

Quantum Bell nonlocality cannot be shared under a special kind of bilateral measurements for high-dimensional quantum states

Tinggui Zhang^{1,2,†}, Qiming Luo¹ and Xiaofen Huang^{1,2}

1 School of Mathematics and Statistics,

Hainan Normal University,

Haikou, 571158, China

2 Key Laboratory of Data Science and Smart Education,

Ministry of Education,

Hainan Normal University,

Haikou, 571158, China

† Correspondence to tinggui333@163.com

(Dated:)

Abstract

Quantum Bell nonlocality is an important quantum phenomenon. Recently, the shareability of Bell nonlocality under unilateral measurements has been widely studied. In this study, we consider the shareability of quantum Bell nonlocality under bilateral measurements. Under a specific class of projection operators, we find that quantum Bell nonlocality cannot be shared for a limited number of times, as in the case of unilateral measurements. Our proof is analytical and our measurement strategies can be generalized to higher dimension cases.

Keywords: Quantum Bell nonlocality; Shareability; Unilateral measurements; Bilateral measurements

PACS numbers: 03.67.-a, 02.20.Hj, 03.65.-w

I. INTRODUCTION

As the source of paradoxes such as the Einstein, Podolsky, Rosen paradox [1] quantum nonlocal correlation was a controversial phenomenon in quantum mechanics. Nowadays it has become a key resource in the blooming areas of quantum information and computing [2–6]. Realizing quantum violations of the Bell-CHSH inequality [7] in various quantum systems has acquired great interest as evidenced by a wide range of studies [8–16]. According to quantum physics, measurement outcomes cannot be predicted with certainty in general [17]. Quantum nonlocality implies that the correlations between the probabilities of measurement outcomes from two distant systems cannot be described by classical probability correlation models. Such nonlocal correlations in multipartite systems have been identified as useful resources in device-independent quantum information processing [18], such as key distribution [19, 20], randomness expansion [21–23] and randomness amplification [24].

Recently, the shareability of quantum Bell nonlocality has been extensively studied [25–31]. By constructing an explicit measurement strategy, the authors in [29] show that, contrary to previous expectations [25, 26], there is no limit on the number of independent Bobs that can have an expected violation of the CHSH inequality with only one Alice. A class of initial two-qubit states, including all pure two-qubit entangled states, that are capable of achieving an unlimited number of CHSH inequality violations has been presented. This fact has recently been illustrated for higher dimensional bipartite pure states [30]. Furthermore, in [29], the open question of whether quantum nonlocality can be shared under bilateral measurements was been raised.

In this study, we focus on quantum Bell nonlocality shareability under bilateral measurements. We consider the following scenario: a nonlocal correlated bipartite state ρ_{AB} is initially shared by the first Alice and first Bob. The first Bob performs a randomly selected measurement, records the measurement outcome, and passes the post-measurement qubit to the second Bob. Then, the first Alice performs a randomly selected measurement, records the measurement outcome, and passes the post-measurement qubit to the second Alice. The problem of interest is whether the quantum state between the second Alice and Bob is still nonlocal. In fact, there have been some numerical and experimental studies on this topic. In Ref. [31], the authors used 17 parameters to verify numerically that two-qubit quantum states do not share quantum nonlocality. Moreover, in Ref. [32], they have studied

the sequential generation of Bell nonlocality between independent observers via recycling the components of entangled systems. They obtained the stronger one-sided monogamy relations than [31]. In Ref.[33], using entangled photon pairs, the authors experimentally verified the case of two Alices and two Bobs where Alice⁽¹⁾ and Bob⁽¹⁾ performed optimal weak measurements and Alice⁽²⁾ and Bob⁽²⁾ performed projective measurements. To adopt the same measurement strength for Alice⁽¹⁾ and Bob⁽¹⁾, they observed double EPR steering simultaneously and showed that double Bell-CHSH inequality violations cannot be obtained. But for high-dimensional quantum states, the method used in [31] is not efficient as too many parameters are involved. Here, we find that Bell nonlocality cannot be shared under bilateral measurements for a specific class of projection measurement operators.

II. BIPARTITE STATE UNDER BILATERAL MEASUREMENT

We considered a measurement scenario where the second Alice (Alice⁽²⁾) attempts to share nonlocal correlations of an entangled pure state with the second Bob (Bob⁽²⁾). First, Alice⁽¹⁾ and Bob⁽¹⁾ share an arbitrary entangled bipartite pure state $|\psi\rangle \in H_A \otimes H_B$, where $\dim(H_A) = s$ and $\dim(H_B) = t$ ($s \leq t$). The state has the Schmidt decomposition form, $|\psi\rangle = \sum_{i=1}^s c_i |i_A\rangle \otimes |i_B\rangle$, where $c_i \in [0, 1]$ and $\sum_i c_i^2 = 1$ and $\{|i_A\rangle_1^s$ and $\{|i_B\rangle_1^t$ are the orthonormal bases of H_A and H_B , respectively. $|\psi\rangle$ is entangled if and only if at least two c_i s are nonzero. Without loss of generality, below we assume that the c_i are arranged in descending order. The density matrix corresponding to $|\psi\rangle$ is denoted as $\rho_{A^1 B^1} = |\psi\rangle\langle\psi|$.

The binary input and output of Alice^(k) (Bob^(k)) are denoted by $X^{(k)}$ ($Y^{(k)}$) and $A^{(k)}$ ($B^{(k)}$) and $k = 1, 2$, respectively. Suppose that Bob⁽¹⁾ performs the measurement according to $Y^{(1)} = y$ with the outcome $B^{(1)} = b$. Averaged over the inputs and outputs of Bob⁽¹⁾, the state shared between Alice⁽¹⁾ and Bob⁽²⁾ is given by the Lüders rule [29]

$$\rho_{A^1 B^2} = \frac{1}{2} \sum_{b,y} (I_s \otimes \sqrt{B_{b|y}^{(1)}}) \rho_{A^1 B^1} (I_s \otimes \sqrt{B_{b|y}^{(1)}}),$$

where $B_{b|y}^{(1)}$ is the positive operator-valued measure (POVM) effect corresponding to outcome b of Bob⁽¹⁾'s measurement for input y , and I_s is the $s \times s$ identity matrix. Next Alice⁽¹⁾ similarly performs the measurement on subsystem A. Then, the state $\rho_{A^2 B^2}$ shared between Alice⁽²⁾ and Bob⁽²⁾ is acquired.

To detect the Bell nonlocality of a state ρ we employ the CHSH inequality [7], $I_{CHSH} =$

$|\langle \mathbb{B} \rangle| \leq 2$, where $\langle \mathbb{B} \rangle = Tr(\mathbb{B}\rho)$, $\mathbb{B} = A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1$, A_i , and B_i and $i = 0, 1$ are Hermitian operators with eigenvalues of $\in [-1, 1]$. If for some binary observables $A_i^{(k)}$ and $B_i^{(k)}$, $i = 0, 1$, $I_{CHSH}^{(k)} \equiv Tr(\mathbb{B}\rho_{A^k B^k}) > 2$, then the state $\rho_{A^k B^k}$ is nonlocally correlated.

A. Two-qubit pure states

We first assume that the initial bipartite pure quantum state is a two-qubit state, $|\psi\rangle \in H_2 \otimes H_2$, with Schmidt decomposition $|\psi\rangle = \sum_{i=1}^2 c_i |i_A\rangle |i_B\rangle$. Namely, $\rho_{A^1 B^1} = |\psi\rangle\langle\psi|$. We employ the POVMs with measurement operators $\{E, I - E\}$, where E has the form $E = \frac{1}{2}(I + \gamma \cdot \sigma_r)$, $r \in R^3$ with $\|r\| = 1$, $\sigma_r = r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3$, σ_i for $i = 1, 2, 3$ are the standard Pauli matrices, and $\gamma \in [0, 1]$ is the sharpness of the measurement.

We set the POVM of Alice⁽¹⁾ to

$$A_{0|0} = \frac{1}{2}(I + (\cos\theta\sigma_1 + \sin\theta\sigma_3)), \quad (1)$$

$$A_{0|1} = \frac{1}{2}(I + (\cos\theta\sigma_1 - \sin\theta\sigma_3)) \quad (2)$$

for $\theta \in (0, \frac{\pi}{4}]$. We also let the POVM of Bob⁽¹⁾ be given by

$$B_{0|0}^{(1)} = \frac{1}{2}(I + \sigma_1), \quad (3)$$

$$B_{0|1}^{(1)} = \frac{1}{2}(I + \gamma_1\sigma_3), \quad (4)$$

where $0 \leq \gamma_1 \leq 1$. Further, we defined the expectation operators $A_x = A_{0|x} - A_{1|x}$ and $B_y = B_{0|y} - B_{1|y}$ and reached the following conclusions:

Lemma 1 *For the quantum state $\rho_{A^2 B^2}$, we have*

$$\begin{aligned} & Tr[\rho_{A^2 B^2}(\sigma_1 \otimes \sigma_1)] \\ &= \frac{1 + \sqrt{1 - \gamma_1^2}}{2} \cos^2(\theta) Tr[\rho_{A^1 B^1}(\sigma_1 \otimes \sigma_1)] \end{aligned}$$

and

$$\begin{aligned} & Tr[\rho_{A^2 B^2}(\sigma_3 \otimes \sigma_3)] \\ &= \frac{1}{2} \sin^2(\theta) Tr[\rho_{A^1 B^1}(\sigma_3 \otimes \sigma_3)]. \end{aligned}$$

Proof: First, after the Bob⁽¹⁾'s measurement we have

$$\rho_{A^1 B^2} = \frac{2 + \sqrt{1 - \gamma_1^2}}{4} \rho_{A^1 B^1} + \frac{1}{4} (I \otimes \sigma_1) \rho_{A^1 B^1} (I \otimes \sigma_1) + \frac{1 - \sqrt{1 - \gamma_1^2}}{4} (I \otimes \sigma_3) \rho_{A^1 B^1} (I \otimes \sigma_3).$$

After Alice⁽¹⁾'s measurement we get

$$\begin{aligned} & \rho_{A^2 B^2} \\ &= \frac{1}{2} \sum_{a,x} (\sqrt{A_{a|x}} \otimes I) \rho_{A^1 B^2} (\sqrt{A_{a|x}} \otimes I) \\ &= \frac{1}{2} \rho_{A^1 B^2} + \frac{1}{2} (\cos(\theta) \sigma_1 \otimes I) \rho_{A^1 B^2} (\cos(\theta) \sigma_1 \otimes I) + \frac{1}{2} (\sin(\theta) \sigma_3 \otimes I) \rho_{A^1 B^2} (\sin(\theta) \sigma_3 \otimes I) \\ &= \frac{2 + \sqrt{1 - \gamma_1^2}}{8} \rho_{A^1 B^1} + \frac{1}{8} (I \otimes \sigma_1) \rho_{A^1 B^1} (I \otimes \sigma_1) + \frac{1 - \sqrt{1 - \gamma_1^2}}{8} (I \otimes \sigma_3) \rho_{A^1 B^1} (I \otimes \sigma_3) \\ &\quad + \frac{2 + \sqrt{1 - \gamma_1^2}}{8} (\cos(\theta) \sigma_1 \otimes I) \rho_{A^1 B^1} (\cos(\theta) \sigma_1 \otimes I) \\ &\quad + \frac{1}{8} (\cos(\theta) \sigma_1 \otimes I) (I \otimes \sigma_1) \rho_{A^1 B^1} (I \otimes \sigma_1) (\cos(\theta) \sigma_1 \otimes I) \\ &\quad + \frac{1 - \sqrt{1 - \gamma_1^2}}{8} (\cos(\theta) \sigma_1 \otimes I) (I \otimes \sigma_3) \rho_{A^1 B^1} (I \otimes \sigma_3) (\cos(\theta) \sigma_1 \otimes I) \\ &\quad + \frac{2 + \sqrt{1 - \gamma_1^2}}{8} (\sin(\theta) \sigma_3 \otimes I) \rho_{A^1 B^1} (\sin(\theta) \sigma_3 \otimes I) \\ &\quad + \frac{1}{8} (\sin(\theta) \sigma_3 \otimes I) (I \otimes \sigma_1) \rho_{A^1 B^1} (I \otimes \sigma_1) (\sin(\theta) \sigma_3 \otimes I) \\ &\quad + \frac{1 - \sqrt{1 - \gamma_1^2}}{8} (\sin(\theta) \sigma_3 \otimes I) (I \otimes \sigma_3) \rho_{A^1 B^1} (I \otimes \sigma_3) (\sin(\theta) \sigma_3 \otimes I). \end{aligned}$$

Then

$$\begin{aligned} & Tr[\rho_{A^2 B^2} (\sigma_1 \otimes \sigma_1)] \\ &= \frac{2 + \sqrt{1 - \gamma_1^2}}{8} Tr[\rho_{A^1 B^1} (\sigma_1 \otimes \sigma_1)] + \frac{1}{8} Tr[\rho_{A^1 B^1} (\sigma_1 \otimes \sigma_1)] - \frac{1 - \sqrt{1 - \gamma_1^2}}{8} Tr[\rho_{A^1 B^1} (\sigma_1 \otimes \sigma_1)] \\ &\quad + \frac{2 + \sqrt{1 - \gamma_1^2}}{8} \cos^2(\theta) Tr[\rho_{A^1 B^1} (\sigma_1 \otimes \sigma_1)] + \frac{1}{8} \cos^2(\theta) Tr[\rho_{A^1 B^1} (\sigma_1 \otimes \sigma_1)] \\ &\quad - \frac{1 - \sqrt{1 - \gamma_1^2}}{8} \cos^2(\theta) Tr[\rho_{A^1 B^1} (\sigma_1 \otimes \sigma_1)] - \frac{2 + \sqrt{1 - \gamma_1^2}}{8} \sin^2(\theta) Tr[\rho_{A^1 B^1} (\sigma_1 \otimes \sigma_1)] \\ &\quad - \frac{1}{8} \sin^2(\theta) Tr[\rho_{A^1 B^1} (\sigma_1 \otimes \sigma_1)] + \frac{1 - \sqrt{1 - \gamma_1^2}}{8} \sin^2(\theta) Tr[\rho_{A^1 B^1} (\sigma_1 \otimes \sigma_1)] \\ &= \frac{1 + \sqrt{1 - \gamma_1^2}}{4} Tr[\rho_{A^1 B^1} (\sigma_1 \otimes \sigma_1)] + \frac{1 + \sqrt{1 - \gamma_1^2}}{4} \cos^2(\theta) Tr[\rho_{A^1 B^1} (\sigma_1 \otimes \sigma_1)] \\ &\quad - \frac{1 + \sqrt{1 - \gamma_1^2}}{4} \sin^2(\theta) Tr[\rho_{A^1 B^1} (\sigma_1 \otimes \sigma_1)] \\ &= \frac{1 + \sqrt{1 - \gamma_1^2}}{2} \cos^2(\theta) Tr[\rho_{A^1 B^1} (\sigma_1 \otimes \sigma_1)]. \end{aligned}$$

Similarly,

$$\begin{aligned}
& Tr[\rho_{A^2B^2}(\sigma_3 \otimes \sigma_3)] \\
= & \frac{2 + \sqrt{1 - \gamma_1^2}}{8} Tr[\rho_{A^1B^1}(\sigma_3 \otimes \sigma_3)] - \frac{1}{8} Tr[\rho_{A^1B^1}(\sigma_3 \otimes \sigma_3)] + \frac{1 - \sqrt{1 - \gamma_1^2}}{8} Tr[\rho_{A^1B^1}(\sigma_3 \otimes \sigma_3)] \\
& - \frac{2 + \sqrt{1 - \gamma_1^2}}{8} \cos^2(\theta) Tr[\rho_{A^1B^1}(\sigma_3 \otimes \sigma_3)] + \frac{1}{8} \cos^2(\theta) Tr[\rho_{A^1B^1}(\sigma_3 \otimes \sigma_3)] \\
& - \frac{1 - \sqrt{1 - \gamma_1^2}}{8} \cos^2(\theta) Tr[\rho_{A^1B^1}(\sigma_3 \otimes \sigma_3)] + \frac{2 + \sqrt{1 - \gamma_1^2}}{8} \sin^2(\theta) Tr[\rho_{AB^1}(\sigma_3 \otimes \sigma_3)] \\
& - \frac{1}{8} \sin^2(\theta) Tr[\rho_{A^1B^1}(\sigma_3 \otimes \sigma_3)] + \frac{1 - \sqrt{1 - \gamma_1^2}}{8} \cos^2(\theta) Tr[\rho_{A^1B^1}(\sigma_3 \otimes \sigma_3)] \\
= & \frac{1}{4} Tr[\rho_{A^1B^1}(\sigma_3 \otimes \sigma_3)] - \frac{1}{4} \cos^2(\theta) Tr[\rho_{A^1B^1}(\sigma_3 \otimes \sigma_3)] \\
& + \frac{1}{4} \sin^2(\theta) Tr[\rho_{A^1B^1}(\sigma_3 \otimes \sigma_3)] \\
= & \frac{1}{2} \sin^2(\theta) Tr[\rho_{A^1B^1}(\sigma_3 \otimes \sigma_3)].
\end{aligned}$$

□

Using the Lemma above, we have the following Theorem:

Theorem 1 *For any initial entangled bipartite pure quantum state $|\psi\rangle \in H_2 \otimes H_2$, $|\psi\rangle = \sum_{i=1}^2 c_i |i_A\rangle |i_B\rangle$. After the bilateral measurements, the expected CHSH value of $\rho_{A^2B^2}$ is less than or equal to 2, that is,*

$$\begin{aligned}
I_{CHSH} &= Tr[\rho_{A^2B^2}((A_0 + A_1) \otimes B_0)] \\
&+ Tr[\rho_{A^2B^2}((A_0 - A_1) \otimes B_1)] \leq 2.
\end{aligned}$$

Proof:

$$\begin{aligned}
I_{CHSH} &= Tr[\rho_{A^2B^2}((A_0 + A_1) \otimes B_0)] \\
&+ Tr[\rho_{A^2B^2}((A_0 - A_1) \otimes B_1)] \\
&= 2 \cos(\theta) Tr[\rho_{A^2B^2}(\sigma_1 \otimes \sigma_1)] \\
&+ 2 \gamma_1 \sin(\theta) Tr[\rho_{A^2B^2}(\sigma_3 \otimes \sigma_3)] \\
&= \cos^3(\theta) (1 + \sqrt{1 - \gamma_1^2}) Tr[\rho_{A^1B^1}(\sigma_1 \otimes \sigma_1)] \\
&+ \gamma_1 \sin^3(\theta) Tr[\rho_{A^1B^1}(\sigma_3 \otimes \sigma_3)].
\end{aligned}$$

Since $Tr[\rho_{A^1B^1}(\sigma_1 \otimes \sigma_1)] \leq 1$ and $Tr[\rho_{A^1B^1}(\sigma_3 \otimes \sigma_3)] \leq 1$ we have

$$\begin{aligned}
I_{CHSH} &= \cos^3(\theta) (1 + \sqrt{1 - \gamma_1^2}) + \gamma_1 \sin^3(\theta) \\
&\leq 2 \cos^3(\theta) + \sin^3(\theta).
\end{aligned}$$

Using

$$f(\theta) = 2\cos^3(\theta) + \sin^3(\theta), \quad 0 < \theta \leq \frac{\pi}{4},$$

we have $f'(\theta) = 3\sin(\theta)\cos(\theta)[\sin(\theta) - 2\cos(\theta)] < 0$, as $\sin(\theta) < 2\cos(\theta)$ for $0 < \theta \leq \frac{\pi}{4}$. Hence, $f(\theta)$ is a decreasing function of θ with $f(\theta) \leq f(0) = 2$. Therefore, $I_{CHSH} \leq 2$. \square

The above Theorem shows that the second Bob shares no quantum nonlocality with the second Alice.

B. Generation for Higher dimensional pure states

For general $d \otimes d$ ($d \geq 3$) entangled pure state ρ is given by $\rho_{A^1B^1} = |\varphi\rangle\langle\varphi|$, where $|\varphi\rangle = \sum_{i=1}^d c_i |ii\rangle$ with $\sum_{i=1}^d c_i^2 = 1$. Let

$$\begin{aligned} A_{0|0} &= \frac{1}{2} \left[I_d + \begin{pmatrix} \cos(\theta)\sigma_3 + \sin(\theta)\sigma_1 & 0 \\ 0 & I_{d-2} \end{pmatrix} \right], \\ A_{0|1} &= \frac{1}{2} \left[I_d + \begin{pmatrix} \cos(\theta)\sigma_3 - \sin(\theta)\sigma_1 & 0 \\ 0 & I_{d-2} \end{pmatrix} \right], \\ B_{0|0} &= \frac{1}{2} \left[I_d + \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_3 \end{pmatrix} \right], \\ B_{0|1} &= \frac{1}{2} \left[I_d + \begin{pmatrix} I_{d-2} & 0 \\ 0 & \gamma_1\sigma_1 \end{pmatrix} \right], \end{aligned}$$

i.e.

$$\begin{aligned} A_0 &= \begin{pmatrix} \cos(\theta)\sigma_3 + \sin(\theta)\sigma_1 & 0 \\ 0 & I_{d-2} \end{pmatrix} \\ A_1 &= \begin{pmatrix} \cos(\theta)\sigma_3 - \sin(\theta)\sigma_1 & 0 \\ 0 & I_{d-2} \end{pmatrix} \\ B_0 &= \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_3 \end{pmatrix} \\ B_1 &= \begin{pmatrix} I_{d-2} & 0 \\ 0 & \gamma_1\sigma_1 \end{pmatrix}. \end{aligned}$$

Suppose Bob and Alice each perform the measurement above and we write it as $\rho_{A^2B^2}$. We can easily obtain the following Lemma with its proof given in the Appendix:

Lemma 2 For the quantum state $\rho_{A^2B^2}$, we have

$$\begin{aligned} \text{Tr}[\rho_{A^2B^2}((A_0 + A_1) \otimes B_0)] &= 2 \cos^3(\theta) c_1^2 - 2 \cos^3(\theta) c_2^2 - (1 + \sqrt{1 - \gamma_1^2}) c_d^2 \\ &+ (1 + \sqrt{1 - \gamma_1^2}) [c_3^2 + c_4^2 + \dots + c_{d-1}^2] \leq 2. \end{aligned} \quad (5)$$

$$\text{Tr}[\rho_{A^2B^2}((A_0 - A_1) \otimes B_1)] = 0.$$

We can naturally draw the following conclusion.

Theorem 2 For any initial entangled bipartite pure quantum state $\rho_{A^1B^1} = |\varphi\rangle\langle\varphi|$. After the bilateral measurements, the expected CHSH value of $\rho_{A^2B^2}$ satisfies

$$\begin{aligned} I_{CHSH}^{(2)} &= \text{Tr}[\rho_{A^2B^2}((A_0 + A_1) \otimes B_0)] \\ &+ \text{Tr}[\rho_{A^2B^2}((A_0 - A_1) \otimes B_1)] \leq 2. \end{aligned}$$

Remark: In Lemma 3, there are only the first three terms for $d = 3$, the last term will appear only when $d \geq 4$.

III. CONCLUSION AND DISCUSSION

In this article, we explored the ability to share the quantum nonlocality of bipartite quantum states under specific measurements. It has been shown that in these cases, quantum nonlocality cannot be shared under bilateral measurements. We have made an attempt in verifying the shareability of quantum nonlocality for high-dimensional quantum states. But now our analysis is only true under the kind of quantum measurements we give. We don't know whether they are optimal or not. Next, we can discuss the selection of optimal measurements for bipartite quantum pure states and some mixed states. For multipartite quantum states, the sharing ability of nonlocality in unilateral POVM measurement is already very weak[30, 34], so it should be weaker than bipartite quantum state in bilateral measurement, and we can continue to study it. In the latest literature [35], by characterising two-valued qubit observables in terms of strength, bias, and directional parameters, the authors investigated generalising the Horodecki criterion to nonprojective qubit observables. Therefore, we may continue to think about a series of problems such as the sharing of network nonlocality [36] or other quantum resources under nonprojective measurement. Ref.[31] discussed that for the qubit case CHSH nonlocality can be shared by bilateral measurements when there is

a bias on the measurements made by Alice and Bob. It is an interesting question whether there are similar results for higher-dimensional quantum states.

Data Availability Statement: Our manuscript has no associated data.

Acknowledgments: We thank Shao-Ming Fei, Naihuan Jing and Yuan-Hong Tao for their helpful discussions. This work was supported by Hainan Provincial Natural Science Foundation of China under Grant No.121RC539 and the National Natural Science Foundation of China under Grant Nos.12126314,12126351,11861031. This project is also supported by the specific research fund of the Innovation Platform for Academicians of Hainan Province under Grant No.YSPTZX202215.

References

- [1] A. Einstein, B. Podolsky, and N. Rosen, Can quantum mechanical description of physical reality be considered complete, *Physical Review* **47**, 777 (1935).
- [2] M. A. Nielsen and I. Chuang, *Quantum computation and quantum information* (2002).
- [3] C. Macchiavello, On the role of entanglement in quantum information, *Physica A: Statistical Mechanics and its Applications* **338**, 68 (2004), proceedings of the conference A Nonlinear World: the Real World, 2nd International Conference on Frontier Science.
- [4] C. H. Bennett and S. J. Wiesner, Communication via one and two-particle operators on Einstein-podolsky-rosen states, *Phys. Rev. Lett.* **69**, 2881 (1992).
- [5] A. K. Ekert, Quantum cryptography based on bell's theorem, *Phys. Rev. Lett.* **67**, 661 (1991).
- [6] A. Ekert, R. Jozsa, and P. Marcer, Quantum algorithms: Entanglement-enhanced information processing [and discussion], *Philosophical Transactions: Mathematical, Physical and Engineering Sciences* **356**, 1769 (1998).
- [7] J. F. Clauser, M. A. Horne, A. Shimony and R. A. Holt, Proposed experiment to test local hidden-variable theories, *Phys. Rev. Lett.* **23**, 880 (1969).
- [8] S. J. Freedman, and J. F. Clauser, Experimental test of local hidden-variable theories, *Phys. Rev. Lett.* **28**, 938 (1972).
- [9] A. Aspect, J. Dalibard, and G. Roger, Experimental test of Bell's inequalities using time-varying analyzers, *Phys. Rev. Lett.* **49**, 1804 (1982).

- [10] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, Violation of Bell's inequality under strict Einstein locality conditions, *Phys. Rev. Lett.* **81**, 5039 (1998).
- [11] M. A. Rowe, D. Kielpinski, V. Meyer, C. A. Sackett, W. M. Itano, C. Monroe, and D. J. Wineland, Experimental violation of a Bell's inequality with efficient detection, *Nature (London)* **409**, 791 (2001).
- [12] D. N. Matsukevich, P. Maunz, D. L. Moehring, S. Olmschenk, and C. Monroe, Bell inequality violation with two remote atomic qubits, *Phys. Rev. Lett.* **100**, 150404 (2008).
- [13] M. Ansmann, H. Wang, R. C. Bialczak, M. Hofheinz, E. Lucero, M. Neeley, A. D. O'Connell, D. Sank, M. Weides, J. Wenner, A. N. Cleland, and J. M. Martinis, Violation of Bell's inequality in Josephson phase qubits, *Nature (London)* **461**, 504 (2009).
- [14] J. Hofmann, M. Krug, N. Ortegel, L. Gerard, M. Weber, W. Rosenfeld, and H. Weinfurter, Heralded entanglement between widely separated atoms, *Science* **337**, 72 (2012).
- [15] M. Giustina, A. Mech, S. Ramelow, B. Wittmann, J. Kofler, J. Beyer, A. Lita, B. Calkins, T. Gerrits, S. W. Nam, R. Ursin, and A. Zeilinger, Bell violation using entangled photons without the fair-sampling assumption, *Nature (London)* **497**, 227 (2013).
- [16] B. G. Christensen, K. T. McCusker, J. B. Altepeter, B. Calkins, T. Gerrits, A. E. Lita, A. Miller, L. K. Shalm, Y. Zhang, S. W. Nam, N. Brunner, C. C. W. Lim, N. Gisin, and P. G. Kwiat, Detection-loophole-free test of quantum nonlocality, and applications, *Phys. Rev. Lett.* **111**, 130406 (2013).
- [17] T. E. Stuart, J. A. Slater, R. Colbeck, R. Renner and W. Tittel, An experimental test of all theories with predictive power beyond quantum theory, *Phys. Rev. Lett.* **109**, 020402 (2012).
- [18] S. Dutta, A. Mukherjee and M. Banik, Operational characterization of multipartite nonlocal correlations, *Phys. Rev. A* **102**, 052218 (2020).
- [19] J. Barrett, L. Hardy, and A. Kent, No signalling and quantum key distribution, *Phys. Rev. Lett.* **95**, 010503(2005).
- [20] A. Acin, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, Device-independent security of quantum cryptography against collective attacks, *Phys. Rev. Lett.* **98**, 230501 (2007).
- [21] R. Colbeck, Quantum and relativistic protocols for secure multi- party computation, Ph.D. thesis, University of Cambridge, 2007, also available as arXiv:0911.3814.
- [22] R. Colbeck and A. Kent, Private randomness expansion with untrusted devices, *J. Phys. A*

- 44**, 095305 (2011).
- [23] S. Pironio, A. Acin, S. Massar, A. Boyer de la Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, Random numbers certified by Bell's theorem, *Nature (London)* **464**, 1021 (2010).
- [24] R. Colbeck and R. Renner, Free randomness can be amplified, *Nat. Phys.* **8**, 450 (2012).
- [25] R. Silva, N. Gisin, Y. Guryanova, and S. Popescu, Multiple observers can share the nonlocality of half of an entangled pair by using optimal weak measurements, *Phys. Rev. Lett.* **114**, 250401 (2015).
- [26] S. Mal, A. Majumdar, and D. Home, Sharing of nonlocality of a single member of an entangled pair of qubits is not possible by more than two unbiased observers on the other wing, *Mathematics* **4**, 48 (2016).
- [27] D. Das, A. Ghosal, S. Sasmal, S. Mal, and A. S. Majumdar, Facets of bipartite nonlocality sharing by multiple observers via sequential measurements, *Phys. Rev. A* **99**, 022305 (2019).
- [28] C. Ren, T. Feng, D. Yao, H. Shi, J. Chen, and X. Zhou, Passive and active nonlocality sharing for a two-qubit system via weak measurements, *Phys. Rev. A* **100**, 052121 (2019).
- [29] Peter J. Brown and Roger Colbeck, Arbitrarily many independent observers can share the nonlocality of a single maximally entangled qubit pair, *Phys. Rev. Lett.* **125**, 090401 (2020).
- [30] T. Zhang and S. M. Fei, Sharing quantum nonlocality and genuine nonlocality with independent observables, *Phys. Rev. A* **103**, 032216 (2021).
- [31] S. Cheng, L. Liu, T. J. Baker, M. J. W. Hall, Limitations on sharing Bell nonlocality between sequential pairs of observers, *Phys. Rev. A* **104**, L060201 (2021).
- [32] S. Cheng, L. Liu, T. J. Baker, M. J. W. Hall, Recycling qubits for the generation of Bell nonlocality between independent sequential observers, *Phys. Rev. A* **105**, 022411 (2022).
- [33] J. Zhu, M. J. Hu, G. C. Guo, C. F. Li, Y. S. Zhang, Einstein-Podolsky-Rosen steering in two-sided sequential measurements with one entangled pair, *Phys. Rev. A* **105**, 032211 (2022).
- [34] S. Saha, D. Das, S. Sasmal, D. Sarkar, K. Mukherjee, A. Roy, S. S. Bhattacharya, Sharing of tripartite nonlocality by multiple observers measuring sequentially at one side, *Quant. Inf. Process.* **18**, 42(2019).
- [35] M. J. W. Hall and S. Cheng, Generalising the Horodecki criterion to nonprojective qubit observables, *J. Phys. A: Math. Theor.* **55**, 045301 (2022).
- [36] W. Hou, X. Liu, and C. Ren, Network nonlocality sharing via weak measurements in the

Appendix

Proof of Lemma 2

By straightforward calculation we have

$$\begin{aligned} \rho_{A^1 B^2} &= \frac{2 + \sqrt{1 - \gamma_1^2}}{4} \rho_{A^1 B^1} + \frac{1}{4} (I \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_3 \end{pmatrix}) \rho_{A^1 B^1} (I \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_3 \end{pmatrix}) \\ &\quad + \frac{1 - \sqrt{1 - \gamma_1^2}}{4} (I \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_1 \end{pmatrix}) \rho_{A^1 B^1} (I \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_1 \end{pmatrix}) \end{aligned}$$

and

$$\begin{aligned} \rho_{A^2 B^2} &= \frac{1}{2} \sum_{a,x} (\sqrt{A_{a|x}} \otimes I) \rho_{A^1 B^2} (\sqrt{A_{a|x}} \otimes I) \\ &= \frac{1}{8} ([I_4 + \begin{pmatrix} \cos(\theta)\sigma_3 + \sin(\theta)\sigma_1 & 0 \\ 0 & I_{d-2} \end{pmatrix}] \otimes I) \rho_{A^1 B^2} ([I_4 + \begin{pmatrix} \cos(\theta)\sigma_3 + \sin(\theta)\sigma_1 & 0 \\ 0 & I_{d-2} \end{pmatrix}] \otimes I) \\ &\quad + \frac{1}{8} ([I_4 - \begin{pmatrix} \cos(\theta)\sigma_3 + \sin(\theta)\sigma_1 & 0 \\ 0 & I_{d-2} \end{pmatrix}] \otimes I) \rho_{A^1 B^2} ([I_4 - \begin{pmatrix} \cos(\theta)\sigma_3 + \sin(\theta)\sigma_1 & 0 \\ 0 & I_{d-2} \end{pmatrix}] \otimes I) \\ &\quad + \frac{1}{8} ([I_4 + \begin{pmatrix} \cos(\theta)\sigma_3 - \sin(\theta)\sigma_1 & 0 \\ 0 & I_{d-2} \end{pmatrix}] \otimes I) \rho_{A^1 B^2} ([I_4 + \begin{pmatrix} \cos(\theta)\sigma_3 - \sin(\theta)\sigma_1 & 0 \\ 0 & I_{d-2} \end{pmatrix}] \otimes I) \\ &\quad + \frac{1}{8} ([I_4 - \begin{pmatrix} \cos(\theta)\sigma_3 - \sin(\theta)\sigma_1 & 0 \\ 0 & I_{d-2} \end{pmatrix}] \otimes I) \rho_{A^1 B^2} ([I_4 - \begin{pmatrix} \cos(\theta)\sigma_3 - \sin(\theta)\sigma_1 & 0 \\ 0 & I_{d-2} \end{pmatrix}] \otimes I) \\ &= \frac{1}{2} \rho_{A^1 B^2} + \frac{1}{4} \left(\begin{pmatrix} \cos(\theta)\sigma_3 + \sin(\theta)\sigma_1 & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes I \right) \rho_{A^1 B^2} \left(\begin{pmatrix} \cos(\theta)\sigma_3 + \sin(\theta)\sigma_1 & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes I \right) \\ &\quad + \frac{1}{4} \left(\begin{pmatrix} \cos(\theta)\sigma_3 - \sin(\theta)\sigma_1 & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes I \right) \rho_{A^1 B^2} \left(\begin{pmatrix} \cos(\theta)\sigma_3 - \sin(\theta)\sigma_1 & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes I \right). \end{aligned}$$

Set $P = \cos(\theta)\sigma_3 + \sin(\theta)\sigma_1$ and $Q = \cos(\theta)\sigma_3 - \sin(\theta)\sigma_1$. $\rho_{A^2B^2}$ can be expressed as

$$\begin{aligned}
\rho_{A^2B^2} = & \frac{2 + \sqrt{1 - \gamma_1^2}}{8} \rho_{A^1B^1} \\
& + \frac{1}{8} \left(I \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_3 \end{pmatrix} \right) \rho_{A^1B^1} \left(I \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_3 \end{pmatrix} \right) \\
& + \frac{1 - \sqrt{1 - \gamma_1^2}}{8} \left(I \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_1 \end{pmatrix} \right) \rho_{A^1B^1} \left(I \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_1 \end{pmatrix} \right) \\
& + \frac{1}{16} \left(\begin{pmatrix} P & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_3 \end{pmatrix} \right) \rho_{A^1B^1} \left(\begin{pmatrix} P & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_3 \end{pmatrix} \right) \\
& + \frac{2 + \sqrt{1 - \gamma_1^2}}{16} \left(\begin{pmatrix} P & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes I \right) \rho_{A^1B^1} \left(\begin{pmatrix} P & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes I \right) \\
& + \frac{1 - \sqrt{1 - \gamma_1^2}}{16} \left(\begin{pmatrix} P & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_1 \end{pmatrix} \right) \rho_{A^1B^1} \left(\begin{pmatrix} P & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_1 \end{pmatrix} \right) \\
& + \frac{1}{16} \left(\begin{pmatrix} Q & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_3 \end{pmatrix} \right) \rho_{A^1B^1} \left(\begin{pmatrix} Q & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_3 \end{pmatrix} \right) \\
& + \frac{2 + \sqrt{1 - \gamma_1^2}}{16} \left(\begin{pmatrix} Q & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes I \right) \rho_{A^1B^1} \left(\begin{pmatrix} Q & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes I \right) \\
& + \frac{1 - \sqrt{1 - \gamma_1^2}}{16} \left(\begin{pmatrix} Q & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_1 \end{pmatrix} \right) \rho_{A^1B^1} \left(\begin{pmatrix} Q & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_1 \end{pmatrix} \right),
\end{aligned}$$

where $\rho_{A^1B^1} = |\varphi\rangle\langle\varphi|$.

Therefore,

$$\begin{aligned}
& Tr[\rho_{A^2 B^2}((A_0 + A_1) \otimes B_0^2)] \\
&= 2Tr[\rho_{A^2 B^2} \left(\begin{pmatrix} \cos(\theta)\sigma_3 & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_3 \end{pmatrix} \right)] \\
&= 2Tr\left[\frac{2 + \sqrt{1 - \gamma_1^2}}{8} \rho_{A^1 B^1} \left(\begin{pmatrix} \cos(\theta)\sigma_3 & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_3 \end{pmatrix} \right) \right. \\
&+ \frac{1}{8} \rho_{A^1 B^1} \left(\begin{pmatrix} \cos(\theta)\sigma_3 & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_3 \end{pmatrix} \right) \\
&+ \frac{1 - \sqrt{1 - \gamma_1^2}}{8} \rho_{A^1 B^1} \left(\begin{pmatrix} \cos(\theta)\sigma_3 & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & -\sigma_3 \end{pmatrix} \right) \\
&+ \frac{1}{16} \rho_{A^1 B^1} \left(\begin{pmatrix} P \cos(\theta)\sigma_3 P & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_3 \end{pmatrix} \right) \\
&+ \frac{2 + \sqrt{1 - \gamma_1^2}}{16} \rho_{A^1 B^1} \left(\begin{pmatrix} P \cos(\theta)\sigma_3 P & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_3 \end{pmatrix} \right) \\
&+ \frac{1 - \sqrt{1 - \gamma_1^2}}{16} \rho_{A^1 B^1} \left(\begin{pmatrix} P \cos(\theta)\sigma_3 P & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & -\sigma_3 \end{pmatrix} \right) \\
&+ \frac{1}{16} \rho_{A^1 B^1} \left(\begin{pmatrix} Q \cos(\theta)\sigma_3 Q & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_3 \end{pmatrix} \right) \\
&+ \frac{2 + \sqrt{1 - \gamma_1^2}}{16} \rho_{A^1 B^1} \left(\begin{pmatrix} Q \cos(\theta)\sigma_3 Q & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_3 \end{pmatrix} \right) \\
&+ \left. \frac{1 - \sqrt{1 - \gamma_1^2}}{16} \rho_{A^1 B^1} \left(\begin{pmatrix} Q \cos(\theta)\sigma_3 Q & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & -\sigma_3 \end{pmatrix} \right) \right] \\
&= Tr\left[\frac{3 + \sqrt{1 - \gamma_1^2}}{2} \rho_{A^1 B^1} \left(\begin{pmatrix} \cos^3(\theta)\sigma_3 & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \sigma_3 \end{pmatrix} \right) \right. \\
&+ \left. \frac{1 - \sqrt{1 - \gamma_1^2}}{2} \rho_{A^1 B^1} \left(\begin{pmatrix} \cos^3(\theta)\sigma_3 & 0 \\ 0 & I_{d-2} \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & -\sigma_3 \end{pmatrix} \right) \right] \\
&= 2 \cos^3(\theta) c_1^2 - 2 \cos^3(\theta) c_2^2 - (1 + \sqrt{1 - \gamma_1^2}) c_d^2 \\
&+ (1 + \sqrt{1 - \gamma_1^2}) [c_3^2 + c_4^2 + \dots + c_{d-1}^2].
\end{aligned}$$

Since $2 \cos^3(\theta) \leq 2$, $-2 \cos^3(\theta) \leq 2$, $(1 + \sqrt{1 - \gamma_1^2}) \leq 2$ and $(1 + \sqrt{1 - \gamma_1^2}) \leq 2$, where $0 < \gamma_1 < 1$, $0 < \theta \leq \frac{\pi}{4}$, and $\sum_{i=1}^d c_i^2 = 1$ we have $Tr[\rho_{A^2 B^2}((A_0 + A_1) \otimes B_0)] \leq 2$.

Similarly, we have

$$\begin{aligned}
& Tr[\rho_{A^2 B^2}((A_0 - A_1) \otimes B_1)] \\
&= 2Tr[\rho_{A^2 B^2} \left(\begin{pmatrix} \sin(\theta)\sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \gamma_1\sigma_1 \end{pmatrix} \right)] \\
&= 2Tr\left[\frac{2 + \sqrt{1 - \gamma_1^2}}{8} \rho_{A^1 B^1} \left(\begin{pmatrix} \sin(\theta)\sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \gamma_1\sigma_1 \end{pmatrix} \right) \right. \\
&\quad + \frac{1}{8} \rho_{A^1 B^1} \left(\begin{pmatrix} \sin(\theta)\sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & -\gamma_1\sigma_1 \end{pmatrix} \right) \\
&\quad + \frac{1 - \sqrt{1 - \gamma_1^2}}{8} \rho_{A^1 B^1} \left(\begin{pmatrix} \sin(\theta)\sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \gamma_1\sigma_1 \end{pmatrix} \right) \\
&\quad + \frac{1}{16} \rho_{A^1 B^1} \left(\begin{pmatrix} P \sin(\theta)\sigma_1 P & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & -\gamma_1\sigma_1 \end{pmatrix} \right) \\
&\quad + \frac{2 + \sqrt{1 - \gamma_1^2}}{16} \rho_{A^1 B^1} \left(\begin{pmatrix} P \sin(\theta)\sigma_1 P & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \gamma_1\sigma_1 \end{pmatrix} \right) \\
&\quad + \frac{1 - \sqrt{1 - \gamma_1^2}}{16} \rho_{A^1 B^1} \left(\begin{pmatrix} P \sin(\theta)\sigma_1 P & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \gamma_1\sigma_1 \end{pmatrix} \right) \\
&\quad + \frac{1}{16} \rho_{A^1 B^1} \left(\begin{pmatrix} Q \sin(\theta)\sigma_1 Q & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & -\gamma_1\sigma_1 \end{pmatrix} \right) \\
&\quad + \frac{2 + \sqrt{1 - \gamma_1^2}}{16} \rho_{A^1 B^1} \left(\begin{pmatrix} Q \sin(\theta)\sigma_1 Q & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \gamma_1\sigma_1 \end{pmatrix} \right) \\
&\quad \left. + \frac{1 - \sqrt{1 - \gamma_1^2}}{16} \rho_{A^1 B^1} \left(\begin{pmatrix} Q \sin(\theta)\sigma_1 Q & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} I_{d-2} & 0 \\ 0 & \gamma_1\sigma_1 \end{pmatrix} \right) \right] \\
&= 0.
\end{aligned}$$