(x_i)$ and $q_i = Q_X(x_i)$. We can consider GIFs of three NES $\widehat{A}, \widehat{B}$ and $\widehat{C}$ and prove using standard methods in any point of causal curves and applying explicit integral N-adapted calculations on $\mathbf{V} \otimes \mathbf{V} \otimes \mathbf{V}$ that

$$\begin{aligned} \widehat{\mathcal{J}}\ [\widehat{A};\widehat{B}] &:= \widehat{\mathcal{S}}[\widehat{A}] - {}_{AB}\widehat{\mathcal{S}} + \widehat{\mathcal{S}}[\widehat{B}] \geq 0, \text{ subadditivity of entropy;} \\ \widehat{\mathcal{S}}\ [\ {}_{AB}\widehat{\mathcal{Z}} \| \ {}_{AB}\widehat{\mathcal{Z}}] &\geq \widehat{\mathcal{S}}\ [\ {}_{A}\widehat{\mathcal{Z}} \| \ {}_{A}\widehat{\mathcal{Z}}], \widehat{\mathcal{S}}\ [\ {}_{ABC}\widehat{\mathcal{Z}} \| \ {}_{ABC}\widehat{\mathcal{Z}}] \geq \widehat{\mathcal{S}}\ [\ {}_{AB}\widehat{\mathcal{Z}} \| \ {}_{AB}\widehat{\mathcal{Z}}], \text{ monotonicity of relative entropy.} \end{aligned}$$

The conditions of strong subadditivity for GIFs and NES entropies are stated by formulas

$${}_A\widehat{\mathcal{S}} - {}_{ABC}\widehat{\mathcal{S}} - {}_{BC}\widehat{\mathcal{S}} \geq {}_A\widehat{\mathcal{S}} - {}_{AB}\widehat{\mathcal{S}} + {}_B\widehat{\mathcal{S}}, \text{ or } {}_{AB}\widehat{\mathcal{S}} + {}_{BC}\widehat{\mathcal{S}} \geq {}_B\widehat{\mathcal{S}} + {}_{ABC}\widehat{\mathcal{S}}.$$

In equivalent form, these formulas can be written as the condition of monotonicity of GIFs and NES mutual information, $\widehat{\mathcal{J}}\ [\widehat{A};\widehat{B}\widehat{C}] \geq \widehat{\mathcal{J}}\ [\widehat{A};\widehat{B}]$.

The inequalities claimed above can be proven for any point along causal curves on $\mathbf{V}$. For three systems, there are involved thermodynamic generating functions generalizing (29) in the form,

$$\begin{aligned} {}_{ABC}\widehat{\mathcal{Z}}[\mathbf{g}(\tau),\ {}_1\mathbf{g}(\tau),\ {}_2\mathbf{g}(\tau)] &= \int {}_1\int {}_2\int (4\pi\tau)^{-6} e^{-\ {}_{ABC}\widehat{f}} \sqrt{|\ \mathbf{g}|}\sqrt{|\ {}_1\mathbf{g}|}\sqrt{|\ {}_2\mathbf{g}|} d^4u\ d^4\ {}_1u\ d^4\ {}_2u \\ & (-\ {}_{ABC}\widehat{f} + 6), \text{ for } \mathbf{V} \otimes \mathbf{V} \otimes \mathbf{V}, \end{aligned} \tag{34}$$

with a normalizing function ${}_{ABC}\widehat{f}(\ u,\ {}_1u,\ {}_2u)$. On such tensor products of nonholonomic Lorentz manifolds, the d-metric structure is ${}_{ABC}\mathbf{g} = \{\mathbf{g} = [q_1, q_2, \mathbf{q}_3, {}_qN],\ {}_1\mathbf{g} = [\ {}_1q_1,\ {}_1q_2,\ {}_1\mathbf{q}_3, {}_{1q}N],\ {}_2\mathbf{g} = [\ {}_2q_1,\ {}_2q_2,\ {}_2\mathbf{q}_3, {}_{2q}N]\}$. We can define a canonical d–connection ${}_{ABC}\widehat{\mathbf{D}} = \widehat{\mathbf{D}} + {}_B\widehat{\mathbf{D}} + {}_C\widehat{\mathbf{D}}$ and respective scalar curvature ${}_{sABC}\widehat{R} = {}_s\widehat{R} + {}_{s1}\widehat{R} + {}_{s2}\widehat{R}$. The resulting entropy function

$$\begin{aligned} {}_{ABC}\widehat{\mathcal{S}} &= \widehat{\mathcal{S}}[\widehat{A},\widehat{B},\widehat{C}] = -\int {}_1\int {}_2\int (4\pi\tau)^{-6} e^{-\ {}_{ABC}\widehat{f}} \sqrt{|\mathbf{g}|}\sqrt{|\ {}_1\mathbf{g}|}\sqrt{|\ {}_2\mathbf{g}|} d^4u\ d^4\ {}_1\underline{u}\ d^4\ {}_2u \\ & \left[\tau\left(\ {}_s\widehat{R} + \ {}_{s1}\widehat{R} + \ {}_{s2}\widehat{R} + |\widehat{\mathbf{D}}\ {}_{ABC}\widehat{f} + \ {}_1\widehat{\mathbf{D}}\ {}_{ABC}\widehat{f} + \ {}_2\widehat{\mathbf{D}}\ {}_{ABC}\widehat{f}|^2\right) + \ {}_{ABC}\widehat{f} - 12\right]. \end{aligned} \tag{35}$$

Similar formulas are considered for relativistic mechanical QGIFs in our partner works [38, 39].

3.2 Density matrix for quantum geometric information and gravitational flows

The goal of this subsection is to study how GIFs of NES can be generalized using basic concepts of QM and information theory. We shall elaborate on QGIFs described in terms of density matrices defined as quantum analogs of state densities of type $\widehat{\sigma}[\mathbf{g}(\tau)] = \widehat{\mathcal{Z}}^{-1} e^{-\beta E}$ with $\widehat{\mathcal{Z}}[\mathbf{g}(\tau)]$ (29).

Let us consider a thermodynamical model $\widehat{\mathcal{A}} = \left[\widehat{\mathcal{W}}; \widehat{\mathcal{Z}}, \widehat{\mathcal{E}}, \widehat{\mathcal{S}}, \widehat{\eta}\right]$ (30). In any point $u \in \mathbf{V}$ along a causal curve covering an open region with such points, we associate a typical Hilbert space $\widehat{\mathcal{H}}_{\mathcal{A}}$. A state vector $\psi_{\mathcal{A}} \in \widehat{\mathcal{H}}_{\mathcal{A}}$ is an infinite dimensional complex vector function. In quantum information theory, such a value is approximated to a vector in complex spaces of finite dimension. A vector $\psi_{\mathcal{A}}$ is a solution of the Schrödinger equation with as a well-defined quantum version of a canonical Hamiltonian $\widehat{\mathcal{H}}_{\mathcal{A}}$, see details in [1, 2] and, for nonholonomic systems, [38, 39].

For QGIFs of NES, the combined Hilbert space is defined as a tensor product, $\widehat{\mathcal{H}}_{\mathcal{AB}} = \widehat{\mathcal{H}}_{\mathcal{A}} \otimes \widehat{\mathcal{H}}_{\mathcal{B}}$, with an associate Hilbert space $\widehat{\mathcal{H}}_{\mathcal{A}}$ considered for a complementary system $\widehat{\mathcal{A}}$. The state vectors for a combined system are written $\psi_{\mathcal{AB}} = \psi_{\mathcal{A}} \otimes \psi_{\mathcal{B}} \in \widehat{\mathcal{H}}_{\mathcal{AB}}$ for $\psi_{\mathcal{A}} = 1_{\mathcal{A}}$ taken as the unity. A pure state $\psi_{\mathcal{AB}} \in \widehat{\mathcal{H}}_{\mathcal{AB}}$ may be not only a tensor product of complex vectors. A quantum system under geometric flow evolution can be also *entangled* and represented by a matrix of dimension $\underline{N} \times \underline{M}$ if $\dim \widehat{\mathcal{H}}_{\mathcal{A}} = \underline{N}$ and $\dim \widehat{\mathcal{H}}_{\mathcal{B}} = \underline{M}$ (we underline symbols for dimensions in order to avoid ambiguities with the N-connection symbol $\mathbf{N}$). A Schmidt decomposition can be performed for a pure state function,

$$\psi_{\mathcal{AB}} = \sum_{\underline{i}} \sqrt{p_{\underline{i}}} \psi_{\mathcal{A}}^{\underline{i}} \otimes \psi_{\mathcal{B}}^{\underline{i}}, \tag{36}$$

for any index $\underline{i} = 1, 2, ...$(up to a finite value). We can consider that a state vector $\psi_{\mathcal{A}}^{\underline{i}}$ which is orthonormal if $< \psi_{\mathcal{A}}^{\underline{i}}, \psi_{\mathcal{A}}^{\underline{j}} >=< \psi_{\mathcal{B}}^{\underline{i}}, \psi_{\mathcal{B}}^{\underline{j}} >= \delta^{\underline{ij}}$, where $\delta^{\underline{ij}}$ is the Kronecker symbol. Considering $p_{\underline{i}} > 0$ and $\sum_{\underline{i}} p_{\underline{i}} = 1$, we treat $p_{\underline{i}}$ as probabilities. In general, such $\psi_{\mathcal{A}}^{\underline{i}}$ and/or $\psi_{\mathcal{B}}^{\underline{i}}$ do not define bases of $\widehat{\mathcal{H}}_{\mathcal{A}}$ and/or $\widehat{\mathcal{H}}_{\mathcal{B}}$.

We define the quantum density matrix for a QGIF of NES $\widehat{\mathcal{A}}$ as $\widehat{\rho}_{\mathcal{A}} := \sum_{\underline{a}} p_{\underline{a}} |\psi_{\mathcal{A}}^{\underline{a}} >< \otimes \psi_{\mathcal{A}}^{\underline{a}}|$ as a Hermitian and positive semi-definite operator with trace $Tr_{\mathcal{H}_{\mathcal{A}}} \widehat{\rho}_{\mathcal{A}} = 1$. The hat symbol is used in order to emphasize that the constructions are associated to canonical nonholonomic variables and respective gravitational systems.

This allows us to compute the *expectation* value of any operator $\widehat{\mathcal{O}}_{\mathcal{A}}$ characterizing additionally such a system,

$$\begin{aligned} < \widehat{\mathcal{O}} >_{\mathcal{AB}} &= < \psi_{\mathcal{AB}}| \widehat{\mathcal{O}} \otimes 1_{\mathcal{B}} |\psi_{\mathcal{AB}} >= \sum_{\underline{i}} p_{\underline{i}} < \psi_{\mathcal{A}}^{\underline{i}} |\widehat{\mathcal{O}}| \psi_{\mathcal{A}}^{\underline{i}} >< \psi_{\mathcal{B}}^{\underline{i}} |1_{\mathcal{B}}| \psi_{\mathcal{B}}^{\underline{i}} >= \\ < \widehat{\mathcal{O}} >_{\mathcal{A}} &= \sum_{\underline{i}} p_{\underline{i}} < \psi_{\mathcal{A}}^{\underline{i}}| \widehat{\mathcal{O}}_{\mathcal{A}} |\psi_{\mathcal{A}}^{\underline{i}} >= Tr_{\mathcal{H}_{\mathcal{A}}} \widehat{\rho}_{\mathcal{A}} \widehat{\mathcal{O}}_{\mathcal{A}}. \end{aligned} \tag{37}$$

Here we note that for arbitrary nonholonomic frame transforms and deformations of d-connection, we can consider a general covariant form when $< \mathcal{O} >_{\mathcal{A}} = Tr_{\mathcal{H}_{\mathcal{A}}} \rho_{\mathcal{A}} \mathcal{O}_{\mathcal{A}}$, or other type nonholonomic variables with tilde (for mechanical like variables) are considered, see [38, 39].

To model both quantum information and geometric flow evolution of bipartite systems we consider GIF and NES of type $\widehat{\mathcal{A}}$, $\widehat{\mathcal{B}}$, and $\widehat{\mathcal{A}}\,\widehat{\mathcal{B}}$. Such quantum systems are with both quantum and geometric entanglement defined by density matrices. In general form, bipartite QGIFs and NES are described by quantum density matrices of type $\widehat{\rho}_{\mathcal{AB}}$.[13] Considering $\widehat{\mathcal{A}}\,\widehat{\mathcal{B}}$ as a bipartite quantum system with Hilbert space $\widehat{\mathcal{H}}_{\mathcal{AB}}$, we can define and parameterize a QGIF NES density matrix:

$$\widehat{\rho}_{\mathcal{AB}} = \sum_{\underline{a},\underline{a}',\underline{b},\underline{b}'} \widehat{\rho}_{\underline{aa}'\underline{bb}'} |\underline{a} >_{\mathcal{A}} \otimes |\underline{b} >_{\mathcal{B}} \ {}_{\mathcal{A}} < \underline{a}'| \otimes \ {}_{\mathcal{B}} < \underline{b}'|,$$

where $|\underline{a} >_{A}$, $\underline{a} = 1, 2, ..., \underline{n}$ is an orthonormal basis of $\mathcal{H}_{\mathcal{A}}$ and $|\underline{b} >_{\mathcal{B}}$, $\underline{b} = 1, 2, ..., \underline{m}$ is an orthonormal basis of $\widehat{\mathcal{H}}_{\mathcal{B}}$.

A *measurement* of the QGIFs and NES $\widehat{\mathcal{H}}$ is characterized by a *reduced density matrix*

$$\widehat{\rho}_{\mathcal{A}} = Tr_{\mathcal{H}_{\mathcal{B}}} \ \widehat{\rho}_{\mathcal{AB}} = \sum_{\underline{a},\underline{a}',\underline{b},\underline{b}} \widehat{\rho}_{\underline{aa}'\underline{bb}} |\underline{a} >_{\mathcal{A}} \ {}_{\mathcal{A}} < \underline{a}'|, \text{ for } |\underline{b} >_{\mathcal{B}} \ {}_{\mathcal{B}} < \underline{b}| = 1.$$

In a similar form, we can define and compute $\widehat{\rho}_{\mathcal{B}} = Tr_{\mathcal{H}_{\mathcal{A}}} \widehat{\rho}_{\mathcal{AB}}$. Using above introduced concepts and formulas, we can elaborate on QGIF models formulated in canonical variables or in a general covariant form.

Let us analyze the properties of quantum density matrix and von Neumann entropy for QGIFs and NES. For such systems, the quantum density matrix $\widehat{\sigma}_{\mathcal{AB}}$ for a state density $\widehat{\sigma}[\mathbf{g}(\tau)] = \widehat{\mathcal{Z}}^{-1} e^{-\beta E}$ can be defined and computed using formulas (37). We obtain

$$\begin{aligned} \widehat{\sigma}_{\mathcal{AB}} &= < \widehat{\sigma} >_{\mathcal{AB}} = < \psi_{\mathcal{AB}} |\widehat{\sigma} \otimes 1_{\mathcal{B}}| \psi_{\mathcal{AB}} >= \sum_{\underline{i}} p_{\underline{i}} < \psi_{\mathcal{A}}^{\underline{i}} |\widehat{\sigma}| \psi_{\mathcal{A}}^{\underline{i}} >< \psi_{\mathcal{B}}^{\underline{i}} |1_{\mathcal{B}}| \psi_{\mathcal{B}}^{\underline{i}} >= \\ \widehat{\sigma}_{\mathcal{A}} &= < \widehat{\sigma} >_{\mathcal{A}} = \sum_{\underline{i}} p_{\underline{i}} < \psi_{\mathcal{A}}^{\underline{i}} |\widehat{\sigma}| \psi_{\mathcal{A}}^{\underline{i}} >= Tr_{\mathcal{H}_{\mathcal{A}}} \widehat{\rho}_{\mathcal{A}} \widehat{\sigma}. \end{aligned} \tag{38}$$

[13] In classical theory of probability, a bipartite system XY by a *joint probability* distribution $P_{X,Y}(x_{\underline{i}}, y_{\underline{j}})$, where $P_X(x_{\underline{i}}) := \sum_{\underline{j}} P_{X,Y}(x_{\underline{i}}, y_{\underline{j}})$.

The density matrix $\widehat{\rho}_{\mathcal{A}}$ is taken for computing the density matrix $\widehat{\sigma}_{\mathcal{A}}$. The values (38) are determined by a state density of the thermodynamical model for GIFs of a classical NES determined by $\widehat{\sigma}$. We can work with quantum density matrices $\widehat{\sigma}_{\mathcal{AB}}$, $\widehat{\sigma}_{\mathcal{A}} = Tr_{\mathcal{H}_{\mathcal{B}}}\ \widehat{\sigma}_{\mathcal{AB}}$ and $\widehat{\sigma}_{\mathcal{B}} = Tr_{\mathcal{H}_{\mathcal{A}}}\widehat{\sigma}_{\mathcal{AB}}$. Such formulas can be written in respective coefficient forms

$$\widehat{\sigma}_{\mathcal{AB}} = \sum_{\underline{a},\underline{a}',\underline{b},\underline{b}'} \widehat{\sigma}_{\underline{a}\underline{a}'\underline{b}\underline{b}'}|\underline{a}>_{\mathcal{A}} \otimes |\underline{b}>_{\mathcal{B}}\ \ _{\mathcal{A}}<\underline{a}'| \otimes\ _{\mathcal{B}}<\underline{b}'| \text{ and } \widehat{\sigma}_{\mathcal{A}} = \sum_{\underline{a},\underline{a}',\underline{b},\underline{b}} \widehat{\sigma}_{\underline{a}\underline{a}'\underline{b}\underline{b}}|\underline{a}>_{\mathcal{A}}\ \ _{\mathcal{A}}<\underline{a}'|.$$

We conclude that QGIFs of NES can be characterized by quantum analogs of entropy values used for classical geometric flows. Such values can be computed using formulas of type (38) for classical conditional and mutual entropy, see details for GIFs and in information theory [1, 2, 38]. For instance,

$$\begin{aligned} {}_q\widehat{\mathcal{W}}_{\mathcal{AB}} &= Tr_{\mathcal{H}_{\mathcal{AB}}}[(\ \widehat{\sigma}_{\mathcal{AB}})(_{\mathcal{AB}}\widehat{\mathcal{W}})] \text{ and } {}_q\widehat{\mathcal{W}}_{\mathcal{A}} = Tr_{\mathcal{H}_{\mathcal{A}}}[(\ \widehat{\sigma}_{\mathcal{A}})(\ _{\mathcal{A}}\widehat{\mathcal{W}})],\ {}_q\widehat{\mathcal{W}}_{\mathcal{B}} = Tr_{\mathcal{H}_{\mathcal{B}}}[(\ \widehat{\sigma}_{\mathcal{B}})(\ _{\mathcal{B}}\widehat{\mathcal{W}})]; \\ {}_q\widehat{\mathcal{S}}_{\mathcal{AB}} &= Tr_{\mathcal{H}_{\mathcal{AB}}}[(\ \widehat{\sigma}_{\mathcal{AB}})(\ _{\mathcal{AB}}\widehat{\mathcal{S}})] \text{ and } {}_q\widehat{\mathcal{S}}_{\mathcal{A}} = Tr_{\mathcal{H}_{\mathcal{A}}}[(\ \widehat{\sigma}_{\mathcal{A}})(\ _{\mathcal{A}}\widehat{\mathcal{S}})],\ {}_q\widehat{\mathcal{S}}_{\mathcal{B}} = Tr_{\mathcal{H}_{\mathcal{B}}}[(\ \widehat{\sigma}_{\mathcal{B}})(\ _{\mathcal{B}}\widehat{\mathcal{S}})]. \end{aligned}$$

Such values describe additional entropic properties of quantum NES with rich geometric structure under QGIFs.

Let us consider quantum generalizations of the concept of W- and thermodynamic entropy of GIFs of NES. We describe such systems in standard QM form for the von Neumann entropy determined by $\widehat{\sigma}_{\mathcal{A}}$ (38) as a probability distribution,

$$_q\widehat{\mathcal{S}}(\ \widehat{\sigma}_{\mathcal{A}}) := Tr\ \ \widehat{\sigma}_{\mathcal{A}} \log\ \widehat{\sigma}_{\mathcal{A}}. \tag{39}$$

We can consider generalizations of this concept of quantum entropy for $\widehat{\mathcal{A}}\ \widehat{\mathcal{B}}$ and $\widehat{\mathcal{C}}$ systems, respectively,

$$_q\widehat{\mathcal{S}}(\ \widehat{\sigma}_{\mathcal{AB}}) := Tr\ \ \widehat{\sigma}_{\mathcal{AB}} \log\ \widehat{\sigma}_{\mathcal{AB}} \text{ and }\ \ _q\widehat{\mathcal{S}}(\ \widehat{\sigma}_{\mathcal{A}}) := Tr\ \ \widehat{\sigma}_{\mathcal{A}} \log\ \ \widehat{\sigma}_{\mathcal{A}},\ \ _q\widehat{\mathcal{S}}(\ \widehat{\sigma}_{\mathcal{B}})\ := Tr\ \ \widehat{\sigma}_{\mathcal{B}} \log\ \widehat{\sigma}_{\mathcal{B}}.$$

The von Neumann entropy for QGIFs of NES, $_q\widehat{\mathcal{S}}(\ \widehat{\sigma}_{\mathcal{A}})$, has a purifying property not existing for classical analogs. For instance, considering bipartite systems $\psi_{\mathcal{AB}} = \sum_{\underline{i}} \sqrt{p_{\underline{i}}}\psi^{\underline{i}}_{\mathcal{A}} \otimes \psi^{\underline{i}}_{\mathcal{B}}$ and $\widehat{\sigma}_{\mathcal{A}} := \sum_{\underline{i}} p_{\underline{i}}|\psi^{\underline{i}}_{\mathcal{A}} > \otimes < \psi^{\underline{i}}_{\mathcal{A}}|$, we compute

$$\widehat{\sigma}_{\mathcal{A}} := \sum_{\underline{a},\underline{a}',\underline{b},\underline{b}} \sum_{\underline{k}} \widehat{\sigma}_{\underline{a}\underline{a}'\underline{b}\underline{b}} p_{\underline{k}}\ _{\mathcal{A}}<\underline{a}'||\psi^{\underline{k}}_{\mathcal{A}}>< \otimes\psi^{\underline{k}}_{\mathcal{A}}||\underline{a}>_{\mathcal{A}},\ \ \widehat{\sigma}_{\mathcal{B}} := \sum_{\underline{a},\underline{a}',\underline{b},\underline{b}} \sum_{\underline{k}} \widehat{\sigma}_{\underline{a}\underline{a}'\underline{b}\underline{b}} p_{\underline{k}}\ _{\mathcal{B}}<\underline{a}'||\psi^{\underline{k}}_{\mathcal{B}}>< \otimes\psi^{\underline{k}}_{\mathcal{B}}||\underline{b}>_{\mathcal{B}}. \tag{40}$$

Using $_q\widehat{\mathcal{S}}(\ \widehat{\sigma}_{\mathcal{A}})$ (39) and respective formulas (38) and (40) for classical conditional and mutual entropy considered for GIFs and NES and in information theory, there are defined and computed respectively

$$\begin{aligned} {}_q\widehat{\mathcal{W}}_{\mathcal{AB}} &= Tr_{\mathcal{H}_{\mathcal{AB}}}[(\ \widehat{\sigma}_{\mathcal{AB}})(\ _{\mathcal{AB}}\widehat{\mathcal{W}})] \text{ and } {}_q\widehat{\mathcal{W}}_{\mathcal{A}} = Tr_{\mathcal{H}_{\mathcal{A}}}[(\ \widehat{\sigma}_{\mathcal{A}})(\ _{\mathcal{A}}\widehat{\mathcal{W}})],\ {}_q\widehat{\mathcal{W}}_{\mathcal{B}} = Tr_{\mathcal{H}_{\mathcal{B}}}[(\ \widehat{\sigma}_{\mathcal{B}})(\ _{\mathcal{B}}\widehat{\mathcal{W}})]; \\ {}_q\widehat{\mathcal{S}}_{\mathcal{AB}} &= Tr_{\mathcal{H}_{\mathcal{AB}}}[(\ \widehat{\sigma}_{\mathcal{AB}})(\ _{\mathcal{AB}}\widehat{\mathcal{S}})] \text{ and } {}_q\widehat{\mathcal{S}}_{\mathcal{A}} = Tr_{\mathcal{H}_{\mathcal{A}}}[(\ \widehat{\sigma}_{\mathcal{A}})(\ _{\mathcal{A}}\widehat{\mathcal{S}})],\ {}_q\widehat{\mathcal{S}}_{\mathcal{B}} = Tr_{\mathcal{H}_{\mathcal{B}}}[(\ \widehat{\sigma}_{\mathcal{B}})(\ _{\mathcal{B}}\widehat{\mathcal{S}})]. \end{aligned}$$

Such values describe complimentary entropic properties of quantum NES systems with rich geometric structure under quantum GIF evolution.

3.3 Geometric flows of nonholonomic Einstein systems with entanglement

We study entanglement of QGIFs and NES using the notion of bipartite entanglement introduced for pure states and density matrices in description of finite-dimensional QM systems [1, 2, 12, 13]. For thermodynamic and QM analogs of gravitational GIFs, we consider a series of relevant entropic values related to G. Perelman's W-entropy. A set of inequalities involving GIFs, NES, and entanglement entropies playing a crucial role in definition and description of such systems will be formulated.

3.3.1 Bipartite entanglement for quantum geometric information flows and gravity

For a NES and various type MGTs, we can consider various canonical, (relativistic) mechanic, continuous or lattice models of QFT, thermofield theory, QGIF models etc. A QM model can be characterized by a pure ground state $|\widehat{\Psi}>$ for a total Hilbert space ${}_{t}\widehat{\mathcal{H}}$. In this section, "hat" variables are used for all types of classical and quantum thermodynamic systems described in canonical nonholonomic variables. The density matrix

$$ {}_{t}\widehat{\rho} = |\widehat{\Psi}><\widehat{\Psi}| \tag{41} $$

can be normalized following the conditions $<\widehat{\Psi}|\widehat{\Psi}>=1$ and total trace ${}_{t}tr({}_{t}\widehat{\rho})=1$. We suppose that such a total quantum system is divided into a two subsystems $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$ associated to some analogous GIF and NES thermodynamic models. For instance, $\widehat{\mathcal{A}}(\mathbf{g}) = \left[\widehat{\mathcal{W}}(\mathbf{g}); \widehat{\mathcal{Z}}(\mathbf{g}), \widehat{\mathcal{E}}(\mathbf{g}), \widehat{\mathcal{S}}(\mathbf{g}), \widehat{\eta}(\mathbf{g})\right]$ (30) is determined by canonical functionals on a d-metric $\mathbf{g}$. Similarly, a second subsystem is generated by a d-metric ${}_{1}\mathbf{g}$ and respective $\widehat{\mathcal{B}}({}_{1}\mathbf{g}) = \left[\widehat{\mathcal{W}}({}_{1}\mathbf{g}); \widehat{\mathcal{Z}}({}_{1}\mathbf{g}), \widehat{\mathcal{E}}({}_{1}\mathbf{g}), \widehat{\mathcal{S}}({}_{1}\mathbf{g}), \widehat{\eta}({}_{1}\mathbf{g})\right]$. We can elaborate on models with the same $\mathbf{g}$ but with $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$ associated to different causal regions of a nonholonomic Lorentz spacetime $\mathbf{V}$. Two subsystems $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}} = \overline{\widehat{\mathcal{A}}}$ are complimentary to each other if there is a common boundary $\partial\widehat{\mathcal{A}} = \partial\widehat{\mathcal{B}}$ of codimension 2 and when the non-singular flow evolution $\widehat{\mathcal{A}}$ transforms into a necessary analytic class of flows on $\overline{\widehat{\mathcal{A}}}$. For such bipartite NES and QGIFs, we define ${}_{t}\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_{\mathcal{AB}} = \widehat{\mathcal{H}}_{\mathcal{A}} \otimes \widehat{\mathcal{H}}_{\mathcal{B}}$.

We introduce the measure of entanglement of a QGIF and NES as the von Neumann entropy ${}_{q}\widehat{\mathcal{S}}$ (39) but defined for the above considered total reduced density matrix $\widehat{\rho}_{\mathcal{A}} = Tr_{\mathcal{H}_{\mathcal{B}}}({}_{t}\widehat{\rho})$. For such canonical nonholonomic systems, it is possible to define and compute the *entanglement entropy* of $\underline{\mathcal{A}}$ as

$$ {}_{qt}\widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}}) := Tr(\widehat{\rho}_{\mathcal{A}} \log \widehat{\rho}_{\mathcal{A}}), \tag{42} $$

when $\widehat{\rho}_{\mathcal{A}}$ is associated to a state density of type $\widehat{\rho}(\beta, \widehat{\mathcal{E}}, \mathbf{g}) = \widehat{\rho}[\mathbf{g}(\tau)] = \widehat{\mathcal{Z}}^{-1} e^{-\beta E}$ with $\widehat{\mathcal{Z}}[\mathbf{g}(\tau)]$ (29). We note that the total entropy ${}_{qt}\widehat{\mathcal{S}} = 0$ for a pure grand state (41) associated to $\mathbf{V}$.

3.3.2 Separable and entangled gravitational and quantum geometric information flows

We can extend for NES in canonical nonholonomic variables the concepts were introduced for QGIFs in our partner works [38, 39] (those papers are based on analogous thermodynamic models standard constructions in quantum information theory [1, 2, 12, 13]). Let us consider $\{|\underline{a}>_{\mathcal{A}}; \underline{a} = 1, 2, ...k_a\} \in \widehat{\mathcal{H}}_{\mathcal{A}}$ and $\{|\underline{b}>_{\mathcal{B}}; \underline{b} = 1, 2, ...k_b\} \in \widehat{\mathcal{H}}_{\mathcal{B}}$ as orthonormal bases when a pure total ground state is parameterized in the form

$$ |\widehat{\Psi}> = \sum_{\underline{ab}} \widehat{C}_{\underline{ab}} |\underline{a}>_{\mathcal{A}} \otimes |\underline{b}>_{\mathcal{B}}, \tag{43} $$

where $\widehat{C}_{\underline{ab}}$ is a complex matrix of dimension $\dim \widehat{\mathcal{H}}_{\mathcal{A}} \times \dim \widehat{\mathcal{H}}_{\mathcal{B}}$. If such coefficients factorize, $\widehat{C}_{\underline{ab}} = \widehat{C}_{\underline{a}} \widehat{C}_{\underline{b}}$, there are defined separable ground states (equivalently, pure product states), when

$$ |\widehat{\Psi}> = |\widehat{\Psi}_{\mathcal{A}}> \otimes |\widehat{\Psi}_{\mathcal{B}}>, \text{ for } |\widehat{\Psi}_{\mathcal{A}}> = \sum_{\underline{a}} \widehat{C}_{\underline{a}} |\underline{a}>_{\mathcal{A}} \text{ and } |\widehat{\Psi}_{\mathcal{B}}> = \sum_{\underline{b}} \widehat{C}_{\underline{b}} |\underline{b}>_{\mathcal{B}}. $$

The entanglement entropy vanishes, ${}_{q}\widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}}) = 0$, if and only if the pure ground state is separable (we omit the label t as total considering that such quantum systems may involve bi- or multi-partition and respective total spaces). For NES and QGIFs, such definitions are motivated because all sub-systems are described by an effective thermodynamics energy, ${}_{\mathcal{A}}\widehat{\mathcal{E}}$ and ${}_{\mathcal{B}}\widehat{\mathcal{E}}$ as in (30). Similar values can be defined and computed for a W-entropy $\widehat{\mathcal{W}}[\mathbf{g}(\tau)]$ (28).

We say that a ground state $|\widehat{\Psi}>$ (43) is *entangled (inseparable)* if $\widehat{C}_{\underline{ab}} \neq \widehat{C}_{\underline{a}} \widehat{C}_{\underline{b}}$. For such a state, the entanglement entropy is positive, ${}_{q}\widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}}) :> 0$. Contrary, certain QGIF and NES are considered un-physical. Using quantum Schmidt decompositions (36), we can prove for any point along a causal curve on $\mathbf{V}$ that

$$ {}_{q}\widehat{\mathcal{S}} = -\sum_{\underline{a}}^{\min(\underline{a},\underline{b})} p_{\underline{a}} \log p_{\underline{a}} \text{ and } {}_{q}\underline{\mathcal{S}}_{|\max} = \log \min(\underline{a}, \underline{b}) \text{ for } \sum_{\underline{a}} p_{\underline{a}} = 1 \text{ and } p_{\underline{a}} = 1/\min(\underline{a}, \underline{b}), \forall a. \tag{44} $$

It should be noted that such quantum entropy is associated to a thermodynamic model for geometric/ information flows and not directly for a curved spacetime and possible geometric evolution. Nevertheless, d-metrics can be used for elaborating on causal physical states defined along timelike curves and respective 3+1 splitting.

An entangled state of NES and QGIFs is a superposition of several quantum states associated to respective GIFs of gravitational systems. This means that an observer having access only to a quantum subsystem $\widehat{\mathcal{A}}$ will find him/ herself in a mixed state when the total ground state $|\widehat{\Psi}>$ is entangled following such conditions: $|\widehat{\Psi}>$: separable $\longleftrightarrow \widehat{\rho}_{\mathcal{A}}$: pure state, or $|\widehat{\Psi}>$: entangled $\longleftrightarrow \widehat{\rho}_{\mathcal{A}}$: mixed state.

We conclude that the von Neumann entanglement entropy for QGIF and NES ${}_{qt}\widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}})$ (42) encodes four types of information data: 1) how the geometric evolution in canonical nonholonomic variables is quantum flow correlated; 2) how much a given QGIF canonical nonholonomic state differs from a separable associated QM state; 3) how NES, or other type MGT models, are subjected to quantum flow evolution; and 4) in which forms such GIFs and NES are modelled in canonical nonholonomic variables. A maximum value of quantum correlations is reached when a given QGIF NES state is a superposition of all possible quantum states with an equal weight. Such constructions are derived for associated thermodynamic models. There are also additional GIF and NES properties which are characterized by the W-entropy, $\widehat{\mathcal{W}}$ (28), and thermodynamic entropy, $\widehat{\mathcal{S}}$ (30), which can be computed in certain quasi-classical QM limits for a 3+1 splitting and along a time like curve, see similar constructions in [41].

3.3.3 Thermofield double states for gravity and geometric information flows

In our works [24, 38, 39], we consider an evolution parameter $\beta = T^{-1}$ is treated as temperature similarly to G. Perelman's approach [27]. This allows us to elaborate on GIF and NES theories as relativistic classical and/or quantum thermofield models. A nontrivial example with entanglement and a thermofield double state is defined by a ground state (43) parameterized in the form

$$|\widehat{\Psi}>= \widehat{Z}^{-1/2}\sum_{\underline{k}} e^{-\beta E_{\underline{k}}/2}|\underline{k}>_{\mathcal{A}}\otimes|\underline{k}>_{\mathcal{B}}, \qquad (45)$$

for a partition function $\widehat{Z}=\sum_{\underline{k}} e^{-\beta E_{\underline{k}}/2}$. Such values are associated to the thermodynamic generating function $\widehat{\mathcal{Z}}\,[\mathbf{g}(\tau)]$ (29) and state density matrix $\widehat{\rho}(\beta,\widehat{\mathcal{E}}\,,\mathbf{g}) = \widehat{\rho}[\mathbf{g}(\tau)] = \widehat{\mathcal{Z}}^{-1}e^{-\beta E}$. Values of energy $\widehat{\mathcal{E}}_{\mathcal{A}}=\{E_{\underline{k}}\}$ is considered quantized with a discrete spectrum for a QGIF and NES $\widehat{\mathcal{A}}=[\widehat{\mathcal{W}};\widehat{\mathcal{Z}},\widehat{\mathcal{E}},\widehat{\mathcal{S}},\widehat{\eta}]$ (30). The thermodynamic values are computed via integrals and measures determine by $\mathbf{g}(\tau)$ and canonical nonholonomic variables. We compute the density matrix for such a thermofield subsystem determining a Gibbs state,

$$\widehat{\rho}_{\mathcal{A}}=\widehat{Z}^{-1}\sum_{\underline{k}} e^{-\beta E_{\underline{k}}/2}|\underline{k}>_{\mathcal{A}}\otimes {}_{\mathcal{A}}<\underline{k}| = \widehat{Z}^{-1}e^{-\beta\widehat{\mathcal{E}}_{\mathcal{A}}}.$$

In above formulas, we consider $\widehat{\mathcal{E}}$ as a (modular) Hamiltonian $\widehat{\mathcal{E}}_{\mathcal{A}}$ such that $\widehat{\mathcal{E}}_{\mathcal{A}}|\underline{k}>_{\mathcal{A}}= E_{\underline{k}}|\underline{k}>_{\mathcal{A}}$.

Thermofield double states are certain entanglement purifications of thermal states with Boltzmann weight $p_k=\widehat{Z}^{-1}\sum_{\underline{k}} e^{-\beta E_{\underline{k}}}$. Transferring state vectors $\{|\underline{k}>_{\mathcal{B}}\}$ from $\widehat{\mathcal{H}}_{\mathcal{A}}$ to $\widehat{\mathcal{H}}_{\mathcal{B}}$, we can purify $\widehat{\mathcal{A}}$ in the extended Hilbert space $\widehat{\mathcal{H}}_{\mathcal{A}}\otimes\widehat{\mathcal{H}}_{\mathcal{B}}$. So, every expectation of local operators in $\widehat{\mathcal{A}}$ can be represented using the thermofield double state $|\widehat{\Psi}>$ (45) of the total system $\widehat{\mathcal{A}}\cup\widehat{\mathcal{B}}$. The entanglement entropy can be treated as a measure of the thermal entropy of the subsystem $\widehat{\mathcal{A}}$,

$$\widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}}) = -tr_{\mathcal{A}}[\widehat{\rho}_{\mathcal{A}}(-\beta\widehat{\mathcal{E}}_{\mathcal{A}}-\log\widehat{Z})] = \beta(<\widehat{\mathcal{E}}_{\mathcal{A}}> - \widehat{\mathcal{F}}_{\mathcal{A}}).$$

In this formula, $\widehat{\mathcal{F}}_{\mathcal{A}}=-\log\widehat{Z}$ is the thermal free energy.

For thermofield values, we omit the label "q" considered, for instance, for ${}_{qt}\widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}})$ (42). Here it should be noted that thermofield GIF and NES configurations are also characterized by W-entropy $\widehat{\mathcal{W}}$ (28), see examples how to compute such values for the gravitational configurations generated by the AFDM in [19].

3.4 Inequalities for entropies of NES QGIFs

We study certain important inequalities and properties of the entanglement entropy (42) using the density matrix $\widehat{\rho}_{\mathcal{A}} = Tr_{\mathcal{H}_{\mathcal{B}}}({}_t\widehat{\rho})$. Proofs with entanglement entropy are similar to those presented in [47]. Concerning geometric analysis technique [27, 29, 30, 31], we refer readers to generalizations for nonholonomic manifolds and applications in modern in modern gravity and particle physics theories in [19, 20, 24].

3.4.1 (Strong) subadditivity of entangled NES systems

There are three important properties of QGIFs and NES related to strong subadditivity property of entanglement and Perelman's entropies.

Entanglement entropy for complementary QGIF and gravitational subsystems: If $\widehat{\mathcal{B}} = \overline{\widehat{\mathcal{A}}}$, we have such a condition for entropies ${}_q\widehat{\mathcal{S}}_{\mathcal{A}} = {}_q\widehat{\mathcal{S}}_{\overline{\mathcal{A}}}$, which can be proven using formulas (44) for a pure ground state wave function. Similar equalities for the W-entropy $\widehat{\mathcal{W}}$ (28) and/or thermodynamic entropy $\widehat{\mathcal{S}}$ (30) can be proven if we use the same d-metric $\mathbf{g}$ and respective normalization on $\widehat{\mathcal{A}}$ and $\overline{\widehat{\mathcal{A}}}$. For quantum models of GIF thermodynamic systems, ${}_q\widehat{\mathcal{S}}_{\mathcal{A}} \neq {}_q\widehat{\mathcal{S}}_{\mathcal{B}}$ if $\widehat{\mathcal{A}} \cup \widehat{\mathcal{B}}$ is a mixed state. In result, we have general inequalities,

$$ {}_q\widehat{\mathcal{S}}_{\mathcal{A}} \neq {}_q\widehat{\mathcal{S}}_{\mathcal{B}} \text{ and } {}_q\widehat{\mathcal{W}}_{\mathcal{A}} \neq {}_q\widehat{\mathcal{W}}_{\mathcal{B}}, $$

which can be also proven in any quasi-classical limit, for instance, in the WKB approximation as in [41]. For some special subclasses of nonholonomic deformations and certain classes of normalizing functions such conditions may transform in equalities.

Subadditivity conditions are satisfied for disjoint subsystems $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$,

$$ {}_q\widehat{\mathcal{S}}_{\mathcal{A}\cup\mathcal{B}} \leq {}_q\widehat{\mathcal{S}}_{\mathcal{A}} + {}_q\widehat{\mathcal{S}}_{\mathcal{B}} \text{ and } | {}_q\widehat{\mathcal{S}}_{\mathcal{A}} - {}_q\widehat{\mathcal{S}}_{\mathcal{B}}| \leq {}_q\widehat{\mathcal{S}}_{\mathcal{A}\cup\mathcal{B}}. \quad (46) $$

Similar conditions hold for the W-entropy $\widehat{\mathcal{W}}$ (28) and respective quantum versions,

$$ {}_q\widehat{\mathcal{W}}_{\mathcal{A}\cup\mathcal{B}} \leq {}_q\widehat{\mathcal{W}}_{\mathcal{A}} + {}_q\widehat{\mathcal{W}}_{\mathcal{B}} \text{ and } | {}_q\widehat{\mathcal{W}}_{\mathcal{A}} - {}_q\widehat{\mathcal{W}}_{\mathcal{B}}| \leq {}_q\widehat{\mathcal{W}}_{\mathcal{A}\cup\mathcal{B}}. $$

Such NES flow evolution and QM scenarios are elaborated for mixed geometric, gravitational and quantum probabilistic information flows.

Strong subadditivity is considered for three disjointed QGIF gravitational subsystems $\widehat{\mathcal{A}}, \widehat{\mathcal{B}}$ and $\widehat{\mathcal{C}}$ and conditions of convexity of a function built from respective density matrices and unitarity of systems [51, 52, 2, 13]. One follow such inequalities:

$$ {}_q\widehat{\mathcal{S}}_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}} + {}_q\widehat{\mathcal{S}}_{\mathcal{B}} \leq {}_q\widehat{\mathcal{S}}_{\mathcal{A}\cup\mathcal{B}} + {}_q\widehat{\mathcal{S}}_{B\cup\mathcal{C}} \text{ and } {}_q\widehat{\mathcal{S}}_{\mathcal{A}} + {}_q\widehat{\mathcal{S}}_{\mathcal{C}} \leq {}_q\widehat{\mathcal{S}}_{\mathcal{A}\cup\mathcal{B}} + {}_q\widehat{\mathcal{S}}_{B\cup\mathcal{C}}. $$

The conditions of subadditivity (46) consist certain particular cases defined by strong subadditivity. Similar formulas can be proven for the W-entropy and small quantum perturbations,

$$ {}_q\widehat{\mathcal{W}}_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}} + {}_q\widehat{\mathcal{W}}_{\mathcal{B}} \leq {}_q\widehat{\mathcal{W}}_{\mathcal{A}\cup\mathcal{B}} + {}_q\widehat{\mathcal{W}}_{B\cup\mathcal{C}} \text{ and } {}_q\widehat{\mathcal{W}}_{\mathcal{A}} + {}_q\widehat{\mathcal{W}}_{\mathcal{C}} \leq {}_q\widehat{\mathcal{W}}_{\mathcal{A}\cup\mathcal{B}} + {}_q\widehat{\mathcal{W}}_{B\cup\mathcal{C}}. $$

3.4.2 Relative entropy and mutual information of NES and QGIFs

The concept of *relative entropy* is defined in canonical nonholonomic variables as in geometric information theories,

$$ \widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}} \shortparallel \widehat{\sigma}_{\mathcal{A}}) = Tr_{\mathcal{H}_{\mathcal{B}}}[\widehat{\rho}_{\mathcal{A}}(\log \widehat{\rho}_{\mathcal{A}} - \log \widehat{\sigma}_{\mathcal{A}})], \quad (47) $$

where $\widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}} \shortparallel \widehat{\rho}_{\mathcal{A}}) = 0$. This value allow us to define a measure of "distance" between two QGIFs and NES with a norm $||\widehat{\rho}_{\mathcal{A}}|| = tr(\sqrt{(\widehat{\rho}_{\mathcal{A}})(\widehat{\rho}^{\dagger}_{\mathcal{A}})})$, see details in reviews [1, 2, 13].

Two QGIFs and NES are characterized by some important formulas and conditions for relative entropy:

- tensor products of density matrices,

$$\widehat{\mathcal{S}}(\ {}_1\widehat{\rho}_{\mathcal{A}} \otimes\ {}_2\widehat{\rho}_{\mathcal{A}} \shortparallel\ {}_1\widehat{\sigma}_{\mathcal{A}} \otimes\ {}_2\widehat{\sigma}_{\mathcal{A}}) = \ \widehat{\mathcal{S}}(\ {}_1\widehat{\rho}_{\mathcal{A}} \shortparallel\ {}_1\widehat{\sigma}_{\mathcal{A}}) + \ \widehat{\mathcal{S}}(\ {}_2\widehat{\rho}_{\mathcal{A}} \shortparallel\ {}_2\widehat{\sigma}_{\mathcal{A}});$$

- positivity, $\widehat{\mathcal{S}}(\ \widehat{\rho}_{\mathcal{A}} \shortparallel \widehat{\sigma}_{\mathcal{A}}) \geq \frac{1}{2}||\ \widehat{\rho}_{\mathcal{A}} - \widehat{\sigma}_{\mathcal{A}}||^2$, i.e. $\widehat{\mathcal{S}}(\ \widehat{\rho}_{\mathcal{A}} \shortparallel \widehat{\sigma}_{\mathcal{A}}) \geq 0$;

- monotonicity, $\widehat{\mathcal{S}}(\ \widehat{\rho}_{\mathcal{A}} \shortparallel \widehat{\sigma}_{\mathcal{A}}) \geq\ \widehat{\mathcal{S}}(tr_s\ \widehat{\rho}_{\mathcal{A}}|tr_s\ \widehat{\sigma}_{\mathcal{A}})$, where tr_s denotes the trace for a subsystem of $\widehat{\mathcal{A}}$.

The positivity formula and (Schwarz) inequality $||X|| \geq tr(XY)/||X||$ result in $2\ \widehat{\mathcal{S}}(\ \widehat{\rho}_{\mathcal{A}} \shortparallel \widehat{\sigma}_{\mathcal{A}}) \geq (\langle \mathcal{O}\rangle_\rho - \langle \mathcal{O}\rangle_\sigma)^2/||\mathcal{O}||^2$, for any expectation value $\langle \mathcal{O}\rangle_\rho$ of an operator $\mathcal{O}$ computed (37) with the density matrix $\widehat{\rho}_{\mathcal{A}}$. The relative entropy $\widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}} \shortparallel \widehat{\sigma}_{\mathcal{A}})$ (47) is related to the entanglement entropy ${}_q\ \widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}})$ (42) using formula $\widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}} \shortparallel 1_{\mathcal{A}}/k_{\mathcal{A}}) = \log k_{\mathcal{A}} -\ {}_q\ \widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}})$, where $1_{\mathcal{A}}$ is the $k_{\mathcal{A}} \times k_{\mathcal{A}}$ unit matrix for a $k_{\mathcal{A}}$-dimensional Hilbert space associated to the region $\widehat{\mathcal{A}}$.

Let us denote by $\widehat{\rho}_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}}$ the density matrix of $\widehat{\mathcal{A}}\cup\widehat{\mathcal{B}}\cup\widehat{\mathcal{C}}$ when $\widehat{\rho}_{\mathcal{A}\cup\mathcal{B}}$ is written for its restriction on $\widehat{\mathcal{A}}\cup\widehat{\mathcal{B}}$ and $\widehat{\rho}_{\mathcal{B}}$ is stated for its restriction on $\widehat{\mathcal{B}}$. Using $tr_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}}[\ \widehat{\rho}_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}}(\mathcal{O}_{\mathcal{A}\cup\mathcal{B}} \otimes 1_{\mathcal{C}}/k_{\mathcal{C}})] = tr_{\mathcal{A}\cup\mathcal{B}}(\ \widehat{\rho}_{\mathcal{A}\cup\mathcal{B}}\mathcal{O}_{\mathcal{A}\cup\mathcal{B}})$, we prove such identities

$$\begin{aligned}
\widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}} \shortparallel 1_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}}/k_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}}) &= \widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}\cup\mathcal{B}} \shortparallel 1_{\mathcal{A}\cup\mathcal{B}}/k_{\mathcal{A}\cup\mathcal{B}}) + \widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}} \shortparallel \widehat{\rho}_{\mathcal{A}\cup\mathcal{B}} \otimes 1_{\mathcal{C}}/k_{\mathcal{C}}),\\
\widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{B}\cup\mathcal{C}} \shortparallel 1_{\mathcal{B}\cup\mathcal{C}}/k_{\mathcal{B}\cup\mathcal{C}}) &= \widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{B}} \shortparallel 1_{\mathcal{B}}/k_{\mathcal{B}}) + \ \widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{B}\cup\mathcal{C}} \shortparallel \widehat{\rho}_{\mathcal{B}} \otimes 1_{\mathcal{C}}/k_{\mathcal{C}});
\end{aligned}$$

and inequalities $\widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}} \shortparallel \widehat{\rho}_{\mathcal{A}\cup\mathcal{B}} \otimes 1_{\mathcal{C}}/k_{\mathcal{C}}) \geq\ \widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{B}\cup\mathcal{C}} \shortparallel \widehat{\rho}_{\mathcal{B}} \otimes 1_{\mathcal{C}}/k_{\mathcal{C}})$,
$\widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}} \shortparallel 1_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}}/k_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}}) + \ \widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{B}} \shortparallel 1_{\mathcal{B}}/k_{\mathcal{B}}) \geq\ \widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}\cup\mathcal{B}} \shortparallel 1_{\mathcal{A}\cup\mathcal{B}}/k_{\mathcal{A}\cup\mathcal{B}}) + \widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{B}\cup\mathcal{C}} \shortparallel 1_{\mathcal{B}\cup\mathcal{C}}/k_{\mathcal{B}\cup\mathcal{C}})$.

The correlation between two QGIFs and NES $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$ (it can be involved also a third system $\widehat{\mathcal{C}}$) is characterized by *mutual information*

$$\widehat{\mathcal{J}}(\widehat{\mathcal{A}}, \widehat{\mathcal{B}}) :=\ \widehat{\mathcal{S}}_{\mathcal{A}} +\ \widehat{\mathcal{S}}_{\mathcal{B}} -\ \widehat{\mathcal{S}}_{\mathcal{A}\cup\mathcal{B}} \geq 0 \text{ and } \widehat{\mathcal{J}}(\widehat{\mathcal{A}}, \widehat{\mathcal{B}}\cup\underline{\mathcal{C}}) \leq \widehat{\mathcal{J}}(\widehat{\mathcal{A}}, \widehat{\mathcal{B}}).$$

Using formula $\widehat{\mathcal{J}}(\widehat{\mathcal{A}}, \widehat{\mathcal{B}}) =\ \widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}\cup\mathcal{B}} \shortparallel \widehat{\rho}_{\mathcal{A}} \otimes \widehat{\rho}_{\mathcal{B}})$, we can prove important inequalities for the entanglement of QGIFs and NES,

$${}_q\widehat{\mathcal{J}}(\widehat{\mathcal{A}}, \widehat{\mathcal{B}}) :=\ {}_q\ \widehat{\mathcal{S}}_{\mathcal{A}} +\ {}_q\ \widehat{\mathcal{S}}_{\mathcal{B}} -\ {}_q\ \widehat{\mathcal{S}}_{\mathcal{A}\cup\mathcal{B}} \geq 0 \text{ and } {}_q\widehat{\mathcal{J}}(\widehat{\mathcal{A}}, \widehat{\mathcal{B}}\cup\underline{\mathcal{C}}) \leq\ {}_q\widehat{\mathcal{J}}(\widehat{\mathcal{A}}, \widehat{\mathcal{B}}), \text{ for } {}_q\widehat{\mathcal{J}}(\widehat{\mathcal{A}}, \widehat{\mathcal{B}}) =\ {}_q\ \widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}\cup\mathcal{B}} \shortparallel \widehat{\rho}_{\mathcal{A}} \otimes \widehat{\rho}_{\mathcal{B}}).$$

The mutual information between two QGIFs and NES is a measure how much the density matrix $\widehat{\rho}_{\mathcal{A}\cup\mathcal{B}}$ differs from a separable state $\widehat{\rho}_{\mathcal{A}} \otimes \widehat{\rho}_{\mathcal{B}}$. Quantum correlations entangle even spacetime disconnected regions of the phase spacetime under geometric flow evolution. For GIF and NES flows in respective regions, $2\widehat{\mathcal{J}}(\widehat{\mathcal{A}}, \widehat{\mathcal{B}}) \geq (\langle \mathcal{O}_{\mathcal{A}}\mathcal{O}_{\mathcal{B}}\rangle - \langle \mathcal{O}_{\mathcal{A}}\rangle\langle \mathcal{O}_{\mathcal{B}}\rangle)^2/||\mathcal{O}_{\mathcal{A}}||^2||\mathcal{O}_{\mathcal{B}}||^2$, for bounded operators $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{B}}$.

3.4.3 The Rényi entropy for NES QGIFs

The concept of Rényi entropy [53] is used for computing the entanglement entropy of QFTs by developing the replica method (see section IV of [13] and further generalizations in [54]). Similar constructions are possible for QGIF and NES because the thermodynamic generating function $\widehat{\mathcal{Z}}[\mathbf{g}(\tau)]$ (29) with $\widehat{\sigma}[\mathbf{g}(\tau)] = \widehat{\mathcal{Z}}^{-1}e^{-\beta E}$ as statistical density $\widehat{\rho}(\beta, \widehat{\mathcal{E}}\ , \mathbf{g})$ used for defining $\widehat{\sigma}_{\mathcal{A}}$ (38) as a probability distribution. We extend the replica method to G. Perelman's thermodynamical model and related classical and quantum information theories for NES. Considering an integer r (replica parameter), the Rényi entropy is

$${}_r\ \widehat{\mathcal{S}}(\widehat{\mathcal{A}}) := \frac{1}{1-r}\log[tr_{\mathcal{A}}(\widehat{\rho}_{\mathcal{A}})^r] \tag{48}$$

for a QGIF and NES determined by a density matrix $\widehat{\rho}_{\mathcal{A}}$. A replica computational formalism is elaborated for an analytic continuation of r to a real number with a defined limit ${}_q\widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}}) = \lim_{r\to 1}\ {}_r\ \widehat{\mathcal{S}}(\widehat{\mathcal{A}})$ and normalization $tr_{\mathcal{A}}(\widehat{\rho}_{\mathcal{A}})$ for $r \to 1$, when (48) reduces to the entanglement entropy (42).

Considering similar formulas proven in [55], there are introduced such important inequalities for the derivative on the replica parameter, ∂_r. We have

$$\partial_r(\ _r\ \widehat{\mathcal{S}}) \leq 0,\ \partial_r\left(\frac{r-1}{r}\ _r\ \widehat{\mathcal{S}}\right) \geq 0,\ \partial_r[(r-1)\ _r\ \widehat{\mathcal{S}}] \geq 0,\ \partial^2_{rr}[(r-1)](\ _r\ \widehat{\mathcal{S}}) \leq 0. \qquad (49)$$

A usual thermodynamical interpretation of such formulas is possible for GIF and NES with a conventional modular Hamiltonian $\widehat{H}_{\mathcal{A}} = \ \widehat{\mathcal{E}}$ and effective statistical density $\widehat{\rho}_{\mathcal{A}} := e^{-2\pi \widehat{H}_{\mathcal{A}}}$. The value $\beta_r = 2\pi r$ is considered as the inverse temperature and the effective "thermal" statistical generation (partition) function is defined $_r\widehat{\mathcal{Z}}(\beta_r) := tr_{\mathcal{A}}(\widehat{\rho}_{\mathcal{A}})^r = tr_{\mathcal{A}}(e^{-\beta_r \widehat{H}_{\mathcal{A}}})$ similarly to $\widehat{\mathcal{Z}}\ [\mathbf{g}(\tau)]$ (29). We compute using canonical relations such statistical mechanics values

$$\begin{aligned}
\text{modular energy :} \quad & _r\widehat{\mathcal{E}}(\beta_r) := -\partial_{\beta_r} \log[\ _r\widehat{\mathcal{Z}}(\beta_r)] \geq 0;\\
\text{modular entropy :} \quad & _r\breve{\mathcal{S}}(\beta_r) := (1-\beta_r\partial_{\beta_r}) \log[\ _r\widehat{\mathcal{Z}}(\beta_r)] \geq 0;\\
\text{modular capacity :} \quad & _r\widehat{\mathcal{C}}(\beta_r) := \beta_r^2 \partial^2_{\beta_r} \log[\ _r\widehat{\mathcal{Z}}(\beta_r)] \geq 0.
\end{aligned}$$

These inequalities are equivalent to the conditions stated in the second line in (49) and characterize the stability of GIFs and NES considered as a thermal system with replica parameter regarded as the inverse temperature for a respective modular Hamiltonian. Such replica criteria of stability define a new direction for the theory of geometric flows, QGIFs, and applications in modern physics and cosmology, see [18, 21, 19, 20, 25, 26, 38, 35] and references therein.

The constructions with the modular entropy can be transformed into models derived for GIFs and associated thermodynamic models and with the Rényi entropy and inversely. Such transforms in canonical nonholonomic can be performed using formulas $_r\breve{\mathcal{S}} := r^2 \partial_r \left(\frac{r-1}{r}\ _r\widehat{\mathcal{S}}\right)$ and, inversely, $_r\widehat{\mathcal{S}} = \frac{r}{r-1}\int_1^r dr' \frac{_{r'}\breve{\mathcal{S}}}{(r')^2}$. The implications of the inequalities for the Rényi entropy were analyzed for the quantum information and gravitational systems with holographic description, see reviews [5, 1, 13]. In this work, the approach is generalized for G. Perelman entropies, QGIFs and NES.

The concept of relative entropy $\widehat{\mathcal{S}}(\ \widehat{\rho}_{\mathcal{A}} \shortparallel \widehat{\sigma}_{\mathcal{A}})$ (47) introduced for nonholonomic geometric information flows can be extended to that of relative Rényi entropy (for a review, see section II.E.3b in [13]). For a system QGIFs with two density matrices $\widehat{\rho}_{\mathcal{A}}$ and $\widehat{\sigma}_{\mathcal{A}}$, we compute

$$\begin{aligned}
_r\widehat{\mathcal{S}}(\ \widehat{\rho}_{\mathcal{A}} \shortparallel \widehat{\sigma}_{\mathcal{A}}) &= \frac{1}{r-1}\log\left[tr\left((\ \widehat{\sigma}_{\mathcal{A}})^{(1-r)/2r}\ \widehat{\rho}_{\mathcal{A}}(\ \widehat{\sigma}_{\mathcal{A}})^{(1-r)/2r}\right)^r\right],\ \text{for } r \in (0,1)\cup(1,\infty); \qquad (50)\\
\text{or }\ _1\widehat{\mathcal{S}}(\ \widehat{\rho}_{\mathcal{A}} \shortparallel\ \widehat{\sigma}_{\mathcal{A}}) &= \widehat{\mathcal{S}}(\ \widehat{\rho}_{\mathcal{A}} \shortparallel\ \widehat{\sigma}_{\mathcal{A}}) \text{ and }\ _\infty\widehat{\mathcal{S}}(\ \widehat{\rho}_{\mathcal{A}} \shortparallel\ \widehat{\sigma}_{\mathcal{A}}) = \log||(\ \widehat{\sigma}_{\mathcal{A}})^{-1/2}\ \widehat{\rho}_{\mathcal{A}}(\widehat{\sigma}_{\mathcal{A}})^{-1/2}||_\infty.
\end{aligned}$$

In any point of causal curves, one prove monotonic properties, $_r\widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}} \shortparallel \widehat{\sigma}_{\mathcal{A}}) \geq\ _r\widehat{\mathcal{S}}(tr_s\ \widehat{\rho}_{\mathcal{A}}|tr_s\ \widehat{\sigma}_{\mathcal{A}})$ and $\partial_r[\ _r\widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}} \shortparallel \widehat{\sigma}_{\mathcal{A}})] \geq 0$, and to reduce the relative Rényi entropy to the Rényi entropy using formula $_r\widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}} \shortparallel 1_{\mathcal{A}}/k_{\mathcal{A}}) = \log k_{\mathcal{A}} -\ _r\underline{\mathcal{S}}(\widehat{\mathcal{A}})$.

The values (50) do not allow a naive extension of the concept of mutual information and interpretation as an entanglement measure of quantum information for QGIF systems and possible applications in gravity theory. There are possible negative values of relative Rényi entropy for $r \neq 1$ even for standard quantum information models [56]. This problem can be solved for various classes of solutions and MGTs if it is introduced the concept of the r-Rényi mutual information [57],

$$_r\widehat{\mathcal{J}}(\widehat{\mathcal{A}},\widehat{\mathcal{B}})\ := \min_{'\sigma_{\mathcal{B}}}\ _r\widehat{\mathcal{S}}(\ \widehat{\rho}_{\mathcal{A}\cup\mathcal{B}} \shortparallel\ \widehat{\rho}_{\mathcal{A}} \otimes\ \widehat{\sigma}_{\mathcal{B}}) \geq 0,$$

for a minimum taken over all $\widehat{\sigma}_{\mathcal{B}}$. Such conditions can be satisfied for corresponding nonholonomic distributions with causality and respective subclasses of generating functions. We obtain the standard definition of mutual information for $r = 1$. In result, we can elaborate a self–consistent geometric–information thermodynamic theory for QGIFs and NES which consists a more general approach than the constructions with area horizon, holographic and conformal field entropic models.

4 Decoupling and integrability of geometric flow equations for NES

The goal of this section is to prove that the system of nonlinear PDEs (25) describing nonholonomic geometric evolution of NES can be decoupled and integrated in very general forms when the generating functions and (effective and/or matter field) sources are determined by geometric data for nonlinear waves, solitonic hierarchies and black hole, BH, configurations. For corresponding parameterizations, the coefficients of such generic off-diagonal metrics and canonical d-connections (in particular, for LC–configurations) may depend on all space like coordinates and run on a temperature like parameter τ. The geometric / physical objects for such effective statistical thermodynamical systems and models of GIFs and QGIFs corresponding to such solutions are determined by generating and integration functions and (effective) matter sources encoding solitonic configurations, see details and examples in [58, 59, 60, 61]. In this work, we restrict our considerations only to stationary configurations or families of such generic off-diagonal solutions which in certain canonical systems of reference do not depend on time like coordinate but may run on a geometric flow parameter τ used also for parametrization of curve flows and related solitonic hierarchies. Various types of locally anisotropic cosmological solutions in MGTs and GR and related inflation and dark matter and dark energy models were studied in [34, 35, 40]. More general GIFs and QGIFs constructions and applications to cosmological thermodynamic models will be elaborated in our further works.

4.1 Why and how GIFs can be encoded into solitonic hierarchies?

Geometric flow and dynamical field equations in MGTs and GR are described by systems of nonlinear PDEs. In general, the solutions of such systems do not possess any geometric/ physical symmetries but we have to prescribe certain symmetries in order to select and study some classes of solutions in explicit form. For nonlinear systems, it is not satisfy a principle of superposition and the solutions can not be expressed as Fourier series on certain finite regions of $\mathbf{V}, T\mathbf{V}$ and $T^*\mathbf{V}$. Nevertheless, any pseudo-Riemannian metric and mentioned type physically important solutions can be encoded into certain classes of solitonic hierarchies with associated bi-Hamilton structures if there are satisfied some very general assumptions on the smooth classes of metrics and connections under consideration, see details in [59, 60, 61] and references therein. On some different solitonic models and alternative approaches see [62, 63]. The constructions can be generalized for MGTs (Einstein-Dirac structures, Finsler like commutative and noncommutative or fractional derivative models,nonsymmetric metrics, string gravity, black hole and black ring deformations etc.) [64, 65, 66, 67, 68, 69, 70, 71]. Recent results on soliton, quasiperiodic and pattern structures and ellipsoid–solitonic deformations in string and modified massive gravity and quantum anomalies can be found in [72, 73, 74].

We model geometric evolution of a a 'prime' metric, $\mathring{\mathbf{g}}$, into a family 'target' d-metrics $\mathbf{g}(\tau)$ (7), with transforms $\mathring{\mathbf{g}} \rightarrow \mathbf{g}(\tau)$, using so-called η-polarization functions when

$$\begin{aligned} g(\tau) &= \eta_\alpha(\tau, x^k, y^b)\mathring{g}_\alpha \mathbf{e}^\alpha[\eta] \otimes \mathbf{e}^\alpha[\eta] = \eta_i(\tau, x^k)\mathring{g}_i dx^i \otimes dx^i + \eta_a(\tau, x^k, y^b)\mathring{h}_a e^a[\eta] \otimes e^a[\eta], \\ \mathbf{e}^\alpha[\eta] &= (dx^i, \mathbf{e}^a = dy^a + \eta^a_i \mathring{N}^a_i dx^i), \end{aligned} \quad (51)$$

where the target N-connection coefficients are parameterized $N^a_i(\tau, u) = \eta^a_i(\tau, x^k, y^b)\mathring{N}^a_i(\tau, x^k, y^b)$.[14] The values $\eta_i(\tau) = \eta_i(\tau, x^k), \eta_a(\tau) = \eta_a(\tau, x^k, y^b)$ and $\eta^a_i(\tau) = \eta^a_i(\tau, x^k, y^b)$ are called respectively geometric flow, or gravitational, polarization functions, or η-polarizations. Any $\mathbf{g}(\tau)$ is subjected to the condition that it defines a solution of the N-adapted Hamilton equations in canonical variables (25), or for relativistic nonholonomic Ricci soliton equations (24) with $\tau = \tau_0$ as modified Einstein equations for NES.

A prime metric $\mathring{\mathbf{g}} = \mathring{g}_{\alpha\beta}(x^i, y^a)du^\alpha \otimes du^\beta$ can be parameterized in a general coordinate form with off-diagonal N-coefficients (6) and represented equivalently in N-adapted form

$$\begin{aligned} \mathring{\mathbf{g}} &= \mathring{g}_\alpha(u)\mathring{\mathbf{e}}^\alpha \otimes \mathring{\mathbf{e}}^\beta = \mathring{g}_i(x)dx^i \otimes dx^i + \mathring{g}_a(x,y)\mathring{\mathbf{e}}^a \otimes \mathring{\mathbf{e}}^a, \\ \text{for} \quad & \mathring{\mathbf{e}}^\alpha = (dx^i, \mathbf{e}^a = dy^a + \mathring{N}^a_i(u)dx^i), \text{ and } \mathring{\mathbf{e}}_\alpha = (\mathring{\mathbf{e}}_i = \partial/\partial y^a - \mathring{N}^b_i(u)\partial/\partial y^b, \ e_a = \partial/\partial y^a). \end{aligned} \quad (52)$$

[14] we do not consider summation on repeating indices if they are not written as a contraction of "up-low" ones

We consider that such a d-metric $\mathring{\mathbf{g}}(\tau_0) = \mathring{g}_\alpha(u)$ can be, or not, a solution of some gravitational field equations in a MGT or GR but under geometric evolution it transform into a target metric (51) subjected to the condition to define an exact or parametric solution of certain geometric flow evolution equations of NES, or some nonholonomic Ricci soliton / (modified) Einstein equations.

4.1.1 Generating Solitonic Hierarchies

To geometric evolution of a d-metric $\mathbf{g}(\tau)$ we can associate a non–stretching curve $\gamma(\tau, \mathbf{l})$ on a (modified) Einstein manifold $\mathbf{V}$, where τ is a real parameter (it can be identified with the geometric flow parameter. The value $\mathbf{l}$ is the arclength of the curve on $\mathbf{V}$ which is defined by such evolution d–vector $\mathbf{Y} = \gamma_\tau$ and tangent d–vector $\mathbf{X} = \gamma_\mathbf{l}$ that $\mathbf{g}(\mathbf{X}, \mathbf{X}) = 1$. Such a curve $\gamma(\tau, \mathbf{l})$ swept out a two–dimensional surface in $T_{\gamma(\tau,\mathbf{l})}\mathbf{V} \subset T\mathbf{V}$, see details in [59, 60, 61]. We consider a coframe $\mathbf{e} \in T^*_\gamma \mathbf{V_N} \otimes (h\mathfrak{p} \oplus v\mathfrak{p})$, which is a N–adapted $(SO(n) \oplus SO(m))$–parallel basis along γ. We use a label $\mathbf{N}$ in order to emphasize that the geometric constructions are performed for nontrivial N-connection structures. In this work, we consider that $n = m = 4$ and model the evolution of 4-d Lorentzian d-metrics. The symbols n and m will be kept in order to distinguish N-adapted decompositions into h- and v-, or cv-components.

We can associate a canonical d-connection $\widehat{\mathbf{D}}$ with a linear connection 1–form is $\widehat{\mathbf{\Gamma}} \in T^*_\gamma \mathbf{V_N} \otimes (\mathfrak{so}(n) \oplus \mathfrak{so}(m))$. Similar 1-forms can be associated to other types of d-connections or to a LC-connection. We parameterize frame bases by 1-forms $\mathbf{e_X} = \mathbf{e}_{h\mathbf{X}} + \mathbf{e}_{v\mathbf{X}}$, where (for $(1, \overrightarrow{0}) \in \mathbb{R}^n$, $\overrightarrow{0} \in \mathbb{R}^{n-1}$ and $(1, \overleftarrow{0}) \in \mathbb{R}^m$, $\overleftarrow{0} \in \mathbb{R}^{m-1}$), for

$$\mathbf{e}_{h\mathbf{X}} = \gamma_{h\mathbf{X}} \rfloor h\mathbf{e} = \begin{bmatrix} 0 & (1, \overrightarrow{0}) \\ -(1, \overrightarrow{0})^T & h\mathbf{0} \end{bmatrix}, \mathbf{e}_{v\mathbf{X}} = \gamma_{v\mathbf{X}} \rfloor v\mathbf{e} = \begin{bmatrix} 0 & (1, \overleftarrow{0}) \\ -(1, \overleftarrow{0})^T & v\mathbf{0} \end{bmatrix}.$$

For a $n+m$ splitting, $\widehat{\mathbf{\Gamma}} = \left[\widehat{\mathbf{\Gamma}}_{h\mathbf{X}}, \widehat{\mathbf{\Gamma}}_{v\mathbf{X}}\right]$, with $\widehat{\mathbf{\Gamma}}_{h\mathbf{X}} = \gamma_{h\mathbf{X}} \rfloor \widehat{\mathbf{L}} = \begin{bmatrix} 0 & (0, \overrightarrow{0}) \\ -(0, \overrightarrow{0})^T & \widehat{\mathbf{L}} \end{bmatrix} \in \mathfrak{so}(n+1)$, where $\widehat{\mathbf{L}} = \begin{bmatrix} 0 & \overrightarrow{v} \\ -\overrightarrow{v}^T & h\mathbf{0} \end{bmatrix} \in \mathfrak{so}(n)$, $\overrightarrow{v} \in \mathbb{R}^{n-1}$, $h\mathbf{0} \in \mathfrak{so}(n-1)$; and $\widehat{\mathbf{\Gamma}}_{v\mathbf{X}} = \gamma_{v\mathbf{X}} \rfloor \widehat{\mathbf{C}} = \begin{bmatrix} 0 & (0, \overleftarrow{0}) \\ -(0, \overleftarrow{0})^T & \widehat{\mathbf{C}} \end{bmatrix} \in \mathfrak{so}(m+1)$, where $\widehat{\mathbf{C}} = \begin{bmatrix} 0 & \overleftarrow{v} \\ -\overleftarrow{v}^T & v\mathbf{0} \end{bmatrix} \in \mathfrak{so}(m)$, $\overleftarrow{v} \in \mathbb{R}^{m-1}$, $v\mathbf{0} \in \mathfrak{so}(m-1)$.

Using the canonical d–connection $\widehat{\mathbf{D}}$, we can define some d-matrices being decomposed with respect to the flow direction: in the h–direction, $\mathbf{e}_{h\mathbf{Y}} = \gamma_\tau \rfloor h\mathbf{e} = \begin{bmatrix} 0 & \left(h\mathbf{e}_\parallel, h\overrightarrow{\mathbf{e}}_\perp\right) \\ -\left(h\mathbf{e}_\parallel, h\overrightarrow{\mathbf{e}}_\perp\right)^T & h\mathbf{0} \end{bmatrix}$, when $\mathbf{e}_{h\mathbf{Y}} \in h\mathfrak{p}$, $\left(h\mathbf{e}_\parallel, h\overrightarrow{\mathbf{e}}_\perp\right) \in \mathbb{R}^n$ and $h\overrightarrow{\mathbf{e}}_\perp \in \mathbb{R}^{n-1}$, and

$$\widehat{\mathbf{\Gamma}}_{h\mathbf{Y}} = \gamma_{h\mathbf{Y}} \rfloor \widehat{\mathbf{L}} = \begin{bmatrix} 0 & (0, \overrightarrow{0}) \\ -(0, \overrightarrow{0})^T & h\varpi_\tau \end{bmatrix} \in \mathfrak{so}(n+1),$$

where $h\varpi_\tau = \begin{bmatrix} 0 & \overrightarrow{\varpi} \\ -\overrightarrow{\varpi}^T & h\widehat{\mathbf{\Theta}} \end{bmatrix} \in \mathfrak{so}(n)$, $\overrightarrow{\varpi} \in \mathbb{R}^{n-1}$, $h\widehat{\mathbf{\Theta}} \in \mathfrak{so}(n-1)$. Similar parameterizations can be performed in the v–direction, $\mathbf{e}_{v\mathbf{Y}} = \gamma_\tau \rfloor v\mathbf{e} = \begin{bmatrix} 0 & \left(v\mathbf{e}_\parallel, v\overleftarrow{\mathbf{e}}_\perp\right) \\ -\left(v\mathbf{e}_\parallel, v\overleftarrow{\mathbf{e}}_\perp\right)^T & v\mathbf{0} \end{bmatrix}$, when $\mathbf{e}_{v\mathbf{Y}} \in v\mathfrak{p}$, $\left(v\mathbf{e}_\parallel, v\overleftarrow{\mathbf{e}}_\perp\right) \in \mathbb{R}^m$ and $v\overleftarrow{\mathbf{e}}_\perp \in \mathbb{R}^{m-1}$, and

$$\widehat{\mathbf{\Gamma}}_{v\mathbf{Y}} = \gamma_{v\mathbf{Y}} \rfloor \widehat{\mathbf{C}} = \begin{bmatrix} 0 & (0, \overleftarrow{0}) \\ -(0, \overleftarrow{0})^T & v\varpi_\tau \end{bmatrix} \in \mathfrak{so}(m+1),$$

where $v\varpi_\tau = \begin{bmatrix} 0 & \overleftarrow{\varpi} \\ -\overleftarrow{\varpi}^T & v\widehat{\mathbf{\Theta}} \end{bmatrix} \in \mathfrak{so}(m)$, $\overleftarrow{\varpi} \in \mathbb{R}^{m-1}$, $v\widehat{\mathbf{\Theta}} \in \mathfrak{so}(m-1)$.

Summarizing the results proven in [59, 60, 61] for parameterizations related to geometric flows of 4-d Lorentzian metrics, we formulate such

Main Results: For any solution of N-adapted Hamilton equations in canonical variables (25), or for relativistic nonholonomic Ricci soliton equations (24), there is a canonical hierarchy of N–adapted flows of curves $\gamma(\tau,\mathbf{l}) = h\gamma(\tau,\mathbf{l}) + v\gamma(\tau,\mathbf{l})$ described by geometric nonholonomic map equations:

- The 0 flows are convective (travelling wave) maps $\gamma_\tau = \gamma_{\mathbf{l}}$ distinguished as $(h\gamma)_\tau = (h\gamma)_{h\mathbf{X}}$ and $(v\gamma)_\tau = (v\gamma)_{v\mathbf{X}}$. The classification of such maps depend on the type of d-connection structure.
- There are +1 flows defined as non–stretching mKdV maps (see details and examples in [59, 60, 61])
$$-(h\gamma)_\tau = \widehat{\mathbf{D}}^2_{h\mathbf{X}}\,(h\gamma)_{h\mathbf{X}} + \frac{3}{2}|\widehat{\mathbf{D}}_{h\mathbf{X}}\,(h\gamma)_{h\mathbf{X}}|^2_{h\mathbf{g}}\ (h\gamma)_{h\mathbf{X}}\,, -(v\gamma)_\tau = \widehat{\mathbf{D}}^2_{v\mathbf{X}}\,(v\gamma)_{v\mathbf{X}} + \frac{3}{2}|\widehat{\mathbf{D}}_{v\mathbf{X}}\,(v\gamma)_{v\mathbf{X}}|^2_{v\mathbf{g}}\ (v\gamma)_{v\mathbf{X}}\,,$$
and the +2,... flows as higher order analogs.
- Finally, the -1 flows are defined by the kernels of the canonical recursion h–operator,
$$h\widehat{\mathfrak{R}} = \widehat{\mathbf{D}}_{h\mathbf{X}}\left(\widehat{\mathbf{D}}_{h\mathbf{X}} + \widehat{\mathbf{D}}^{-1}_{h\mathbf{X}}\,(\overrightarrow{v}\cdot)\,\overrightarrow{v}\right) + \overrightarrow{v}\rfloor\widehat{\mathbf{D}}^{-1}_{h\mathbf{X}}\left(\overrightarrow{v}\wedge\widehat{\mathbf{D}}_{h\mathbf{X}}\right),$$
and of the canonical recursion v–operator, $v\widehat{\mathfrak{R}} = \widehat{\mathbf{D}}_{v\mathbf{X}}\left(\widehat{\mathbf{D}}_{v\mathbf{X}} + \widehat{\mathbf{D}}^{-1}_{v\mathbf{X}}\,(\overleftarrow{v}\cdot)\,\overleftarrow{v}\right) + \overleftarrow{v}\rfloor\widehat{\mathbf{D}}^{-1}_{v\mathbf{X}}\left(\overleftarrow{v}\wedge\widehat{\mathbf{D}}_{v\mathbf{X}}\right)$, inducing non–stretching maps $\widehat{\mathbf{D}}_{h\mathbf{Y}}\,(h\gamma)_{h\mathbf{X}} = 0$ and $\widehat{\mathbf{D}}_{v\mathbf{Y}}\,(v\gamma)_{v\mathbf{X}} = 0$.

The canonical recursion d-operator $\widehat{\mathfrak{R}} = (h\widehat{\mathfrak{R}}, v\widehat{\mathfrak{R}})$ is related to respective bi-Hamiltonian structures in our case determined by geometric flows and respective GIF models.

4.1.2 Examples of solitonic space like stationary distributions and nonlinear waves

Using the Main Results from previous section, we conclude that the geometric flow evolution of any d-metric on a Lorentz manifold and related models on (co) tangent Lorentz bundles can be encoded into solitonic hierarchies. In this work, we shall study stationary exact and parametric solutions $\mathbf{g}(\tau) = \mathbf{g}(\tau, x^i, y^3) = [h\mathbf{g}(\tau, x^i), v\mathbf{g}(\tau, x^i, y^3)]$, with Killing symmetry on $\partial_4 = \partial_t$ when in adapted coordinates the coefficients of d-metrics do not depend on the time like coordinate $y^4 = t$.

Stationary solitonic distributions:
We shall use distributions $\iota = \iota(r,\vartheta,\varphi)$ as solutions of a respective class of solitonic 3-d equations
$$\begin{aligned}
\partial^2_{rr}\iota + \epsilon\partial_\varphi(\partial_\vartheta\iota + 6\iota\partial_\varphi\iota + \partial^3_{\varphi\varphi\varphi}\iota) = 0,\ \partial^2_{rr}\iota + \epsilon\partial_\vartheta(\partial_\varphi\iota + 6\iota\partial_\vartheta\iota + \partial^3_{\vartheta\vartheta\vartheta}\iota) &= 0,\\
\partial^2_{\vartheta\vartheta}\iota + \epsilon\partial_\varphi(\partial_r\iota + 6\iota\partial_\varphi\iota + \partial^3_{\varphi\varphi\varphi}\iota) = 0, \partial^2_{\vartheta\vartheta}\iota + \epsilon\partial_r(\partial_\varphi\iota + 6\iota\partial_r\iota + \partial^3_{rrr}\iota) &= 0,\\
\partial^2_{\varphi\varphi}\iota + \epsilon\partial_r(\partial_\vartheta\iota + 6\iota\partial_r\iota + \partial^3_{rrr}\iota) = 0,\ \partial^2_{\varphi\varphi}\iota + \epsilon\partial_\vartheta(\partial_r\iota + 6\iota\partial_\vartheta\iota + \partial^3_{\vartheta\vartheta\vartheta}\iota) &= 0,
\end{aligned} \tag{53}$$
for $\epsilon = \pm 1$. The label "v" states that such a function is defined as a "solitonic distribution" when in N-adapted frames a function $\iota(u)$ is a solution of an equation (53) and does not depend on the time coordinate. These equations and their solutions can be redefined via frame/coordinate transforms for stationary generating functions parameterized in non-spherical coordinates and labeled in the form $\iota = \iota(x^i, y^3)$.

Generating nonlinear solitonic waves:
Stationary geometric flow evolution on a Lorentz manifold can be characterized by 3-d solitonic waves with explicit dependence flow parameter τ defined by functions $\iota(\tau, u)$ as solutions of such nonlinear PDEs:
$$\iota = \begin{cases}
\iota(\tau,\vartheta,\varphi) & \text{as a solution of} \quad \partial^2_{\tau\tau}\ \iota + \epsilon\frac{\partial}{\partial\varphi}[\partial_\vartheta\ \iota + 6\ \iota\frac{\partial}{\partial\varphi}\ \iota + \frac{\partial^3}{(\partial\varphi)^3}\ \iota] = 0;\\
\iota(\vartheta,\tau,\varphi) & \text{as a solution of} \quad \partial^2_{\vartheta\vartheta}\ \iota + \epsilon\frac{\partial}{\partial\varphi}[\partial_t\ \iota + 6\ \iota\frac{\partial}{\partial\varphi}\ \iota + \frac{\partial^3}{(\partial\varphi)^3}\ \iota] = 0;\\
\iota(\tau,r,\varphi) & \text{as a solution of} \quad \partial^2_{\tau\tau}\ \iota + \epsilon\frac{\partial}{\partial\varphi}[\partial_r\ \iota + 6\ \iota\frac{\partial}{\partial\varphi}\ \iota + \frac{\partial^3}{(\partial\varphi)^3}\ \iota] = 0;\\
\iota(r,\tau,\varphi) & \text{as a solution of} \quad \partial^2_{rr}\iota + \epsilon\frac{\partial}{\partial\varphi}[\partial_\tau\iota + 6\iota\frac{\partial}{\partial\varphi}\iota + \frac{\partial^3}{(\partial\varphi)^3}\iota] = 0;\\
\iota(\tau,\varphi,\vartheta) & \text{as a solution of} \quad \partial^2_{\tau\tau}\iota + \epsilon\frac{\partial}{\partial\vartheta}[\partial_\varphi\iota + 6\iota\frac{\partial}{\partial\vartheta}\iota + \frac{\partial^3}{(\partial\vartheta)^3}\iota] = 0;\\
\iota(\varphi,\tau,\vartheta) & \text{as a solution of} \quad \partial^2_{\varphi\varphi}\ \iota + \epsilon\frac{\partial}{\partial\vartheta}[\partial_\tau\ \iota + 6\ \iota\frac{\partial}{\partial\vartheta}\ \iota + \frac{\partial^3}{(\partial\vartheta)^3}\ q\iota] = 0.
\end{cases} \tag{54}$$

Applying general frame/coordinate transforms on respective solutions (54), we construct solitonic waves parameterized by functions labled in the form $\iota = \iota(\tau, x^i), = \iota(\tau, x^1, y^3)$, or $= \iota(\tau, x^2, y^3)$.

In a similar form, we can consider other types of solitonic stationary configurations determined, for instance, by sine-Gordon and various types of nonlinear wave configurations characterized by geometric curve flows as the equations outlined in Main Results. Any such solitonic hierarchy configuration, nonlinear wave and solitonic distribution of type $\iota(\tau, u)$ (54) or $\iota = \iota(x^i, y^3)$ (53) can be can be used as generating functions for certain classes of nonholonomic deformations of stationary, or cosmological metrics, and as generating sources. for geometric flow and MGT generic off-diagonal solutions, see details in [59, 60, 61, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74]. In this work, we denote such d-metrics of type $\mathbf{g}(\tau)$ (7) or (51) as d-tensor functionals of type

$$\mathbf{g}(\tau) = \mathbf{g}[\iota(\tau, u)] = \mathbf{g}[\iota] = (g_i[\iota], g_a[\iota]) \tag{55}$$

with polarization functions $\eta_i(\tau) = \eta_i(\tau, x^k) = \eta_i[\iota], \eta_a(\tau) = \eta_a(\tau, x^k, y^b) = \eta_a[\iota]$ and $\eta_i^a(\tau) = \eta_i^a(\tau, x^k, y^b) = \eta_i^a[\iota]$. In general, a functional dependence $[\iota]$ may be on multiple types of solitonic hierarchies (for instance, on some different solutions of equations of type (54), (53)) which can be written conventionally in the form $[\iota] = [\ {}_1\iota,\ {}_2\iota, ...]$ where the left label is use for numbering the type of solitonic hierarchies. We shall construct in explicit form such stationary solutions for geometric flows of NES in next section.

4.1.3 Table 1 with ansatz for stationary geometric flows and solitonic hierarchies

In this work, we use brief notations of partial derivatives $\partial_\alpha q = \partial q / \partial u^\alpha$ when a function $q(x^k, y^a)$,

$$\begin{aligned} \partial_1 q &= q^\bullet = \partial q/\partial x^1, \partial_2 q = q' = \partial q/\partial x^2, \partial_3 q = \partial q/\partial y^3 = \partial q/\partial \varphi = q^\diamond, \partial_4 q = \partial q/\partial t = \partial_t q = q^*, \\ \partial^2_{33} &= \partial^2 q/\partial \varphi^2 = \partial^2_{\varphi\varphi} q = q^{\diamond\diamond}, \partial^2_{44} = \partial^2 q/\partial t^2 = \partial^2_{tt} q = q^{**}. \end{aligned}$$

Partial derivatives on a flow parameter will be written in the form $\partial_\tau = \partial/\partial\tau$.

Ansatz for geometric and curve flows of d-metrics with stationary solitonic hierarchies:

Using frame transforms, the τ-evolution of any d–metric $\mathbf{g}(\tau)$ of type (7), (51) and (55) can be parameterized for respective spherical symmetric coordinates $u^\alpha = (r, \theta, y^3 = \varphi, t)$ or some general local coordinates $(x^k, y^4 = t)$ and a common geometric flow evolution and/or curve flows parameter τ,

$$\begin{aligned} g_i(\tau) &= e^{\psi(\tau, r, \theta)} = e^{\psi[\ {}_1\iota]},\ g_a(\tau) = \omega(\tau, r, \theta, y^b) h_a(\tau, r, \theta, \varphi) = \omega[\ {}_3\iota] h_a[\ {}_2\iota], \\ N_i^3(\tau) &= w_i(\tau, r, \theta, \varphi) = w_i[\ {}_2\iota],\quad N_i^4(\tau) = n_i(\tau, r, \theta, \varphi) = n_i[\ {}_4\iota,\ {}_2\iota], \end{aligned} \tag{56}$$

taking $\omega = 1$ for a large class of stationary configurations. The AFDM results in more simple and explicit (still very general classes) of solutions if we work with nonholonomic configurations possessing at least one Killing symmetry, for instance, on $\partial_4 = \partial_t$ for stationary solutions.

We can construct for physically important systems on nonlinear PDEs (25) or (24) various classes of exact and parametric off-diagonal solutions generically depending on τ and all spacetime coordinates (x^k, y^a) but that would result in hundreds of pages with a cumbersome formulas for respective geometric techniques, see details and examples in [59, 60, 61, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74] and references therein.

Ansatz for flow evolution of (effective) sources with stationary solitonic hierarchies:

Using nonholonomic frame transforms and tetradic (vierbein) fields, we can introduce effective sources for geometric flows of NES (25) or (24) which in N–adapted form are parameterized in the form

$${}^{eff}\widehat{\mathbf{\Upsilon}}_{\mu\nu}(\tau) = \mathbf{e}^{\mu'}_{\ \mu}(\tau) \mathbf{e}_\nu^{\ \nu'}(\tau) [\ \widehat{\mathbf{\Upsilon}}_{\mu'\nu'}(\tau) + \frac{1}{2}\ \partial_\tau \mathbf{g}_{\mu'\nu'}(\tau)] = [\ {}_h\widehat{\mathbf{\Upsilon}}(\tau, x^k)\delta^i_j, \widehat{\mathbf{\Upsilon}}(\tau, x^k, y^c)\delta^a_b].$$

Such families of vielbein transforms $\mathbf{e}^\mu_{\ \mu'}(\tau) = \mathbf{e}^\mu_{\ \mu'}(\tau, u^\gamma)$ and their dual $\mathbf{e}_\nu^{\ \nu'}(\tau, u^\gamma)$, when $\mathbf{e}^\mu = \mathbf{e}^\mu_{\ \mu'} du^{\mu'}$ can be chosen in the form (3) and/or any frame transforms of a N-connection structure (1). The values ${}_h\widehat{\mathbf{\Upsilon}}(\tau, x)$

and $\widehat{\mathbf{\Upsilon}}(\tau, x, y)$ can be taken as functionals of certain solutions of solitonic equations and then considered as generating data for (effective) matter sources and certain forms compatible with solitonic hierarchies for d-metrics (56). Prescribing geometric and physically motivated data, we impose certain nonholonomic frame constraints on geometric evolution and self-similar configurations of stationary NES. In such cases, we write

$$\widehat{\Im}[\iota] = \ ^{eff}\widehat{\mathbf{\Upsilon}}^{\mu}{}_{\nu}(\tau) = [\ \ _{h}\widehat{\Im}[\ _{1}\iota] = \ \ _{h}\widehat{\mathbf{\Upsilon}}(\tau, r, \theta)\delta^{i}_{j}, \ \ _{v}\widehat{\Im}[\ _{2}\iota] = \widehat{\mathbf{\Upsilon}}(\tau, r, \theta, \varphi)\delta^{a}_{b}]. \qquad (57)$$

There are used "hat" symbols in order to emphasize that such values are considered systems on nonlinear PDEs involving a canonical d-connection.

In canonical nonholonomic variables with functional dependence of d-metrics and effective sources on some prescribed classes of solitonic hierarchie, the system of nonholonomic entropic R. Hamilton equations (25) can be written in the form (24) but with geometric objects depending additionally on a temperature like parameter τ and for effective source (57),

$$\widehat{\mathbf{R}}_{\alpha\beta}[\iota] = \widehat{\Im}_{\alpha\beta}[\iota]. \qquad (58)$$

We note that such geometric evolution equations are for an undetermined normalization function $f(\tau) = f(\tau, u^{\gamma})$ which can be defined explicitly for respective classes of exact or parametric solutions for prescribed solitonic hierarchies. For self-similar point $\tau = \tau_0$ configurations with $\partial_{\tau}\mathbf{g}_{\mu\nu}(\tau_0) = 0$, this system of nonlinear PDEs transforms into the canonical nonholonomic Ricci soliton equations (23).

Let us summarize in Table 1 below the data on nonholonomic 2+2 variables and corresponding ansatz which allow us to transform relativistic geometric flow equations and/or nonholonomic Ricci solitons (and equivalent gravitational field equations in MGTs and GR) into respective systems of nonlinear ordinary differential equations, ODEs, and partial differential equations, PDEs, determined by respective solitonic hierarchies. All formulas will be proven in next sections, see details in [59, 68, 69, 70, 71, 72, 73, 74]. Our goal is to show that such systems of nonlinear PDEs can be decoupled in general forms for generating functions and effective sources determined by solitonic hierarchies.

Table 1: Geometric flows and solitonic modified Einstein eqs as systems of nonlinear PDEs
and the Anholonomic Frame Deformation Method, **AFDM**,
for constructing generic off-diagonal exact, parametric, and stationary solutions

diagonal ansatz: PDEs $\rightarrow$ **ODE**s		AFDM: **PDE**s **with decoupling; generating functions**
radial coordinates $u^{\alpha} = (r, \theta, \varphi, t)$	$u = (x, y)$:	2+2 splitting, $u^{\alpha} = (x^1, x^2, y^3, y^4 = t)$; flow parameter τ
LC-connection $\mathring{\nabla}$	[connections]	$\mathbf{N} : T\mathbf{V} = hT\mathbf{V} \oplus vT\mathbf{V}$, locally $\mathbf{N} = \{N^a_i(x, y)\}$ canonical connection distortion $\widehat{\mathbf{D}} = \nabla + \widehat{\mathbf{Z}}$
diagonal ansatz $g_{\alpha\beta}(u)$ $= \begin{pmatrix} \mathring{g}_1 & & & \\ & \mathring{g}_2 & & \\ & & \mathring{g}_3 & \\ & & & \mathring{g}_4 \end{pmatrix}$	$\mathring{\mathbf{g}} \Leftrightarrow \mathbf{g}(\tau)$	$g_{\alpha\beta}(\tau, x^i, y^a)$ general frames / coordinates $g_{\alpha\beta}(\tau) = \begin{bmatrix} g_{ij} + N^a_i N^b_j h_{ab} & N^b_i h_{cb} \\ N^a_j h_{ab} & h_{ac} \end{bmatrix}$, 2 x 2 blocks $\mathbf{g}_{\alpha\beta}(\tau) = [g_{ij}(\tau), h_{ab}(\tau)]$, $\mathbf{g}(\tau) = \mathbf{g}_i(\tau, x^k)dx^i \otimes dx^i + \mathbf{g}_a(\tau, x^k, y^b)\mathbf{e}^a \otimes \mathbf{e}^b$
$\mathring{g}_{\alpha\beta} = \mathring{g}_{\alpha}(r)$ for BHs	[coord.frames]	$g_{\alpha\beta}(\tau) = g_{\alpha\beta}(\tau, r, \theta, y^3 = \varphi)$ stationary configurations
coord.tranfsorms $e_{\alpha} = e^{\alpha'}_{\ \alpha}\partial_{\alpha'}$, $e^{\beta} = e^{\ \beta}_{\beta'}du^{\beta'}, \mathring{g}_{\alpha\beta} = \mathring{g}_{\alpha'\beta'}e^{\alpha'}_{\ \alpha}e^{\beta'}_{\ \beta}$ $\mathring{\mathbf{g}}_{\alpha}(x^k, y^a) \rightarrow \mathring{g}_{\alpha}(r)$, or $\mathring{g}_{\alpha}(t)$, $\mathring{N}^a_i(x^k, y^a) \rightarrow 0$.	[N-adapt. fr.]	$\begin{cases} \mathbf{g}_i(\tau, r, \theta) = \mathbf{g}_i[\ _1\iota], \mathbf{g}_a(\tau, r, \theta, \varphi) = \mathbf{g}_a[\ _2\iota], & \text{d-metrics} \\ N^3_i(\tau) = w_i[\ _2\iota], N^4_i = n_i[\ _4\iota,\ _2\iota], & \text{N-connections} \end{cases}$
$\mathring{\nabla}$, $Ric = \{\mathring{R}_{\ \beta\gamma}\}$	Ricci tensors	$\widehat{\mathbf{D}}$, $\widehat{\mathcal{R}}ic = \{\widehat{\mathbf{R}}_{\ \beta\gamma}\}$
$^{m}\mathcal{L}[\phi] \rightarrow {}^{m}\mathbf{T}_{\alpha\beta}[\phi]$	sources	$\widehat{\Im}[\iota] = \widehat{\mathbf{\Upsilon}}^{\mu}{}_{\nu}(\tau) = \mathbf{e}^{\mu}_{\ \mu'}\mathbf{e}_{\nu}^{\ \nu'}\widehat{\mathbf{\Upsilon}}^{\mu'}{}_{\nu'}$ $= [\ \ _h\widehat{\Im}[\ _1\iota]\delta^i_j, \widehat{\Im}[\ _2\iota]\delta^a_b]$, stationary conf.
trivial equations for $\mathring{\nabla}$-torsion	LC-conditions	$\widehat{\mathbf{D}}_{\mid\widehat{\mathcal{T}}\rightarrow 0} = \nabla$ extracting new classes of solutions in GR

In this paper, we consider a physically important cases when $\mathring{\mathbf{g}}$ (52) defines a BH solution (for instance, a vacuum Kerr, or Schwarzschild, Kerr-(anti) de Sitter metric). For diagonalizable via coordinate transforms prime metrics, we can always find a coordinate system when $\mathring{N}^b_i = 0$. To study non-singular noholonomic deformations is convenient to construct exact solutions with nontrivial functions $\eta_{\alpha} = (\eta_i, \eta_a), \eta^a_i$, and nonzero coefficients $\mathring{N}^b_i(u)$. For a d-metric (51), we can analyze the conditions of existence and geometric/ physical properties of some target and/or prime solutions, for instance, when $\eta_{\alpha} \rightarrow 1$ and $N^a_i \rightarrow \mathring{N}^a_i$. The values $\eta_{\alpha} = 1$ and/or $\mathring{N}^a_i = 0$ can be imposed as some special nonholonomic constraints.

4.2 Decoupling of GIF flow equations into stationary solitonic hierarchies

In this subsection, we prove that the system of nonlinear PDEs (58) describing geometric flow evolution of stationary NES encoding solitonic hierarchies can be decoupled in general form. Such geometric and information flow systems possess an important type of nonlinear symmetries relating generating functions and effective generating source which will be applied for computing effective thermodynamic values.

4.2.1 Canonical Ricci d-tensors for geometric flows of NES encoding solitonic hierarchies

We can chose certain systems of reference/ coordinates when coefficients of the d-metrics (56) and derived geometric objects do not depend on $y^4 = t$ with respect to a class of N-adapted frames. Using parameterizations for a d-metric with $\omega = 1$ and a source $\widehat{\Im}[\iota] = [\ _h\widehat{\Im}[\ _1\iota],\ \ _v\widehat{\Im}[\ _2\iota]]$ (57) , we obtain such nontrivial N–adapted coefficients of the Ricci d-tensor, which allow to write the geometric flow modified Einstein equations (58) in the form

$$
\begin{aligned}
\widehat{\mathbf{R}}_1^1[\ \iota] &= \widehat{\mathbf{R}}_2^2[\ \iota] = -\ _h\widehat{\Im}[\ _1\iota] \text{ i.e.} \frac{g_1^{\bullet} g_2^{\bullet}}{2g_1} + \frac{(g_2^{\bullet})^2}{2g_2} - g_2^{\bullet\bullet} + \frac{g_1' g_2'}{2g_2} + \frac{(g_1')^2}{2g_1} - g_1'' = -2g_1 g_2\ _h\widehat{\Im}; & (59)\\
\widehat{\mathbf{R}}_3^3[\ \iota] &= \widehat{\mathbf{R}}_4^4[\ \iota] = -\ _v\widehat{\Im}[\ _2\iota] \text{ i.e. } \frac{(h_4^{\diamond})^2}{2h_4} + \frac{h_3^{\diamond}\ h_4^{\diamond}}{2h_3} - h_4^{\diamond\diamond} = -2h_3 h_4\ \ _v\widehat{\Im}; & (60)\\
\widehat{\mathbf{R}}_{3k}(\tau) &= -w_k \left[\left(\frac{h_4^{\diamond}}{2h_4} \right)^2 + \frac{h_3^{\diamond}}{2h_3} \frac{h_4^{\diamond}}{2h_4} - \frac{h_4^{\diamond\diamond}}{2h_4} \right] + \frac{h_4^{\diamond}}{2h_4} \left(\frac{\partial_k h_3}{2h_3} + \frac{\partial_k h_4}{2h_4} \right) - \frac{\partial_k h_4^{\diamond}}{2h_4} = 0, & (61)\\
\widehat{\mathbf{R}}_{4k}(\tau) &= \frac{h_4}{2h_4} n_k^{\diamond\diamond} + \left(\frac{3}{2} h_4^{\diamond} - \frac{h_4}{h_3} h_3^{\diamond} \right) \frac{n_k^{\diamond}}{2h_3} = 0. & (62)
\end{aligned}
$$

This system of nonlinear PDEs has a very important decoupling property: Using (59), we can find g_1 (or, inversely, g_2) for any prescribed functional of solitonic hierarchies encoded into a h-source $\ _h\widehat{\Im}[\ _1\iota]$ and any given coefficient $g_2(\tau, r, \theta) = g_2[\ \iota]$ (or, inversely, $g_1(\tau, r, \theta) = g_1[\ \iota]$) when the solitonic hierarchies for the coefficients of a h-metric can be different from the geometric/ solitonic data for effective sources. Then we can integrate on y^3 in (60) and define $h_3(\tau, r, \theta, \varphi)$ as a solution of first order PDE for any prescribed v-source $\ _v\widehat{\Im}[\ _2\iota]$ and given coefficient $h_4(\tau, r, \theta, \varphi) = h_4[\ _2\iota]$. Inversely, we can define $h_4(\tau, r, \theta, \varphi)$ if $h_3(\tau, r, \theta, \varphi) = h_3[\ _2\iota]$ is given but in such cases we have to solve a second order PDE. The coefficients of v-metrics involve, in general, different types of solitonic hierarchies comparing to those prescribed for the effective v-source. If the values of h_3 and h_4 are defined, the equations (61) transform into a system of algebraic linear equations for $w_k(\tau, r, \theta, \varphi) = w_k[\ \iota]$. We have to integrate two times on y^3 in (62) in order to compute $n_k(\tau, r, \theta, \varphi) = n_k[\ \iota]$ for any defined h_3 and h_4. The solitonic hierarchies encoded in the coefficients of a N-connection are different (in general) from those encoded in the coefficients of d-metric and nontrivial effective sources.

Using the decoupling property of nonlinear systems of type (59)–(62), we can integrate such PDEs step by step by prescribing respectively the effective sources, the h-coefficients, g_i, and v–coefficients, h_a, for geometric flowd of d-metrics $[\mathbf{g}_i(\tau) = g_i(\tau), \mathbf{g}_a(\tau) = g_a(\tau)]$ and for the N-connection coefficients, $N_i^a(\tau) = [w_i(\tau), n_i(\tau)]$, see formulas in (51) and/or (56). The geometric evolution of such solutions involves a prescribed nonholonomic constraint on $\partial_\tau \mathbf{g}_{\mu'\nu'}(\tau)$ included in $\widehat{\Im}[\iota]$.

4.2.2 Nonlinear symmetries for solitonic generating functions and sources

Introducing the coefficients $\alpha_i = (\partial_\varphi h_4)\ (\partial_i \varpi),\ \beta = (\partial_\varphi h_4)\ (\partial_\varphi \varpi),\ \gamma = \partial_\varphi \left(\ln |h_4|^{3/2} / |h_3| \right)$, where

$$\varpi = \ln |\partial_3 h_4 / \sqrt{|h_3 h_4|}| \qquad (63)$$

for nonsingular values for $\partial_3 h_a \neq 0$ and $\partial_t \varpi \neq 0$,[15] we obtain

$$\psi^{\bullet\bullet} + \psi'' = 2\ \ _h\widehat{\Im}[\ _1\iota], \quad \varpi^{\diamond}\ h_4^{\diamond} = 2h_3 h_4\ \ _v\widehat{\Im}[\ _2\iota],\ \beta w_i - \alpha_i = 0, \quad n_k^{\diamond\diamond} + \gamma n_k^{\diamond} = 0. \qquad (64)$$

[15]we can construct nontrivial solutions if such conditions are not satisfied; we omit in this work considerations for more special geometric evolution models; it is possible to introduce such frame/coorinate transforms when necessary type conditions are satisfied

Such a system can be integrated in explicit form (see details in [59, 68, 69, 70, 71, 72, 73, 74] and a series of examples will be provided in next sections) if there are prescribed a generating function $\Psi(\tau) = \Psi(\tau, x^i, y^3) = \Psi[\ \iota] := e^{\varpi}$ and generating sources $\ _h\widehat{\Im}$ and $\ _v\widehat{\Im}$.[16]

We have a system of two equations for ϖ in (63) and (64) involving four functions ($h_3, h_4,\ _v\widehat{\Im}$, and Ψ). By straightforward computation we can check that there an important nonlinear symmetry which allows to redefine the generating function and effective source (in particular, to introduce a family of effective cosmological constants $\Lambda(\tau) \neq 0, \Lambda(\tau_0) = const$, not depending on spacetime coordinates u^α). We can consider nonlinear transforms $(\Psi(\tau),\ _v\widehat{\Im}(\tau)) \iff (\Phi(\tau), \Lambda(\tau))$ defined by formulas

$$\Lambda(\ \Psi^2[\ _1\iota])^\diamond = |\ _v\widehat{\Im}[\ \ _2\iota]|(\Phi^2[\ \iota])^\diamond, \text{ or } \Lambda\ \Psi^2[\ _1\iota] = \Phi^2[\ \iota]|\ _v\widehat{\Im}[\ \ _2\iota]| - \int dy^3\ \Phi^2[\ \iota]|\ _v\widehat{\Im}[\ _2\iota]|^\diamond, \qquad (65)$$

which allow us to introduce families of new generating functions $\Phi(\tau, x^i, y^3) = \Phi[\ \iota]$ and families of (effective) cosmological constants. The values $\Lambda(\tau)$ can be chosen from certain physical considerations when the geometric/physical data for Φ encode nonlinear symmetries and solitonic hierarchies for $_v\widehat{\Im}$, and Ψ. Solutions with $\Lambda = 0$ have to be studied by applying special methods, see details and examples in [73, 35, 34]. Using nonlinear symmetries, we can describe nonlinear systems of PDEs by two equivalent sets of generating data (Ψ, Υ) or (Φ, Λ) but in all cases the symmetries of solitonic hierarchies are encoded into functionals with respective partial derivations on ∂_3 and/or integration on dy^3. To generate certain classes of solutions, we can work with effective cosmological constants but for other ones we have to consider generating sources. Such alternatives are convenient for constructing more general classes of exact solutions, prescribe necessary types of solitonic symmetries, and to elaborate on realistic physical models. Modules in formulas (65) should be taken in certain forms resulting in physically motivated nonlinear symmetries, relativistic causal models which are compatible with observational data.

4.3 Integrability of geometric flow equations with solitonic hierarchies

We integrate in explicit form and study properties of some classes of generic off-diagonal stationary solutions of (58) determined by generated functions and sources with solitonic hierarchies.

4.3.1 Stationary solutions for off-diagonal metrics and N–coefficients

By straightforward computations we can prove that integrating "step by step" the system (59)–(62) represented in the form (64) (see similar details and rigorous proofs in [73, 35]) one generates exact stationary solutions of geometric flow and/or modified Einstein equations if the d–metric and respective N–connection coefficients are computed

$$\begin{aligned}
g_i(\tau) &= e^{\ \psi(\tau,x^k)} \text{ as a solution of 2-d Poisson eqs. } \psi^{\bullet\bullet} + \psi'' = 2\ \ \ _h\widehat{\Im}[\ _1\iota]; \\
g_3[\ _3\iota] &= h_3(\tau, r, \theta, \varphi) = -\frac{(\Psi^\diamond[\iota])^2}{4(\ _v\widehat{\Im}[\ _2\iota])^2 h_4[\ _4\iota]} = -\frac{(\partial_3\Psi)^2}{4(\ _v\widehat{\Im})^2\left(h_4^{[0]}(\tau, x^k) - \int dy^3 (\Psi^2)^\diamond/4\ _v\widehat{\Im}\right)} \qquad (66)\\
&= -\frac{(\Phi^2)(\Phi^2)^\diamond}{h_4|\Lambda(\tau)\int dy^3\ _v\widehat{\Im}[\Phi^2]^\diamond|} = -\frac{[\partial_3(\Phi^2)]^2}{4[h_4^{[0]}(\tau,x^k) - \Phi^2/4\Lambda(\tau)]|\int dy^3\ \ _v\widehat{\Im}\partial_3[\Phi^2]|}; \\
g_4[\ _4\iota] &= h_4(\tau, r, \theta, \varphi) = h_4^{[0]}(\tau, x^k) - \int dy^3 \frac{(\Psi^2)^\diamond}{4\ _v\widehat{\Im}} = h_4^{[0]}(\tau, x^k) - \Phi^2/4\Lambda(\tau); \\
N_i^3[\iota] &= w_i(\tau, r, \theta, \varphi) = \frac{\partial_i\ \Psi}{\partial_3\Psi} = \frac{\partial_i\ \Psi^2}{\partial_3\Psi^2} = \frac{\partial_i[\int dy^3\ \ _v\widehat{\Im}(\Phi^2)^\diamond]}{_v\widehat{\Im}(\Phi^2)^\diamond};
\end{aligned}$$

[16] The LC–conditions (10) for stationary configurations transform into a system of 1st order PDEs,

$$\partial_\varphi w_i = (\partial_i - w_i\partial_\varphi)\ln\sqrt{|h_3|}, (\partial_i - w_i\partial_\varphi)\ln\sqrt{|h_4|} = 0, \partial_k w_i = \partial_i w_k, \partial_\varphi n_i = 0, \partial_i n_k = \partial_k n_i,$$

imposing additional constraints on off-diagonal coefficients of metrics of type (56).

$$
\begin{aligned}
N_k^4[\ _5\iota] &= n_k(\tau,r,\theta,\varphi) = \ _1n_k(\tau,x^i) + \ _2n_k(\tau,x^i)\int dy^3 \frac{(\Psi^\diamond)^2}{_v\widehat{\Im}^2|h_4^{[0]}(\tau,x^i) - \int dy^3(\Psi^2)^\diamond/4\ _v\widehat{\Im}|^{5/2}} \\
&= \ _1n_k(\tau,x^i) + \ _2n_k(\tau,x^i)\int dy^3 \frac{(\Phi^\diamond)^2}{4|\Lambda(\tau)\int dy^3\ _v\widehat{\Im}[\Phi^2]^\diamond||h_4|^{5/2}}.
\end{aligned}
$$

In these formulas, there are stated different sets of solitonic hierarchies which is motivated by the facts that there are integration functions $h_3^{[0]}(\tau,x^k)$, $\ _1n_k(\tau,x^i)$, and $\ _2n_k(\tau,x^i)$ encoding (non) commutative parameters and integration constants but also nonlinear evolution scenarios on τ. Such values, together with symmetries of solitonic hierarchies generating geometric evolution data (Ψ,Υ), or (Φ,Λ), related by nonlinear differential / integral transforms (65) can be stated in explicit form following certain topology/ symmetry / asymptotic conditions. The coefficients (66) define generic off-diagonal stationary solitonic solutions with associated bi Hamilton structures if the corresponding anholonomy coefficients are not trivial. Such geometric flow solutions are with nontrivial nonholonomically induced d-torsion solitonic hierarchies determined by evolution of N-adapted coefficients of d-metric structures. We can impose additional nonholonomic constraints (10) in order to extract LC-configurations under geometric flow evolution.

4.3.2 Quadratic line elements for off-diagonal stationary solitonic hierarchies

We can consider as a generating function any coefficient $h_4[\ _4\iota] = h_4^{[0]} - \Phi^2/4\Lambda, h_4^\diamond(\tau) \neq 0$ and write formulas $\Phi^2(\tau) = 4\Lambda\left(h_4[\ _4\iota] - h_4^{[0]}\right)$, $(\Phi^2)^\diamond = 4\Lambda(h_4)^\diamond$ and $(\Phi^\diamond)^2 = \Lambda(h_4)^\diamond(\frac{h_4}{h_4^{[0]}} - 1)$. Using (65), find $(\Psi^2)^\diamond = 4\left|\ _v\widehat{\Im}[\ _2\iota]\right|(h_4)^\diamond$ and $\Psi^2 = 4\left|\ _v\widehat{\Im}\right|h_4 - 4\int dy^3\left|\ _v\widehat{\Im}\right|^\diamond h_4$, which allows to construct functionals $\Psi[\ _v\widehat{\Im}, h_4, h_4^{[0]}]$ and $\Phi[\Lambda, h_4, h_4^{[0]}]$. Then, we can introduce such values into respective formulas for h_a, N_i^b and $\ _v\widehat{\Im}$ in (66) and and express possible generating functions and the d–metric (7) with stationary data (56) in terms of h_4, integration functions and effective sources for geometric evolutions.

$$
\begin{aligned}
g_3[\ _3\iota] &= h_3(\tau,r,\theta,\varphi) = -\frac{(\Phi^2)(\Phi^2)^\diamond}{h_4|\Lambda(\tau)\int dy^3\ _v\widehat{\Im}[\Phi^2]^\diamond|} = \frac{4|(h_4)^\diamond|}{|\int dy^3\ _v\widehat{\Im}(h_4)^\diamond|}; \\
g_4[\ _4\iota] &= h_4(\tau,r,\theta,\varphi) = h_4^{[0]}(\tau,x^k) - \int dy^3\frac{(\Psi^2)^\diamond}{4\ _v\widehat{\Im}} = h_4^{[0]}(\tau,x^k) - \Phi^2/4\Lambda(\tau); \\
N_i^3[\iota] &= w_i(\tau,r,\theta,\varphi) = \frac{\partial_i\ \Psi}{\partial_3\Psi} = \frac{\partial_i\ \Psi^2}{\partial_3\Psi^2} = \frac{\partial_i[\int dy^3\ \ _v\widehat{\Im}(\Phi^2)^\diamond]}{_v\widehat{\Im}(\Phi^2)^\diamond} = \frac{\partial_i\left(\left|\ _v\widehat{\Im}\right|h_4 - \int dy^3\left|\ _v\widehat{\Im}\right|^\diamond h_4\right)}{\left|\ _v\widehat{\Im}[\ _2\iota]\right|h_4^\diamond}; \\
N_k^4[\ _5\iota] &= n_k(\tau,r,\theta,\varphi) = \ _1n_k(\tau,x^i) + \ _2n_k(\tau,x^i)\int dy^3\frac{(\Phi^\diamond)^2}{4|\Lambda(\tau)\int dy^3\ _v\widehat{\Im}[\Phi^2]^\diamond||h_4|^{5/2}} = \\
&= \ _1n_k(\tau,x^i) + \ _2\widetilde{n}_k(\tau,x^i)\int dy^3\frac{(h_4)^\diamond(1 - h_4/h_4^{[0]})}{|\Lambda\int dy^3\ _v\widehat{\Im}(h_4)^\diamond||h_4|^{5/2}}.
\end{aligned}
$$

In result, we can express the respective quadratic elements in three equivalent forms:

$$
\begin{aligned}
ds^2 &= e^{\ \psi(\tau,x^k)}[(dx^1)^2 + (dx^2)^2] + \qquad (67) \\
&\left\{
\begin{array}{cc}
\begin{array}{c}
-\frac{4|(h_4)^\diamond|}{|\int dy^3\ _v\widehat{\Im}(h_4)^\diamond|}[dy^3 + \frac{\partial_i\left(\left|\ _v\widehat{\Im}\right|h_4 - \int dy^3\left|\ _v\widehat{\Im}\right|^\diamond h_4\right)}{\left|\ _v\widehat{\Im}[\ _2\iota]\right|h_4^\diamond}dx^i] - \\
h_4[dt + (\ \ _1n_k(\tau,x^i) + \ _2\widetilde{n}_k(\tau,x^i)\int dy^3\frac{(h_4)^\diamond(1-h_4/h_4^{[0]})}{|\Lambda\int dy^3\ _v\widehat{\Im}(h_4)^\diamond||h_4|^{5/2}})dx^k],
\end{array} & \begin{array}{c}\text{gener. funct.}h_4, \\ \text{source } _v\widehat{\Im},\text{ or }\Lambda;\end{array} \\
\text{or} & \\
\begin{array}{c}
\frac{\partial_\varphi(\Psi^2)}{4(\ _v\widehat{\Im})^2(h_4^{[0]} - \int dy^3\frac{(\Psi^2)^\diamond}{4\ _v\widehat{\Im}})}[dy^3 + \frac{\partial_i\ \Psi}{\partial_3\Psi}dx^i] - (h_4^{[0]} - \int dy^3\frac{(\Psi^2)^\diamond}{4\ \ _v\widehat{\Im}}) \\
[dt + (_1n_k + \ _2n_k\int dy^3\frac{(\Psi^\diamond)^2}{4(\ _v\widehat{\Im})^2|h_4^{[0]} - \int dy^3\frac{(\Psi^2)^\diamond}{4\ \ _v\widehat{\Im}}|^{5/2}})dx^k],
\end{array} & \begin{array}{c}\text{gener. funct.}\Psi, \\ \text{source } _v\widehat{\Im};\end{array} \\
\text{or} & \\
\begin{array}{c}
-\frac{[(\Phi^2)^\diamond]^2}{4|\Lambda\int dy^3\ \ _v\widehat{\Im}[(\Phi)^2]^\diamond|\ (h_4^{[0]} - \frac{\Phi^2}{4\Lambda})}[dy^3 + \frac{\partial_i[\int dy^3\ \ _v\widehat{\Im}(\Phi^2)^\diamond]}{_v\widehat{\Im}\ (\Phi^2)^\diamond}dx^i] - (h_4^{[0]} - \frac{\Phi^2}{4\Lambda}) \\
[dt + (_1n_k + \ _2n_k\int dy^3\frac{[(\Phi^2)^\diamond]^2}{|\ 4\Lambda\int dy^3\ \ _v\widehat{\Im}[(\Phi)^2]^\diamond|}|h_4^{[0]} - \frac{\Phi^2}{4\Lambda}|^{-5/2})dx^k],
\end{array} & \begin{array}{c}\text{gener. funct.}\Phi \\ \text{effective }\Lambda\text{ for } _v\widehat{\Im}.\end{array}
\end{array}
\right.
\end{aligned}
$$

Formulas (66) and (67) encode solitonic hierarchies determined by generating functions but a generating source ${}_{v}\widehat{\Im}$ and effective cosmological constant Λ do not involve (in general) any solitonic behaviour. Nonlinear symmetries (65) mix different solitonic structures of generating functions and any functional for source.[17]

4.3.3 Off-diagonal Levi-Civita stationary solitonic hierarchies

The equations (10) for zero torsion conditions, see also footnote 16, can be solved for a special class of generating functions and sources. For instance, we can take a $\Psi(\tau) = \check{\Psi}(\tau, x^i, \varphi)$ for which $(\partial_i \check{\Psi})^{\diamond} = \partial_i(\check{\Psi}^{\diamond})$ and fix ${}_{v}\widehat{\Im}(\tau, x^i, y^3) = {}_{v}\widehat{\Im}[\check{\Psi}] = {}_{v}\check{\Im}(\tau)$, or ${}_{v}\widehat{\Im} = const$, modifying the nonlinear symmetries (65) to $\Lambda\ \check{\Psi}^2 = \check{\Phi}^2|\ {}_{v}\widehat{\Im}| - \int dy^3\ \Phi^2|\ {}_{v}\widehat{\Im}|^{\diamond}, \check{\Phi}^2 = -4\Lambda \check{h}_4(\tau, x^i, y^3), \check{\Psi}^2 = \int dy^3\ \ {}_{v}\widehat{\Im}\check{h}_4^{\diamond}$. The coefficient $h_4(\tau) = \check{h}_4(\tau, x^i, y^3)$ can be considered also as generating function when h_3 and N-connection coefficients are computed using certain nonlinear symmetries and nonholonomic constraints. For zero torsion solitonic hierarchies, we find some functions $\check{A}(\tau) = \check{A}(\tau, x^i, y^3)$ and $n(\tau) = n(\tau, x^i)$ when the coefficients of N-connection are

$$w_i(\tau) = \partial_i \check{A}(\tau) = \frac{\partial_i(\int dy^3\ {}_{v}\check{\Im}\ \check{h}_4^{\diamond}])}{{}_{v}\check{\Im}\ \check{h}_4^{\diamond}} = \frac{\partial_i \check{\Psi}}{\check{\Psi}^{\diamond}} = \frac{\partial_i[\int dy^3\ \ {}_{v}\check{\Im}(\check{\Phi}^2)^{\diamond}]}{{}_{v}\check{\Im}(\check{\Phi}^2)^{\diamond}} \text{ and } n_k(\tau) = \check{n}_k(\tau) = \partial_k n(\tau, x^i).$$

In result, we can construct new classes of off-diagonal zero torsion stationary solutions encoding solitonic hierarchies and defined as subclasses of solutions (67),

$$ds^2 = e^{\ \psi(\tau,x^k)}[(dx^1)^2 + (dx^2)^2] - \left\{ \begin{array}{ll} \frac{(\check{h}_4^{\diamond})^2}{|\int dy^3\ \ {}_{v}\check{\Im}\check{h}_4^{\diamond}|\ \check{h}_4}[dy^3 + (\partial_i \check{A})dx^i] + \check{h}_4\left[dt + (\partial_k n)dx^k\right], & \text{gener. funct.}\check{h}_4, \\ \text{or} & \text{source } {}_{v}\check{\Im}, \text{ or } \Lambda; \\ \frac{\partial_\varphi(\check{\Psi}^2)}{4\ (\ {}_{v}\check{\Im})^2(h_4^{[0]} - \int dy^3 \frac{(\check{\Psi}^2)^{\diamond}}{4\ {}_{v}\check{\Im}})}[dy^3 + (\partial_i \check{A})dx^i] + & \text{gener. funct.}\check{\Psi}, \\ (h_4^{[0]} - \int dy^3 \frac{(\check{\Psi}^2)^{\diamond}}{4\ \ {}_{v}\check{\Im}})\left[dt + (\partial_k n)dx^k\right], & \text{source } {}_{v}\check{\Im}; \\ \text{or} & \\ \frac{[(\check{\Phi}^2)^{\diamond}]^2}{4|\Lambda \int dy^3\ \ {}_{v}\check{\Im}(\check{\Phi}^2)^{\diamond}|\ (h_4^{[0]} - \frac{\check{\Phi}^2}{4\Lambda})}[dy^3 + (\partial_i \check{A})dx^i] & \text{gener. funct. } \check{\Phi} \\ +(h_4^{[0]} - \frac{\check{\Phi}^2}{4\Lambda})\left[dt + (\partial_k n)dx^k\right], & \text{effective } \Lambda \text{ for } {}_{v}\check{\Im}. \end{array} \right. \quad (68)$$

For any value of flow parameter τ, such stationary metrics are generic off-diagonal and define new classes of solutions which are different, for instance, from the Kerr metric (defined by rotation coordinates, or other equivalent ones). We may check if the anholonomy coefficients $C_{\alpha\beta}^{\gamma} = \{C_{ia}^b = \partial_a N_i^b, C_{ji}^a = \mathbf{e}_j N_i^a - \mathbf{e}_i N_j^a\}$ are not zero for solitonic values of $N_i^3 = \partial_i \check{A}$ and $N_k^4 = \partial_k n$ and understand if certain metrics are or not generic off-diagonal. We can fix and analyze certain nonholonomic solitonic configurations determined, for instance, by data $(\ {}_{v}\check{\Im}, \check{\Psi}, h_4^{[0]}, \check{n}_k)$, with $w_i = \partial_i \check{A} \to 0$ and $\partial_k n \to 0$.

5 Stationary geometric flows of BH and solitonic hierarchies

The goal of this section is to provide applications of the anholonomic frame deformation method (AFDM, outlined in previous section) for constructing in explicit form exact and parametric stationary generic off-diagonal solutions describing solitonic geometric flow deformations of prime BH metrics.

[17] We can consider nonholonomic solitonic deformations of a primary d-metric $\mathring{\mathbf{g}}$ into a target stationary one $\mathbf{g}(\tau) = [g_\alpha(\tau) = \eta_\alpha(\tau)\mathring{g}_\alpha,\ \eta_i^a(\tau)\mathring{N}_i^a]$ with Killing symmetry on ∂_t and respective bi-Hamilton structures. Formulas for the coefficients of d-metrics and N-connections presented above can be re-written equivalently in terms of η–polarization functions, η_α and η_i^a, determined by generation and integration functions and respective sources and encoding primary data $[\mathring{g}_\alpha, \mathring{N}_i^a]$. For instance, we can consider a d-metric $\mathring{\mathbf{g}}$ for a BH solution in GR and study hierarchies of solitonic deformations by geometric flows or for certain nonholonomic Ricci solition configurations which result in a stationary target d-metric $\mathbf{g}(\tau)$. Off-diagonal nonholonomic deformations of the metric and (non) linear connection structures and sources may preserve the singular structure of a primary metric with certain possible deformations of the horizons, for certain classes of solutions encoding nonsingular solitonic hierarchies and nonsingular distributions of effective sources. For more general classes of solutions with singular solitonic configurations, deformations of horizons, nonlinear symmetries etc., there are possible scenarios eliminating the singular structure, generating new symmetries, or changing the topology of target solutions.

5.1 Table 2: AFDM for constructing solitonic stationary flows of NES

We consider $h_4(\tau) = h_4(\tau, x^i, y^3) = h_4(\tau, r, \theta, \varphi)$ (67) as a generating function (it can be also determined by a family of solitonic hierarchies, $h_4(\tau) = h_4[\ _4\iota])$ and construct a deformation procedure for constructing a class of off–diagonal stationary solutions with Killing symmetry on ∂_t determined by solitonic hierarchies $\widehat{\Im}[\iota] = [\ _h\widehat{\Im}[\ _1\iota],\ \ _v\widehat{\Im}[\ _2\iota]]$ (57) and a parametric running cosmological constant $\Lambda(\tau)$,

$$
\begin{aligned}
ds^2 &= e^{\ \psi(\tau,x^k)}[(dx^1)^2 + (dx^2)^2] - \frac{[h_4^\diamond(\tau)]^2}{|\int dy^3\ \ _v\widehat{\Im}(\tau)(h_4{}^\diamond(\tau))|\ h_4}[dy^3 + \frac{\partial_i(\int d\varphi\ \ _v\widehat{\Im}(\tau) h_4^\diamond])}{_v\widehat{\Im}(\tau)\ h_4^\diamond(\tau)} dx^i] \\
&\quad + h_4(\tau)[dt + (\ _1n_k + 4\ _2n_k \int d\varphi \frac{(h_4^\diamond(\tau))^2}{|\int dy^3\ \ _v\widehat{\Im}(\tau)(h_4^\diamond(\tau))|\ (h_4(\tau))^{5/2}}) dx^k].
\end{aligned}
$$

Such solutions involve different types of solitonic hierarchies and, in general, are with nontrivial nonholonomically induced torsion which can be nonholonomically constrained to LC-configurations (68).

Table 2: Off-diagonal stationary flows with solitonic hierarchies

Exact solutions of $\widehat{\mathbf{R}}_{\mu\nu}(\tau) = \widehat{\Im}_{\mu\nu}(\tau)$ (58) transformed into a system of nonlinear PDEs (59)-(62)

d-metric ansatz with Killing symmetry $\partial_4 = \partial_t$ Effective matter sources	$ds^2 = g_i(\tau)(dx^i)^2 + g_a(\tau)(dy^a + N_i^a(\tau)dx^i)^2,$ for $g_i = e^{\psi(\tau,r,\theta)},\ \ g_a = h_a(\tau,r,\theta,\varphi),\ N_i^3 = w_i(\tau,r,\theta,\varphi),\ \ N_i^4 = n_i(\tau,r,\theta,\varphi),$ $\widehat{\Im}^\mu{}_\nu(\tau) = [\ _h\widehat{\Im}(\tau,r,\theta)\delta^i_j,\ \ _v\widehat{\Im}(\tau,r,\theta,\varphi)\delta^a_b]; x^1 = r, x^2 = \theta, y^3 = \varphi, y^4 = t$
Nonlinear PDEs (64)	$\psi^{\bullet\bullet} + \psi'' = 2\ \ _h\widehat{\Im}[\ _1\iota];$ $\varpi^\diamond\ h_4^\diamond = 2h_3h_4\ \ _v\widehat{\Im}[\ _2\iota];$ $\beta w_i - \alpha_i = 0;$ $n_k^{\diamond\diamond} + \gamma n_k^\diamond = 0;$ for $\varpi = \ln \vert\partial_3 h_4/\sqrt{\vert h_3h_4\vert}\vert,$ $\alpha_i = (\partial_\varphi h_4)\ (\partial_i\varpi),\ \beta = (\partial_\varphi h_4)\ (\partial_\varphi\varpi),$ $\gamma = \partial_\varphi \left(\ln \vert h_4\vert^{3/2}/\vert h_3\vert\right),$ $\partial_1 q = q^\bullet, \partial_2 q = q', \partial_3 q = \partial q/\partial\varphi = q^\diamond$
Generating functions: $h_4[\ _4\iota]$, $\Psi(\tau,r,\theta,\varphi) = e^\varpi, \Phi[\iota]$; integration functions: $h_4^{[0]}(\tau,x^k)$, $_1n_k(\tau,x^i),\ \ _2n_k(\tau,x^i)$	$(\Psi^2)^\diamond = -\int dy^3\ \ _v\widehat{\Im}h_4{}^\diamond, \Phi^2 = -4\Lambda(\tau)h_4,$ see nonlinear symmetries (65); $h_4(\tau) = h_4^{[0]} - \Phi^2/4\Lambda(\tau), h_4^\diamond \neq 0, \Lambda(\tau) \neq 0 = const$
Off-diag. solutions, d–metric N-connec.	$g_i(\tau) = e^{\ \psi(\tau,x^k)}$ as a solution of 2-d Poisson eqs. $\psi^{\bullet\bullet} + \psi'' = 2\ \ _h\widehat{\Im}(\tau);$ $h_3(\tau) = -(\Psi^\diamond)^2/4(\ _v\widehat{\Im})^2 h_4,$ see (66); $h_4(\tau) = h_4^{[0]} - \int dy^3(\Psi^2)^\diamond/4\ _v\widehat{\Im} = h_4^{[0]} - \Phi^2/4\Lambda(\tau);$ $w_i(\tau) = \partial_i\ \Psi/\ \partial_\varphi\Psi = \partial_i\ \Psi^2/\ \vert\partial_\varphi\Psi^2\vert;$ $n_k(\tau) = \ _1n_k +\ _2n_k \int dy^3(\Psi^\diamond)^2/(\ _v\widehat{\Im})^2\vert h_4^{[0]} - \int dy^3(\Psi^2)^\diamond/4\ _v\widehat{\Im}\vert^{5/2}.$
LC-configurations (16)	$\partial_\varphi w_i = (\partial_i - w_i\partial_\varphi)\ln\sqrt{\vert h_3(\tau)\vert}, (\partial_i - w_i\partial_\varphi)\ln\sqrt{\vert h_4(\tau)\vert} = 0,$ $\partial_k w_i(\tau) = \partial_i w_k(\tau), \partial_\varphi n_i(\tau) = 0, \partial_i n_k(\tau) = \partial_k n_i(\tau); \Psi = \check{\Psi}[\iota], (\partial_i\check{\Psi})^\diamond = \partial_i(\check{\Psi}^\diamond)$ and $\ _v\widehat{\Im}(\tau,x^i,\varphi) = \ _v\widehat{\Im}[\check{\Psi}] = \ _v\check{\Im},$ or $\ _v\widehat{\Im} = const.$
N-connections, zero torsion	$w_i(\tau) = \partial_i\check{A}(\tau) = \begin{cases} \partial_i(\int d\varphi\ \check{\Im}\ \check{h}_4{}^\diamond])/\check{\Im}\ \check{h}_4{}^\diamond; \\ \partial_i\check{\Psi}/\check{\Psi}^\diamond; \\ \partial_i(\int dy^3\ \check{\Im}(\check{\Phi}^2)^\diamond)/(\check{\Phi})^\diamond\check{\Im}; \end{cases}$ and $n_k(\tau) = \check{n}_k(\tau) = \partial_k n(\tau,x^i).$
polarization functions $\mathring{\mathbf{g}} \to \widehat{\mathbf{g}}$=$[g_\alpha = \eta_\alpha\mathring{g}_\alpha,\ \eta_i^a\mathring{N}_i^a]$	$ds^2 = \eta_1[\ _1\iota]\mathring{g}_1(r,\theta)[dx^1(r,\theta)]^2 + \eta_2[\ _2\iota]\mathring{g}_2(r,\theta)[dx^2(r,\theta)]^2 + \eta_3[\ _3\iota]\mathring{g}_3(r,\theta)$ $[d\varphi + \eta_i^3[\ _5\iota]\mathring{N}_i^3(r,\theta)dx^i(r,\theta)]^2 + \eta_4[\ _4\iota]\mathring{g}_4(r,\theta)[dt + \eta_i^4[\ _6\iota]\mathring{N}_i^4(r,\theta)dx^i(r,\theta)]^2,$
Prime metric defines a BH Example of a prime metric	$[\mathring{g}_i(r,\theta), \mathring{g}_a = \mathring{h}_a(r,\theta); \mathring{N}_k^3 = \mathring{w}_k(r,\theta), \mathring{N}_k^4 = \mathring{n}_k(r,\theta)]$ diagonalizable by frame/ coordinate transforms. $\mathring{g}_1 = (1 - r_g/r)^{-1}, \mathring{g}_2 = r^2, \mathring{h}_3 = r^2\sin^2\theta, \mathring{h}_4 = (1 - r_g/r), r_g = const$ the Schwarzschild solution, or any BH solution.
Solutions for polarization funct.	$\eta_i(\tau) = e^{\ \psi(\tau,x^k)}/\mathring{g}_i; \eta_3\mathring{h}_3 = -\frac{4[(\vert\eta_4\mathring{h}_4\vert^{1/2})^\diamond]^2}{\vert\int dy^3\ \ _v\widehat{\Im}[(\eta_4\mathring{h}_4)]^\diamond\vert};$ $\eta_4(\tau) = \eta_4(\tau,r,\theta,\varphi) = \eta_4[\ _4\iota]$ as a generating function; $\eta_i^3(\tau)\ \mathring{N}_i^3 = \frac{\partial_i\ \int dy^3\ \ _v\widehat{\Im}(\eta_4\ \mathring{h}_4)^\diamond}{_v\widehat{\Im}\ (\eta_4\ \mathring{h}_4)^\diamond}; \eta_k^4\ (\tau)\mathring{N}_k^4 = \ _1n_k + 16\ \ _2n_k \int dy^3 \frac{\left([(\eta_4\mathring{h}_4)^{-1/4}]^\diamond\right)^2}{\vert\int dy^3\ \ _v\widehat{\Im}[(\eta_4\ \mathring{h}_4)]^\diamond\vert}$
Polariz. funct. with zero torsion	$\eta_i(\tau) = e^{\ \psi(\tau,x^k)}/\mathring{g}_i; \eta_4 = \check{\eta}_4(\tau,r,\theta,\varphi)$ as a generating function; $\eta_3(\tau) = -\frac{4[(\vert\eta_4\mathring{h}_4\vert^{1/2})^\diamond]^2}{\mathring{g}_3\vert\int dy^3\ \ _v\check{\Im}[(\check{\eta}_4\mathring{h}_4)]^\diamond\vert}; \eta_i^3(\tau) = \frac{\partial_i\check{A}}{\mathring{w}_k}, \eta_k^4(\tau) = \frac{\partial_k n}{\mathring{n}_k}$

5.2 Nonlinear PDEs for geometric flows with stationary solitonic hierarchies

The goal of this subsection is to study explicit examples for constructing exact and parametric solutions encoding solitonic hierarchies for geometric flow modified Einstein equations (58) transformed into systems of nonlinear PDEs with decoupling (64).

5.2.1 Parametric stationary solutions with solitonic sources

We shall write that in a geometric flow source $\ _{v}\widehat{\Im}(\tau)$ there is, for instance, a term with left label "0" written $\ _{0}^{int}\widehat{\Im}(\tau)=\ _{v}^{int}\widehat{\Im}[\ _{2}\iota]$ if the corresponding term in $\ ^{eff}\widehat{\mathbf{\Upsilon}}^{\mu}{}_{\nu}(\tau)$ (57) is defined as a stationary functional on a solitonic hierarchy $[\ _{2}\iota]$. If it is written $\ _{v}^{int}\widehat{\Im}(\tau)$ without a left label "0", such a term correspond to a general $\ ^{eff}\widehat{\mathbf{\Upsilon}}^{\mu}{}_{\nu}(\tau)$ (without any solitonic specification) encoding contributions from a distortion tensor $\widehat{\mathbf{Z}}$ (9). In this work, an effective source term $\ _{v}^{fl}\widehat{\Im}$ determined by geometric flows of the d-metric, $\partial_{\tau}\mathbf{g}_{\alpha'}(\tau)$, in (57) is introduced. It is solitonic if the d-metric coefficients are solitonic. We can consider solitonic hierarchies for Ricci soliton configurations with $\ _{v}^{fl}\widehat{\Im}=0$.

For this class of solutions, we consider a source (57) (the left label a is used for "additive stationary")

$$\ _{v}^{a}\widehat{\Im}(\tau)=\ _{v}^{a}\widehat{\Im}[\iota]\ =\ _{v}^{a}\widehat{\Im}(\tau,x^{i},y^{3})=\ _{v}^{fl}\widehat{\Im}[\ _{1}\iota]+\ _{v}^{int}\widehat{\Im}[\ _{2}\iota]+\ _{v}^{int}\widehat{\Im}[\ _{3}\iota], \quad (69)$$

where it is considered that we prescripe an effective solitonic hierarchy for matter fields even, in general, such gravitational interactions can be of non-solitonic type. The second equation (64) with source $\ _{v}\widehat{\Im}[\iota]=\ _{v}^{a}\widehat{\Im}[\iota]$ can be integrated on y^{3}. In result, we construct off-diagonal metrics and generalized connections encoding solitonic hierarchic determined, , by a generating function $h_{4}(\tau,r,\theta,\varphi)$ with Killing symmetry on ∂_{t}, by effective sources $\ ^{a}\widehat{\Im}[\iota]\ =(\ _{h}^{a}\widehat{\Im}[\ _{1}\iota],\ \ _{v}^{a}\widehat{\Im}[\ _{2}\iota])$ and effective cosmological constant

$$\ ^{a}\Lambda(\tau)=\ ^{fl}\Lambda(\tau)+\ ^{m}\Lambda(\tau)+\ _{0}^{int}\Lambda(\tau) \quad (70)$$

related to $\ _{v}^{a}\widehat{\Im}[\iota]$ (69) via nonlinear symmetry transforms (65).

Applying the method summarized in Table 2, we construct such a class of quadratic elements defining stationary generic off-diagonal solutions determined by effective sources encoding solitonic hierarchies,

$$\begin{aligned} ds^{2} &= e^{\ \psi[\ _{1}\iota]}[(dx^{1})^{2}+(dx^{2})^{2}]-\frac{[h_{4}{}^{\diamond}(\tau)]^{2}}{|\int dy^{3}\ \ _{v}^{a}\widehat{\Im}[\iota]h_{4}{}^{\diamond}(\tau)|\ h_{4}(\tau)}\left[dy^{3}+\frac{\partial_{i}(\int dy^{3}\ \ _{v}^{a}\widehat{\Im}[\iota]\ h_{4}{}^{\diamond}(\tau))}{\ _{v}^{a}\widehat{\Im}[\iota]\ h_{4}{}^{\diamond}(\tau)}dx^{i}\right] \\ & +h_{4}(\tau)\left[dt+\left(\ _{1}n_{k}(\tau)+4\ _{2}n_{k}(\tau)\int dy^{3}\frac{[h_{4}{}^{\diamond}(\tau)]^{2}}{|\int dy^{3}\ \ _{v}^{a}\widehat{\Im}[\iota]h_{4}^{\diamond}(\tau)|\ [h_{4}(\tau)]^{5/2}}\right)dx^{k}\right]. \end{aligned} \quad (71)$$

Such solutions can be constrained to LC-configurations. The formulas (71) can be re-defined equivalently in terms of generating functions $\Psi(\tau,r,\theta,\varphi)$ or $\Phi(\tau,r,\theta,\varphi)$ which can be of a general (non–solitonic) character.

5.2.2 Modified and Einstein gravity with stationary solitonic generating functions

We can generate generic off-diagonal stationary solutions using generating functionals encoding solitonic hierarchies $\Phi(\tau)=\Phi[i]$ characterized by nonlinear symmetries of type (65) and general effective sources $\ _{v}\widehat{\Im}(\tau)$ which can be of non-solitonic character. The second equation into (64) transforms into

$$\varpi^{\diamond}(\tau)[\ \Phi[i],\Lambda(\tau)]\ h_{4}^{\diamond}(\tau)[\Phi[i],\Lambda(\tau)]=2h_{3}(\tau)[\Phi[i],\Lambda(\tau)]h_{4}(\tau)[\Phi[i],\Lambda(\tau)]\ \ _{v}\widehat{\Im}(\tau),$$

which can be solved together with other equations (59)-(62) following the AFDM, see Table 2.

The solutions for such stationary configurations determined by general nonlinear functionals for generating functions can be written in all forms (67). For simplicity, we present here the quadratic line element only the third type parametrization

$$\begin{aligned} ds^{2} &= e^{\ \psi(\tau,x^{k})}[(dx^{1})^{2}+(dx^{2})^{2}]-\frac{(\Phi[i])^{2}[(\Phi[i])^{2}]^{\diamond}}{|\Lambda(\tau)\int dy^{3}\ _{v}\widehat{\Im}(\tau)[(\Phi[i])^{2}]^{\diamond}|\ (h_{4}^{[0]}-\frac{(\Phi[i])^{2}}{4\Lambda(\tau)})} \\ & [dy^{3}+\frac{\partial_{i}\left(\int dy^{3}\ _{v}\widehat{\Im}(\tau)[(\Phi[i])^{2}]^{\diamond}\right)}{\ _{v}\widehat{\Im}(\tau)\ [(\Phi[i])^{2}]^{\diamond}}dx^{i}]+(h_{4}^{[0]}(\tau,x^{k})-\frac{(\Phi[i])^{2}}{4\Lambda(\tau)})[dt+(\ _{1}n_{k}(\tau,x^{k})+ \\ & \ _{2}n_{k}(\tau,x^{k})\int dy^{3}\frac{[(\Phi[i])^{2}]^{\diamond}}{|\Lambda(\tau)\int dy^{3}\ \ _{v}\widehat{\Im}(\tau)\ [(\Phi[i])^{2}]^{\diamond}|}|h_{4}^{[0]}(\tau,x^{k})-\frac{(\Phi[i])^{2}}{4\Lambda(\tau)}|^{-5/2})dx^{k}]. \end{aligned} \quad (72)$$

For zero torsion constraints, there are extracted LC-configurations,

$$ds^2 = e^{\psi(\tau)}[(dx^1)^2+(dx^2)^2] - \frac{(\check{\Phi}[i])^2[(\check{\Phi}[i])^2]^{\diamond}}{|\Lambda(\tau)\int dy^3\ {}_v\widehat{\Im}(\tau)[(\check{\Phi}[i])^2]^{\diamond}|\ (h_4^{[0]}-\frac{(\check{\Phi}[i])^2}{4\Lambda(\tau)})}[dy^3+(\partial_i\ \check{A}(\tau))dx^i]$$
$$+(h_4^{[0]}(\tau,x^k)-\frac{(\check{\Phi}[i])^2}{4\Lambda(\tau)})\left[dt+(\partial_k n(\tau))dx^k\right], \quad (73)$$

where $\check{A}(\tau)$ and $n(\tau)$ are also generating functions.

5.2.3 Small N-adapted stationary solitonic flow deformations

We can study important physical properties of some classes of solutions if there are considered small parametric deformations from certain well-known solutions (for instance, from a black hole, BH, configuration of Kerr or Schwarzschild type).

Let us consider a prime pseudo–Riemannian d-metric $\mathring{\mathbf{g}} = [\mathring{g}_i, \mathring{g}_a, \mathring{N}_b^j]$ (52) when $\partial_3\mathring{g}_4 = \mathring{g}_4^{\diamond} \neq 0$. It can be diagonalized via coordinate transforms. Our goal is to formulate a geometric formalism for small generic off–diagonal parametric deformations of $\mathring{\mathbf{g}}$ into certain target stationary metrics of type g (51)

$$ds^2 = \eta_i(\varepsilon,\tau)\mathring{g}_i(dx^i)^2+\eta_a(\varepsilon,\tau)\mathring{g}_a(\mathbf{e}^a)^2, \quad (74)$$
$$\mathbf{e}^3 = dy^3+\ {}^w\eta_i(\varepsilon,\tau)\mathring{w}_i dx^i, \mathbf{e}^4 = dt+\ {}^n\eta_i(\varepsilon,\tau)\mathring{n}_i dx^i,$$

where the coefficients $[g_\alpha = \eta_\alpha\mathring{g}_\alpha, {}^w\eta_i\mathring{w}_i,\ {}^n\eta_i n_i]$ depend on a small parameter ε, $0\leq\varepsilon\ll 1$, and on evolution parameter τ (in this work, it is used both for the geometric and curve flow evolution). We suppose that (74) define a solution of entropic flow evolution equations reduced to the system of nonlinear PDEs with decoupling (64). Some ε -deformations are parameterised in the form

$$\eta_i(\varepsilon,\tau) = 1+\varepsilon\upsilon_i(\tau,x^k), \eta_a = 1+\varepsilon\upsilon_a(\tau,x^k,y^3) \text{ for the coefficients of d-metrics ;} \quad (75)$$
$${}^w\eta_i(\varepsilon,\tau) = 1+\varepsilon\ {}^w\upsilon_i(\tau,x^k,y^3),\ \ {}^n\eta_i(\tau,x^k,y^3) = 1+\varepsilon\ \ {}^n\upsilon_i(\tau,x^k,y^3) \text{for the coefficients of N-metrics ,}$$

where $g_4(\tau) = \eta_4(\tau)\mathring{g}_4 = \ \eta_4(\tau,r,\theta,\varphi)\mathring{g}_4(r,\theta,\varphi) = [1+\varepsilon\upsilon(\tau,r,\theta,\varphi)]\mathring{g}_4$, for $\upsilon = \upsilon_4(\tau,r,\theta,\varphi)$ and $g_4^{\diamond}(\tau)\neq 0$, as a generating function.

Deformations of h-components of a stationary d-metric are written ${}_\varepsilon g_i = \mathring{g}_i(1+\varepsilon\upsilon_i) = e^{\psi(\tau,x^k)}$ for a solution of the 2-d Laplace equation in (64). For $\psi(\tau) = \ {}^0\psi(\tau,x^k)+\varepsilon\ {}^1\psi(\tau,x^k)$ and ${}_h\widehat{\Im}(\tau)(\tau) = {}_h^0\widehat{\Im}(\tau,x^k)+\varepsilon\ {}_h^1\widehat{\Im}(\tau,x^k)$, we compute the deformation polarization functions in the form $\upsilon_i = e^{\ {}^0\psi}\ {}^1\psi/\mathring{g}_i\ \ {}_h^0\widehat{\Im}$. In these formulas, the generating and source functions are solutions of ${}^0\psi^{\bullet\bullet}+\ {}^0\psi'' = \ {}_h^0\widehat{\Im}$ and ${}^1\psi^{\bullet\bullet}+\ {}^1\psi'' = \ {}_h^1\widehat{\Im}$. Using such ε-decomposition of polarization functions of type (75), we obtain ε-decomposition of the target stationary d-metric and N-connection coefficients[18] and compute

$$\mathring{g}_i\eta_i(\tau) = e^{\ \psi(\tau,x^k)} \text{ as a solution of 2-d Poisson equations}$$
$${}_\varepsilon g_i(\tau) = [1+\varepsilon e^{\ {}^0\psi}\ {}^1\psi/\mathring{g}_i\ \ {}_h^0\widehat{\Im}]\mathring{g}_i, \text{ also constructed as a solution of 2-d Poisson equations for } {}^1\psi$$
$$\mathring{g}_3\eta_3(\tau) = -\frac{4[(|\eta_4(\tau)\mathring{g}_4|^{1/2})^{\diamond}]^2}{|\int dy^3\ \ {}_v\widehat{\Im}(\tau)[\eta_4(\tau)\mathring{g}_4]^{\diamond}|}$$
$$\text{i.e. } {}_\varepsilon g_3(\tau) = [1+\varepsilon\ \upsilon_3]\mathring{g}_3 \text{ for } \upsilon_3(\tau,x^i,y^3) = 2\frac{(\upsilon\mathring{g}_4)^{\diamond}}{\mathring{g}_4^{\diamond}} - \frac{\int dy^3\ \ {}_v\widehat{\Im}(\tau)(\upsilon\mathring{g}_4)^{\diamond}}{\int dy^3\ \ {}_v\widehat{\Im}(\tau)\mathring{g}_4^{\diamond}}.$$

In these formulas, a new system of coordinates $[x^i(r,\theta,\varphi), y^3(r,\theta,\varphi)]$ is used in order to satisfy the condition $(\mathring{g}_4^{\diamond})^2 = \mathring{g}_3|\int dy^3\ {}_v\widehat{\Im}(\tau)\mathring{g}_4^{\diamond}|$, which allow to find $\mathring{g}_3$ for any prescribed values $\mathring{g}_4$ and ${}_v\widehat{\Im}(\tau)$.

[18] the vertical components for ε-decompositions of generating solutions are computed in a similar form

The N-connection coefficients are computed

$$\eta_i^3(\tau)\mathring{w}_i = \frac{\partial_i \int dy^3 \ {}_v\widehat{\Im}(\tau)[\eta_4(\tau)\ \mathring{g}_4]^\diamond}{{}_v\widehat{\Im}(\tau)\ [\eta_4(\tau)\ \mathring{g}_4]^\diamond}$$

$$\text{i.e. } {}_\varepsilon w_i(\tau) = [1+\varepsilon\ {}^w\upsilon_i(\tau)]\mathring{w}_i \text{ for } {}^w\upsilon_i(\tau,x^i,y^3) = \frac{\partial_i \int dy^3 \ {}_v\widehat{\Im}(\tau)(\upsilon\mathring{g}_4)^\diamond}{\partial_i \int dy^3 \ {}_v\widehat{\Im}(\tau)\mathring{g}_4^\diamond} - \frac{(\upsilon\mathring{g}_4)^\diamond}{\mathring{g}_4^\diamond},$$

when $\mathring{w}_i = \partial_i \int dy^3 \ {}_v\widehat{\Im}(\tau)\mathring{g}_4^\diamond / \ {}_v\widehat{\Im}(\tau)\mathring{g}_4^\diamond$ is defined for some prescribed ${}_v\widehat{\Im}(\tau)$ and $\mathring{g}_4^\diamond$;

$$\eta_k^4(\tau)\mathring{n}_k = {}_1n_k(\tau) + 16\ {}_2n_k(\tau)\int dy^3 \frac{\left([(\eta_4(\tau)\mathring{g}_4)^{-1/4}]^\diamond\right)^2}{|\int dy^3 \ {}_v\widehat{\Im}(\tau)(\eta_4\ (\tau)\mathring{g}_4)^\diamond|}$$

$$\text{i.e. } {}_\varepsilon n_i(\tau) = [1+\varepsilon\ {}^n\upsilon_i(\tau)]\mathring{n}_k = 0 \text{ for } {}^n\upsilon_i(\tau,x^i,y^3) = 0,$$

if the integration functions are chosen ${}_1n_k(\tau)=0$ and ${}_2n_k(\tau)=0$.

The values with a "circle" are prescribed for a chosen prime solution which can be a 4-d Kerr metric but subjected to some additional frame and coordinated transform to satisfy the conditions $\mathring{g}_4^\diamond \neq 0$ and relations of $\mathring{g}_4^\diamond$ to $\mathring{g}_3$ and $\mathring{w}_i$ as we considered above. Fixing a small value ε, we can compute such deformations for stationary configurations and state well-defined conditions of stability if the prime metric is stable. We conclude that ε–deformed quadratic elements can be written in a general form

$$\begin{aligned} ds^2_{\varepsilon t} &= {}_\varepsilon g_{\alpha\beta}(\tau,x^k,y^3)du^{\alpha_s}du^{\beta_s} \\ &= {}_\varepsilon g_i\left(\tau,x^k\right)[(dx^1)^2+(dx^2)^2] + \ {}_\varepsilon h_3(\tau,x^k,y^3)\ [dy^3 + \ {}_\varepsilon w_i(\tau,x^k,y^3)dx^i]^2 + \ {}_\varepsilon g_4(\tau,x^k,y^3)dt^2. \end{aligned}$$

We can impose additional constraints in order to extract LC–configurations with zero torsion.

5.3 BHs in (off-) diagonal stationary media with solitonic hierarchies

We can construct and describe new classes of classes of generic off-diagonal stationary solutions in terms of η–polarization functions introduced in formulas (51) and following the AFDM summarized in Tables 1 and 2. As a primary metric we consider a primary BH d-metric (for instance, it can be a Schwarzschild or Kerr metric) defined by geometric data $\mathring{\mathbf{g}} = [\mathring{g}_i(r,\theta,\varphi), \mathring{g}_a = \mathring{h}_a(r,\theta,\varphi); \mathring{N}_k^3 = \mathring{w}_k(r,\theta,\varphi), \mathring{N}_k^4 = \mathring{n}_k(r,\theta,\varphi)]$ (52) which can be diagonalized by frame/ coordinate transforms. The stationary target metrics $\mathbf{g}$ are generated by nonholonomic η–deformations, $\mathring{\mathbf{g}} \to \mathbf{g}(\tau) = [g_i(\tau,x^k) = \eta_i(\tau)\mathring{g}_i, g_b(\tau,x^k,y^3) = \eta_b(\tau)\mathring{g}_b, N_i^a(\tau,x^k,y^3) = \ \eta_i^a(\tau)\mathring{N}_i^a]$, and constrained to the conditions to define exact and parametric solutions of the system of nonlinear PDEs with decoupling (64). The quadratic line elements corresponding to d-metrics $\mathbf{g}$ are parameterized in some forms similar to (51),

$$ds^2 = \eta_i(\tau,r,\theta,\varphi)\mathring{g}_i(r,\theta,\varphi)[dx^i(r,\theta,\varphi)]^2 + \eta_a(\tau,r,\theta,\varphi)\mathring{g}_a(r,\theta,\varphi)[d\varphi + \eta_k^a(\tau,r,\theta,\varphi)\mathring{N}_k^a(r,\theta,\varphi)dx^k(r,\theta,\varphi)]^2, \quad (76)$$

with summation on repeating contracted low-up indices. The values $\eta_\alpha(\tau)$ and $\eta_i^a(\tau)$ are determined by solitonic flows and nonlinear interactions.

5.3.1 Stationary solutions generated by solitonic sources

Considering effective sources determined by solitonic hierarchies $\widehat{\mathbf{\Upsilon}}(\tau,r,\theta,\varphi) = \ {}_v\widehat{\Im}[\ {}_2\iota]$ (57), we compute the coefficients for solutions of type (76) following formulas from Table 2,

$$\begin{aligned} \eta_i(\tau) &= \frac{e^{\ \psi(\tau,x^k)}}{\mathring{g}\quad_i}; \eta_3(\tau) = -\frac{4[(|\eta_4(\tau)\mathring{h}_4|^{1/2})^\diamond]^2}{\mathring{h}_3|\int dy^3 \ {}_v\widehat{\Im}[\ {}_2\iota](\eta_4(\tau)\mathring{h}_4)^\diamond|}; \quad (77) \\ \eta_4(\tau) &= \eta_4(\tau,r,\theta,\varphi) \text{ as a generating function;} \\ \eta_i^3(\tau) &= \frac{\partial_i \int dy^3 \ {}_v\widehat{\Im}[\ {}_2\iota](\eta_4(\tau)\mathring{h}_4)^\diamond}{\mathring{w}_i \ {}_v\widehat{\Im}[\ {}_2\iota]\ (\eta_4(\tau)\mathring{h}_4)^\diamond};\ \eta_k^4\ (\tau) = \frac{{}_1n_k}{\mathring{n}_k} + 16\ \frac{{}_2n_k}{\mathring{n}_k}\int dy^3 \frac{\left([(\eta_4(\tau)\mathring{h}_4)^{-1/4}]^\diamond\right)^2}{|\int dy^3 \ {}_v\widehat{\Im}[\ {}_2\iota](\eta_4\ (\tau)\mathring{h}_4)^\diamond|}, \end{aligned}$$

for integration functions ${}_1n_k(\tau,r,\theta)$ and ${}_2n_k(\tau,r,\theta)$.

In (77), the gravitational polarization $\eta_4(r,\theta,\varphi)$ is taken as a (non) singular generating function which following nonlinear symmetries (65) can be related to other type generating functions,

$$\begin{aligned}\Phi^2(\tau) &= -4\ \Lambda(\tau)h_4(\tau) = -4\ \Lambda\eta_4(\tau,r,\theta,\varphi)\mathring{h}_4(\tau,r,\theta,\varphi),\\ (\Psi^2)^{\diamond}(\tau) &= -\int d\varphi\ \ {}_v\widehat{\Im}[\ {}_2\iota][\eta_4(\tau,r,\theta,\varphi)\mathring{h}_4(\tau,r,\theta,\varphi)]^{\diamond}.\end{aligned}$$

It should be noted that the values Φ, h_4 and η_4 may not encode soltionic hierarchies but Ψ and other coefficients of such d-metric are solitonic ones if they are computed using ${}_v\widehat{\Im}[\ {}_2\iota]$. We can constrain the coefficients (77) to a subclass of data generating target stationary off-diagonal metrics of type (68) with zero torsion.

The nonlinear functionals for the soliton v-source and (effective) cosmological constant considered above can be changed into additive functionals ${}_v\widehat{\Im}\ \rightarrow\ {}_v^a\widehat{\Im}$ and $\Lambda\rightarrow\ {}^a\Lambda$ as ${}_v^a\widehat{\Im}[\iota]$ (69) and ${}^a\Lambda$ (70). The singular behaviour of such solutions is generated by some prime BH data $\mathring{\mathbf{g}}=[\mathring{g}_i,\mathring{g}_a,\mathring{N}_b^j]$ (52) which can be preserved or changed for different classes of generating and integration functions. For certain classes of generating functions and sources and small nonholonomic deformations, the same type of singularity is preserved. Similar stationary configurations can be computed for general solitonic hierarchies. The constructions depend on the type of explicit geometric evolution or dynamical model we construct (for instance, with one type solitonic wave, nonlinear superposition of solitonic waves on τ, solitonic stationary distributions etc.). Such generic off-diagonal stationary entropic solutions can be considered as certain conventional nonholonomically deformed BH configurations imbedded into some aether (non) singular media with flows and off-diagonal interactions determined by stationary solitonic and non-solitonic fields modeling dark, usual matter quasiperiodic distributions and patern forming structures.

5.3.2 BH solutions deformed by solitonic generating functions

Solutions with entropic η–polarizations (76) can be constructed with coefficients of the d-metrics determined by nonlinear generating functionals $\Phi[i]$, or any additive functionals ${}^a\Phi[i]$, including terms with integration functions $h_4^{[0]}(\tau,r,\theta)$ for $h_4[i]$. Such configurations are defined also by some prescribed data ${}_v\widehat{\Im}\ (\tau,r,\theta,\varphi)$ and $\Lambda(\tau)$, which are not obligatory of solitonic nature. Using nonlinear symmetries (65), we can compute (recurrently) corresponding nonlinear functionals, $\eta_4(\tau,r,\theta,\varphi)$ (for simplicity, we omit here similar formulas for additive functionals ${}^a\eta_4(\tau,r,\theta,\varphi)$) and related polarization functions,

$$\begin{aligned}\eta_4[i] &= -\Phi^2[i]/4\Lambda(\tau)\mathring{h}_4(r,\theta,\varphi),\\ [\Psi^2(\tau)]^{\diamond} &= -\int d\varphi\ \ {}_v\widehat{\Im}(\tau,r,\theta,\varphi)h_4^{\diamond}(\tau) = -\int d\varphi\ \ {}_v\widehat{\Im}(\tau,r,\theta,\varphi)[\ \eta_4[i]\mathring{h}_4(r,\theta,\varphi)]^{\diamond}.\end{aligned}$$

Using such formulas for Table 2, the coefficients of d-metric (76) are computed

$$\begin{aligned}\eta_i(\tau) &= \frac{e^{\ \psi(\tau,x^k)}}{\mathring{g}_i};\eta_3 = -\frac{4[(|\ \eta_4[i]\mathring{h}_4|^{1/2})^{\diamond}]^2}{\mathring{h}_3|\int dy^3\ \ {}_v\widehat{\Im}(\tau)\ \eta_4[i]\mathring{h}_4)^{\diamond}|}; \qquad (78)\\ \eta_4(\tau) &= \eta_4(\tau,r,\theta,\varphi) = \eta_4[i] \text{ as a generating function};\\ \eta_i^3(\tau) &= \frac{\partial_i\ \int d\varphi\ \ {}_v\widehat{\Im}(\tau)(\eta_4[i]\ \mathring{h}_4)^{\diamond}}{\mathring{w}_i\ \ {}_v\widehat{\Im}(\tau)\ (\eta_4[i]\ \mathring{h}_4)^{\diamond}};\eta_k^4(\tau) = \frac{{}_1n_k(\tau)}{\mathring{n}_k} + 16\ \ \frac{{}_2n_k(\tau)}{\mathring{n}_k}\int d\varphi\frac{\left([(\eta_4[i]\mathring{h}_4)^{-1/4}]^{\diamond}\right)^2}{|\int dy^3\ \ {}_v\widehat{\Im}(\tau)(\eta_4[i]\mathring{h}_4)^{\diamond}|},\end{aligned}$$

for integrating functions ${}_1n_k\tau,r,\theta)$ and ${}_2n_k\tau,r,\theta)$.

Using (78), target stationary off-diagonal metrics (68) with zero torsion can be generated by polarization functions subjected to additional nonholonomic constraints and integrability conditions,

$$\begin{aligned}\eta_i(\tau) &= \frac{e^{\ \psi(\tau,x^k)}}{\mathring{g}_i};\ \eta_3(\tau) = -\frac{4[(|\ \check{\eta}_4[i]\mathring{h}_4|^{1/2})^{\diamond}]^2}{\mathring{h}_3|\int dy^3\ \ {}_v\check{\Im}(\tau)(\check{\eta}_4[i]\mathring{h}_4)^{\diamond}|};\\ \eta_4(\tau) &= \check{\eta}_4(\tau,r,\theta,\varphi) = \check{\eta}_4[i] \text{ as a generating function };\eta_i^3(\tau) = \frac{\partial_i\ \check{A}(\tau)}{\mathring{w}_k},\eta_k^4(\tau) = \frac{\partial_k n(\tau)}{\mathring{n}_k},\end{aligned}$$

for an integrating functions $n(\tau, r, \theta)$ and a generating function $\check{A}(\tau, r, \theta, \varphi)$.

The solutions construced in this subsection describe certain nonholonomically deformed BH configurations self-consistently imbedded into a solitonic gravitational evolution media modeling certain aether properties for nonholonomic dark energy distributions.

5.3.3 Stationary BH deformations by solitonic sources & solitonic generating functions

More general classes of stationary solitonic deformations of BHs can be constructed using nonlinear functionals both for the generating functions and sources. Nonlinear superpositions of solutions of type (76) and (78) can be performed if the coefficients of d-metric are computed

$$\begin{aligned} \eta_i(\tau) &= \frac{e^{\ \psi(\tau,x^k)}}{\mathring{g}_i}; \eta_3(\tau) = -\frac{4[(|\eta_4[\ _4\iota]\mathring{h}_4|^{1/2})^\diamond]^2}{\mathring{h}_3|\int d\varphi\ \ _v\widehat{\Im}[\iota](\eta_4[\ _4\iota]\mathring{h}_4)^\diamond|}; \qquad (79)\\ \eta_4(\tau) &= \eta_4(\tau,r,\theta,\varphi) = \eta_4[\ _4\iota] \text{ as a generating function}; \eta_i^3(\tau) = \frac{\partial_i \int d\varphi\ \ _v\widehat{\Im}[\iota](\eta_4[\ _4\iota]\ \mathring{h}_4)^\diamond}{\mathring{w}_i\ \ _v\widehat{\Im}[\iota]\ (\eta_4[\ _4\iota]\ \mathring{h}_4)^\diamond};\\ \eta_k^4(\tau) &= \frac{\ _1n_k(\tau)}{\mathring{n}_k} + 16\ \frac{\ _2n_k(\tau)}{\mathring{n}_k}\int d\varphi \frac{\left([(\eta_4[\ _4\iota]\mathring{h}_4)^{-1/4}]^\diamond\right)^2}{|\int dy^3\ \ _v\widehat{\Im}[\iota](\eta_4[\ _4\iota]\ \mathring{h}_4)^\diamond|}, \end{aligned}$$

where $_1n_k(\tau, x^k)$ and $_2n_k(\tau, x^k)$ are integration functions.

In (79), we consider a nonlinear generating functional $\Phi[\ _4\iota]$ and prescribed nonlinear functional $_v\widehat{\Im}[\iota]$ and running constant $\Lambda(\tau)$ related via nonlinear symmetries generalizing (65). This allows us to compute corresponding nonlinear functionals $\eta_3(\tau, r, \theta, \varphi) = \eta_3[\iota,\ _4\iota, ...]$ and polarization functions,

$$\eta_4(\tau) = -\ \Phi^2[\ _4\iota]/4\ \Lambda(\tau)\mathring{h}_4(r,\theta,\varphi),\ (\ \Psi^2(\tau))^\diamond = -\int d\varphi\ \ _v\widehat{\Im}[\iota] h_4^\diamond[\ _4\iota] = -\int d\varphi\ \ _v\widehat{\Im}[\iota][\ \eta_4(\tau)\mathring{h}_4(r,\theta,\varphi)]^\diamond. \quad (80)$$

Imposing additional conditions for a zero torsion, target stationary metrics (68) are generated.

A corresponding quadratic line element can be written for generating data $(\Phi[\ _4\iota], \Lambda(\tau))$:

$$\begin{aligned} ds^2 &= e^{\ \psi(\ _4\iota)}[(dx^1)^2 + (dx^2)^2] - \frac{(\Phi[\ _4\iota])^2[(\Phi[\ _4\iota])^2]^\diamond}{|\Lambda(\tau)\int dy^3\ \ _v\widehat{\Im}[\iota][(\Phi[\ _4\iota])^2]^\diamond|\ (h_4^{[0]} - \frac{(\Phi[\ _4\iota])^2}{4\Lambda(\tau)})} \qquad (81)\\ & [dy^3 + \frac{\partial_i\left(\int dy^3\ \ _v\widehat{\Im}[\iota][(\Phi[\ _4\iota])^2]^\diamond\right)}{_v\widehat{\Im}[\iota]\ [(\Phi[\ _4\iota])^2]^\diamond}dx^i] + (h_4^{[0]}(\tau,x^k) - \frac{(\Phi[\ _4\iota])^2}{4\Lambda(\tau)})[dt + (\ _1n_k(\tau,x^k) + \\ & \ _2n_k(\tau,x^k)\int dy^3 \frac{[(\Phi[\ _4\iota])^2]^\diamond}{|\Lambda(\tau)\int dy^3\ \ _v\widehat{\Im}[\iota]\ [(\Phi[\ _4\iota])^2]^\diamond|}|h_4^{[0]}(\tau,x^k) - \frac{(\Phi[\ _4\iota])^2}{4\Lambda(\tau)}|^{-5/2})dx^k]. \end{aligned}$$

The data for a primary BH can be extracted using nonlinear symmetries (80), when $\mathring{g}_i = e^{\ \psi(\tau,x^k)}/\eta_i(\tau)$ and $\mathring{h}_4(r,\theta,\varphi) = -\ \Phi^2[\ _4\iota]/4\ \Lambda(\tau)\eta_4(\tau)$ certain values for a τ_0 are such way prescribed that the integration functions $h_4^{[0]}(\tau, x^k),_1 n_k(\tau, x^k)$ and $_2n_k(\tau, x^k)$ encode a prime d-metric $\mathring{\mathbf{g}} = [\mathring{g}_i, \mathring{g}_a, \mathring{N}_b^j]$ (52) and describes certain evoluton for $\tau > \tau_0$ parameterized in the form (81). This class of stationary solutions with gravitational polarizations (79) describes nonholonomic solitonic hierarchies deformations of a BH self-consistently imbedded into solitonic gravitational (dark energy) backgrounds and solitonic dark and/or standard matter.

5.4 Off–diagonal deformations of Kerr metrics by solitonic flow sources

In this section, we study how effective sources for geometric flows with solitonic hierarchies flows result in generic off–diagonal deformations and generalizations of the 4-d Kerr metric and construct such new classes of exact solutions of systems of nonlinear PDEs (64).

5.4.1 The Kerr BH solution in nonholonomic variables

To apply the AFDM is necessary to define some special classes of nonholonomic variables which allow decoupling and integration of certain systems of equations describing N-adapted nonholonomic deformations, for instance, of a Kerr metric black hole, BH, solution as prime d-metric $\mathring{\mathbf{g}} = [\mathring{g}_i, \mathring{g}_a = \mathring{h}_a; \mathring{N}_k^3 = \mathring{w}_k, \mathring{N}_k^4 = \mathring{n}_k]$ (52). We cite here [42] as a standard monograph on GR with necessary details on geometry of BHs and [73, 35, 19, 20] for examples of nonholonomic deformations of BH solutions in geometric flows and MGTs.

Let consider a 4-d ansatz for a prime metric,

$$ds^2_{[0]} = \mathring{g}_{\alpha\beta} du^\alpha du^\beta = Y^{-1} e^{2h}(d\rho^2 + dz^2) + Y(d\varphi + Adt)^2 - \rho^2 Y^{-1} dt^2.$$

This nonlinear quadratic line element is determined by three functions (h, Y, A) on coordinates $x^i = (\rho, z)$. It defines the Kerr solution of the vacuum Einstein equations (for rotating BHs) if the coefficients are chosen

$$\begin{aligned} Y &= \frac{1 - (p\widehat{x}_1)^2 - (q\widehat{x}_2)^2}{(1 + p\widehat{x}_1)^2 + (q\widehat{x}_2)^2}, \ A = 2M\frac{q}{p}\frac{(1 - \widehat{x}_2)(1 + p\widehat{x}_1)}{1 - (p\widehat{x}_1) - (q\widehat{x}_2)}, \\ e^{2h} &= \frac{1 - (p\widehat{x}_1)^2 - (q\widehat{x}_2)^2}{p^2[(\widehat{x}_1)^2 + (\widehat{x}_2)^2]}, \ \rho^2 = M^2(\widehat{x}_1^2 - 1)(1 - \widehat{x}_2^2), \ z = M\widehat{x}_1\widehat{x}_2. \end{aligned}$$

For $M = const$ and $\rho = 0$, we obtain result a horizon $\widehat{x}_1 = 0$ and the "north / south" segments of the rotation axis, $\widehat{x}_2 = +1/-1$. For our purposes, it is convenient to write this Kerr metric in the form

$$ds^2_{[0]} = (dx^1)^2 + (dx^2)^2 + Y(\mathbf{e}^3)^2 - \rho^2 Y^{-1}(\mathbf{e}^4)^2, \qquad (82)$$

where the coordinates $x^1(\widehat{x}_1, \widehat{x}_2)$ and $x^2(\widehat{x}_1, \widehat{x}_2)$ are defined for any

$$(dx^1)^2 + (dx^2)^2 = M^2 e^{2h}(\widehat{x}_1^2 - \widehat{x}_2^2)Y^{-1}\left(\frac{d\widehat{x}_1^2}{\widehat{x}_1^2 - 1} + \frac{d\widehat{x}_2^2}{1 - \widehat{x}_2^2}\right)$$

and the v-coordinates are changed $y^3 = \varphi + \widehat{y}^3(x^1, x^2, t)$, $y^4 = t + \widehat{y}^4(x^1, x^2)$. We can consider an N-adapted basis $\mathbf{e}^3 = dy^3 + (\partial_i \widehat{y}^3)dx^i$ and $\mathbf{e}^4 = dt + (\partial_i \widehat{y}^4)dx^i$, for some functions $\widehat{y}^a$, $a = 3, 4$, with $\partial_t \widehat{y}^3 = -A(x^k)$.

We can use the Kerr metric in the so–called Boyer–Linquist coordinates $(r, \vartheta, \varphi, t)$, for $r = m_0(1 + p\widehat{x}_1), \widehat{x}_2 = \cos\vartheta$, which are more convenient for applying of the AFDM. Such coordinates are be related to parameters p, q involving the total BH mass, m_0 and the total angular momentum, am_0, for the asymptotically flat, stationary and anti-symmetric Kerr spacetime. Considering $m_0 = Mp^{-1}$ and $a = Mqp^{-1}$ with $p^2 + q^2 = 1$ and $m_0^2 - a^2 = M^2$, we write the metric (82) as a d-metric

$$\begin{aligned} ds^2_{[0]} &= (dx^{1'})^2 + (dx^{2'})^2 + (\overline{C} - \overline{B}^2/\overline{A})(\mathbf{e}^{3'})^2 + \overline{A}(\mathbf{e}^{4'})^2, \qquad (83) \\ \mathbf{e}^{3'} &= dy^{3'} = d\varphi, \mathbf{e}^{4'} = dt + d\varphi\overline{B}/\overline{A} = dy^{4'} - \partial_{i'}(\widehat{y}^{4'} + \varphi\overline{B}/\overline{A})dx^{i'}, \end{aligned}$$

with coordinate functions $x^{1'}(r, \vartheta)$, $x^{2'}(r, \vartheta)$, $y^{3'} = \varphi$, $y^{4'} = t + \widehat{y}^{4'}(r, \vartheta, \varphi) + \varphi\overline{B}/\overline{A}, \partial_\varphi \widehat{y}^{4'} = -\overline{B}/\overline{A}$. In formulas (83), $(dx^{1'})^2 + (dx^{2'})^2 = \Xi\left(\Delta^{-1}dr^2 + d\vartheta^2\right)$, and the coefficients are defined in the form

$$\begin{aligned} \overline{A} &= -\Xi^{-1}(\Delta - a^2\sin^2\vartheta), \overline{B} = \Xi^{-1}a\sin^2\vartheta\left[\Delta - (r^2 + a^2)\right], \\ \overline{C} &= \Xi^{-1}\sin^2\vartheta\left[(r^2 + a^2)^2 - \Delta a^2\sin^2\vartheta\right], \text{ and } \Delta = r^2 - 2m_0 + a^2, \ \Xi = r^2 + a^2\cos^2\vartheta. \end{aligned}$$

The quadratic linear elements (82) and/or (83) can be written as a stationary prime metric (52) with coefficients

$$\begin{aligned} \mathring{g}_1 &= 1, \mathring{g}_2 = 1, \mathring{g}_3 = Y, \mathring{g}_4 = -\rho^2 Y^{-1}, \mathring{N}_i^a = \partial_i \widehat{y}^a, \text{ or} \qquad (84) \\ \mathring{g}_{1'} &= 1, \mathring{g}_{2'} = 1, \mathring{g}_{3'} = \overline{C} - \overline{B}^2/\overline{A}, \ \mathring{g}_{4'} = \overline{A}, \mathring{N}_{i'}^3 = \mathring{w}_{i'} = 0, \mathring{N}_{i'}^4 = \mathring{n}_{i'} = -\partial_{i'}(\widehat{y}^{3'} + \varphi\overline{B}/\overline{A}), \end{aligned}$$

Such d-metrics define BH solutions of the vacuum Einstein equations with zero sources.

5.4.2 Nonholonomic evolution of Kerr metrics with induced (or zero) torsion

We consider the coefficients (84) as a prime metric $\mathring{\mathbf{g}}$ when $\mathring{g}_{1'} = 1, \mathring{g}_{2'} = 1, \mathring{g}_{3'} = \overline{C} - \overline{B}^2/\overline{A}$ together with some coordinate transforms $g_{4'} = \widehat{A}(\overline{A}, y^3) \to \overline{A}$, $\mathring{N}^3_{i'} = \mathring{w}_{i'}(r,\theta,\varphi) \to 0$ and $\mathring{N}^4_{i'} = \mathring{n}_{i'} = -\partial_{i'}(\widehat{y}^{3'} + \varphi \overline{B}/\overline{A}) \to 0$, to a local coordinate system when $\mathring{g}_4^{\diamond} \neq 0$.[19] This allows us to construct nonholonomic deformations following the geometric formalism outlined in section 5.2.3 and Table 2.

For general η–deformations (74) and constraints $n_i = 0$, the solitonic flow modifications of the Kerr metric are computed

$$\begin{aligned} ds^2 &= e^{\,\psi(\tau, x^{k'})}[(dx^{1'})^2 + (dx^{2'})^2] - \frac{4[(|\eta_4[\ _4\iota]\overline{A}|^{1/2})^{\diamond}]^2}{|\int dy^3\ _v\widehat{\Im}[\iota][\eta_4[\ _4\iota]\overline{A}]^{\diamond}|}(\overline{C} - \frac{\overline{B}^2}{\overline{A}})(\mathbf{e}^{3'})^2 + \eta_{4'}[\ _4\iota]\overline{A}(\mathbf{e}^{4'})^2, \\ \mathbf{e}^{3'} &= dy^{3'} + \frac{\partial_{i'}\ \int dy^3\ _v\widehat{\Im}[\iota][\eta_{4'}[\ _4\iota]\overline{A}]^{\diamond}}{_v\widehat{\Im}[\iota]\ [\eta_{4'}[\ _4\iota]\overline{A}]^{\diamond}} dx^{i'}, \mathbf{e}^{4'} = dt, \end{aligned} \tag{85}$$

where $\eta_{4'}(\tau) = \eta_{4'}(\tau, x^{k'}, y^{3'}) = \eta_4[\ _4\iota]$ is a generating function and $_v\widehat{\Im}(\tau) = \ _v\widehat{\Im}[\iota]$ is a flow generating source as in (79) and $\psi(\tau, x^{k'})$ is a solution of a 2-d Poisson equation (66).

5.4.3 Small parametric modifications of BHs and effective entropic flow sources

We study models of geometric and courve flows for nonholonomic distributions describing ε-deformations described by formulas (75). Such deformations of a prime Kerr metric (84) with $\overline{A}(r,\theta) \to \widehat{A}[x^i(r,\theta), y^3]$ for $\mathring{g}_4^{\diamond} \neq 0$ result in stationary target metrics of type (74). The corresponding quadratic line elements are written in the form

$$\begin{aligned} ds^2 &= [1 + \varepsilon e^{\,^0\psi} \frac{^1\psi}{\mathring{g}_i}\ ^0_h\widehat{\Im}]\mathring{g}_i(dx^i)^2 + \left[1 + \varepsilon\ (2\frac{[\upsilon \mathring{g}_4]^{\diamond}}{\mathring{g}_4^{\diamond}} - \frac{\int dy^3\ _v\widehat{\Im}([\upsilon \mathring{g}_4])^{\diamond}}{\int dy^3\ _v\widehat{\Im}\mathring{g}_4^{\diamond}})\right] \mathring{g}_3(\mathbf{e}^3)^2 + [1 + \varepsilon\upsilon]\mathring{g}_4(\mathbf{e}^4)^2, \\ \mathbf{e}^3 &= dy^3 + [1 + \varepsilon(\frac{\partial_i\ \int dy^3\ _v\widehat{\Im}(\upsilon \mathring{g}_4)^{\diamond}}{\partial_i\ \int dy^3\ _v\widehat{\Im}\mathring{g}_4^{\diamond}} - \frac{(\upsilon \mathring{g}_4)^{\diamond}}{\mathring{g}_4^{\diamond}})]\ \mathring{w}_i dx^i, \mathbf{e}^4 = dx^4 = dt, \end{aligned} \tag{86}$$

where $^0\psi(\tau) = \ ^0\psi(\tau, x^k)$ and $^1\psi(\tau) = \ ^1\psi(\tau, x^k)$ are solutions of 2-d Poisson equations with a generating h-source $_h\widehat{\Im}(\tau) = \ _h\widehat{\Im}(\tau, x^k) = \ ^0_h\widehat{\Im}(\tau, x^k) + \varepsilon\ ^1_h\widehat{\Im}(\tau, x^k)$ as described in section 5.2.3. In this formula, $_v\widehat{\Im}(\tau) = \ _v\widehat{\Im}[\iota]$ is a generating v-source for which a ε-decomposition is possible and $\upsilon_4 = \upsilon(\tau) = \upsilon(\tau, x^k, y^3) = \upsilon[\ _4\iota]$ is a generating function. The formula (86) is for a N-adapted system of references and space coordinates $[x^i(r,\theta,\varphi), y^3(r,\theta,\varphi)]$ for which the condition $(\mathring{g}_4^{\diamond})^2 = \mathring{g}_3|\int dy^3\ _v\widehat{\Im}\mathring{g}_4^{\diamond}|$ allows to compute $\mathring{g}_3$ and $\mathring{w}_i = \partial_i\ [dy^3\ \ _v\widehat{\Im}\mathring{g}_4^{\diamond}]/\ _v\widehat{\Im}\mathring{g}_4^{\diamond}$ when there are prescribed some $_v\widehat{\Im}$ and $\mathring{g}_4^{\diamond} \neq 0$. For simplicity, we fix the conditions $_1n_k(\tau) = 0$ and $_2n_k(\tau) = 0$ for which $N_i^4 = n_i = 0$ but a non-zero $N_i^3 = w_i(\varepsilon, \tau, x^k, y^3)$ results in trivial nonholonomic torsion and anholonomy coefficients. We can impose additional constraints on $\upsilon(\tau)$ and sources which allow us to extract LC–configurations as described in footnote 16.

Using nonlinear symmetries of type (80) with $\eta_4(\tau) = -\ \Phi^2[\ _4\iota]/4\ \Lambda(\tau)\mathring{g}_4$ for (86), we can use as a generating function determined by solitonic hierarchies the value

$$\varepsilon\upsilon[\ _4\iota] = -\left(1 + \Phi^2[\ _4\iota]/4\ \Lambda(\tau)\overline{A}\right) \text{ or } \Phi[\ _4\iota] \simeq 2\sqrt{|\Lambda(\tau)\overline{A}|}(1 - \frac{\varepsilon}{2}\upsilon[\ _4\iota]). \tag{87}$$

Other types solitonic hierarchies can be encoded into generated sources $_h\widehat{\Im}(\tau, x^k) = \ ^0_h\widehat{\Im}(\tau, x^k) + \varepsilon\ ^1_h\widehat{\Im}(\tau, x^k)$ and $_v\widehat{\Im}(\tau) = \ _v\widehat{\Im}[\iota]$. The geometric solitonic data $(\Lambda(\tau), \upsilon[\ _4\iota],\ ^0_h\widehat{\Im} + \varepsilon\ ^1_h\widehat{\Im}, _v\widehat{\Im}[\iota])$ determine this class of parametric solutions for geometric flows and respective Perelman's thermodynamic for GIFs and QGIFs.

[19] we can define nonholonomic deformations with $\mathring{g}_4^{\diamond} = 0$ and/or $g_4^{\diamond} \neq 0$ when the solutions are constructed on certain hypersurfaces and certain models are with singular geometric evolution as we considered in our previous works [73, 35]

6 Computing Perelman's thermodynamic values for stationary geometric flows and QGIFs

We show how G. Pereman's W-entropy and related thermodynamic values can be computed for nonholonomic Einstein systems, NES, describing stationary solitonic and nonholonomically deformed black hole, BH, solutions under geometric flow evolution. There are provided formulas for extensions of main concepts and physical values to GIFs and QGIFs with entanglement elaborated in sections 3-5. In this work, there are studied only stationary configurations (see [40] as a partner work on QGIFs and applications in modern cosmology and [19, 21] for locally anisotropic cosmological solutions and related geometric thermodynamic models). Similar constructions can be performed for any type of stationary exact and parametric solitonic solutions parameterized in Table 2 and/or (in nonholonomic geometric dual form) for solutions with Killing symmetry on a time like vector.

6.1 Fixed values for normalization and integration functions

To study geometric evolution of NES when $\widehat{\mathbf{R}}_{\alpha\beta} = \widehat{\mathbf{\Upsilon}}_{\alpha\beta}$ and $\ _s\widehat{R} = \widehat{\mathbf{\Upsilon}}^a_a$, we can chose a constant value for the normalizing function, $\ \widehat{f}(\tau) = \ \widehat{f}_0 = const = 0$, in (26). This prescribes a geometric vertical scale for flow evolution determined by data $(\Phi(\tau),\ \Lambda(\tau))$ of such physical models and related via nonlinear symmetries (65) to a generating source $\ _v\widehat{\Im}(\tau)$ (such a h-scale is determined by a 2-d Poisson equation as described in section 5.2.3). Fixing additionally certain constants for integration functions, we can simplify substantially the formulas for G. Perelman's thermodynamic values.[20] In result, the formulas for the F- and W-functionals in canonical geometric variables (see respectively (27)) are written

$$\widehat{\mathcal{F}} = \frac{1}{8\pi^2}\int \tau^{-2}\sqrt{|\mathbf{g}[\Phi(\tau)]|}\delta^4 u[\ _h\Lambda(\tau) + \Lambda(\tau)],\ \widehat{\mathcal{W}} = \frac{1}{4\pi^2}\int \tau^{-2}\sqrt{|\mathbf{g}[\Phi(\tau)]|}\delta^4 u(\tau[\ _h\Lambda(\tau) + \Lambda(\tau)]^2 - 1), \quad (88)$$

where $\sqrt{|\mathbf{g}[\Phi(\tau)]|} = \sqrt{|q_1 q_2 \mathbf{q}_3(\ _qN)|} = 2e^{\ \psi(\tau)}\,|\Phi(\tau)|\sqrt{\frac{|[\Phi^2(\tau)]^\diamond|}{|\Lambda(\tau)\int dy^3\ _v\widehat{\Im}(\tau)[\Phi^2(\tau)]^\diamond|}}$ is computed for d-metrics parameterized in the form (13) with

$$q_1(\tau) = q_2(\tau) = e^{\ \psi(\tau)}, \mathbf{q}_3(\tau) = -\frac{4[\Phi^2(\tau)]^\diamond}{|\int dy^3\ _v\widehat{\Im}(\tau)[\Phi^2(\tau)]^\diamond|}, [\ _qN(\tau)]^2 = h_4(\tau, x^k, y^3) = -\frac{\Phi^2(\tau)}{4\Lambda(\tau)}$$

for $h_4^{[0]} = 0$. The N-adapted differential $\delta^4 u = dx^1 dx^2 \mathbf{e}^3 \mathbf{e}^4 = dx^1 dx^2 [dy^3 + w_i(\tau)dx^i][dt + n_i(\tau)dx^i]$ is taken for respective values of N-connection coefficients when $N_i^a = [w_i(\tau) = \frac{\partial_i(\int dy^3\ _v\widehat{\Im}(\tau)[\Phi^2(\tau)]^\diamond)}{_v\widehat{\Im}(\tau)[\Phi^2(\tau)]^\diamond}, n_i(\tau) = 0]$ for fixed integration functions $_1n_k(\tau) = 0$ and $_2n_k(\tau) = 0$.

The thermodynamic generating function (29) corresponding to $\widehat{\mathcal{W}}$ (88) and fixed $\ \widehat{f}$–normalization is

$$\widehat{\mathcal{Z}}[\mathbf{g}(\tau)] = \frac{1}{4\pi^2}\int \tau^{-2} d\mathcal{V}(\tau), \quad (89)$$

where the effective integration volume functional $d\mathcal{V}(\tau) = d\mathcal{V}(\psi(\tau), \Phi(\tau),\ _v\widehat{\Im}(\tau), \Lambda(\tau))$,

$$d\mathcal{V}(\tau) = e^{\ \psi(\tau)}\,|\Phi(\tau)|\sqrt{\frac{|[\Phi^2(\tau)]^\diamond|}{|\Lambda(\tau)\int dy^3\ _v\widehat{\Im}(\tau)[\Phi^2(\tau)]^\diamond|}} dx^1 dx^2 \left[dy^3 + \frac{\partial_i\left(\int dy^3\ _v\widehat{\Im}(\tau)[\Phi^2(\tau)]^\diamond\right)}{_v\widehat{\Im}(\tau)\ [\Phi^2(\tau)]^\diamond} dx^i\right] dt \quad (90)$$

is completely determined by data $(\psi(\tau), \Phi(\tau),\ _v\widehat{\Im}(\tau), \Lambda(\tau))$. These formulas allow us to compute analogous thermodynamic values for stationary configurations,

$$\widehat{\mathcal{E}}\ (\tau) = -\frac{\tau^2}{4\pi^2}\int\left([\ _h\Lambda(\tau) + \Lambda(\tau)] - \frac{2}{\tau}\right)\tau^{-2} d\mathcal{V}(\tau),\ \widehat{\mathcal{S}}(\tau)\ = -\frac{1}{4\pi^2}\int(\tau[\ _h\Lambda(\tau) + \Lambda(\tau)] - 2)\,\tau^{-2} d\mathcal{V}(\tau). \quad (91)$$

[20] Computing such values in a convenient system of reference/coordinates, we can consider changing to any system of reference and curved (co)tangent Lorentz manifolds and other type normalizations for their geometric evolution.

In this work, we omit and do not provide applications of cumbersome formulas for computing flow fluctuations $\widehat{\eta}$ (30).

Using geometric thermodynamic values (91) for $\widehat{f}_0 = 0$, we can compute in canonical variables the respective free energy and relative entropy (31). If two d-metrics are defined by different classes of solutions defined by respective geometric data $\ _1\mathbf{g}(\ _1\psi,\ _1\Phi,\ _v^1\widehat{\Im},\ _1\Lambda)$ and $\mathbf{g}(\psi, \Phi,\ _v\widehat{\Im}, \Lambda)$, we obtain

$$\begin{aligned}
\widehat{\mathcal{F}}\ (\ _1\psi,\ _1\Phi,\ _v^1\widehat{\Im},\ _1\Lambda) &= \widehat{\mathcal{S}}(\ _1\psi,\ _1\Phi,\ _v^1\widehat{\Im},\ _1\Lambda) - \beta^{-1}\widehat{\mathcal{S}}(\ _1\psi,\ _1\Phi,\ _v^1\widehat{\Im},\ _1\Lambda) \text{ and} \qquad (92)\\
\widehat{\mathcal{S}}(\ \ _1\psi,\ _1\Phi,\ _v^1\widehat{\Im},\ _1\Lambda) \shortparallel \psi, \Phi,\ _v\widehat{\Im}, \Lambda) &= \beta[\widehat{\mathcal{F}}((\ _1\psi,\ _1\Phi,\ _v^1\widehat{\Im},\ _1\Lambda)) - \widehat{\mathcal{F}}(\psi, \Phi,\ _v\widehat{\Im}, \Lambda)],\\
\text{where } \widehat{\mathcal{E}}(\ _1\psi,\ _1\Phi,\ _v^1\widehat{\Im},\ _1\Lambda) &= -\frac{\tau^2}{4\pi^2}\int \left(\left[\ _h^1\Lambda(\tau) +\ _1\Lambda(\tau)\right] - \frac{2}{\tau}\right)\tau^{-2} d\ _1\mathcal{V}(\tau),\\
\widehat{\mathcal{S}}(\ _1\psi,\ _1\Phi,\ _v^1\widehat{\Im},\ _1\Lambda) &= -\frac{1}{4\pi^2}\int \left(\tau\left[\ _h^1\Lambda(\tau) +\ _1\Lambda(\tau)\right] - 2\right)\tau^{-2} d\ _1\mathcal{V}(\tau)
\end{aligned}$$

for $d\ _1\mathcal{V}(\tau) = e^{\ _1\psi(\tau)}\mid\ _1\Phi(\tau)|\sqrt{\frac{|[\ _1\Phi^2(\tau)]^\diamond|}{|\ _1\Lambda(\tau)\int dy^3\ _v^1\widehat{\Im}(\tau)[\ _1\Phi^2(\tau)]^\diamond|}} dx^1 dx^2 [dy^3 + \frac{\partial_i\left(\int dy^3\ _v^1\widehat{\Im}(\tau)[\ _1\Phi^2(\tau)]^\diamond\right)}{_v^1\widehat{\Im}(\tau)\ [\ _1\Phi^2(\tau)]^\diamond} dx^i]dt$, Similar values for $\mathbf{g}(\psi, \Phi,\ _v\widehat{\Im}, \Lambda)$ are given by formulas (90) and (91). Here we emphasize that the free energy and relative entropy values (92) can be defined and computed for the same class of NES but subjected to different types geometric flows.

A density state (see definition in footnote 9) is a functional $\widehat{\rho}[\mathbf{g}(\tau)] = \widehat{\mathcal{Z}}^{-1}(\tau)e^{-\beta E}$ where $\widehat{\mathcal{Z}}$ for stationary solitonic solutions are computed following formula (89) (we use equivalently two symbols ρ and/or σ). In QGIF theory, there are considered also the geometric evolution densities $\widehat{\rho}[\ _1\mathbf{g}]$ and $\widehat{\rho}'[\ _1\mathbf{g}]$, where the left label 1 is used in order to distinguish two d-metrics $\mathbf{g}$ and $\ _1\mathbf{g}$ which in this work may define two different geometric flows NES (in principle, such systems can be similar ones). Conventionally, we consider two stationary configurations for GIFs and NES systems, $\widehat{A}(\tau) = \widehat{A}(\psi(\tau), \Phi(\tau),\ _v\widehat{\Im}(\tau), \Lambda(\tau))$ and $\widehat{B}(\tau) = \widehat{B}(\ _1\psi(\tau),\ _1\Phi(\tau),\ _v^1\widehat{\Im}(\tau),\ _1\Lambda(\tau))$. Such two systems thermodynamic geometric flow models are elaborated on $\mathbf{V}\otimes\mathbf{V}$ when the normalizing function is fixed $\ _{AB}\widehat{f}(u,\ _1u) = 0$. The respective generating function (32) and entropy (33) are computed

$$\begin{aligned}
_{AB}\widehat{\mathcal{Z}}(\tau) &= \frac{1}{16\pi^4}\int\ _1\int \tau^{-4} d\mathcal{V}(\tau) d\ _1\mathcal{V}(\tau), \text{ for } \mathbf{V}\otimes\mathbf{V} \qquad (93)\\
_{AB}\widehat{\mathcal{S}}(\tau) &= \widehat{\mathcal{S}}\ [\widehat{A}, \widehat{B}] = -\frac{1}{16\pi^4}\int (\tau\ [\ _h\Lambda(\tau) + \Lambda(\tau)] - 2)\left(\tau[\ _h^1\Lambda(\tau) +\ _1\Lambda(\tau)] - 2\right)\tau^{-4} d\mathcal{V}(\tau) d\ _1\mathcal{V}(\tau). \qquad (94)
\end{aligned}$$

In similar forms, the three partite thermodynamic generation function and entropy (see formulas (34) and entropy (35)) of GIF and NES stationary systems $\widehat{A}(\tau) = \widehat{A}(\psi(\tau), \Phi(\tau),\ _v\widehat{\Im}(\tau), \Lambda(\tau))$, $\widehat{B}(\tau) = \widehat{B}(\ _1\psi(\tau),\ _1\Phi(\tau),\ _v^1\widehat{\Im}(\tau),\ _1\Lambda(\tau))$ and $\widehat{C}(\tau) = \widehat{C}(\ _2\psi(\tau),\ _2\Phi(\tau),\ _v^2\widehat{\Im}(\tau),\ _2\Lambda(\tau))$ are considered for a fixed normalizing function $\ _{ABC}\widehat{f}(u,\ _1u,\ _2u) = 0$. They are characterized by respective formulas,

$$\begin{aligned}
_{ABC}\widehat{\mathcal{Z}}(\tau) &= \frac{1}{64\pi^6}\int\ \ _1\int \tau^{-4} d\mathcal{V}(\tau) d\ _1\mathcal{V}(\tau) d\ _2\mathcal{V}(\tau), \text{ for } \mathbf{V}\otimes\mathbf{V}\otimes\mathbf{V} \qquad (95)\\
_{ABC}\widehat{\mathcal{S}}(\tau) &= \widehat{\mathcal{S}}\ [\widehat{A}, \widehat{B}, \widehat{C}]\ = -\frac{1}{64\pi^6}\int (\tau\ [\ _h\Lambda(\tau) + \Lambda(\tau)] - 2)\left(\tau[\ _h^1\Lambda(\tau) +\ _1\Lambda(\tau)] - 2\right)\\
&\quad \left(\tau[\ _h^2\Lambda(\tau) +\ _2\Lambda(\tau)] - 2\right)\tau^{-6} d\mathcal{V}(\tau) d\ _1\mathcal{V}(\tau) d\ _2\mathcal{V}(\tau). \qquad (96)
\end{aligned}$$

Using formulas (93), (94) and (95), (96) we can elaborate on GIF models for stationary d-metrics. For such a solution $\widehat{A}(\tau) = \widehat{A}[\mathbf{g}(\psi(\tau), \Phi(\tau),\ _v\widehat{\Im}(\tau), \Lambda(\tau))]$, we can associate a quantum system $\mathcal{A}(\tau)$ when the density matrix

$$\widehat{\rho}_{\mathcal{A}}(\tau) := \widehat{\mathcal{Z}}^{-1}(\tau)e^{-\tau^{-1}\widehat{\mathcal{E}}} \qquad (97)$$

is determined by values $\widehat{\mathcal{Z}}(\psi(\tau), \Phi(\tau),\ _v\widehat{\Im}(\tau), \Lambda(\tau))$ (89) and $\widehat{\mathcal{E}}(\psi(\tau), \Phi(\tau),\ _v\widehat{\Im}(\tau), \Lambda(\tau))$ (91). In result, we can compute the entanglement entropy (42) for stationary configurations

$$_q\widehat{\mathcal{S}}[\ \widehat{\rho}_{\mathcal{A}}(\psi(\tau), \Phi(\tau),\ _v\widehat{\Im}(\tau), \Lambda(\tau))] := Tr[\ \widehat{\rho}_{\mathcal{A}}\ (\psi(\tau), \Phi(\tau),\ _v\widehat{\Im}(\tau), \Lambda(\tau)) \log \widehat{\rho}_{\mathcal{A}}(\psi(\tau), \Phi(\tau),\ _v\widehat{\Im}(\tau), \Lambda(\tau))], \qquad (98)$$

when $\widehat{\rho}_{\mathcal{A}}$ is computed using formulas (97). This entanglement entropy is a QGIF version of the G. Perelman thermodynamic entropy $\widehat{\mathcal{S}}(\tau)$ (91).

For stationary QGIF systems, we can compute the Rényi entropy (48) using the replica method with $\widehat{\rho}_{\mathcal{A}}(\psi(\tau), \Phi(\tau), \ _{v}\widehat{\Im}(\tau), \Lambda(\tau))$ (97). Considering an integer replica parameter r, the Rényi entropy for the mentioned class of stationary flow configurations with 3+1 splitting (13),

$$ _{r}\ \widehat{\mathcal{S}}_{\mathcal{A}}(\tau) = \ _{r}\ \widehat{\mathcal{S}}(\widehat{\mathcal{A}}) := \frac{1}{1-r} \log[tr_{\mathcal{A}}(\widehat{\rho}_{\mathcal{A}}(\tau))^{r}] \quad (99)$$

for stationary solitonic QGIF system determined by the matrix $\widehat{\rho}_{\mathcal{A}}\ (\tau)$ (97) and associated thermodynamic model $\widehat{\mathcal{A}}(\tau) = \left[\widehat{\mathcal{Z}}(\tau),\ \widehat{\mathcal{E}}(\tau), \widehat{\mathcal{S}}(\tau)\right]$. Applying a standard computational formalism elaborated for an analytic continuation of r to a real number with a well defined limit $\ _{q}\widehat{\mathcal{S}}(\widehat{\rho}_{\mathcal{A}}(\tau)) = \lim_{r\to 1}\ \ _{r}\ \widehat{\mathcal{S}}(\widehat{\mathcal{A}}(\tau))$ and normalization $tr_{\mathcal{A}}(\widehat{\rho}_{\mathcal{A}})$ for $r \to 1$. For such limits and stationary solitonic flow d-metrics, the Rényi entropy reduces to the entanglement entropy (98). In result, we can formulate QGIFs models from section 3 for stationary configurations.

6.2 Thermodynamic values for stationary solitonic generating functions and generating sources

Such values are computed for a 3+1 spitting (13) determined by stationary solitonic d-metric (72) (for LC-configurations, we can consider (73)), when

$$\begin{aligned} q_1 &= q_2[\ _{h}i] = e^{\ \psi[\ _{h}i]}, \mathbf{q}_3[\ _{h}i] = -\frac{4[(\Phi[\ _{4}i])^2]^{\diamond}}{|\int dy^3\ _{v}\widehat{\Im}[i][(\Phi[\ _{4}i])^2]^{\diamond}|},\ [\ _{q}N(\tau)]^2 = h_4[\ _{4}i] = -\frac{(\Phi[\ _{4}i])^2}{4\Lambda(\tau)} \\ \text{and } N_i^a &= [w_i(\tau) = \frac{\partial_i \left(\int dy^3\ _{v}\widehat{\Im}[i][\Phi^2[\ _{4}i]]^{\diamond}\right)}{_{v}\widehat{\Im}[i][\Phi^2[\ _{4}i]]^{\diamond}}, n_i(\tau) = 0]. \end{aligned}$$

Using these coefficients and prescribing solitonic hierarchies for the effective volume (90), we obtain

$$\begin{aligned} d\mathcal{V}[\ _{h}i,\ _{4}i, i] &= e^{\ \psi[\ _{h}i]} |\Phi[\ _{4}i]| \sqrt{\frac{|[\Phi^2[\ _{4}i]]^{\diamond}|}{|\Lambda(\tau)\int dy^3\ _{v}\widehat{\Im}[i][\Phi^2[\ _{4}i]]^{\diamond}|}} dx^1 dx^2 \\ & \left[dy^3 + \frac{\partial_i \left(\int dy^3\ _{v}\widehat{\Im}[i][\Phi^2[\ _{4}i]]^{\diamond}\right)}{_{v}\widehat{\Im}[i]\ [\Phi^2[\ _{4}i]]^{\diamond}} dx^i \right] dt \end{aligned}$$

and respective thermodynamic generating function (89) $\widehat{\mathcal{Z}}[\ _{h}i,\ _{4}i, i] = \frac{1}{4\pi^2}\int \tau^{-2} d\mathcal{V}[\ _{h}i,\ _{4}i, i]$.

The value $\widehat{\mathcal{Z}}[\ _{h}i,\ _{4}i, i]$ determine the thermodynamic values (91) for geometric flows of such solitonic hierarchies,

$$\begin{aligned} \widehat{\mathcal{E}}\ [\ _{h}i,\ _{4}i, i] &= -\frac{\tau^2}{4\pi^2}\int \left([\ _{h}\Lambda(\tau) + \Lambda(\tau)] - \frac{2}{\tau}\right) \tau^{-2} d\mathcal{V}[\ _{h}i,\ _{4}i, i], \\ \widehat{\mathcal{S}}[\ _{h}i,\ _{4}i, i] &= -\frac{1}{4\pi^2}\int \left(\tau\ [\ _{h}\Lambda(\tau) + \Lambda(\tau)] - 2\right) \tau^{-2} d\mathcal{V}[\ _{h}i,\ _{4}i, i]. \end{aligned}$$

In a similar form, using and $d\mathcal{V}[\ _{h}i,\ _{4}i, i]$, we compute two partite and three partite thermodynamic values (93), (94) and (95), (96).

To any GIF solitonic stationary system $\widehat{A}[\ _{h}i,\ _{4}i, i] = \widehat{A}[\mathbf{g}(\psi[\ _{h}i], \Phi[\ _{4}i],\ _{v}\widehat{\Im}[i], \Lambda(\tau))]$, we can associate a quantum system $\mathcal{A}[\ _{h}i,\ _{4}i, i]$ when the density matrix $\widehat{\rho}_{\mathcal{A}}[\ _{h}i,\ _{4}i, i] := \widehat{\mathcal{Z}}^{-1}[\ _{h}i,\ _{4}i, i] e^{-\tau^{-1}\widehat{\mathcal{E}}[\ _{h}i,\ _{4}i, i]}$ allows us to compute the entanglement entropy (42) for stationary solitonic configurations

$$ _{q}\widehat{\mathcal{S}}[\ \widehat{\rho}_{\mathcal{A}}(\psi[\ _{h}i], \Phi[\ _{4}i],\ _{v}\widehat{\Im}[i], \Lambda(\tau))] := Tr[\ \widehat{\rho}_{\mathcal{A}}(\psi[\ _{h}i], \Phi[\ _{4}i],\ _{v}\widehat{\Im}[i], \Lambda(\tau)) \log \widehat{\rho}_{\mathcal{A}}(\psi[\ _{h}i], \Phi[\ _{4}i],\ _{v}\widehat{\Im}[i], \Lambda(\tau))].$$

For such stationary solitonic QGIF systems, the Rényi entropy (48) is computed using the replica method for the density matric $\widehat{\rho}_{\mathcal{A}}[\ _{h}i,\ _{4}i,i]$,

$$ _{r}\ \widehat{\mathcal{S}}_{\mathcal{A}}[\ _{h}i,\ _{4}i,i] = \ _{r}\ \widehat{\mathcal{S}}(\widehat{\mathcal{A}}[\ _{h}i,\ _{4}i,i]) := \frac{1}{1-r}\log[tr_{\mathcal{A}}(\widehat{\rho}_{\mathcal{A}}([\ _{h}i,\ _{4}i,i]))^{r}],$$

characterizing an associated thermodynamic model $\widehat{\mathcal{A}}[\ _{h}i,\ _{4}i,i]) = \left[\widehat{\mathcal{Z}}[\ _{h}i,\ _{4}i,i]),\ \widehat{\mathcal{E}}[\ _{h}i,\ _{4}i,i],\widehat{\mathcal{S}}[\ _{h}i,\ _{4}i,i]\right]$.

6.3 Perelman's thermodynamics for BHs deformed by stationary solitonic hierarchies

Such GIF and QGIF systems encode stationary solitonic d-metrics (81), or LC-configurations (85), when the d-metric $\mathring{\mathbf{g}} = [\mathring{g}_i, \mathring{g}_a, \mathring{N}_b^j]$ (52) defines a Kerr BH solution (84). The solitonic generating function (80) can be written in the form $\Phi[\ _{4}\iota] = 2\sqrt{|\Lambda(\tau)\ \eta_4[\ _{4}\iota]\overline{A}|}$ which results in such an effective volume form

$$d\mathcal{V}[\ _{h}i,\ \eta_4[\ _{4}\iota],i,\overline{A}] = 2e^{\ \psi[\ _{h}i]}\sqrt{\frac{|\eta_4[\ _{4}\iota]\overline{A}|\ \eta_4[\ _{4}\iota]|^{\diamond}|}{|\int dy^3\ _{v}\widehat{\Im}[i]|\ \eta_4[\ _{4}\iota]|^{\diamond}|}}dx^1dx^2\left[dy^3 + \frac{\partial_i\left(\int dy^3\ _{v}\widehat{\Im}[i]|\ \eta_4[\ _{4}\iota]|^{\diamond}\right)}{_{v}\widehat{\Im}[i]\ |\ \eta_4[\ _{4}\iota]|^{\diamond}}dx^i\right]dt$$

and corresponding GIF thermodynamic generating function (89)

$$\widehat{\mathcal{Z}}[\ _{h}i,\ \eta_4[\ _{4}\iota],i,\overline{A}] = \frac{1}{4\pi^2}\int \tau^{-2}d\mathcal{V}[\ _{h}i,\ \eta_4[\ _{4}\iota],i,\overline{A}].$$

Both the primary BH and solitonic data are encoded also in the thermodynamic values (91),

$$\begin{aligned}
\widehat{\mathcal{E}}[\ _{h}i,\ \eta_4[\ _{4}\iota],i,\overline{A}] &= -\frac{\tau^2}{4\pi^2}\int\left([\ _{h}\Lambda(\tau)+\Lambda(\tau)] - \frac{2}{\tau}\right)\tau^{-2}d\mathcal{V}[\ _{h}i,\ \eta_4[\ _{4}\iota],i,\overline{A}],\\
\widehat{\mathcal{S}}[\ _{h}i,\ \eta_4[\ _{4}\iota],i,\overline{A}] &= -\frac{1}{4\pi^2}\int\left(\tau\,[\ _{h}\Lambda(\tau)+\Lambda(\tau)] - 2\right)\tau^{-2}d\mathcal{V}[\ _{h}i,\ \eta_4[\ _{4}\iota],i,\overline{A}]
\end{aligned}$$

and (in similar forms via $d\mathcal{V}[\ _{h}i,\ \eta_4[\ _{4}\iota],i,\overline{A}]$) in two partite and three partite thermodynamic values (93), (94) and (95), (96).

Such a BH deformed GIF solitonic stationary system $\widehat{A}[\ _{h}i,\ \eta_4[\ _{4}\iota],i,\overline{A}] = \widehat{A}[\mathbf{g}(\psi[\ _{h}i],\eta_4[\ _{4}\iota],\ _{v}\widehat{\Im}[i],\Lambda(\tau))]$ can be associated to a quantum system $\mathcal{A}[\ _{h}i,\ \eta_4[\ _{4}\iota],i,\overline{A}]$ characterized by a respective density matrix

$$\widehat{\rho}_{\mathcal{A}}[\ _{h}i,\ \eta_4[\ _{4}\iota],i,\overline{A}] := \widehat{\mathcal{Z}}^{-1}[\ _{h}i,\ \eta_4[\ _{4}\iota],i,\overline{A}]e^{-\tau^{-1}\widehat{\mathcal{E}}[\ _{h}i,\ \eta_4[\ _{4}\iota],i,\overline{A}]}\ .$$

This value can be used constructing a QGIF system and computing the entanglement entropy (42) for stationary BH solitonic deformations

$$_{q}\widehat{\mathcal{S}}[\ \widehat{\rho}_{\mathcal{A}}(\psi[\ _{h}i],\eta_4[\ _{4}\iota],i],\ _{v}\widehat{\Im}[i],\Lambda(\tau))] := Tr[\ \widehat{\rho}_{\mathcal{A}}(\psi[\ _{h}i],\eta_4[\ _{4}\iota],\ _{v}\widehat{\Im}[i],\Lambda(\tau))\log\widehat{\rho}_{\mathcal{A}}(\psi[\ _{h}i],\eta_4[\ _{4}\iota],\ _{v}\widehat{\Im}[i],\Lambda(\tau))]$$

and the Rényi entropy (48), $_{r}\ \widehat{\mathcal{S}}_{\mathcal{A}}[\ _{h}i,\eta_4[\ _{4}\iota],i] = \ _{r}\ \widehat{\mathcal{S}}(\widehat{\mathcal{A}}[\ _{h}i,\eta_4[\ _{4}\iota],i]) := \frac{1}{1-r}\log[tr_{\mathcal{A}}(\widehat{\rho}_{\mathcal{A}}([\ _{h}i,\eta_4[\ _{4}\iota],i]))^{r}]$.

6.4 Small parametric stationary solitonic BH deformations and geometric flow thermodynamics

The d-metrics for such parametric solutions are described by quadratic elements (86) and generating functions $\Phi[\ _{4}\iota,\overline{A}] \simeq 2\sqrt{|\Lambda(\tau)\overline{A}|}(1-\frac{\varepsilon}{2}\upsilon[\ _{4}\iota])$ (87) and a primary BH metric (84). Using the respective effective volume form

$$\begin{aligned}
d\mathcal{V}[\ _{h}i,\ \varepsilon\upsilon[\ _{4}\iota],i,\overline{A}] &= 2e^{\ \psi[\ _{h}i]}\left|(1-\frac{\varepsilon}{2}\upsilon[\ _{4}\iota])\right|\sqrt{\frac{|\ \overline{A}|\ \upsilon[\ _{4}\iota]|^{\diamond}|}{|\int dy^3\ _{v}\widehat{\Im}[i]|\ \upsilon[\ _{4}\iota]|^{\diamond}|}}dx^1dx^2\\
&\quad\left[dy^3 + \frac{\partial_i\left(\int dy^3\ _{v}\widehat{\Im}[i]|\ \upsilon[\ _{4}\iota]|^{\diamond}\right)}{_{v}\widehat{\Im}[i]\ |\ \upsilon[\ _{4}\iota]|^{\diamond}}dx^i\right]dt,
\end{aligned}$$

we compute corresponding thermodynamic generating function (89) and canonical energy and entropy (91) for stationary geometric solitonic flow parametric deformations

$$\begin{aligned}
\widehat{\mathcal{Z}}[\ _{h}i,\ \varepsilon\upsilon[\ _{4}\iota],i,\overline{A}] &= \frac{1}{4\pi^2}\int \tau^{-2}d\mathcal{V}[\ _{h}i,\ \varepsilon\upsilon[\ _{4}\iota],i,\overline{A}] \text{ and} \\
\widehat{\mathcal{E}}\ [\ _{h}i,\ \varepsilon\upsilon[\ _{4}\iota],i,\overline{A}] &= -\frac{\tau^2}{4\pi^2}\int \left([\ _{h}\Lambda(\tau)+\Lambda(\tau)]-\frac{2}{\tau}\right)\tau^{-2}d\mathcal{V}, \\
\widehat{\mathcal{S}}[\ _{h}i,\ \varepsilon\upsilon[\ _{4}\iota],i,\overline{A}] &= -\frac{1}{4\pi^2}\int \left(\tau\left[\ _{h}\Lambda(\tau)+\Lambda(\tau)\right]-2\right)\tau^{-2}d\mathcal{V}[\ _{h}i,\ \varepsilon\upsilon[\ _{4}\iota],i,\overline{A}]
\end{aligned}$$

and (in similar form but for $d\mathcal{V}[\ _{h}i,\ \varepsilon\upsilon[\ _{4}\iota],i,\overline{A}]$) two partite and three partite thermodynamic values (93), (94) and (95), (96).

For any BH parametric deformed GIF solitonic stationary system

$$\widehat{A}[\ _{h}i,\ \varepsilon\upsilon[\ _{4}\iota],i,\overline{A}] = \widehat{A}[\mathbf{g}(\psi[\ _{h}i],\varepsilon\upsilon[\ _{4}\iota],\ _{v}\widehat{\Im}[i],\Lambda(\tau),\overline{A})],$$

we can associate to a quantum system $\mathcal{A}[\ _{h}i,\ \varepsilon\upsilon[\ _{4}\iota],i,\overline{A}]$ and compute the density matrix

$$\widehat{\rho}_{\mathcal{A}}[\ _{h}i,\ \varepsilon\upsilon[\ _{4}\iota],i,\overline{A}] := \widehat{\mathcal{Z}}^{-1}[\ _{h}i,\ \varepsilon\upsilon[\ _{4}\iota],i,\overline{A}]e^{-\tau^{-1}\widehat{\mathcal{E}}[\ _{h}i,\ \varepsilon\upsilon[\ _{4}\iota],i,\overline{A}]}.$$

A corresponding QGIF stationary solitonic system with small parameter is characterized by the entanglement entropy (42)

$$\begin{aligned}
&{}_{q}\widehat{\mathcal{S}}[\ \widehat{\rho}_{\mathcal{A}}(\psi[\ _{h}i],\ \varepsilon\upsilon[\ _{4}\iota],i],\ _{v}\widehat{\Im}[i],\Lambda(\tau)),\overline{A}] := \\
&Tr[\ \widehat{\rho}_{\mathcal{A}}(\psi[\ _{h}i],\ \varepsilon\upsilon[\ _{4}\iota],\ _{v}\widehat{\Im}[i],\Lambda(\tau),\overline{A})\log\widehat{\rho}_{\mathcal{A}}(\psi[\ _{h}i],\ \varepsilon\upsilon[\ _{4}\iota],\ _{v}\widehat{\Im}[i],\Lambda(\tau),\overline{A})],
\end{aligned}$$

when, for instance, the Rényi entropy (48),

$$_{r}\ \widehat{\mathcal{S}}_{\mathcal{A}}[\ _{h}i,\ \varepsilon\upsilon[\ _{4}\iota],i,\overline{A}] = \ _{r}\ \widehat{\mathcal{S}}(\widehat{\mathcal{A}}[\ _{h}i,\ \varepsilon\upsilon[\ _{4}\iota],i,\overline{A}]) := \frac{1}{1-r}\log[tr_{\mathcal{A}}(\widehat{\rho}_{\mathcal{A}}([\ _{h}i,\ \varepsilon\upsilon[\ _{4}\iota],i,\overline{A}]))^{r}].$$

All formulas and inequalities for entropies of NES QGIFs can be computed for such small parametric stationary deformations. For instance, prescribing a solitonic wave, all values and equations can be computed in explicit form for such a BH solitonic stationary configuration, see a number of examples in our previous works [58, 59, 60, 61, 64, 65, 66, 68, 69, 70, 71, 72, 73, 74].

Finally, we emphasize that it is not possible to define and compute the Bekenstein–Hawking entropy for the exact and parametric stationary solitonic and/or BH solutions constructed in section 5. Such geometric flows of NES systems are characterized by respective thermodynamic values, GIF and QGIF models computed in section 6.

7 Outlook, conclusions, and discussion

This is the third our work on the theory of classical and quantum geometric information flows (respectively, GIFs and QGIFs), see [38, 39] on QGIF of relativistic classical and quantum mechanical systems. On entanglement and QGIF of Einstein-Maxwell and Kaluza–Klein gravity theories, we cite the forth partner work [40]. In a more general context, such papers belong to a series of articles [36, 37, 18, 19, 20, 21, 24, 25, 26] on generalized (relativistic, modified, or noncommutative and nonassociative, supersymmetric etc.) Ricci flows and applications to modified gravity theories, MGTs, and general relativity, GR. The key idea of this and partner articles is that all such theories and fundamental physical equations, their symmetries and solutions, and associated thermodynamic and information models can be derived from certain types of nonholonomically deformed Perelman–Lyapunov type F- and W- functionals [27]. In our research, we do not attempt to use such results for formulating and providing proofs for certain relativistically generalized Thurston–Poincaré conjectures (as it was performed due to G. Perelman [27] and R. Hamilton [28], see reviews in [29, 30, 31]). The goal of this work is to elaborate on applications in quantum information theory of the concept W-entropy (Perelman's W-functional can be treated as a "minus" entropy) and associated thermodynamic models of geometric flows containing MGTs and GR, as certain particular nonholonomic Ricci soliton configurations.

In this paper, relativistic versions of Perelman's functionals and associated thermodynamic models are formulated in canonical nonholonomic variables (with "hats" on geometric objects). In result, any exact and parametric solution in gravity theories and geometric flow models[21] can be derived and characterized thermodynamically considering a respective W-entropy. Such constructions are more general than those for gravitational thermodynamics and black hole / (anti) de Sitter physics [14, 15, 16, 17] (with a conventional hypersurface-area entropy) intensively elaborated during last two decades with new concepts of entanglement, entropic and holographic gravity [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

In the framework of the theory of relativistic geometric flows of metrics on Lorentz spacetime manifolds, various MGTs and GR are modelled as certain nonholonomic Einstein structures, NES, running on a temperature like parameter and on a time like parameter. Such theories are described equivalently as some self-similar systems (i.e. nonholonomic Ricci solitons) which are characterized by a corresponding W-entropy and other type associated relativistic thermodynamic parameters. In this article, we formulate an approach to the theory of GIFs and QGIFs and NES with a temperature like parameter. We apply and develop for such gravitational flow evolution models and geometric thermodynamical systems, the standard concepts and methods of information theory and quantum physics and gravity [45, 44, 47, 1, 48, 49, 50, 2, 12, 13]. The constructions are generalized for the Shannon/von Neumann/conditional/relative entropy determined by thermodynamic generating functions and density matrices encoding geometric data for GIFs and NES, and characterized by respective Perelman W-entropy. The concept of quantum geometric flow and gravitational entanglement and main properties (inequalities) are formulated and studied for new classes of theories of QGIFs for NES.

In sections 4-6, we shown how to construct in explicit form stationary generic off-diagonal solutions for relativistic geometric flows, nonholonomic Ricci solitons and generalized gravitational field equations. Such configurations are not characterized, in general, by certain entropy-area, holographic or duality conditions and can not described as some modified Bekenstein-Hawking BH thermodynamic systems. In our works, it is developed an alternative and more general way when stationary and cosmological solutions in geometric flow evolution theories, MGTs and GR, GIFs and QGIFs, can be defined and characterized by nonholonomic deformations of Perelman's W-entropy and associated statistical thermodynamic models.

Finally, we note that further developments of our approach will involve explicit examples with computations of the W-entropy and QGIF and NES entanglement, Rényi and other type entropies for various classes of stationary and cosmological type solutions in MGTs and quantum gravity models, spinor and noncommutative variables, see some previous our results in [18, 34, 35].

Acknowledgments: This research develops former programs partially supported by IDEI, PN-II-ID-PCE-2011-3-0256, CERN and DAAD and extended to adjunct positions and collaborations with California State University at Fresno, the USA, and Yu. Fedkovych Chernivtsi National University, Ukraine.

References

[1] J. Preskill, lecture notes, http://www.theory.caltech.edu/~preskill/ph219/index.html#lecture

[2] E. Witten, A mini-introduction to information theory, arXiv: 1805.11965

[3] S. Ryu and T. Takayanagi, Holographic derivation of entanglement entropy from AdS/CFT, Phys. Rev. Lett. 96 (2006) 181602; arXiv: hep-th/0603001

[4] M. Van Raamsdonk, Building up spacetime with quantum entanglement, Gen. Rel. Grav. 42 (2010) 2323 [Int. J. Mod. Phys. D 19 (2010) 2429]; arXiv: 1005.3035

[21] in principle, being generic off-diagonal, with generalized nonlinear and linear connections, various types of effective and matter field sources, and depending on all modified spacetime/ phase space coordinates, see various examples and applications in modern cosmology and astrophysics [33, 19, 20, 21, 25, 34, 35, 24, 26]

[5] L. Bubuianu, S. Vacaru, E. V. Veliev, Nonassociative black ellipsoids distorted by R-fluxes and four dimensional thin locally anisotropic accretion disks, Eur. Phys. J. C 81 (2021) 1145; arXiv: 2108.04689

[6] T. Faulkner, M. Guica, T. Harman, R. C. Myers and M. Van Raamsdonk, Gravitation from entanglement and holographic CFTs, JHEP 1403 (2015) 051; arXiv: 1312.7856

[7] B. Swingle, Entanglement renormalization and holography, Phys. Rev. D 86 (2012) 065007, arXiv: 0905.1317

[8] T. Jacobson, Entanglement equilibrium and the Einstein equation, Phys. Rev. Lett. 116 (2016) 201101, arXiv: 1505.04753

[9] M. Taylor and W. Woodhead, Non-conformal entanglement entropy, JHEP 2018 (2018) 4; arXiv: 1704.08269

[10] F. Pastawski, B. Yoshida, D. Harlow and J. Preskill, Holographic quantum error-correcting codes: Toy models for the bulk/boundary correspondence, JHEP 1506 (2015) 149; arXiv:1503.06237

[11] H. Casini, M. Huerta and R. C. Myers, Towards a derivation of holographic entanglement entropy, JHEP 1105 (2011) 036; arXiv:1102.0440

[12] L. Aolita, F. de Melo, L. Davidovich, Opens-system dynamics of entanglement, Rep. Progr. Phys. 78 (2015) 042001; arXiv: 1402.3713

[13] T. Nishioka, Entanglement entropy: holography and renormalization group, Rev. Mod. Phys. 90 (2018) 03500; arXiv: 1801.10352

[14] J. D. Bekenstein, Black holes and the second law, Nuovo Cimento Letters 4 (1972) 737-740

[15] J. D. Bekenstein, Black holes and entropy, Phys. Rev. D 7 (1973) 2333-2346

[16] J. M. Bardeen, B. Carter and S. W. Hawking, The four laws of black hole mechanics, Commun. Math. Phys. 31 (1973) 161

[17] S. W. Hawking, Particle creation by black holes, Commun. Math. Phys. 43 (1975) 199-220

[18] S. Vacaru, Spectral functionals, nonholonomic Dirac operators, and noncommutative Ricci flows, J. Math. Phys. 50 (2009) 073503; arXiv: 0806.3814 [math-ph]

[19] V. Ruchin, O. Vacaru and S. Vacaru, Perelman's W-entropy and Statistical and Relativistic Thermodynamic Description of Gravitational Fields, Eur. Phys. J. C 77 (2017) 184; arXiv: 1312.2580

[20] T. Gheorghiu, V. Ruchin, O. Vacaru and S. Vacaru, Geometric flows and Perelman thermodynamics for black ellipsoids in R2 and Einstein gravity theories, Annals of Physics, NY, 369 (2016) 1-35; arXiv: 1602.08512

[21] S. Rajpoot and S. Vacaru, On supersymmetric geometric flows and R2 inflation from scale invariant supergravity, Annals of Physics, NY, 384 (2017) 20-60; arXiv: 1606.06884

[22] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, Modified gravity theories in nutshell: inflation, bounce and late-time evolution, Phys. Rep. 692 (2017) 1-104; arXiv: 1705.11098

[23] S. Basilakos, A. P. Kouretsis, E. N. Saridakis and P. Stavrinos, Resembling dark energy and modified gravity with Finsler-Randers cosmology, Phys. Rev. D 83 (2013) 123510; arXiv: 1311.5915

[24] L. Bubuianu and S. Vacaru, Black holes with MDRs and Bekenstein-Hawking and Perelman entropies for Finsler-Lagrange-Hamilton spaces, Annals of Physics, NY, 404 (2019) 10-38; arXiv: 1812.02590

[25] S. Vacaru, Entropy functionals and thermodynamics of geometric flows, stationary quasi-periodic Ricci solitons, and gravity, Annals of Physics, NY, 423 (2020) 16833; arXiv: 1903.04920v4

[26] S. Vacaru, L. Bubuianu, Off-diagonal cosmological solutions in emergent gravity theories and Grigory Perelman entropy for geometric flows, Eur. Phys. J. C 81 (2021) 81; arXiv: 1904.05149v3

[27] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv: math. DG/0211159

[28] R. S. Hamilton, Three-manifolds with postive Ricci curvature, J. Diff. Geom. 17 (1982) 255-306

[29] H. -D. Cao and H. -P. Zhu, A complete proof of the Poincaré and geometrization conjectures - application of the Hamilton–Perelman theory of the Ricci flow, Asian J. Math. 10 (2006) 165-495; see also a preprint version: H. -D. Cao and H. -P. Zhu, Hamilton-Perelman's proof of the Poincaré conjecture and the geometrization conjectures, arXiv: math/0612069

[30] J. W. Morgan and G. Tian, Ricci flow and the Poincaré conjecture, AMS, Clay Mathematics Monogaphs, vol. 3 (2007); arXiv: math/ 0607607

[31] B. Kleiner and J. Lott, Notes on Perelman's papers, Geometry & Topology 12 (2008) 2587-2855; arXiv: math/0605667

[32] D. Friedan, Nonlinear models in $2+\varepsilon$ dimensions, Phys. Rev. Lett. 45 (1980) 1057-1060

[33] S. Vacaru, On general solutions in Einstein gravity, Int. J. Geom. Meth. Mod. Phys. 8 (2011) 9-21; arXiv: 0909.3949v1 [gr-qc] and 1106.4660 [physics.gen-ph]

[34] S. Vacaru, Space-time quasicrystal structures and inflationary and late time evolution dynamics in accelerating cosmology, Class. Quant. Grav. 35 (2018) 245009; arXiv: 1803.04810

[35] L. Bubuianu and S. Vacaru, Deforming black hole and cosmological solutions by quasiperiodic and/or pattern forming structures in modified and Einstein gravity, Eur. Phys. J. C78 (2018) 393; arXiv: 1706.02584

[36] S. Vacaru, Deformation quantization of almost Kaehler models and Lagrange-Finsler spaces, J. Math. Phys. 48 (2007) 123509; arXiv: 0707.1519 [gr-qc]

[37] S. Vacaru, Branes and quantization for an A-model complexification of Einstein gravity in almost Kaehler variables, Int. J. Geom. Meth. Mod. Phys. 6 (2009) 873-909; arXiv: 0810.4692

[38] S. Vacaru, Geometric information flows and G. Perelman entropy for relativistic classical and quantum mechanical systems, Eur. Phys. J. C80 (2020) 639; arXiv: 1905.12399

[39] S. Vacaru and L. Bubuianu, Classical and quantum geometric information flows and entanglement of relativistic mechanical systems, Quantum Inf. Process. 18 (2019) 376; arXiv: 1905.13015

[40] Iuliana Bubuianu, S. Vacaru and E. V. Veliev, Kaluza–Klein gravity & cosmology emerging from G. Perelman's entropy functionals and quantum geometric information flows, Eur. Phys. J. Plus 136 (2021) 149; arXiv: 1907.05847v3

[41] M. C. Palmer, M, Takahashi, and H. F. Westman, Localized qubits in curved spaceteimes, Annals of Physics 327 (2012) 1078-1131; arXiv: 1108.3896

[42] C. W. Misner, K. S. Thorne and J. A. Wheeler, Gravitation (Freeman, 1973)

[43] J. M. Overduin and P. S. Wesson, Kaluza-Klein gravity, Phys. Rep. 283 (1997) 303-380; arXiv: gr-qc/ 9805018

[44] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, 2000)

[45] T. M. Cover and J. A. Thomas, Elements of Information Theory (John Wiley & Sons, 1991)

[46] M. M. Wilde, Quantum Information Theory (Cambridge University Press, 2013)

[47] M. A. Nielsen and I. L. Chuang, Quantum computation and quantum information (Cambridge University Press, 2010)

[48] C. Weedbrook, S. Pirandola, R. García–Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro and S. Lloyd, Gaussian quantum information, Reviews of Modern Physics 84 (2012) 621; arXiv: 1110.3234

[49] M. Hayashi, Quantum Information Theory (Springer, 2017)

[50] J. Watrous, The theory of quantum information (Cambridge University Press, 2018)

[51] E. H. Lieb and M. B. Urskai, Proof of the strong subadditivity of quantum-mechanical entropy, J. Math. Phys. 14 (1973) 1938-1941

[52] H. Narnhofer and W. E. Thirring, From relative entropy to entropy, Fizika 17 (1985) 257-265

[53] A. Rényi, On measures of entropy and information, in: Fourth Berkeley Symposium on Mathematical Statistics and Probablility (1961), pp. 547-561

[54] N. Bao, M. Moosa and I. Shehzad, The holographyc dual of Rényi relative entropy, arXiv: 1904.08433

[55] K. Zyczkowski, Rényi extrapolation of Shannon entropy, Open Systems & Information Dynamics 10 (2003) 297-310

[56] G. Adesso, D. Girolami and A. Serafini, Measuring Gaussian quantum information and correlation using the Rényi entropy of order 2, Phys. Rev. Lett. 109 (2012) 190502

[57] S. Beingi, Sandwiched Rényi divergence satisfied data processing inequality, J. Math. Phys. 54 (2013) 122202

[58] S. Vacaru, Anholonomic soliton-dilaton and black hole solutions in general relativity, JHEP 04 (2001) 009; arXiv: gr-qc/0005025

[59] S. Vacaru, Curve flows and solitonic hierarchies generated by Einstein metrics, Acta Applicandae Mathematicae [ACAP] 110 (2010) 73-107; arXiv: 0810.0707

[60] S. Anco and S. Vacaru, Curve flows in Lagrange-Finsler geometry, bi-Hamiltonian structures and solitons, J. Geom. Phys. 59 (2009) 79-103; arXiv: math-ph/0609070

[61] S. Vacaru, Generic Off-Diagonal Solutions and Solitonic Hierarchies in Einstein and Modified Gravity, Mod. Phys. Lett. A 30 (2015) 1550090; arXiv: 1308.6180

[62] B. B. Kadomtsev and V. I. Petviashvili, On stability of solitary waves in weakly dispersive media, Sov. Phys. Dokl. 15 (1970) 539-541 [Russian translation: Doklady Akademii Nauk SSSR 192 (1970) 753-756]

[63] V. Belinski and E. Verdaguer, Gravitational Solitons (Cambriged Universtity Press, 2001)

[64] S. Vacaru and F. C. Popa, Dirac spinor waves and solitons in anisotropic Taub-NUT spaces, Class. Quant. Gravity, 18 (2001) 4921-4938; arXiv: hep-th/0105316

[65] S. Vacaru and D. Singleton, Warped solitonic deformations and propagation of black holes in 5D vacuum gravity, Class. Quant. Grav. 19 (2002) 3583-3602; arXiv: hep-th/0112112

[66] S. Vacaru, Black holes, ellipsoids, and nonlinear waves in pseudo-Finsler spaces and Einstein gravity, Int. J. Theor. Physics 52 (2013) 1654-1681; arXiv: 0905.4401

[67] D. Baleanu and S. Vacaru, Fractional curve flows and solitonic hierarchies in gravity and geometric mechanics, J. Math. Phys. 52 (2011) 053514; arXiv: 1007.2866

[68] S. Rajpoot and S. Vacaru, Black Ring and Kerr Ellipsoid – Solitonic Configurations in Modified Finselr Gravity, Int. J. Geom. Meth. Mod. Phys. 12 (2015) 1550102; arXiv: 1506.08696

[69] S. Vacaru, Ricci flows and solitonic pp-waves, Int. J. Mod. Phys. A21 (2006) 4899-4912; arXiv: hep-th/ 0602063

[70] S. Vacaru, Nonholonomic Ricci flows and parametric deformations of the solitonic pp-waves and Schwarzschild solutions, Electronic Journal of Theoretical Physics (EJTP) 6, N21 (2009) 63-93; arXiv: 0705.0729

[71] S. Vacaru, Nonholonomic Ricci flows, exact solutions in gravity, and symmetric and nonsymmetric metrics, Int. J. Theor. Phys. 48 (2009) 579-606; arXiv: 0806.3812

[72] L. Bubuianu, K. Irwin and S. Vacaru, Heterotic supergravity with internal almost-Kaehler spaces; instantons for SO(32), or E8 x E8, gauge groups; and deformed black holes with soliton, quasiperiodic and/or pattern-forming structures, Class. Quant. Grav. 34 (2017) 075012; arXiv: 1611.002

[73] T. Gheorghiu, O. Vacaru, S. Vacaru, Off-diagonal deformations of Kerr black holes in Einstein and modified massive gravity and higher dimensions, Eur. Phys. J. C 74 (2014) 3152; arXiv: 1312.4844

[74] S. Vacaru, Hidden symmetries for ellipsoid-solitonic deformations of Kerr-Sen black holes and quantum anomalies, Eur. Phys. J. C 73 (2013) 2287; arXiv: 1106.1033