# Uncertainty of quantum channels via modified generalized variance and modified generalized Wigner-Yanase-Dyson skew information 

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#### Abstract

Uncertainty relation is a fundamental issue in quantum mechanics and quantum information theory. By using modified generalized variance (MGV), and modified generalized Wigner-Yanase-Dyson skew information (MGWYD), we identify the total and quantum uncertainty of quantum channels. The elegant properties of the total uncertainty of quantum channels are explored in detail. In addition, we present a trade-off relation between the total uncertainty of quantum channels and the entanglement fidelity, and establish the relationships between the total uncertainty and entropy exchange/coherent information. Detailed examples are given to the explicit formulas of the total uncertainty and the quantum uncertainty of quantum channels. Moreover, utilizing a realizable experimental measurement scheme by using the Mach-Zehnder interferometer proposed in [65], we discuss how to measure the total/quantum uncertainty of quantum channels for pure states.


Keywords: Quantum uncertainty • Quantum channel • Variance • Skew information

## 1 Introduction

The quantum uncertainty is closely related to quantum measurements. As an extremely important issue in the quantum physics, the uncertainty relation has been initially put forward by Heisenberg [1] and Robertson [2]. Uncertainty is usually quantified by variance or entropies. The total uncertainty of the observable in quantum states is quantified by variance, which is a mixture of classical uncertainty and quantum uncertainty [3-5]. Recently, Gudder [6] has introduced the modified variance of arbitrary operator (not necessarily Hermitian). In addition, Dou and Du [7, 8] have proposed a Heisenberg-type

[^0]uncertainty relation and a Schrödinger-type uncertainty relation based on the modified variance. Sun and Li [9] have proposed the total uncertainty of quantum channels in terms of the modified variance.

The quantum uncertainty can be also described by skew information. The skew information has been originally proposed by Wigner and Yanase [10], which is termed as Wigner-Yanase (WY) skew information. While a more general quantity has been suggested by Dyson, which is now called the Wigner-Yanase-Dyson (WYD) skew information [10]. This quantity has been further generalized in [11] and termed as generalized Wigner-Yanase-Dyson (GWYD) skew information. The relation among the WYD skew information, GWYD skew information and uncertainty relations have been studied extensively [12-14]. As is well known, the observables and Hamiltonians in quantum mechanics are assumed to be Hermitian operators mathematically. The framework of non-Hermitian quantum mechanics, however, has also been taken into account and has attracted much attention [15]. Besides, many operators such as quantum gates [16], generalized quantum gates [17] and the Kraus operators of a quantum channel [16] are not necessarily Hermitian. Therefore, it is desirable to introduce the corresponding definitions of the above-mentioned skew information for pseudo-Hermitian and/or PT-symmetric quantum mechanics [18-23]. By considering an arbitrary operator which may be non-Hermitian, Dou and Du [8] have introduced the modified Wigner-Yanase-Dyson (MWYD) skew information, while Wu, Zhang and Fei $[24,25]$ have further introduced the modified generalized Wigner-Yanase-Dyson (MGWYD) skew information.

Quantum channels characterize the general evolutions of quantum systems [16, 26], which include the quantum measurements as special cases. The past few years have witnessed a great deal of researches on quantum channels [27-40]. The interaction between a quantum system and the external environment would lead to information loss and disturbance on the quantum states. Until now, many kinds of formulations have been established on the trade-off relations between the information and the disturbance [41-50]. Schumacher [51] has introduced the entanglement fidelity which provides a measure of how well the entanglement between a quantum system and an auxiliary system is preserved under the quantum channel.

The main purpose of this paper is to provide a quantification of the total as well as the quantum uncertainties of quantum channels in terms of the modified generalized variance (MGV) and MGWYD skew information, respectively. We give an extension of the trade-off relation between the total uncertainty of quantum channels and the entanglement fidelity. By recalling two important information quantities, the entropy exchange [51] and coherent information [52], we also generalize the relations between the total uncertainty of quantum channels and the entropy exchange/coherent information introduced in [9] to a more general case.

The rest of the paper is formulated as follows. In Section 2, we introduce the
total uncertainty of quantum channels based on MGV and prove that it satisfies several fundamental properties. In Section 3, we generalize a trade-off relation between the total uncertainty of quantum channels and the entanglement fidelity. Furthermore, by using the general trade-off relation, we also explore the relationship between the total uncertainty of quantum channels and the entropy exchange/coherent information. In Section 4, we calculate the total and quantum uncertainty of some typical quantum channels, and derive the explicit formulas of the uncertainties of the quantum channel in Example 5 for two special classes of states, the Werner states and isotropic states. We also illustrate the trade-off relation (18), inequalities (19) and (20) by using Example 6. We use an experimentally feasible protocol given in [65] to measure the total/quantum uncertainty of quantum channels for pure states. Some concluding remarks are given in Section 5.

## 2 The total uncertainty of quantum channels based on MGV

Let $\mathcal{H}$ be a d-dimensional Hilbert space. Denote by $\mathcal{B}(\mathcal{H}), \mathcal{S}(\mathcal{H})$ and $\mathcal{D}(\mathcal{H})$ the set of all bounded linear operators, Hermitian operators and density operators on $\mathcal{H}$, respectively. For $\rho \in \mathcal{D}(\mathcal{H})$ and $K \in \mathcal{B}(\mathcal{H})$, the modified generalized variance (MGV) of the bounded linear operator $K$ in $\rho$ is defined by [24],

$$
\begin{align*}
V^{\alpha, \beta}(\rho, K) & =\frac{1}{2}\left(\operatorname{tr} \rho K_{0}^{\dagger} K_{0}+\operatorname{tr} \rho^{\alpha+\beta} K_{0} \rho^{1-\alpha-\beta} K_{0}^{\dagger}\right) \\
& =\frac{1}{2}\left(\operatorname{tr} \rho K^{\dagger} K+\operatorname{tr} \rho^{\alpha+\beta} K \rho^{1-\alpha-\beta} K^{\dagger}\right)-|\operatorname{tr} \rho K|^{2}, \quad \alpha, \beta \geq 0, \alpha+\beta \leq 1 \tag{1}
\end{align*}
$$

with $K_{0}=K-\operatorname{tr} \rho K$. Note that the modified generalized variance $V^{\alpha, \beta}(\rho, K)$ is nonnegative and $V^{\alpha, \beta}(\rho, K)$ is concave in $\rho$. Analogizing the idea in Ref. [4], $V^{\alpha, \beta}(\rho, K)$ can be also split into quantum and classical parts as [24],

$$
\begin{equation*}
V^{\alpha, \beta}(\rho, K)=Q^{\alpha, \beta}(\rho, K)+C^{\alpha, \beta}(\rho, K), \quad \alpha, \beta \geq 0, \alpha+\beta \leq 1 \tag{2}
\end{equation*}
$$

where

$$
\begin{align*}
Q^{\alpha, \beta}(\rho, K) & =\frac{1}{2} \operatorname{tr}\left(\left[\rho^{\alpha}, K_{0}\right]^{\dagger}\left[\rho^{\beta}, K_{0}\right] \rho^{1-\alpha-\beta}\right) \\
& =\frac{1}{2}\left(\operatorname{tr} \rho K_{0}^{\dagger} K_{0}-\operatorname{tr} \rho^{\alpha} K_{0} \rho^{1-\alpha} K_{0}^{\dagger}-\operatorname{tr} \rho^{\beta} K_{0} \rho^{1-\beta} K_{0}^{\dagger}+\operatorname{tr} \rho^{\alpha+\beta} K_{0} \rho^{1-\alpha-\beta} K_{0}^{\dagger}\right) \\
& =\frac{1}{2}\left(\operatorname{tr} \rho K^{\dagger} K-\operatorname{tr} \rho^{\alpha} K \rho^{1-\alpha} K^{\dagger}-\operatorname{tr} \rho^{\beta} K \rho^{1-\beta} K^{\dagger}+\operatorname{tr} \rho^{\alpha+\beta} K \rho^{1-\alpha-\beta} K^{\dagger}\right) \tag{3}
\end{align*}
$$

where $\alpha, \beta \geq 0, \alpha+\beta \leq 1$, which is called modified generalized Wigner-Yanase-Dyson
(MGWYD) skew information quantifying the quantum uncertainty of $K$ in $\rho$, and

$$
\begin{align*}
C^{\alpha, \beta}(\rho, K) & =V^{\alpha, \beta}(\rho, K)-Q^{\alpha, \beta}(\rho, K) \\
& =\frac{1}{2}\left(\operatorname{tr} \rho^{\alpha} K_{0} \rho^{1-\alpha} K_{0}^{\dagger}+\operatorname{tr} \rho^{\beta} K_{0} \rho^{1-\beta} K_{0}^{\dagger}\right) \\
& =\frac{1}{2}\left(\operatorname{tr} \rho^{\alpha} K \rho^{1-\alpha} K^{\dagger}+\operatorname{tr} \rho^{\beta} K \rho^{1-\beta} K^{\dagger}\right)-|\operatorname{tr} \rho K|^{2}, \quad \alpha, \beta \geq 0, \alpha+\beta \leq 1 \tag{4}
\end{align*}
$$

which quantifies the classical uncertainty of $K$ in $\rho$.
The WY skew information [10] $Q(\rho, H)=-\frac{1}{2} \operatorname{tr}[\sqrt{\rho}, H]^{2}$ has been introduced as a measure of information content of observables not commuting with (skew to) the conserved observable $H$ in the state $\rho$. However, as we have mentioned, some important operators are non-Hermitian. With respect to the WY skew information, $Q(\rho, H)=$ $-\frac{1}{2} \operatorname{tr}\left[\sqrt{\rho}, H^{\dagger}\right][\sqrt{\rho}, H]$, the modified Wigner-Yanase skew information $Q(\rho, K)=-\frac{1}{2} \operatorname{tr}[\sqrt{\rho}$, $\left.K^{\dagger}\right][\sqrt{\rho}, K]$ has been defined in $[7,53]$ (up to a constant factor in [53]), which may be interpreted as a measure of information content of observables skew to the Hamiltonian $K$ in the state $\rho$ in PT-symmetric quantum mechanics. In this regard, we can say that the generalized quantity $Q^{\alpha, \beta}(\rho, K)$ provides a family of quantifiers of such information content.

It is proved that [14] $Q^{\alpha, \beta}(\rho, H)=-\frac{1}{2} \operatorname{tr}\left(\left[\rho^{\alpha}, H\right]\left[\rho^{\beta}, H\right] \rho^{1-\alpha-\beta}\right)$ is convex in $\rho$ when $\alpha, \beta \in[0,1]$ with $\alpha+2 \beta \leq 1$ and $2 \alpha+\beta \leq 1$. By using the Morozova-Chentsov function of a regular metric, the WYD skew information has been extended to the metric adjusted skew information (of a state with respect to a conserved observable) [54], which is a nonnegative quantity bounded by the variance (of an observable in a state) that vanishes for observables commuting with the state. Note that $2 Q^{\alpha, \beta}(\rho, H)$ is a metric adjusted skew information [14], where the Morozova-Chentsov function is

$$
\begin{aligned}
c(x, y) & =\frac{1}{(x-y)^{2}}\left[\left(x^{\alpha}-y^{\alpha}\right)\left(x^{1-\alpha}-y^{1-\alpha}\right)+\left(x^{\beta}-y^{\beta}\right)\left(x^{1-\beta}-y^{1-\beta}\right)\right. \\
& \left.-\left(x^{\alpha+\beta}-y^{\alpha+\beta}\right)\left(x^{1-\alpha-\beta}-y^{1-\alpha-\beta}\right)\right] .
\end{aligned}
$$

The MGWYD skew information $Q^{\alpha, \beta}(\rho, K)$ is also convex in $\rho$ when $\alpha, \beta \in[0,1]$ with $\alpha+2 \beta \leq 1$ and $2 \alpha+\beta \leq 1$ [25]. By using the Lieb's theorem [16,55], it can be easily obtained that $C^{\alpha, \beta}(\rho, K)$ is concave in $\rho$ for $\alpha, \beta \geq 0, \alpha+\beta \leq 1$. These facts demonstrate that $Q^{\alpha, \beta}(\rho, K)$ and $C^{\alpha, \beta}(\rho, K)$ could serve as the quantifiers of quantum and classical uncertainty, respectively.

Consider a quantum channel $\Phi$ given by Kraus operators $\left\{K_{i}\right\}, \Phi(\rho)=\sum_{i} K_{i} \rho K_{i}^{\dagger}$. We introduce the total uncertainty of the quantum channel $\Phi$ based on MGV,

$$
\begin{equation*}
V^{\alpha, \beta}(\rho, \Phi)=\sum_{i} V^{\alpha, \beta}\left(\rho, K_{i}\right), \quad \alpha, \beta \geq 0, \alpha+\beta \leq 1 . \tag{5}
\end{equation*}
$$

By using Eq. (1), $V^{\alpha, \beta}(\rho, \Phi)$ can be further rewritten as,

$$
\begin{align*}
V^{\alpha, \beta}(\rho, \Phi) & =\frac{1+\sum_{i} \operatorname{tr} \rho^{\alpha+\beta} K_{i} \rho^{1-\alpha-\beta} K_{i}^{\dagger}}{2}-\sum_{i}\left|\operatorname{tr} \rho K_{i}\right|^{2} \\
& =\frac{1+\operatorname{tr} \rho^{\alpha+\beta} \Phi\left(\rho^{1-\alpha-\beta}\right)}{2}-\sum_{i}\left|\operatorname{tr} \rho K_{i}\right|^{2}, \quad \alpha, \beta \geq 0, \alpha+\beta \leq 1 . \tag{6}
\end{align*}
$$

When $\alpha+\beta=1$, $V^{\alpha, \beta}(\rho, \Phi)$ reduces to $V(\rho, \Phi)$, and Eq. (6) reduces to the following equation in [9], i.e.,

$$
\begin{equation*}
V(\rho, \Phi)=\frac{1+\operatorname{tr} \rho \Phi(\mathbf{1})}{2}-\sum_{i}\left|\operatorname{tr} \rho K_{i}\right|^{2} \tag{7}
\end{equation*}
$$

where $\mathbf{1}$ is the identity operator and $\Phi(\mathbf{1})=\sum_{i} K_{i} K_{i}^{\dagger}$.
Let $\left\{E_{i}\right\}$ and $\left\{F_{j}\right\}$ be two sets of Kraus operators of the quantum channel $\Phi$. Then there exists a unitary matrix $U=\left(u_{i j}\right)$ such that $E_{i}=\sum_{j} u_{i j} F_{j}$ for any $i$ [16]. From (6) we can easily check that $V^{\alpha, \beta}(\rho, \Phi)$ is independent of the choice of the Kraus operators of $\Phi$, namely, $V^{\alpha, \beta}(\rho, \Phi)$ is well-defined. For each $\alpha, \beta$ with $\alpha, \beta \geq 0, \alpha+\beta \leq 1$, we can prove that the uncertainty $V^{\alpha, \beta}(\rho, \Phi)$ has the following elegant properties:
(i) (Non-negativity) $V^{\alpha, \beta}(\rho, \Phi) \geq 0$, with the equality holds if and only if $\rho^{\frac{\alpha+\beta}{2}} K_{i} \rho^{\frac{1-\alpha-\beta}{2}}$ $=K_{i} \sqrt{\rho}=\left(\operatorname{tr} \rho K_{i}\right) \sqrt{\rho}$ for any $i$.
(ii) (Linearity) $V^{\alpha, \beta}(\rho, \Phi)$ is positive-real-linear with respect to the channel $\Phi$, i.e., $V^{\alpha, \beta}\left(\rho, \lambda_{1} \Phi_{1}+\lambda_{2} \Phi_{2}\right)=\lambda_{1} V^{\alpha, \beta}\left(\rho, \Phi_{1}\right)+\lambda_{2} V^{\alpha, \beta}\left(\rho, \Phi_{2}\right)$ for any $\lambda_{1}, \lambda_{2} \geq 0$ and any quantum channels $\Phi_{1}$ and $\Phi_{2}$.
(iii) (Concavity) $V^{\alpha, \beta}(\rho, \Phi)$ is concave with respect to $\rho$, i.e., $V^{\alpha, \beta}\left(\sum_{j} \lambda_{j} \rho_{j}, \Phi\right) \geq$ $\sum_{j} \lambda_{j} V^{\alpha, \beta}\left(\rho_{j}, \Phi\right)$, where $\lambda_{j} \geq 0$ for each $j$ with $\sum_{j} \lambda_{j}=1$.
(iv) (Unitary invariance) $V^{\alpha, \beta}\left(U \rho U^{\dagger}, U \Phi U^{\dagger}\right)=V^{\alpha, \beta}(\rho, \Phi)$ for any unitary operators $U$, where $U \Phi U^{\dagger}(\rho)=\sum_{i}\left(U K_{i} U^{\dagger}\right) \rho\left(U K_{i} U^{\dagger}\right)^{\dagger}$ with $\Phi(\rho)=\sum_{i} K_{i} \rho K_{i}^{\dagger}$.
(v) (Ancillary independence) $V^{\alpha, \beta}\left(\rho^{A} \otimes \rho^{B}, \Phi^{A} \otimes \mathcal{I}^{B}\right)=V^{\alpha, \beta}\left(\rho^{A}, \Phi^{A}\right)$, where $\rho^{A}$ and $\rho^{B}$ are any states of systems $A$ and $B$, respectively, and $\mathcal{I}^{B}$ is the identity channel on system $B$.
(vi) (Increasing under partial trace) $V^{\alpha, \beta}\left(\rho^{A B}, \Phi^{A} \otimes \mathcal{I}^{B}\right) \leq V^{\alpha, \beta}\left(\rho^{A}, \Phi^{A}\right)$, where $\rho^{A B}$ is any bipartite state on joint system $A B,\left(\rho^{A}\right)^{\alpha+\beta}=\operatorname{tr}_{B}\left[\left(\rho^{A B}\right)^{\alpha+\beta}\right]$ is the reduced state on system $A$, and $\mathcal{I}^{B}$ is the identity channel on system $B$.
(vii) (Subadditivity) $V^{\alpha, \beta}\left(\rho^{A B}, \Phi^{A} \otimes \mathcal{I}^{B}+\mathcal{I}^{A} \otimes \Phi^{B}\right) \leq V^{\alpha, \beta}\left(\rho^{A}, \Phi^{A}\right)+V^{\alpha, \beta}\left(\rho^{B}, \Phi^{B}\right)$, where $\left(\rho^{A}\right)^{\alpha+\beta}=\operatorname{tr}_{B}\left[\left(\rho^{A B}\right)^{\alpha+\beta}\right]$ and $\left(\rho^{B}\right)^{\alpha+\beta}=\operatorname{tr}_{A}\left[\left(\rho^{A B}\right)^{\alpha+\beta}\right]$ are the reduced states with respect to the subsystems $A$ and $B, \Phi^{A}$ and $\Phi^{B}$ are the channels on systems $A$ and $B$, and $\mathcal{I}^{A}$ and $\mathcal{I}^{B}$ are the identity channels on systems $A$ and $B$, respectively.

The above properties can be proved in the following way. From the definition, $V^{\alpha, \beta}(\rho, \Phi) \geq 0$ is obvious. $V^{\alpha, \beta}(\rho, \Phi)=0$ if and only if $V^{\alpha, \beta}\left(\rho, K_{i}\right)=0$ for any $i$,
i.e., $\operatorname{tr}\left(\rho^{\alpha+\beta} K_{i 0} \rho^{1-\alpha-\beta} K_{i 0}^{\dagger}\right)=\operatorname{tr}\left(\rho K_{i 0}^{\dagger} K_{i 0}\right)=0$, which is equivalent to $\rho^{\frac{\alpha+\beta}{2}} K_{i} \rho^{\frac{1-\alpha-\beta}{2}}=$ $K_{i \sqrt{\rho}}=\left(\operatorname{tr} \rho K_{i}\right) \sqrt{\rho}$. Therefore, item (i) is proved.

By rewriting (6) as

$$
V^{\alpha, \beta}(\rho, \Phi)=\frac{1+\sum_{i} \operatorname{tr} \rho^{\alpha+\beta} K_{i} \rho^{1-\alpha-\beta} K_{i}^{\dagger}}{2}-\sum_{l, m=1}^{d} \operatorname{tr}[\Phi(|l\rangle\langle m| \rho)|m\rangle\langle l| \rho],
$$

where $\alpha, \beta \geq 0, \alpha+\beta \leq 1$ and $\{|l\rangle\}_{l=1}^{d}$ is an orthonormal basis in $\mathcal{H}$, it is not difficult to show that $V^{\alpha, \beta}(\rho, \Phi)$ is positive-real-linear in $\Phi$. Hence, item (ii) holds.

Let $X$ be a matrix, and $0 \leq t \leq 1$. By using the Lieb's theorem $[16,55]$, the function $f(A, B)=\operatorname{tr}\left(X^{\dagger} A^{t} X B^{1-t}\right)$ is jointly concave in positive matrices $A$ and $B$, which implies that $f\left(\sum_{j} \lambda_{j} A_{j}, \sum_{j} \lambda_{j} B_{j}\right) \geq \sum_{j} \lambda_{j} f\left(A_{j}, B_{j}\right)$, where $\lambda_{j} \geq 0$ with $\sum_{j} \lambda_{j}=1$, and $A_{j}$, $B_{j}$ be positive matrices for each $j$. Taking $A_{j}=B_{j}=\rho_{j}, t=\alpha+\beta$, and $X=K_{i}$, we obtain $\operatorname{tr}\left[\left(\sum_{j} \lambda_{j} \rho_{j}\right)^{\alpha+\beta} K_{i}\left(\sum_{j} \lambda_{j} \rho_{j}\right)^{1-\alpha-\beta} K_{i}^{\dagger}\right] \geq \sum_{j} \lambda_{j} \operatorname{tr} \rho_{j}^{\alpha+\beta} K_{i} \rho_{j}^{1-\alpha-\beta} K_{i}^{\dagger}$ for each i. Note that $\left|\operatorname{tr}\left(\sum_{j} \lambda_{j} \rho_{j}\right) K_{i}\right|^{2} \leq \sum_{j} \lambda_{j}\left|\operatorname{tr} \rho_{j} K_{i}\right|^{2}$ holds for each $i$. Summing over $i$ on both sides of the two inequalities, item (iii) follows immediately.

Noting that $\left(U \rho U^{\dagger}\right)^{\alpha+\beta}=U \rho^{\alpha+\beta} U^{\dagger}$ and $\left(U \rho U^{\dagger}\right)^{1-\alpha-\beta}=U \rho^{1-\alpha-\beta} U^{\dagger}$ for $\alpha, \beta \geq 0$ with $\alpha+\beta \leq 1$, by Eq. (6) and the cyclicity of the trace, item (iv) can be derived.

Suppose that $\Phi^{A}(\rho)=\sum_{i} K_{i}^{A} \rho K_{i}^{A^{\dagger}}$. Direct calculation shows that

$$
\begin{aligned}
& \operatorname{tr}\left(\rho^{A} \otimes \rho^{B}\right)^{\alpha+\beta}\left(\Phi^{A} \otimes \mathcal{I}^{B}\right)\left(\left(\rho^{A} \otimes \rho^{B}\right)^{1-\alpha-\beta}\right) \\
= & \operatorname{tr}\left(\left(\rho^{A}\right)^{\alpha+\beta} \otimes\left(\rho^{B}\right)^{\alpha+\beta}\right)\left(\Phi^{A}\left(\left(\rho^{A}\right)^{1-\alpha-\beta}\right) \otimes\left(\rho^{B}\right)^{1-\alpha-\beta}\right) \\
= & \operatorname{tr}\left(\left(\rho^{A}\right)^{\alpha+\beta}\left(\Phi^{A}\left(\left(\rho^{A}\right)^{1-\alpha-\beta}\right) \otimes \rho^{B}\right)\right. \\
= & \operatorname{tr}\left(\rho^{A}\right)^{\alpha+\beta} \Phi^{A}\left(\left(\rho^{A}\right)^{1-\alpha-\beta}\right),
\end{aligned}
$$

and $\operatorname{tr}\left(\rho^{A} \otimes \rho^{B}\right)\left(K_{i}^{A} \otimes \mathbf{1}^{B}\right)=\operatorname{tr}\left(\rho^{A} K_{i}^{A} \otimes \rho^{B}\right)=\operatorname{tr}\left(\rho^{A} K_{i}^{A}\right)$. Thus, it follows from Eq. (6) that item(v) holds.

For item (vi), let $W$ be any operator on $\mathcal{H}_{A} \otimes \mathcal{H}_{B}$, then $\operatorname{tr}\left(\left(F_{A} \otimes \mathbf{1}_{B}\right) W\right)=\operatorname{tr}\left(F_{A}\right.$. $\left.\operatorname{tr}_{B} W\right)$ for any $F_{A}$ on $\mathcal{H}_{A}$, where $\mathbf{1}_{\mathbf{B}}$ denotes the identity operator on $\mathcal{H}_{B}$ [56]. Thus

$$
\begin{aligned}
& V^{\alpha, \beta}\left(\rho^{A B}, \Phi^{A} \otimes \mathcal{I}^{B}\right) \\
= & \frac{1+\sum_{i} \operatorname{tr}\left(\rho^{A B}\right)^{\alpha+\beta}\left(K_{i}^{A} \otimes \mathbf{1}_{B}\right)\left(\rho^{A B}\right)^{1-\alpha-\beta}\left(K_{i}^{A^{\dagger}} \otimes \mathbf{1}_{B}\right)}{2}-\sum_{i}\left|\operatorname{tr} \rho^{A B}\left(K_{i}^{A} \otimes \mathbf{1}_{B}\right)\right|^{2} \\
\leq & \frac{1+\sum_{i} \operatorname{tr}\left(\rho^{A}\right)^{\alpha+\beta} K_{i}^{A}\left(\rho^{A}\right)^{1-\alpha-\beta} K_{i}^{A^{\dagger}}}{2}-\sum_{i}\left|\operatorname{tr} \rho^{A} K_{i}^{A^{\dagger}}\right|^{2} \\
= & V^{\alpha, \beta}\left(\rho^{A}, \Phi^{A}\right), \quad \alpha, \beta \geq 0, \alpha+\beta \leq 1,
\end{aligned}
$$

where $K_{i}^{A^{\dagger}}$ are the Kraus operators of the channel $\Phi^{A}$, and the last inequality follows from Lieb's concavity theorem [55].

Combining items (ii) and (vi), item (vii) follows immediately.
The quantum uncertainty of $\Phi$ in $\rho$ is defined by

$$
\begin{equation*}
Q^{\alpha, \beta}(\rho, \Phi)=\sum_{i} Q^{\alpha, \beta}\left(\rho, K_{i}\right), \quad \alpha, \beta \geq 0, \alpha+\beta \leq 1 \tag{8}
\end{equation*}
$$

By using Eq. (3), $Q^{\alpha, \beta}(\rho, \Phi)$ can be further rewritten as,

$$
\begin{align*}
Q^{\alpha, \beta}(\rho, \Phi) & =\frac{1}{2} \sum_{i}\left(\operatorname{tr} \rho K_{i}^{\dagger} K_{i}-\operatorname{tr} \rho^{\alpha} K_{i} \rho^{1-\alpha} K_{i}^{\dagger}-\operatorname{tr} \rho^{\beta} K_{i} \rho^{1-\beta} K_{i}^{\dagger}+\operatorname{tr} \rho^{\alpha+\beta} K_{i} \rho^{1-\alpha-\beta} K_{i}^{\dagger}\right) \\
& =\frac{1}{2}\left[1-\operatorname{tr} \rho^{\alpha} \Phi\left(\rho^{1-\alpha}\right)-\operatorname{tr} \rho^{\beta} \Phi\left(\rho^{1-\beta}\right)+\operatorname{tr} \rho^{\alpha+\beta} \Phi\left(\rho^{1-\alpha-\beta}\right)\right] \tag{9}
\end{align*}
$$

where $\alpha, \beta \geq 0, \alpha+\beta \leq 1$, which could be viewed as a family of coherence measures with respect to a channel $\Phi$ under certain restrictive conditions [25]. When $\alpha=\beta=\frac{1}{2}$, $Q^{\alpha, \beta}(\rho, \Phi)$ reduces to

$$
Q(\rho, \Phi)=\sum_{i} Q\left(\rho, K_{i}\right)=\frac{1+\operatorname{tr} \rho \Phi(\mathbf{1})}{2}-\operatorname{tr} \sqrt{\rho} \Phi(\sqrt{\rho})
$$

which has been introduced in [9,53] (up to a constant factor in [53]). Based on the interpretation of $Q^{\alpha, \beta}(\rho, K), Q^{\alpha, \beta}(\rho, \Phi)$ can be interpreted as a family of quantifiers of the information content of channels skew to $\Phi$ in the state $\rho$.

The quantity $Q(\rho, \Phi)$ (denoted as $I(\rho, \Phi)$ in [53]), which arises naturally from algebraic and geometric manipulation of state-channel interaction, has intrinsic informationaltheoretical meaning. In fact, it can be intuitively interpreted as the asymmetry, coherence, noncommutativity, quantumness and quantum uncertainty (of state $\rho$ with respect to channel $\Phi$ ) [53]. By replacing the commutator with anti-commutator in $Q(\rho, \Phi)$, the quantity $J(\rho, \Phi)$, the dual to $Q(\rho, \Phi)(I(\rho, \Phi))$, has also been defined and interpreted as the symmetry, incoherence, commutativity, classicality and classical uncertainty (of state $\rho$ with respect to channel $\Phi$ ) [53]. The symmetry-asymmetry complementarity relations have been derived and applied to quantification of the degree of symmetry and waveparticle duality $[25,53]$. The generalized quantity $Q^{\alpha, \beta}(\rho, \Phi)$ can therefore serve as a family of quantifiers of the measures, similar to the interpretations of $Q(\rho, \Phi)$. Note that in this paper, we follow the lines of [9] and identify $Q^{\alpha, \beta}(\rho, \Phi)$ with quantum uncertainty, whereas the classical uncertainty $C^{\alpha, \beta}(\rho, \Phi)$ is identified by fixing the difference of the total uncertainty (quantified by MGV $V^{\alpha, \beta}(\rho, \Phi)$ ) and the quantum uncertainty, which is a little different from the formulations in [53].

Remark 2.1 Note that $Q(\rho, \Phi)$ is ancillary independent, decreasing under partial trace and superadditive [53]. The ancillary independence (in a strong version, which is in fact invariance under partial trace) and additivity of $V(\rho, \Phi)$ has been proved in [9]. However, for $V^{\alpha, \beta}(\rho, \Phi)$ with $\alpha, \beta \geq 0$ and $\alpha+\beta \leq 1$, we can only prove that it is ancillary independent (in the sense of [53]), increasing under partial trace and subadditive.

We define the classical uncertainty of $\Phi$ in $\rho$ as

$$
\begin{equation*}
C^{\alpha, \beta}(\rho, \Phi)=\sum_{i} C^{\alpha, \beta}\left(\rho, K_{i}\right), \quad \alpha, \beta \geq 0, \alpha+\beta \leq 1 \tag{10}
\end{equation*}
$$

By using Eq. (4), $C^{\alpha, \beta}(\rho, \Phi)$ can be further rewritten as,

$$
\begin{align*}
C^{\alpha, \beta}(\rho, \Phi) & =\frac{1}{2} \sum_{i}\left(\operatorname{tr} \rho^{\alpha} K_{i} \rho^{1-\alpha} K_{i}^{\dagger}+\operatorname{tr} \rho^{\beta} K_{i} \rho^{1-\beta} K_{i}^{\dagger}\right)-\sum_{i}\left|\operatorname{tr} \rho K_{i}\right|^{2} \\
& =\frac{1}{2}\left[\operatorname{tr} \rho^{\alpha} \Phi\left(\rho^{1-\alpha}\right)+\operatorname{tr} \rho^{\beta} \Phi\left(\rho^{1-\beta}\right)\right]-\sum_{i}\left|\operatorname{tr} \rho K_{i}\right|^{2}, \tag{11}
\end{align*}
$$

where $\alpha, \beta \geq 0, \alpha+\beta \leq 1$. When $\alpha=\beta=\frac{1}{2}, C^{\alpha, \beta}(\rho, \Phi)$ reduces to $C(\rho, \Phi)$, and Eq. (11) reduces to the following equation in [9], i.e.,

$$
C(\rho, \Phi)=\sum_{i} C\left(\rho, K_{i}\right)=\operatorname{tr} \sqrt{\rho} \Phi(\sqrt{\rho})-\sum_{i}\left|\operatorname{tr} \rho K_{i}\right|^{2}
$$

It can also be verified that the quantities $Q^{\alpha, \beta}(\rho, \Phi)$ and $C^{\alpha, \beta}(\rho, \Phi)$ defined in Eqs. (8) and (10) are independent of the choice of the Kraus operators of $\Phi$. Consequently we have

$$
\begin{equation*}
V^{\alpha, \beta}(\rho, \Phi)=Q^{\alpha, \beta}(\rho, \Phi)+C^{\alpha, \beta}(\rho, \Phi), \quad \alpha, \beta \geq 0, \alpha+\beta \leq 1, \tag{12}
\end{equation*}
$$

which means that the total uncertainty of a quantum channel can also be decomposed into the quantum and classical counterparts. As is shown in [9], we can similarly verify that $C^{\alpha, \beta}(\rho, \Phi)=0$ when $\rho=|\psi\rangle\langle\psi|$ is a pure state, and in this case

$$
\begin{align*}
V^{\alpha, \beta}(\rho, \Phi)=Q^{\alpha, \beta}(\rho, \Phi) & =\frac{1}{2}\left(1-\sum_{i}\langle\psi| K_{i}|\psi\rangle\langle\psi| K_{i}^{\dagger}|\psi\rangle\right) \\
& =\frac{1}{2}\left(1-\sum_{i}\left\langle K_{i}\right\rangle\left\langle K_{i}^{\dagger}\right\rangle\right) \\
& =\frac{1}{2}\left(1-\sum_{i}\left|\left\langle K_{i}\right\rangle\right|^{2}\right) \tag{13}
\end{align*}
$$

where $\left\langle K_{i}\right\rangle$ denotes the expectation value of $K_{i}$ for each $i$. Suppose that $\rho$ is a state on quantum system $B$ and $A$ is an ancillary system. Define a pure state $\left|\psi^{A B}\right\rangle$ for the joint system $A B$ such that $\operatorname{tr}_{A}\left|\psi^{A B}\right\rangle\left\langle\psi^{A B}\right|=\rho$. By Eq. (13) and property (vi) of $V^{\alpha, \beta}(\rho, \Phi)$, we have

$$
V^{\alpha, \beta}(\rho, \Phi) \geq V^{\alpha, \beta}\left(\left|\psi^{A B}\right\rangle\left\langle\psi^{A B}\right|, \mathcal{I}^{A} \otimes \Phi\right)=Q^{\alpha, \beta}\left(\left|\psi^{A B}\right\rangle\left\langle\psi^{A B}\right|, \mathcal{I}^{A} \otimes \Phi\right)
$$

It shows that the total uncertainty of a quantum channel $\Phi$ in a state $\rho$ is no less than the quantum uncertainty of this channel in the corresponding purified states $\left|\psi^{A B}\right\rangle$.

## 3 A trade-off relation between MGV-based total uncertainty of quantum channels and entanglement fidelity

In this section, we first recall the concepts of entanglement fidelity, entropy exchange, and coherent information. We then derive a trade-off relation between the MGV-based total uncertainty of quantum channels and the entanglement fidelity. Furthermore, we exploit this trade-off relation to explore the relations between the MGV-based total uncertainty of quantum channels and entropy exchange/coherent information.

For a quantum state $\rho$ and a quantum channel $\Phi$ on system $B$, the entanglement fidelity is defined as $[16,51]$

$$
\begin{align*}
F_{e}(\rho, \Phi) & =F\left(\left|\psi^{A B}\right\rangle, \mathcal{I}^{A} \otimes \Phi\left(\left|\psi^{A B}\right\rangle\left\langle\psi^{A B}\right|\right)\right)^{2} \\
& =\left\langle\psi^{A B}\right| \mathcal{I}^{A} \otimes \Phi\left(\left|\psi^{A B}\right\rangle\left\langle\psi^{A B}\right|\right)\left|\psi^{A B}\right\rangle, \tag{14}
\end{align*}
$$

where $A$ is an auxiliary system, $\left|\psi^{A B}\right\rangle$ is a purification of $\rho$ satisfying $\operatorname{tr}_{A}\left|\psi^{A B}\right\rangle\left\langle\psi^{A B}\right|=\rho$, and $F\left(\sigma_{1}, \sigma_{2}\right)=\operatorname{tr} \sqrt{\sqrt{\sigma_{1}} \sigma_{2} \sqrt{\sigma_{1}}}$ is the quantum fidelity between quantum states $\sigma_{1}$ and $\sigma_{2}$. The entanglement fidelity provides a quantification of to what extent the entanglement of $\left|\psi^{A B}\right\rangle$ can be preserved under the quantum channel $\Phi . F_{e}(\rho, \Phi)$ does not depend on the ways of purification and can be rewritten as [16],

$$
\begin{equation*}
F_{e}(\rho, \Phi)=\sum_{i}\left|\operatorname{tr} \rho K_{i}\right|^{2} . \tag{15}
\end{equation*}
$$

For a quantum state $\rho$ and a quantum channel $\Phi$ on system $B$, the entropy exchange is defined as [51]

$$
\begin{equation*}
S_{e}(\rho, \Phi)=S\left(\rho^{A^{\prime} B^{\prime}}\right)=-\operatorname{tr} \rho^{A^{\prime} B^{\prime}} \log \rho^{A^{\prime} B^{\prime}} \tag{16}
\end{equation*}
$$

where $\rho^{A^{\prime} B^{\prime}}=\mathcal{I}^{A} \otimes \Phi\left(\left|\psi^{A B}\right\rangle\left\langle\psi^{A B}\right|\right),\left|\psi^{A B}\right\rangle$ is a purification of $\rho$ with auxiliary system $A$, $S(\sigma)=-\operatorname{tr} \sigma \log \sigma$ is the von Neumann entropy of a quantum state $\sigma$, and the logarithm ' $\log$ ' is taken to be base 2. The entropy exchange quantifies the amount of information exchanged between system $B$ and the environment under the action of channel $\Phi$. Correspondingly, the coherent information is defined as [52]

$$
\begin{equation*}
I_{c}(\rho, \Phi)=S\left(\rho^{\prime}\right)-S_{e}=S\left(\rho^{\prime}\right)-S\left(\rho^{A^{\prime} B^{\prime}}\right), \tag{17}
\end{equation*}
$$

where $\rho^{\prime}=\Phi(\rho)$ is the state of system $B$ under the action of channel $\Phi . I_{c}(\rho, \Phi)$ identifies how much quantum information is transmitted when the quantum channel is applied.

Utilizing Eqs. (6) and (15), we obtain the following trade-off relation between the MGV-based total uncertainty of quantum channels and the entanglement fidelity,

$$
\begin{align*}
V^{\alpha, \beta}(\rho, \Phi)+F_{e}(\rho, \Phi) & =\frac{1+\operatorname{tr} \rho^{\alpha+\beta} \Phi\left(\rho^{1-\alpha-\beta}\right)}{2} \\
& \leq \frac{1+\lambda_{\max }\left(\Phi\left(\rho^{1-\alpha-\beta}\right)\right) \operatorname{tr} \rho^{\alpha+\beta}}{2} \tag{18}
\end{align*}
$$

where $\alpha, \beta \geq 0, \alpha+\beta \leq 1$, and $\lambda_{\max }(\cdot)$ denotes the maximum spectrum of a matrix. This trade-off relation is an extension of Eq. (4) in [9], showing that the entanglement
fidelity $F_{e}(\rho, \Phi)$ cannot be too large if the total uncertainty of $\Phi$ in $\rho$ is large. We find that if the nonzero eigenvalues of $\Phi\left(\rho^{1-\alpha-\beta}\right)$ are all equal, then the inequality (18) is saturated. Note that if $\alpha+\beta=1$ and $\Phi$ is unital, we get

$$
V(\rho, \Phi)+F_{e}(\rho, \Phi)=1,
$$

which has been established in Ref. [9], and may be viewed as a demonstration of certain information conservation.

For pure states $\rho=|\psi\rangle\langle\psi|$, we have $\left.\operatorname{tr} \rho^{\alpha+\beta} \Phi\left(\rho^{1-\alpha-\beta}\right)=\sum_{i}\left|\langle\psi| K_{i}\right| \psi\right\rangle\left.\right|^{2}=\sum_{i}\left|\operatorname{tr} \rho K_{i}\right|^{2}$ $=F_{e}(\rho, \Phi)$. Thus Eq. (18) can be rewritten as the following trade-off relation,

$$
2 V^{\alpha, \beta}(|\psi\rangle\langle\psi|, \Phi)+F_{e}(|\psi\rangle\langle\psi|, \Phi)=1 .
$$

It can be seen that the total uncertainty $V^{\alpha, \beta}(|\psi\rangle\langle\psi|, \Phi)$ (or equivalently, the quantum uncertainty $Q^{\alpha, \beta}(|\psi\rangle\langle\psi|, \Phi)$, which is due to Eq. (13)) characterizes the information loss associated with the entanglement of the initial purified state $\left|\psi^{A B}\right\rangle$.

We now apply these trade-off relations to deduce the connections between the total uncertainty of quantum channels and the entropy exchange/coherent information as a consequence of the above trade-off relation. Recall that the quantum Fano inequality is $S_{e}(\rho, \Phi) \leq H\left(F_{e}(\rho, \Phi)\right)+\left(1-F_{e}(\rho, \Phi)\right) \log \left(d^{2}-1\right)$, where $H(\cdot)$ is the binary Shannon entropy, i.e., $H(p)=-p \log p-(1-p) \log (1-p)$ for $0 \leq p \leq 1$ [16]. Combining this inequality, Eq. (18), and the facts that $H\left(F_{e}(\rho, \Phi)\right) \leq 1$ and $\log \left(d^{2}-1\right) \leq 2 \log d$, we have

$$
\begin{aligned}
S_{e}(\rho, \Phi) & \leq H\left(\frac{1+\operatorname{tr} \rho^{\alpha+\beta} \Phi\left(\rho^{1-\alpha-\beta}\right)-2 V^{\alpha, \beta}(\rho, \Phi)}{2}\right) \\
& +\left(\frac{2 V^{\alpha, \beta}(\rho, \Phi)+1-\operatorname{tr} \rho^{\alpha+\beta} \Phi\left(\rho^{1-\alpha-\beta}\right)}{2}\right) \log \left(d^{2}-1\right) \\
& \leq 1+\left(2 V^{\alpha, \beta}(\rho, \Phi)+1-\operatorname{tr} \rho^{\alpha+\beta} \Phi\left(\rho^{1-\alpha-\beta}\right)\right) \log d,
\end{aligned}
$$

where $\alpha, \beta \geq 0$ and $\alpha+\beta \leq 1$. Therefore, the MGV-based total uncertainty of quantum channels $V^{\alpha, \beta}(\rho, \Phi)$ can serve as an upper bound of the entropy exchange $S_{e}(\rho, \Phi)$,

$$
\begin{equation*}
1+\left(2 V^{\alpha, \beta}(\rho, \Phi)+1-\operatorname{tr} \rho^{\alpha+\beta} \Phi\left(\rho^{1-\alpha-\beta}\right)\right) \log d \geq S_{e}(\rho, \Phi), \tag{19}
\end{equation*}
$$

where $\alpha, \beta \geq 0, \alpha+\beta \leq 1$, and $d$ is the dimension of the system Hilbert space. This demonstrates that if the information exchange with environment is large, the total uncertainty of quantum channels cannot be arbitrarily small. Note that when $\alpha+\beta=1$ and $\Phi$ is unital, we get

$$
1+2 V(\rho, \Phi) \log d \geq S_{e}(\rho, \Phi)
$$

which has been established in Ref. [9].
The quantum Fano inequality reveals that if the entropy exchange for a process is large, then the entanglement fidelity for the process must necessarily be small, indicating
that the entanglement between $A$ and $B$ has not been well preserved [16]. This inequality is saturated if $\rho^{A^{\prime} B^{\prime}}=\operatorname{diag}\left(p_{1}, p_{2}, \cdots, p_{d^{2}}\right)$, where $p_{1}=F_{e}(\rho, \Phi)$ and $p_{i}=\left(1-p_{1}\right) /\left(d^{2}-1\right)$ for $i=2, \cdots, d^{2}[16]$. Note that $H\left(F_{e}(\rho, \Phi)\right)$ attains its maximum value of 1 at $F_{e}(\rho, \Phi)=$ $1 / 2$, while $\log \left(d^{2}-1\right) \leq 2 \log d$ cannot be saturated for any finite number $d$, but the values of $\log \left(d^{2}-1\right)$ and $2 \log d$ can be arbitrarily close when $d$ is sufficiently large. This implies that if $\rho^{A^{\prime} B^{\prime}}=\operatorname{diag}\left(1 / 2,1 / 2\left(d^{2}-1\right), \cdots, 1 / 2\left(d^{2}-1\right)\right)$ and $d$ is very large, we have

$$
S_{e}(\rho, \Phi)=1+\frac{1}{2} \log \left(d^{2}-1\right) \approx 1+\log d
$$

and in this case, $V^{\alpha, \beta}(\rho, \Phi)=\frac{1}{2} \operatorname{tr} \rho^{\alpha+\beta} \Phi\left(\rho^{1-\alpha-\beta}\right)$. Thus, we can say that Eq. (19) is saturated approximately in a high-dimensional Hilbert space (the dimension $d$ is large enough) when $\rho^{A^{\prime} B^{\prime}}=\operatorname{diag}\left(1 / 2,1 / 2\left(d^{2}-1\right), \cdots, 1 / 2\left(d^{2}-1\right)\right)$.

From Eqs. (17) and (19), we obtain

$$
1+\left(2 V^{\alpha, \beta}(\rho, \Phi)+1-\operatorname{tr} \rho^{\alpha+\beta} \Phi\left(\rho^{1-\alpha-\beta}\right)\right) \log d+I_{c}(\rho, \Phi) \geq S\left(\rho^{\prime}\right)
$$

where $\alpha, \beta \geq 0, \alpha+\beta \leq 1$.
Combining Eq. (19) and the above inequality, we obtain

$$
\begin{aligned}
& 2+2\left(2 V^{\alpha, \beta}(\rho, \Phi)+1-\operatorname{tr} \rho^{\alpha+\beta} \Phi\left(\rho^{1-\alpha-\beta}\right)\right) \log d+I_{c}(\rho, \Phi) \\
\geq & S_{e}(\rho, \Phi)+S\left(\rho^{\prime}\right) \\
\geq & S(\rho)+2 S_{e}(\rho, \Phi) \\
\geq & S(\rho),
\end{aligned}
$$

where the second inequality follows from the fact that $S\left(\rho^{\prime}\right)-S(\rho) \geq S_{e}(\rho, \Phi)[16,52]$, and we can further obtain

$$
\begin{equation*}
2\left(2 V^{\alpha, \beta}(\rho, \Phi)+1-\operatorname{tr} \rho^{\alpha+\beta} \Phi\left(\rho^{1-\alpha-\beta}\right)\right) \log d+I_{c}(\rho, \Phi) \geq S(\rho)-2, \tag{20}
\end{equation*}
$$

where $\alpha, \beta \geq 0, \alpha+\beta \leq 1$, which gives a relation between the MGV-based total uncertainty of quantum channels and the coherent information. This is a manifestation that the total uncertainty of quantum channels cannot be arbitrarily small when the coherent information is very small. Note that when $\alpha+\beta=1$ and $\Phi$ is unital, we get

$$
4 V(\rho, \Phi) \log d+I_{c}(\rho, \Phi) \geq S(\rho)-2
$$

which has been established in Ref. [9].
From the proof of Eq. (20), it can be easily seen that it is saturated approximately in a high-dimensional Hilbert space (the dimension $d$ is large enough) when $\rho^{A^{\prime} B^{\prime}}=$ $\operatorname{diag}\left(1 / 2,1 / 2\left(d^{2}-1\right), \cdots, 1 / 2\left(d^{2}-1\right)\right)$ and $\Phi$ preserves the von Neumann entropy (i.e., $\left.S\left(\rho^{\prime}\right)=S(\rho)\right)$.

## 4 Examples and an experimental protocol

In this section, we calculate the total uncertainty $V^{\alpha, \beta}(\rho, \Phi)$ and the quantum uncertainty $Q^{\alpha, \beta}(\rho, \Phi)$ for several typical quantum channels, and illustrate the trade-off relation (18), inequalities (19) and (20) for specific channels and states.

A qubit state can be written as $\rho=\frac{1}{2}(\mathbf{1}+\mathbf{r} \cdot \boldsymbol{\sigma})$, where $\mathbf{r}=\left(r_{1}, r_{2}, r_{3}\right)$ is the Bloch vector satisfying $|\mathbf{r}| \leq 1, \boldsymbol{\sigma}=\left(\sigma_{1}, \sigma_{2}, \sigma_{3}\right)$ with $\sigma_{j}(j=1,2,3)$ the Pauli matrices, and $\mathbf{r} \cdot \boldsymbol{\sigma}=\sum_{j=1}^{3} r_{j} \sigma_{j}$. Denote $r=|\mathbf{r}|$. Then the eigenvalues of $\rho$ are $\lambda_{1,2}=(1 \mp r) / 2$ and [57]

$$
\rho^{\kappa}=\left(\begin{array}{cc}
\frac{\lambda_{1}^{\kappa}+\lambda_{2}^{\kappa}}{2}+\frac{r_{3}\left(\lambda_{2}^{\kappa}-\lambda_{1}^{\kappa}\right)}{2} & \frac{\left(-r_{1}+i r_{2}\right)\left(\lambda_{1}^{\kappa}-\lambda_{2}^{\kappa}\right)}{2 r} \\
\frac{\left(-r_{1}-i r_{2}\right)\left(\lambda_{1}^{\kappa}-\lambda_{2}^{\kappa}\right)}{2 r} & \frac{\lambda_{1}^{\kappa}+\lambda_{2}^{\kappa}}{2}-\frac{r_{3}\left(\lambda_{2}^{\kappa}-\lambda_{1}^{\kappa}\right)}{2 r}
\end{array}\right) .
$$

Example 1 Consider the amplitude damping channel $\Phi(\rho)=\sum_{i=1}^{2} K_{i} \rho K_{i}^{\dagger}$ with the Kraus operators

$$
K_{1}=\left(\begin{array}{cc}
1 & 0 \\
0 & \sqrt{1-p}
\end{array}\right), \quad K_{2}=\left(\begin{array}{cc}
0 & \sqrt{p} \\
0 & 0
\end{array}\right), \quad 0 \leq p \leq 1 .
$$

The total uncertainty and quantum uncertainty of $\Phi$ for the qubit state $\rho$ are

$$
\begin{align*}
& V^{\alpha, \beta}(\rho, \Phi) \\
= & {\left[\frac{1}{4}\left(\lambda_{1}^{\alpha+\beta}+\lambda_{2}^{\alpha+\beta}\right)\left(\lambda_{1}^{1-\alpha-\beta}+\lambda_{2}^{1-\alpha-\beta}\right)-\frac{p r^{2}}{4}+\frac{(p+\sqrt{1-p}-1)}{2} r_{3}^{2}\right.} \\
& -\frac{2 p r_{3}-p+2 \sqrt{1-p}}{4}+\frac{p r_{3}}{4 r}\left(\lambda_{1}^{1-\alpha-\beta} \lambda_{2}^{\alpha+\beta}-\lambda_{2}^{1-\alpha-\beta} \lambda_{1}^{\alpha+\beta}+\lambda_{2}-\lambda_{1}\right) \\
& \left.+\frac{\left[2-p-p r_{3}^{2}+2 \sqrt{1-p}\left(r^{2}-r_{3}^{2}\right)\right]}{8 r^{2}}\left(\lambda_{1}^{\alpha+\beta}-\lambda_{2}^{\alpha+\beta}\right)\left(\lambda_{1}^{1-\alpha-\beta}-\lambda_{2}^{1-\alpha-\beta}\right)\right] \tag{21}
\end{align*}
$$

and

$$
\begin{align*}
Q^{\alpha, \beta}(\rho, \Phi)= & \frac{1}{2}\left[\frac{(1-\sqrt{1-p})\left(r_{1}^{2}+r_{2}^{2}\right)+p r_{3}^{2}}{2 r^{2}}\left(\lambda_{1}^{1-\alpha-\beta}+\lambda_{2}^{1-\alpha-\beta}\right)\right. \\
& \left.+\frac{p r_{3}}{2 r}\left(\lambda_{1}^{1-\alpha-\beta}-\lambda_{2}^{1-\alpha-\beta}\right)\right]\left(\lambda_{1}^{\alpha}-\lambda_{2}^{\alpha}\right)\left(\lambda_{1}^{\beta}-\lambda_{2}^{\beta}\right), \tag{22}
\end{align*}
$$

respectively, where $\alpha, \beta \geq 0$ and $\alpha+\beta \leq 1$.
Example 2 Consider the phase damping channel $\Phi(\rho)=\sum_{i=1}^{2} K_{i} \rho K_{i}^{\dagger}$ with the Kraus operators

$$
K_{1}=\left(\begin{array}{cc}
1 & 0 \\
0 & \sqrt{1-p}
\end{array}\right), \quad K_{2}=\left(\begin{array}{cc}
0 & 0 \\
0 & \sqrt{p}
\end{array}\right), \quad 0 \leq p \leq 1 .
$$

The total uncertainty and quantum uncertainty of $\Phi$ for the qubit state $\rho$ are

$$
\begin{align*}
& V^{\alpha, \beta}(\rho, \Phi) \\
= & {\left[\frac{1}{4}\left(\lambda_{1}^{\alpha+\beta}+\lambda_{2}^{\alpha+\beta}\right)\left(\lambda_{1}^{1-\alpha-\beta}+\lambda_{2}^{1-\alpha-\beta}\right)-\frac{\left[\sqrt{1-p}+(1-\sqrt{1-p}) r_{3}^{2}\right]}{2}\right.} \\
& \left.+\frac{\left[(1-\sqrt{1-p}) r_{3}^{2}+\sqrt{1-p} r^{2}\right]}{4 r^{2}}\left(\lambda_{1}^{\alpha+\beta}-\lambda_{2}^{\alpha+\beta}\right)\left(\lambda_{1}^{1-\alpha-\beta}-\lambda_{2}^{1-\alpha-\beta}\right)\right] \tag{23}
\end{align*}
$$

and

$$
\begin{align*}
& Q^{\alpha, \beta}(\rho, \Phi) \\
= & \frac{1}{2} \frac{(1-\sqrt{1-p})\left(r_{1}^{2}+r_{2}^{2}\right)}{2 r^{2}}\left(\lambda_{1}^{\alpha}-\lambda_{2}^{\alpha}\right)\left(\lambda_{1}^{\beta}-\lambda_{2}^{\beta}\right)\left(\lambda_{1}^{1-\alpha-\beta}+\lambda_{2}^{1-\alpha-\beta}\right), \tag{24}
\end{align*}
$$

respectively, where $\alpha, \beta \geq 0$ and $\alpha+\beta \leq 1$.
Example 3 For the depolarizing channel

$$
\Phi(\rho)=(1-3 p) \rho+p \sum_{j=1}^{3} \sigma_{j} \rho \sigma_{j}, \quad 0 \leq p \leq \frac{1}{3},
$$

the total uncertainty and quantum uncertainty of $\Phi$ for the qubit state $\rho$ are given by

$$
\begin{align*}
V^{\alpha, \beta}(\rho, \Phi)= & \frac{1}{4}\left[2+\left(\lambda_{1}^{\alpha+\beta}+\lambda_{2}^{\alpha+\beta}\right)\left(\lambda_{1}^{1-\alpha-\beta}+\lambda_{2}^{1-\alpha-\beta}\right)\right. \\
& \left.+(1-4 p)\left(\lambda_{1}^{\alpha+\beta}-\lambda_{2}^{\alpha+\beta}\right)\left(\lambda_{1}^{1-\alpha-\beta}-\lambda_{2}^{1-\alpha-\beta}\right)\right]-\left(1-3 p+p r^{2}\right) \tag{25}
\end{align*}
$$

and

$$
\begin{equation*}
Q^{\alpha, \beta}(\rho, \Phi)=p\left(\lambda_{1}^{\alpha}-\lambda_{2}^{\alpha}\right)\left(\lambda_{1}^{\beta}-\lambda_{2}^{\beta}\right)\left(\lambda_{1}^{1-\alpha-\beta}+\lambda_{2}^{1-\alpha-\beta}\right), \tag{26}
\end{equation*}
$$

respectively, where $\alpha, \beta \geq 0$ and $\alpha+\beta \leq 1$.
Example 4 The completely decoherent channel [58,59] is given by $\Phi(\rho)=M \circ \rho$, where $M$ is a correlation matrix which is positive semidefinite with all the diagonal elements 1 , and $\circ$ denotes the Hardamard product of matrices. For the following $2 \times 2$ correlation matrix,

$$
M=\left(\begin{array}{ll}
1 & \theta \\
\theta & 1
\end{array}\right), \quad-1 \leq \theta \leq 1,
$$

we have the total uncertainty and quantum uncertainty of $\Phi$ for the qubit state $\rho$,

$$
\begin{align*}
V^{\alpha, \beta}(\rho, \Phi)= & {\left[\frac{1}{4}\left(\lambda_{1}^{\alpha+\beta}+\lambda_{2}^{\alpha+\beta}\right)\left(\lambda_{1}^{1-\alpha-\beta}+\lambda_{2}^{1-\alpha-\beta}\right)-\frac{r_{3}^{2}+\theta-r_{3}^{2} \theta}{2}\right.} \\
& \left.+\frac{1}{r^{2}}\left(\lambda_{1}^{\alpha+\beta}-\lambda_{2}^{\alpha+\beta}\right)\left(\lambda_{1}^{1-\alpha-\beta}-\lambda_{2}^{1-\alpha-\beta}\right)\left(\theta r^{2}+r_{3}^{2}-\theta r_{3}^{2}\right)\right] \tag{27}
\end{align*}
$$

and

$$
\begin{align*}
Q^{\alpha, \beta}(\rho, \Phi)= & \frac{\left(\lambda_{1}^{\alpha+\beta}+\lambda_{2}^{\alpha+\beta}\right)\left(\lambda_{1}^{1-\alpha-\beta}+\lambda_{2}^{1-\alpha-\beta}\right)+2}{4} \\
& -\frac{\left(\lambda_{1}^{\alpha}+\lambda_{2}^{\alpha}\right)\left(\lambda_{1}^{1-\alpha}+\lambda_{2}^{1-\alpha}\right)+\left(\lambda_{1}^{\beta}+\lambda_{2}^{\beta}\right)\left(\lambda_{1}^{1-\beta}+\lambda_{2}^{1-\beta}\right)}{4} \\
& -\frac{r_{3}^{2}+\theta\left(r_{1}^{2}+r_{2}^{2}\right)}{4 r^{2}}\left[\left(\lambda_{1}^{\alpha}-\lambda_{2}^{\alpha}\right)\left(\lambda_{1}^{1-\alpha}-\lambda_{2}^{1-\alpha}\right)+\left(\lambda_{1}^{\beta}-\lambda_{2}^{\beta}\right)\left(\lambda_{1}^{1-\beta}-\lambda_{2}^{1-\beta}\right)\right] \\
& +\frac{r_{3}^{2}+\theta\left(r_{1}^{2}+r_{2}^{2}\right)}{4 r^{2}}\left(\lambda_{1}^{\alpha+\beta}-\lambda_{2}^{\alpha+\beta}\right)\left(\lambda_{1}^{1-\alpha-\beta}-\lambda_{2}^{1-\alpha-\beta}\right), \tag{28}
\end{align*}
$$

respectively, where $\alpha, \beta \geq 0$ and $\alpha+\beta \leq 1$.
In the next example we plot the total uncertainty $V^{\alpha, \beta}(\rho, \Phi)$ and quantum uncertainty $Q^{\alpha, \beta}(\rho, \Phi)$ for two kinds of important quantum states, the Werner and isotropic states.

Example 5 Fix an orthonormal basis $\{|l\rangle\}_{l=1}^{d}$ of a $d$-dimensional Hilbert space $\mathcal{H}$, $\{|l\rangle\langle m|\}_{l, m=1}^{d}$ is an orthonormal basis of $\mathcal{B}(\mathcal{H})$, with the Hilbert-Schmidt inner product $\left\langle A^{\dagger} \mid B\right\rangle:=\operatorname{tr} A B$. For simplicity, denote by $\left\{X_{i}\right\}_{i=1}^{d^{2}}$ the orthonormal basis of $\mathcal{B}(\mathcal{H})$. It can be verified that $\Phi(\rho)=\sum_{i=1}^{d^{2}} X_{i} \rho X_{i}^{\dagger} / d$ is a quantum channel. For this quantum channel we have

$$
\begin{equation*}
V^{\alpha, \beta}(\rho, \Phi)=\frac{1}{2 d}\left[d+\operatorname{tr} \rho^{\alpha+\beta} \operatorname{tr} \rho^{1-\alpha-\beta}-2 \operatorname{tr} \rho^{2}\right] \tag{29}
\end{equation*}
$$

and

$$
\begin{equation*}
Q^{\alpha, \beta}(\rho, \Phi)=\frac{1}{2 d}\left[d+\operatorname{tr} \rho^{1-\alpha-\beta} \operatorname{tr} \rho^{\alpha+\beta}-\operatorname{tr} \rho^{1-\alpha} \operatorname{tr} \rho^{\alpha}-\operatorname{tr} \rho^{1-\beta} \operatorname{tr} \rho^{\beta}\right], \tag{30}
\end{equation*}
$$

where $\alpha, \beta \geq 0$ and $\alpha+\beta \leq 1$.
Now we derive the total uncertainty and quantum uncertainty in Eqs. (29) and (30) for two classes of states, the Werner state and the isotropic state, respectively. First consider the Werner state,

$$
\rho_{w}=\left(\begin{array}{cccc}
\frac{1}{3} p & 0 & 0 & 0 \\
0 & \frac{1}{6}(3-2 p) & \frac{1}{6}(4 p-3) & 0 \\
0 & \frac{1}{6}(4 p-3) & \frac{1}{6}(3-2 p) & 0 \\
0 & 0 & 0 & \frac{1}{3} p
\end{array}\right)
$$

where $p \in[0,1]$. Note that $\rho_{w}$ is separable when $p \in\left[0, \frac{1}{3}\right]$. According to Eqs. (29) and (30), we obtain

$$
\begin{align*}
& V^{\alpha, \beta}\left(\rho_{w}, \Phi\right) \\
= & \frac{1}{8}\left[3+6 p-\frac{8 p^{2}}{3}+3^{\alpha+\beta} p^{1-\alpha-\beta}(1-p)^{\alpha+\beta}+3^{1-\alpha-\beta} p^{\alpha+\beta}(1-p)^{1-\alpha-\beta}\right] \tag{31}
\end{align*}
$$

and

$$
\begin{align*}
& Q^{\alpha, \beta}\left(\rho_{w}, \Phi\right) \\
= & \frac{1}{8}\left[3-2 p-3^{1-\alpha} p^{\alpha}(1-p)^{1-\alpha}-3^{\alpha} p^{1-\alpha}(1-p)^{\alpha}-3^{1-\beta} p^{\beta}(1-p)^{1-\beta}\right. \\
& \left.-3^{\beta} p^{1-\beta}(1-p)^{\beta}+3^{1-\alpha-\beta} p^{\alpha+\beta}(1-p)^{1-\alpha-\beta}+3^{\alpha+\beta} p^{1-\alpha-\beta}(1-p)^{\alpha+\beta}\right] . \tag{32}
\end{align*}
$$

Fig. 1 illustrates the values of $V^{\alpha, \beta}\left(\rho_{w}, \Phi\right)$ and $Q^{\alpha, \beta}\left(\rho_{w}, \Phi\right)$ given in Eqs. (31) and (32) with $\alpha=\frac{1}{5}$ and $\beta=\frac{3}{10}$. Direct calculation shows that $V^{\alpha, \beta}\left(\rho_{w}, \Phi\right)-Q^{\alpha, \beta}\left(\rho_{w}, \Phi\right)$, i.e., the classical uncertainty $C^{\alpha, \beta}\left(\rho_{w}, \Phi\right)$ reaches its maximum value of 0.9375 when $p=\frac{3}{4}, \alpha=0.120797$ and $\beta=0.650832$. Interestingly, the linear entropy (mixedness) $1-\operatorname{tr} \rho_{w}^{2}=2 p-\frac{4}{3} p^{2}$ of the Werner state attains its maximum value of $\frac{3}{4}$ when $p=\frac{3}{4}$.

In Fig. 2, we plot the surfaces of total uncertainty $V^{\alpha, \beta}\left(\rho_{w}, \Phi\right)$, quantum uncertainty $Q^{\alpha, \beta}\left(\rho_{w}, \Phi\right)$ in Eqs. (31) and (32), and their gap, the classical uncertainty $C^{\alpha, \beta}\left(\rho_{w}, \Phi\right)$, for fixed values of $p$.


Figure 1: The $y$-axis is the values of $V^{\alpha, \beta}\left(\rho_{w}, \Phi\right)$ and $Q^{\alpha, \beta}\left(\rho_{w}, \Phi\right)$. Dashed (solid) line represents the value of $V^{\alpha, \beta}\left(\rho_{w}, \Phi\right)\left(Q^{\alpha, \beta}\left(\rho_{w}, \Phi\right)\right)$ in Eq. (31) (Eq. (32)) with $\alpha=\frac{1}{5}$ and $\beta=\frac{3}{10}$. Note that the Werner state $\rho_{w}$ degenerates to a pure state when $p=0$ and in this case $V^{\alpha, \beta}\left(\rho_{w}, \Phi\right)=Q^{\alpha, \beta}\left(\rho_{w}, \Phi\right)$.


Figure 2: Surfaces of $V^{\alpha, \beta}\left(\rho_{w}, \Phi\right)$ and $Q^{\alpha, \beta}\left(\rho_{w}, \Phi\right)$ with fixed $p$ : (a) $p=\frac{1}{4}$; (b) $p=\frac{1}{2}$; (c) $p=\frac{3}{4}$; (d) $p=1$, where the red (blue) surface represents the value of $V^{\alpha, \beta}\left(\rho_{w}, \Phi\right)$ $\left(Q^{\alpha, \beta}\left(\rho_{w}, \Phi\right)\right)$ in Eq. (31) (Eq. (32)). Surfaces of the gap $C^{\alpha, \beta}\left(\rho_{w}, \Phi\right)$ in (a)-(d) with fixed $p:$ (e) $p=\frac{1}{4} ;(\mathbf{f}) p=\frac{1}{2} ;(\mathbf{g}) p=\frac{3}{4} ;(\mathbf{h}) p=1$, where the green surface represents the value of $C^{\alpha, \beta}\left(\rho_{w}, \Phi\right)$.

Now consider the isotropic state,

$$
\rho_{i s o}=\left(\begin{array}{cccc}
\frac{1}{6}(2 F+1) & 0 & 0 & \frac{1}{6}(4 F-1) \\
0 & \frac{1}{3}(1-F) & 0 & 0 \\
0 & 0 & \frac{1}{3}(1-F) & 0 \\
\frac{1}{6}(4 F-1) & 0 & 0 & \frac{1}{6}(2 F+1)
\end{array}\right) \text {, }
$$

where $F \in[0,1]$, which is separable when $F \in\left[0, \frac{1}{2}\right]$. According to Eqs. (29) and (30), we obtain

$$
\begin{align*}
V^{\alpha, \beta}\left(\rho_{\text {iso }}, \Phi\right)= & \frac{1}{24}\left[19-2 F-8 F^{2}+3^{2-\alpha-\beta} F^{1-\alpha-\beta}(1-F)^{\alpha+\beta}\right. \\
& \left.+3^{1+\alpha+\beta} F^{\alpha+\beta}(1-F)^{1-\alpha-\beta}\right] \tag{33}
\end{align*}
$$

and

$$
\begin{align*}
& Q^{\alpha, \beta}\left(\rho_{i s o}, \Phi\right) \\
= & \frac{1}{8}\left[1+2 F-3^{1-\alpha} F^{1-\alpha}(1-F)^{\alpha}-3^{\alpha} F^{\alpha}(1-F)^{1-\alpha}-3^{1-\beta} F^{1-\beta}(1-F)^{\beta}-\right. \\
& \left.3^{\beta} F^{\beta}(1-F)^{1-\beta}+3^{\alpha+\beta} F^{\alpha+\beta}(1-F)^{1-\alpha-\beta}+3^{1-\alpha-\beta} F^{1-\alpha-\beta}(1-F)^{\alpha+\beta}\right] . \tag{34}
\end{align*}
$$

Fig. 3 illustrates the uncertainties $V^{\alpha, \beta}\left(\rho_{i s o}, \Phi\right)$ and $Q^{\alpha, \beta}\left(\rho_{i s o}, \Phi\right)$ in Eqs. (30) and (31) with $\alpha=\frac{1}{5}$ and $\beta=\frac{3}{10}$, respectively. By calculation, it is found that $V^{\alpha, \beta}\left(\rho_{i s o}, \Phi\right)-$ $Q^{\alpha, \beta}\left(\rho_{\text {iso }}, \Phi\right)$ reaches its maximum value of 0.9375 when $F=\frac{1}{4}, \alpha=0.210985$ and $\beta=0.479723$. Interestingly, the linear entropy (mixedness) $1-\operatorname{tr} \rho_{i s o}^{2}=\frac{2}{3}+\frac{2}{3} F-\frac{4}{3} F^{2}$ of the isotropic state attains its maximum value of $\frac{3}{4}$ when $F=\frac{1}{4}$. In Fig. 4, we plot the surfaces of the total uncertainty $V^{\alpha, \beta}\left(\rho_{i s o}, \Phi\right)$ and the quantum uncertainty $Q^{\alpha, \beta}\left(\rho_{i s o}, \Phi\right)$ in Eqs. (33) and (34), respectively, and their gap, the classical uncertainty $C^{\alpha, \beta}\left(\rho_{i s o}, \Phi\right)$, for fixed values of $F$.


Figure 3: The $y$-axis shows the values of $V^{\alpha, \beta}\left(\rho_{i s o}, \Phi\right)$ and $Q^{\alpha, \beta}\left(\rho_{i s o}, \Phi\right)$. Dashed (solid) line represents the value of $V^{\alpha, \beta}\left(\rho_{i s o}, \Phi\right)\left(Q^{\alpha, \beta}\left(\rho_{i s o}, \Phi\right)\right)$ in Eq. (33) (Eq. (34)) with $\alpha=\frac{1}{5}$ and $\beta=\frac{3}{10}$, respectively. The isotropic state $\rho_{\text {iso }}$ degenerates into a pure state when $F=1$ and in this case $V^{\alpha, \beta}\left(\rho_{i s o}, \Phi\right)=Q^{\alpha, \beta}\left(\rho_{i s o}, \Phi\right)$.

$\Phi)$ and $Q^{\alpha, \beta}\left(\rho_{i s o}, \Phi\right)$ with fixed $F:(\mathbf{a}) F=0 ;(\mathbf{b}) F=\frac{1}{4}$; Figure 4: Surfaces of $V^{\alpha, \beta}\left(\rho_{\text {iso }}, \Phi\right)$
(c) $F=\frac{1}{2}$. (d) $F=\frac{3}{4}$, where the red (blue) surface represents the value of $V^{\alpha, \beta}\left(\rho_{i s o}, \Phi\right)$ $\left(Q^{\alpha, \beta}\left(\rho_{i s o}, \Phi\right)\right)$ in Eq. (33) (Eq. (34)). Surfaces of the gap $C^{\alpha, \beta}\left(\rho_{i s o}, \Phi\right)$ in (a)-(d) with fixed $F$ : (e) $F=0 ;(\mathbf{f}) F=\frac{1}{4} ;(\mathbf{g}) F=\frac{1}{2} ;(\mathbf{h}) F=\frac{3}{4}$, where the green surface represents the value of $C^{\alpha, \beta}\left(\rho_{i s o}, \Phi\right)$.

By plotting the animation of Fig. 2 and Fig. 4 for $p \in[0,1]$ and $F \in[0,1]$, respectively, it can be seen that for the Werner state $\rho_{w}$, the maximum value of classical uncertainty of the channel $C^{\alpha, \beta}\left(\rho_{w}, \Phi\right)$ increases with the increase of $p$ when $p \in\left[0, \frac{3}{4}\right]$, and decreases with the increase of $p$ when $p \in\left[\frac{3}{4}, 1\right]$, while for the isotropic state $\rho_{i s o}$, the maximum value of $C^{\alpha, \beta}\left(\rho_{i s o}, \Phi\right)$ increases with the increase of $F$ when $F \in\left[0, \frac{1}{4}\right]$, and decreases with the increase of $F$ when $F \in\left[\frac{1}{4}, 1\right]$. For either the Werner state $\rho_{w}$ or the isotropic state $\rho_{i s o}$, the classical uncertainty of the channel is maximal when the linear entropy (mixedness) is maximal.
Example 6 Consider the von Neumann measurement $\Pi(\rho)=\sum_{i} \Pi_{i} \rho \Pi_{i}$, where $\Pi_{i}=$ $|i\rangle\langle i|$ with $\{|i\rangle\}_{i=1}^{d}$ being an orthonormal basis of $\mathcal{H}$. It is easy to derive that

$$
\begin{equation*}
V^{\alpha, \beta}(\rho, \Pi)=\frac{1}{2}\left[1+\sum_{i}\langle i| \rho^{\alpha+\beta}|i\rangle\langle i| \rho^{1-\alpha-\beta}|i\rangle\right]-\sum_{i}\langle i| \rho|i\rangle^{2} \tag{35}
\end{equation*}
$$

and

$$
\begin{align*}
Q^{\alpha, \beta}(\rho, \Pi)= & \frac{1}{2}\left[1+\sum_{i}\langle i| \rho^{\alpha+\beta}|i\rangle\langle i| \rho^{1-\alpha-\beta}|i\rangle-\sum_{i}\langle i| \rho^{\beta}|i\rangle\langle i| \rho^{1-\beta}|i\rangle\right. \\
& \left.-\sum_{i}\langle i| \rho^{1-\alpha}|i\rangle\langle i| \rho^{\alpha}|i\rangle\right] \tag{36}
\end{align*}
$$

where $\alpha, \beta \geq 0$ and $\alpha+\beta \leq 1$.
Now we use Example 6 to illustrate Eqs. (18), (19) and (20) for two classes of states, the Werner state and the isotropic state, respectively. Under this channel, the
left hand side of inequality (18) is $\frac{1}{2}\left(1+\sum_{i}\langle i| \rho^{\alpha+\beta}|i\rangle\langle i| \rho^{1-\alpha-\beta}|i\rangle\right)$, which is equal to $\frac{3}{4}+\frac{p}{6}+\frac{1}{4}\left[3^{\alpha+\beta-1} p^{1-\alpha-\beta}(1-p)^{\alpha+\beta}+3^{-\alpha-\beta} p^{\alpha+\beta}(1-p)^{1-\alpha-\beta}\right]$ for the Werner state $\rho_{w}$. The right hand side of (18), however, is $\frac{1}{2}\left(1+\max _{i} \rho_{i i}^{1-\alpha-\beta} \operatorname{tr} \rho^{\alpha+\beta}\right)$, which is equal to $\frac{3}{4}+\frac{1}{4}\left(3^{1-\alpha-\beta} p^{\alpha+\beta}(1-p)^{1-\alpha-\beta}+3^{\alpha+\beta-1} p^{1-\alpha-\beta}(1-p)^{\alpha+\beta}\right)$ when $p \in\left[0, \frac{3}{4}\right)$, and $\frac{p}{2}+\frac{1}{2}(1+$ $\left.3^{\alpha+\beta-1} p^{1-\alpha-\beta}(1-p)^{\alpha+\beta}\right)$ when $p \in\left(\frac{3}{4}, 1\right]$ for the Werner state $\rho_{w}$. When $p=\frac{3}{4}$, the left and right hand side of (18) are both 1 . For the isotropic state $\rho_{i s o}$, the left hand side of (18) turns out to be $\frac{11}{12}-\frac{F}{6}+\frac{1}{4}\left[3^{-\alpha-\beta} F^{1-\alpha-\beta}(1-F)^{\alpha+\beta}+3^{\alpha+\beta-1} F^{\alpha+\beta}(1-F)^{1-\alpha-\beta}\right]$, while the right hand side is $1+\frac{1}{2}\left(3^{\alpha+\beta-1} F^{\alpha+\beta}(1-F)^{1-\alpha-\beta}-F\right)$ when $F \in\left[0, \frac{1}{4}\right)$, and $\frac{3}{4}+\frac{1}{4}\left(3^{1-\alpha-\beta} F^{1-\alpha-\beta}(1-F)^{\alpha+\beta}+3^{\alpha+\beta-1} F^{\alpha+\beta}(1-F)^{1-\alpha-\beta}\right)$ when $F \in\left(\frac{1}{4}, 1\right]$. When $F=\frac{1}{4}$, the left and right hand side of (18) are both 1 .

The left and right hand side of Eq. (19) for the Werner state $\rho_{w}$ are

$$
\begin{equation*}
1+\left(2 V^{\alpha, \beta}\left(\rho_{w}, \Phi\right)+1-\operatorname{tr} \rho_{w}^{\alpha+\beta} \Phi\left(\rho_{w}^{1-\alpha-\beta}\right)\right) \log 4=3+\frac{8}{3} p-\frac{16}{3} p^{2} \tag{37}
\end{equation*}
$$

and

$$
\begin{equation*}
S_{e}\left(\rho_{w}, \Phi\right)=(p-1) \log (1-p)-p \log \frac{p}{3}, \tag{38}
\end{equation*}
$$

respectively, while the left and right hand side of Eq. (20) are given by

$$
\begin{align*}
& 2\left(2 V^{\alpha, \beta}\left(\rho_{w}, \Phi\right)+1-\operatorname{tr} \rho_{w}^{\alpha+\beta} \Phi\left(\rho_{w}^{1-\alpha-\beta}\right)\right) \log 4+I_{c}\left(\rho_{w}, \Phi\right) \\
= & 4+\frac{16}{3} p-\frac{32}{9} p^{2}+(1-p) \log (1-p)+\frac{p}{3} \log \frac{p}{3}-\frac{3-2 p}{3} \log \frac{3-2 p}{6} \tag{39}
\end{align*}
$$

and

$$
\begin{equation*}
S\left(\rho_{w}\right)-2=(p-1) \log (1-p)-p \log \frac{p}{3}-2, \tag{40}
\end{equation*}
$$

respectively. Fig. 5 illustrates the quantities in Eqs. (37)-(40) for the Werner state $\rho_{w}$ as a function of $p$.


Figure 5: The $y$-axis shows the uncertainty and its lower bounds. Dashed red (solid red) line represents the value of Eq. (37) (Eq. (38)) for the Werner state $\rho_{w}$; Dashed blue (solid blue) line represents the value of Eq. (39) (Eq. (40)) for the Werner state $\rho_{w}$.

Now consider the isotropic state $\rho_{i s o}$. Direct computation shows that the quantities on the two sides of Eq. (19) are

$$
\begin{equation*}
1+\left(2 V^{\alpha, \beta}\left(\rho_{i s o}, \Phi\right)+1-\operatorname{tr} \rho_{i s o}^{\alpha+\beta} \Phi\left(\rho_{i s o}^{1-\alpha-\beta}\right)\right) \log 4=\frac{35}{9}+\frac{8}{9} F-\frac{16}{9} F^{2} \tag{41}
\end{equation*}
$$

and

$$
\begin{equation*}
S_{e}\left(\rho_{i s o}, \Phi\right)=(F-1) \log \frac{1-F}{3}-F \log F, \tag{42}
\end{equation*}
$$

respectively, and for Eq. (20), we have

$$
\begin{align*}
& 2\left(2 V^{\alpha, \beta}\left(\rho_{i s o}, \Phi\right)+1-\operatorname{tr} \rho_{i s o}^{\alpha+\beta} \Phi\left(\rho_{i s o}^{1-\alpha-\beta}\right)\right) \log 4+I_{c}\left(\rho_{i s o}, \Phi\right) \\
= & \frac{52}{9}+\frac{16}{9} F(1-2 F)+\frac{1-F}{3} \log \frac{1-F}{3}+F \log F-\frac{1+2 F}{3} \log \frac{1+2 F}{6} \tag{43}
\end{align*}
$$

and

$$
\begin{equation*}
S\left(\rho_{i s o}\right)-2=(F-1) \log \frac{1-F}{3}-F \log F-2 . \tag{44}
\end{equation*}
$$

Fig. 6 illustrates the quantities in Eqs. (41)-(44) for the isotropic state $\rho_{i s o}$ as a function of $F$. Comparing Fig. 5 with Fig. 6, it can be seen that for the Werner state $\rho_{w}$ (the


Figure 6: The $y$-axis shows the uncertainty and its lower bounds. Dashed red (solid red) line represents the value of Eq. (41) (Eq. (42)) for the isotropic state $\rho_{i s o}$; Dashed blue (solid blue) line represents the value of Eq. (43) (Eq. (44)) for the isotropic state $\rho_{\text {iso }}$.
isotropic state $\rho_{\text {iso }}$ ), the gap between Eq. (39) (Eq.(43)) and Eq. (40) (Eq.(44)) is greater than the one between Eq. (37) (Eq.(41)) and Eq. (38) (Eq.(42)).

Metric-adjusted skew information is known to be measurable by measuring the linear response function at thermal equilibrium [60]. In recent years, uncertainty relations for unitary operators have been investigated both theoretically and experimentally [61-63], and the problem of measuring arbitrary non-Hermitian operators has attracted much attention [64-66]. We discuss how to measure the total/quantum uncertainty of quantum channels $V^{\alpha, \beta}(\rho, \Phi) / Q^{\alpha, \beta}(\rho, \Phi)$ for pure states by using an experimental protocol.

We now focus on the case in which $\rho$ is a pure state, i.e., $\rho=|\psi\rangle\langle\psi|$. By using Eq. (13), the problem then reduces to how to measure the magnitude of the expectation values of the non-Hermitian operators $K_{i}$.

An arbitrary operator $K \in \mathcal{B}(\mathcal{H})$ can be decomposed as $K=U R$, where $U$ is unitary and $R=\sqrt{K^{\dagger} K}$ is positive semidefinite. By utilizing the scheme proposed in [65], we can measure the magnitude of the expectation value $|\langle K\rangle|$ of a non-Hermitian operator $K$ by using the Mach-Zehnder interferometer.

Given a quantum channel $\Phi$ with Kraus operators $\left\{K_{i}\right\}$. Using the above scheme in [65], we can fix the values of $\left|\left\langle K_{i}\right\rangle\right|$ for each $i$, and by Eq.(13), both of $V^{\alpha, \beta}(\rho, \Phi)$ and $Q^{\alpha, \beta}(\rho, \Phi)$ can be obtained as $\frac{1}{2}\left(1-\sum_{i}\left|\left\langle K_{i}\right\rangle\right|^{2}\right)$.

## 5. Conclusions and discussions

We have defined a family of total uncertainties of quantum channels based on the modified generalized variance (MGV) introduced in [24], which includes the total uncertainty introduced in [9] as a special case. Following the idea in Ref. [9], we have divided the total uncertainty into quantum and classical parts and associated the quantum part with the coherence of quantum states with respect to quantum channels based on modified generalized Wigner-Yanase-Dyson (MGWYD) skew information [25].

In addition, we have formulated the relationship between the MGV-based total uncertainty of quantum channels and the entanglement fidelity, extending the results in [9] to a more general case. As a consequence, we have also figured out the link between the MGV-based total uncertainty of quantum channels and the entropy exchange/coherent information, proving that the MGV-based total uncertainty of quantum channels provides an upper bound on the entropy exchange. Moreover, we have calculated the total uncertainty and quantum uncertainty of some special quantum channels. In particular, we have computed the total uncertainty and quantum uncertainty for a quantum channel with respect to the Werner state and the isotropic state in Example 5. It has been found that for these two different classes of states, there exist subtle similarity, that is, under this quantum channel, both the classical uncertainty and the linear entropy (mixedness) attain their maximum values for the same state, i.e., at the same value of the parameter $p$ or $F$. The uncertainty relations Eqs. (19) and (20) have also been computed for this channel and such two classes of states. Finally, we have used an experimental protocol formulated in [65] to measure the uncertainty of quantum channels for pure states. Our results may shed some new light on the study of uncertainties of quantum channels and provide new insights into the quantum-classical interplay.

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## Competing interests

The authors declare no competing interests.

## Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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