Local-dimension-invariant Calderbank-Shor-Steane Codes with an Improved Distance Promise

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Quantum computers will need effective error-correcting codes. Current quantum processors require precise control of each particle, so having fewer particles to control might be beneficial. Although traditionally quantum computers are considered as using qubits (2-level systems), qudits (systems with more than 2-levels) are appealing since they can have an equivalent computational space using fewer particles, meaning fewer particles need to be controlled. In this work we prove how to construct codes with parameters $[[2^N, 2^N - 1 - 2N, \geq 3]]_q$ for any choice of prime q and natural number N. This is accomplished using the technique of local-dimension-invariant (LDI) codes. Generally LDI codes have the drawback of needing large local-dimensions to ensure the distance is at least preserved, and so this work also reduces this requirement by utilizing the structure of CSS codes, allowing for the aforementioned code family to be imported for any local-dimension choice.

Mistakes are bound to happen. Quantum computers, much like their classical counterparts, must overcome and correct their errors in order to perform reliable computations. Building a modest sized quantum computer currently is a significant challenge. If, however, qudits, particles with q-levels (or local-dimension q), are used in the place of qubits, particles with just 2-levels, we can drastically reduce the number of *particles* that need protection while still having a similarly sized computational space [1]. This relationship can also be captured in the formula for the number of logical codewords for a qudit stabilizer code, q^k, which states that the number of codewords grows exponentially with the local-dimension q as the base, so increasing q increases the amount of information protected per particle [2]. As of this time there are currently some qudit quantum computers which are in early development [3–6]. We aim here to provide more options for error-correcting codes for qudit systems, as well as show that the local-dimension-invariant framework can provide at least one family with the distance always being promised.

A common tool used for correcting errors in the quantum case are stabilizer codes, the quantum analog of classical linear codes [7]. For these codes, there are four parameters that are traditionally used to specify the effectiveness of the code, written as $[[n, k, d]]_q$. n specifies the number of physical qudits, or particles, that the code uses to protect k logical qudits. The level of protection is specified by d, which is the distance of the code. This provides a limit on the rate of errors we are allowed in our system while still ensuring that the information is preserved [8]. Lastly, q is the number of levels the system has which are being used, also known as the local-dimension, and is assumed throughout this work to be some prime number. The choice of the local-dimension always being a prime number is primarily to ensure unique multiplicative inverses–it is likely that this restriction can be at least somewhat loosened. An ideal quantum code would have both a high number of protected qudits, k, and distance, d. Unfortunately, the Generalized Quantum Hamming Bound, given by:

$$\sum_{j=0}^{\lfloor \frac{d-1}{2} \rfloor} \binom{n}{j} (q^2 - 1)^j \leq q^{n-k}, \tag{1}$$

shows that there's a trade-off between k and d–and some dependence on n and q–at least in the case of non-degenerate codes [2]. In general, the higher the distance is, the fewer the number of logical qudits that are protected. While this bound does not always hold in the case of degenerate codes, for some classes of codes this bound is still true [7, 9].

While most codes have been designed for a given local-dimension value, Chau showed that both the 5-qubit [10] and 9-qubit [11] codes could be transformed into qudit codes with the same parameters and minimal modifications to the code. These examples were extended into a framework allowing all qudit codes with q levels to be transformed into valid qudit codes over p levels, however, it only showed that the distance could be preserved once there are sufficiently many levels; beyond some critical value p^*, which is a function of n, k, d, and q [12]. These works paved the path to using codes over various local-dimensions, however, this latter work left a large gap between q (the initial dimension)

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and p^* where statements about the distance cannot be made without manually determining the distance. In this report we consider the subclass of stabilizer codes known as Calderbank-Shor-Steane (CSS) codes which allows for a significantly reduced local-dimension requirement for this class in order for the distance to be preserved. Following this we consider one family of a qubit CSS code and show that at least one local-dimension-invariant (LDI) representation can be used with the same parameters regardless of the local-dimension.

I. BACKGROUND

In this section we provide a brief background on qudit stabilizer operators. For a more full review see [2]. In order to allow for easier discussion of qubits and qudits, the general term "particle" or "register" is used. Throughout this work we set $\mathbb{Z}_q = \{0, 1, \ldots, q-1\}$, where q is a prime number.

For qudits the standard Pauli matrices are replaced by the generalized Paulis X_q and Z_q, which act as follows:

$$X_q|j\rangle = |j \oplus 1\rangle, \quad Z_q|j\rangle = \omega^j|j\rangle, \quad \omega = e^{2\pi i/q}, \quad j \in \mathbb{Z}_q, \tag{2}$$

where $\oplus$ is addition mod q. For notational clarity, we will drop the subscript on these operators. These matrices, like their qubit counterparts, form a complete basis for the set of Unitary operators in $SU(\mathbb{C}^q)$, which means that any error can be decomposed as a linear combination of powers of these operators [13]. These operators also form a group, which over a single particle is indicated by $\mathbb{P}_q$. This group structure is preserved over tensor products of these operators, so a generalized Pauli acting on n particles will be in the group $\mathbb{P}_q^n$.

The generalized Pauli operators follow the following commutation relation: $X^aZ^b = \omega^{-ab}Z^bX^a$. A stabilizer code with $n-k$ generators is equivalent to a commuting subgroup of size q^{n-k} of $\mathbb{P}_q^n$ which does not contain a nontrivial multiple of the identity operator.

Upon quotienting out the leading scalar, the Pauli operators can be written as vectors using the symplectic representation. As we will be varying the local-dimension we use the slightly more flexible ϕ representation which also permits specification of the local-dimension. This mapping carries $\phi_q : \mathbb{P}_q^n \mapsto \mathbb{Z}_q^{2n}$ and is defined by:

$$\phi_q\left(\bigotimes_{t=1}^{n} X^{a_t}Z^{b_t}\right) = \left(\bigoplus_{t=1}^{n} a_t\right) \bigoplus \left(\bigoplus_{t=1}^{n} b_t\right), \tag{3}$$

where in the above $\bigoplus$ is a direct sum symbol. This mapping follows the law of composition of $\phi_q(p_1 \circ p_2) = \phi_q(p_1) \oplus \phi_q(p_2)$, where $\oplus$ is entry-wise addition mod q and p_1 and p_2 are two generalized Pauli operators. We will write a vertical bar between the vector for the X powers and the vector for the Z powers mostly for ease of reading. The special case of ϕ_∞ which follows the same relations, but for this representation one does not take modulo any value but carries computations over the integers, was proven in [12]. As an example of the difference, consider the following:

$$\phi_2(X \otimes Z^{-1} \otimes I \otimes XZ) = (1\ 0\ 0\ 1 \mid 0\ 1\ 0\ 1), \quad \phi_\infty(X \otimes Z^{-1} \otimes I \otimes XZ) = (1\ 0\ 0\ 1 \mid 0\ -1\ 0\ 1). \tag{4}$$

In the ϕ representation the commutator of two generalized Paulis, p_1 and p_2, is written and computed as $\phi(p_1) \odot \phi(p_2) = \vec{a}^{(1)} \cdot \vec{b}^{(2)} - \vec{b}^{(1)} \cdot \vec{a}^{(2)}$. This is not formally a commutator, but when this is zero, or zero modulo the local-dimension, the two operators commute, while otherwise it is a measure of number of times an X operator passed a Z operator without a corresponding Z operator passing an X operator.

The distance of a stabilizer code is a measure of its error-protecting capabilities. The standard choice of the depolarizing channel is considered here, meaning that the distance of the code is measured in terms of the Pauli weight of the error, given by the number of non-identity operators in the Pauli error. To aid in determining which error might have occurred the *syndrome* values are computed by finding the commutator of the error with each of the generators for the stabilizer code.

A stabilizer code is written in the ϕ representation as a matrix whose rows are a set of $n-k$ generators for the subgroup. There are some operations that we may perform which must preserve the parameters of the code, these include, in the ϕ representation:

- Row swaps, corresponding to relabelling the generators.
- Swapping columns i and $i+n$ with j and $j+n$, corresponding to relabelling particles.
- Multiplying a row by any number in $\{1, \ldots, q-1\}$, corresponding to composing that generator with itself.
- Adding row i to row j, corresponding to composing the operators.

- Swapping column i with -1 times column $i+n$, corresponding to a discrete-fourier transform (DFT) on particle i; the qudit analog of the Hadamard gate.

We neglect the phase gate, $\sqrt{Z}$, since we do not use it here, but it would also preserve the parameters of the code. Note though that the SUM gate (qudit CNOT gate) will not usually preserve the distance of the code so we do not allow ourselves to perform that operation on our codes [14]. Now that the tools have been presented we proceed to our results.

II. LOCAL-DIMENSION-INVARIANT CSS CODES

We now proceed to a new result related to local-dimension-invariant (LDI) codes in the case of Calderbank-Shor-Steane (CSS) codes. CSS codes have a set of independent generators only using X operators or Z operators for each generator. While not the only way to generate local-dimension-invariant codes, the prescriptive method from [12] provides one such method. Any stabilizer code $\mathcal{S}$ can be put into canonical form, which is given by:

$$\mathcal{S} = \left[I_{n-k} \;\; X_2 \;\mid\; Z_1 \;\; Z_2 \right]. \tag{5}$$

The first $(n-k) \times (n-k)$ block is the $n-k$ dimensional identity matrix, while X_2 is a $(n-k) \times k$ matrix, Z_1 is a $(n-k) \times (n-k)$ matrix, and Z_2 is a $(n-k) \times k$ matrix, where the last three matrices' entries are determined from the operations performed to make the first $(n-k) \times (n-k)$ block the identity matrix. Then the prescriptive method is to then transform it into:

$$\mathcal{S}' = \left[I_{n-k} \;\; X_2 \;\mid\; Z_1 + L \;\; Z_2 \right] \tag{6}$$

where L is a $(n-k) \times (n-k)$ matrix whose nonzero entries are given by: $L_{ij} = \phi_\infty(s_i) \odot \phi_\infty(s_j)$, when $i > j$. $\mathcal{S}'$ satisfies $\phi_\infty(s_i') \odot \phi_\infty(s_j') = 0$ and $\mathcal{S}' = \mathcal{S} \mod q$. This merely generates a set of commuting operators, but ensuring the distance of the code is a far more challenging task.

Figure 1 illustrates the initial distance promises proven for arbitrary non-degenerate codes, as well as the improvements shown in this work. For local-dimensions p with $q < p < p^*$, the distance of the code is uncertain. For this uncertainty region the distance could be lower, it could be the same, or it could even be higher, however, it must be determined manually. It is conjectured that for $q \leq p$ the distance can be at least preserved, however, this is only known to be possible for a few codes, such as the 5, 7, and 9 particle codes [10–12]. This work adds to this collection an infinite family. While the distance is promised to be at least the same for $p^* < p$, the caveat is that the current bound for p^* for an arbitrary non-degenerate stabilizer code is very large: $p^* = B^{2(d-1)}(2(d-1))^{d-1}$ with B being the largest entry in the LDI form, which is bounded by $B \leq (2 + k(q-1))(q-1)$ [12]. This value is typically large, so reductions to it is crucial to make the technique of more practical use. We show next that in the class of CSS codes we may reduce this cutoff value roughly quadratically: $p^*_{CSS} \approx \sqrt{p^*}$.

Theorem 1. *For all primes p, $p^*_{CSS} < p$, with $p^*_{CSS} = B^{d-1}(d-1)^{(d-1)/2}$, the distance of a local-dimension-invariant non-degenerate CSS code with parameters $[[n,k,d]]_q$ used over p levels with parameters $[[n,k,d']]_p$, has $d' \geq d$.*

In order to prove Theorem 1 we begin by recalling a pair of definitions used to categorize undetectable errors [12]. Undetectable errors are those Pauli operators whose syndrome values for all generators are congruent to zero, but are not themselves part of the subgroup formed by the generators of the code. The following definitions form a complete description of all undetectable errors:

Definition 2. *An unavoidable error is an error that commutes with all stabilizers and produces the $\vec{0}$ syndrome over the integers.*

These were dubbed such as no matter the local-dimension these errors will always be unable to be detected, this contrasts with the other category:

Definition 3. *An artifact error is an error that commutes with all stabilizers but produces at least one syndrome that is only zero modulo the base.*

For errors within this category the fact that these errors are not able to be detected is a feature of the local-dimension. If this code, and this error, were considered over a different local-dimension then the error might be able to be detected, meaning the inability to notice this error is merely an artifact of the local-dimension of the system.

With these definitions, the proof of Theorem 1 follows using largely the same argument as Theorem 16 from [12].

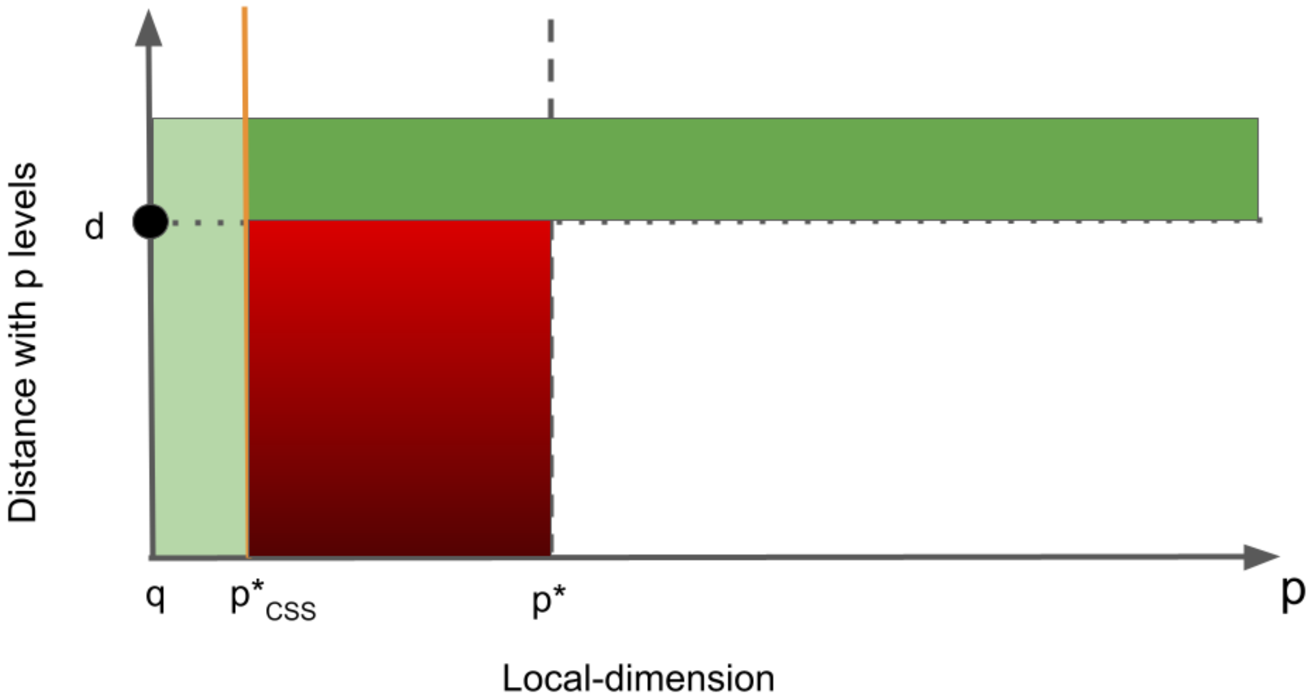

Figure 1. This schematic image illustrates the effect of finding a reduced expression for the cutoff value for the distance promise. The light green area on the left is the set of local-dimension values which must have their distance manually checked, the red region is the set of local-dimension values which must have their distance manually checked for a CSS code if the general p^* expression is used, whereas when p^*_{CSS} is used that region automatically has the distance promised.

Proof. We begin by recalling that Example 15 from [12] showed that LDI codes, using the given method, preserve CSS structure. CSS codes take the structure:

$$\begin{bmatrix} \mathcal{X} & \mid & 0 \\ 0 & \mid & \mathcal{Z} \end{bmatrix}. \tag{7}$$

The distance preservation property may then be considered separately within the two blocks $\mathcal{X}$ and $\mathcal{Z}$. Each block only has nonzero valued syndrome values for Z and X terms in a Pauli respectively. Given this, the Pauli weight for any error activating syndromes in just the $\mathcal{X}$ portion is the same as the Hamming weight for those operators. This is likewise the case for the $\mathcal{Z}$ portion. This then means that in order to have an undetectable error it must be undetectable within the $\mathcal{X}$ portion and within the $\mathcal{Z}$ portion.

Next, undetectable errors correspond to nontrivial kernels of matrix minors. Consider building up the errors composed solely of Z operators by their Hamming weight w. The kernel is nontrivial if the $w \times w$ matrix minor corresponding to this Hamming string has a determinant that is congruent to zero modulo the local-dimension. To avoid tracking the locations of the Hamming weights we allow ourselves to arbitrarily permute the entries within each portion of the CSS code. The determinant for a minor can be zero in two different ways:

- If the determinant is 0 over the integers then this error is either an unavoidable error or an error whose existence did not occur due to the choice of the local-dimension.
- If the determinant is not 0 over the integers, but is merely some multiple of the local-dimension, then this corresponds to an artifact error.

This means that we only have to worry about this second case lowering the distance upon changing the local-dimension. If the determinant of this minor can be guaranteed to be smaller than the local-dimension then we are promised that the distance will at least remain the same. We may bound the determinant of a $w \times w$ matrix minor using Hadamard's inequality, which we evaluate at $d-1$ since we only need to ensure no artifact errors are induced prior to weight d. This then provides:

$$p^*_{CSS} = B^{d-1}(d-1)^{(d-1)/2}. \tag{8}$$

Then so as long as the local-dimension is larger than this no Z error will result in a distance lower than d. This same argument can be made for X errors, which completes the proof. $\square$

In essence this is based on needing only to perform the prior distance promise proof over each of the two blocks in these codes and only needing to worry about the Pauli weight of the error in each block being the same as the Hamming weight. This is slightly better than a quadratic improvement over the general p^* bound. This is a step toward decreasing the value of p^* significantly enough to make this method of practical use. Let us consider the utility of this improvement for the qubit Hamming family, with parameters $[[2^N-1, 2^N-1-2N, 3]]_2$. The first member of this family, H_3, is known as the Steane code with parameters $[[7,1,3]]_2$ [15]. An LDI form, with parameters $[[7,1,\geq 3]]_q$, for this code was found in [12]:

$$\phi_\infty(H_3) = \left[\begin{array}{ccccccc|ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{9}$$

We return this code to CSS form by applying DFTs on particles 4, 5, and 6:

$$\phi_\infty(H_3) = \left[\begin{array}{ccccccc|ccccccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 & 0 \end{array}\right]. \tag{10}$$

Using the bound for p^* without knowing the code is CSS but knowing that the maximal entry is 1, we find $p^* = 16$. This leaves a handful of primes still to verify the distance over. If instead we use Theorem 1 with the known maximal entry of 1 we obtain $p^* = 1^2 2^{2/2}$, which is 2. This means that the distance of this code is always at least preserved. While this was shown in that prior work, this provides a quicker way to obtain this fact and illustrates the improvements obtained here.

The prior description of an LDI representation for the Steane code used the prescriptive method for generating a local-dimension-invariant form for the code, however, we could have equivalently performed the following alteration:

$$\phi_2(H_3) = \left[\begin{array}{ccccccc|ccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{array}\right]. \tag{11}$$

We then put the code into an LDI form by flipping the signs of some of the entries, producing:

$$\phi_\infty(H_3) = \left[\begin{array}{ccccccc|ccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \end{array}\right]. \tag{12}$$

This also satisfies the same distance promise–always having distance at least 3, regardless of the local-dimension. In fact, we can extend this method of flipping the signs of some entries in this code to that of the whole family of codes within the $[[2^N-1, 2^N-1-2N, 3]]_2$ family. We show that this sign flipping can always generate an LDI representation for the code. In particular, we find that for this family we can prove a tight bound on the value of the maximal entry:

Lemma 4. *For the qubit quantum Hamming code family with parameters $[[2^N-1, 2^N-1-2N, 3]]_2$, there is an LDI representation such that $B = 1$ for all members of the family.*

Recall that this family is generated by each column being one of the nonzero binary strings of length N in each the X component and the Z component. We will take the register placement of each string to be the same in the X component and the Z component.

Proof. We begin by noting that for the $N = 3$ case we already have the Steane code discussed just above (equation (12)). We will now prove inductively that we may always generate an LDI form such that $B = 1$. Let H_N^∞ be the N-th family member such that all generators in H_N^∞ commute with those of H_N. Next, consider for $N \geq 4$, one can write the next member of the family in terms of the prior member as:

$$\phi_2(H_{N+1}) = \begin{bmatrix} 1 & 1^{\otimes 2^N-1} & 0^{\otimes 2^N-1} & | & 0 & 0 & 0 \\ 0 & H_N & H_N & | & 0 & 0 & 0 \\ 0 & 0 & 0 & | & 1 & 1^{\otimes 2^N-1} & 0^{\otimes 2^N-1} \\ 0 & 0 & 0 & | & 0 & H_N^\infty & H_N^\infty \end{bmatrix}, \tag{13}$$

with the superscript $\otimes$ indicating repetition of that value. However, this version of the code is not in an LDI representation. We make the following sign changes and then verify that this version is an LDI representation:

$$\phi_\infty(H_{N+1}^\infty) = \begin{bmatrix} 1 & 1^{\otimes 2^N-1} & 0^{\otimes 2^N-1} & | & 0 & 0 & 0 \\ 0 & H_N & H_N & | & 0 & 0 & 0 \\ 0 & 0 & 0 & | & 1 & (-1\ 1)^{\otimes 2^{N-1}-2} \bigoplus (-1) & 0^{\otimes 2^N-1} \\ 0 & 0 & 0 & | & 0 & H_N^\infty & H_N^\infty \end{bmatrix}, \tag{14}$$

where $\bigoplus$ indicates a direct sum, tacking that value onto the end of the repetition. By the inductive hypothesis all operators in H_N commute with those of H_N^∞. Next, the first row commutes with the first nontrivial row in the Z component as there are an equal number of -1 and $+1$. The sum of each row in H_N^∞ is 0, again due to alternating signs, and so the first row commutes with all rows in H_N^∞. We must lastly ensure that $(-1\ 1)^{\otimes 2^{N-1}-2} \bigoplus (-1)$ commutes with each row in H_N. We will denote $v = (-1\ 1)^{\otimes 2^{N-1}-2} \bigoplus (-1)$. Consider the most recently added row in H_N. This will be given by $1^{\otimes 2^{N-1}} \bigoplus 0^{\otimes 2^{N-1}-1}$, which will commute with v. The following rows will be those of $(0 \bigoplus H_{N-2} \bigoplus H_{N-2})^{\otimes 2}$, for which each row commutes with v as for each register in H_{N-2} there will be one time where it is added, $+1$ in v, and one time where it will be subtracted, -1 in v. □

While the ability to write this family in an LDI representation only using $\{0, \pm 1\}$ as the entries is of limited interest, applying this result with Theorem 1 we obtain:

Corollary 5. *All qubit Hamming codes have an LDI representation that has distance at least* 3*, meaning that this generates a* $[[2^N - 1, 2^N - 2N - 1, \geq 3]]_q$ *code family for all* $N \geq 3$ *and* q *a prime.*

Proof. This family is a non-degenerate CSS code family. This result then follows from the CSS distance promise for LDI codes and the above Lemma showing that $B = 1$ may be achieved for this family. □

This shows that the local-dimension-invariant form is at least in some cases able to provide tight expressions that allow for the full importing of code families for larger local-dimension systems. Additionally, this provides another qudit code family with two parameters N and q.

III. DISCUSSION

While few qudit quantum computer prototypes are currently being built this provides another avenue for expanding computational power. While currently the size of the local-dimension must be large to promise the distance of an arbitrary LDI code, we have provided a case where this can be at least quadratically improved. Beyond this improvement we have also constructed a new qudit code family that is directly imported from a qubit code family. While there already exists qudit quantum Hamming families with paramters $[[n, n-2m, 3]]_q$ for $m \geq 2$ in the cases of $gcd(m, q^2-1) = 1,\ n = (q^{2m}-1)/(q^2-1)$ and $gcd(m, q-1) = 1,\ n = (q^m-1)/(q-1)$, this new code family fills in values of n which are not covered by these [2]. Additionally, while there exists $[[n, n-4, 3]]_q$ codes for odd prime power q values and $4 \leq n \leq q^2+1$, this provides options for n beyond q^2+1 [16]. Lastly, while arguably Maximally-Distance-Separable (MDS) codes are optimal, there are a number of values of n for which there are no known MDS codes for a given value of q [17]. While the family presented here is not MDS, perhaps the analysis used in this work can be applied to help fill in missing parameter choices. This work has only concerned itself with preserving the distance of codes in LDI representations, but investigating whether other desirable properties are preserved is also an important direction.

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