Public Verifiable Measurement-Only Blind Quantum Computation Based On Entanglement Witnesses

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Abstract

Recently Sato et al. proposed an public verifiable blind quantum computation (BQC) protocol by inserting a third-party arbiter. However, it isn't true public verifiable in a sense, because the arbiter is determined in advance and participates in the whole process. In this paper, a public verifiable protocol for measurement-only BQC is proposed. The fidelity between arbitrary states and the graph states of **2**-colorable graphs is estimated by measuring the entanglement witnesses of the graph states, so as to verify the correctness of the prepared graph states. Compared with the previous protocol, our protocol is public verifiable in the true sense by allowing other random clients to execute the public verification. It also has greater advantages in the efficiency, where the number of local measurements is $O(n^3 \log n)$ and graph states' copies is $O(n^2 \log n)$.

Keywords: Blind quantum computation, Public verifiability, Graph state, Entanglement witness

1 Introduction

Blind quantum computation (BQC) allows any client (known as Alice) with weak quantum ability to delegate her computing tasks to a quantum server (known as Bob) without leaking her privacy. BQC is divided into two categories: circuit-based BQC (CBQC)[1–5] and measurement-based BQC (MBQC)[6–17]. CBQC realizes blindness through quantum circuits, where the client needs to have the ability to operate some quantum gates. In MBQC, the client only needs to prepare and measure quantum states. Recently, Morimae and Fuji[7] proposed a new type of BQC, called measurement-only BQC (MOBQC), where the server prepare the resource state, while the client just should perform single-qubit measurements.

As more and more BQC protocols are proposed, the *verifiability* of BQC has attracted much attention. In a verifiable BQC protocol, each party can verify whether other party is honest. Although Broadbent et al. [6] have explored the possibility of verifiability in their protocol, it's not complete. Based on the former, Fitzsimons et al.^[8] proposed a relatively complete verifiable version. In this protocol, the verifier encodes the computation task (including the verification mechanism) into a series of single qubits, and then executes BQC. According to the results, it can be verified whether the computation has been correctly executed. In addition, several BQC protocols [2, 8, 9] verifies the correctness of the input of BQC by checking the trap qubits randomly hidden in the input state. For MOBQC^[7], it's proposed to verify graph states^[10–14]. Stabilizer testing[14] is a verification technology of the graph state without setting traps. The server generates graph states and send them to the client, and the latter then directly measure stabilizers on the sent graph states to verifies the correctness. However, these verifiable MOBQC protocols [10, 12–14] using stabilizer test are of high resource consumption, which is an obstacle to the development of scalable quantum computation. In 2021, Xu et al. [15] proposed a verifiable BQC protocol based on entanglement witnesses, which effectively reduces the resource consumption of verification by measuring entanglement witnesses^[18] that can detect the graph states.

The above verifiable protocols only allow Alice to verify Bob's honesty, which is called *private verifiability*. However, private verifiability has the following problems: Alice can detect any dishonest behavior of Bob, but the detection results can not make any third party trusted; even if Bob is honest, he can be framed by Alice. In 2016, on the basis of unconditionally verifiable BQC protocol[8], Kentaro[16] proposed the concept of *public verifiability* and provided a corresponding protocol based on classical cryptography. In 2019, Sato et al.[17] chose to insert a trusted third party as the arbiter to build an arbitrable BQC protocol which realizes public verifiability in a sense. However, the public verifiability depends on the arbiter, which is determined in advance and participates in the whole process, thus the protocol isn't true public verifiable in a sense.

In this paper, inspired by the verifiable mechanism based on entanglement witnesses, we propose a public verifiable MOBQC protocol. The third-party verifier is randomly selected from other clients rather than a specific arbiter, so as to achieve public verifiability in the true sense. In addition, 2-colorable graphs and entanglement witnesses are introduced to reduce resource consumption. Compared with the number of local measurements $(O(n^{2n+5}))$ and of copies of the resource states $(O(n^{2n+5}2^n))$ of Sato et al.[17], our protocol have obvious advantages $(O(n^3 \log n)$ and $O(n^2 \log n)$ respectively). We also consider the communication error and give some error mitigation schemes.

The rest of this paper is organized as follows: In Sect. 2, we briefly introduce 2-colorable graph states, entanglement witnesses, and MOBQC. The protocol is presented in Sect. 3 and analyzed in Sect. 4. Error propagation and mitigation is analyzed in Sect. 5. The paper concludes with Sect. 6.

2 Preliminaries

In this section, we briefly introduce 2-colorable graph states and the entanglement witnesses of them, and then review the basic steps of measurement-only BQC.

2.1 2-colorable graph state

Given an undirected simple graph G with n vertices $i \in V$ and several edges $(i, j) \in E = V \times V$, if all vertices of it can be divided into at least m disjoint subsets S_1, S_2, \dots, S_m , where there is no edge between any pair of vertices in any $S_j, j = 1, 2, \dots, m$, then we call G an m-colorable graph. We use n qubits to represent vertices of G, and the graph state $|G\rangle$ corresponding to G is defined as $|G\rangle = \left(\prod_{(i,j)\in E} U_{ij}\right)|+\rangle^{\otimes n}$, where $|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$ is the initial state of each vertex and U_{ij} is the controlled-Z gate $|0\rangle \langle 0| \otimes I + |1\rangle \langle 1| \otimes Z$ performed on vertices i and j, where I is identity operator and Z is Pauli operator σ_z . On the other hand, there are n stabilizer $g_i = X_i \prod_{k \in N(i)} Z_k$ of $|G\rangle$, i.e., $g_i |G\rangle = |G\rangle$, where $i = 1, 2, \dots, n, N(i)$ is the adjacency points set of vertex i and X_i, Z_j are the Pauli operators σ_x, σ_z performed on i, j respectively.

In this paper, we only consider 2-colorable graph states which are widely used as resource states of BQC, such as brickwork state[6] and Raussendorf-Harrington-Goyal (RHG) state[19], of which the preparation and verification are of research value. An example of a 2-colorable graph is shown in Figure 1.



Fig. 1 A 2-colorable graph as an example, where red vertices belong to S_1 and green vertices belong to S_2 .

2.2 Entanglement witness

An entanglement witness W is an observable measurement which satisfies: (1) For all separable states ρ_s , $tr(W\rho_s) \ge 0$; (2) At least one entangled state ρ_e satisfies $tr(W\rho_e) < 0$, where $tr(\cdot)$ represents matrix trace; then we say that ρ_e is detected by W. For an *n*-qubit graph state and some states close to it, based on the colorability of the graph, some witnesses with constant measurement times are proposed. The following $W^{(2)}$ is a witness of a 2-colorable graph state $|G\rangle[18]$:

$$W^{(2)} = 3I - 2\left[\prod_{i \in S_1} \frac{g_i + I}{2} + \prod_{i \in S_2} \frac{g_i + I}{2}\right],\tag{1}$$

where S_1, S_2 are two divided sets of the graph. According to the structure of the witness, for a given 2-colorable graph state, only two measuring settings are needed, and the *j*-th setting is observable $\prod_{i \in S_j} g_i$. The two settings corresponding to the two-colorable graph in Figure 1 are shown in Figure 2. For the *j*-th measuring setting $\prod_{i \in S_j} g_i$, we only need to measure the qubits corresponding to S_j , and then measure the qubits corresponding to another subset $\overline{S_j}$ according to the adjacency relationship with S_j . Therefore, a setting $\prod_{i \in S_j} g_i$

only needs O(n) local measurement times.



Fig. 2 The two settings corresponding to the graph in Figure 1, where all red vertices belong to S_1 and all green vertices belong to S_2 . (a)(b) are the observables $\prod_{i \in S_1} g_i, \prod_{i \in S_2} g_i$ corresponding to S_1, S_2 respectively.

2.3 Measurement-only blind quantum computation

In MOBQC, the server Bob only needs to prepare the general resource state, and the client Alice only needs to perform quantum measurement. The protocol steps are as follows: Bob prepares the general resource state and then sends the prepared state particles to Alice via quantum channel, then Alice measures the sent particles on the basis determined by her algorithm. The verification of this model is generally aimed at the correctness of the resource state. Bob is often required to prepare multiple copies of the resource state, and some of which are used for verification and one of the rest is used for calculation, as shown in Figure 3.



Fig. 3 The verification process of MOBQC.

3 Public Verifiable Measurement-only Blind Quantum Computation based on entanglement witnesses

3.1 Verification algorithm

Inspired from Xu et al.'s verification mechanism[15], we present a verification algorithm to verify the correctness of the prepared graph state. Given a target graph state $|G\rangle$ corresponding to a 2-colorable graph G, and an unknown state ϱ to be verified. The two divided subsets of G are denoted as S_1, S_2 , and the verification process is shown in Algorithm 1.

In Algorithm 1, the condition constant C is determined to make sure the fidelity between the prepared state ρ and the required state $|G\rangle$ is high enough. Considering the fidelity estimation process, C isn't fixed, but varies with the order of the verifier, i.e., C will be different for the third-party verifier from the client in our protocol. Therefore, we set C as a pending parameter so as to ensure the scalability of the verification. Based on the above, Algorithm 1 can be applied to public verification.

3.2 Proposed protocol

In Sato et al.'s protocol[17], Charlie, the third-party arbiter, can arbitrate in case of a dispute between the server Bob and the client Alice. However, the verifier (Charlie) is determined in advance and participates in the whole process, which isn't a true third party independent with Bob and Alice. To achieve a true public verification, i.e., any third party can participate in verification, we removed Charlie's role of third-party verifier, but only retained its storage capacity of quantum state, therefore it's renamed the storage center. The third party that participates in the public verification will be selected from other clients Alice₂, Alice₃, \cdots , Alice_l randomly, where l is the total number

Algorithm 1 Verification Algorithm: verify the correctness of the prepared graph state ρ

Require: 2K *n*-qubit registers, a required graph state $|G\rangle$, and a constant $0 \le C \le 2K$.

Ensure: The 2K registers store the same prepared *n*-qubit state ρ .

- 1: Select K registers independently, evenly and randomly from 2K registers and mark it as the 1-st group, then mark the rest K registers as the 2-nd group.
- 2: Measure observable $\prod_{i=1}^{n} g_i$ on each register in the *j*-th group.
- 3: Calculate M_j^ℓ by M_j^ℓ = ∏_{i∈S_j} x_i ∏_{k∈N(i)} z_{k+1}/2, where x_i, z_j are the results of measuring observables X_i, Z_j, and count the number of registers satisfy M_j^ℓ = 0 as K_j.
 4: if K₁ + K₂ ≤ C then
- 5: Accept.
- 6: **else**
- 7: Reject.
- 8: end if

of clients. As shown in Figure 4, there are three parties in the protocol: the server Bob is responsible for preparing the graph states; the set $A = \{A|ice_1, A|ice_2, \dots, A|ice_l\}$ is a set of clients of quantum computation, which are all legal users [20, 21] registered with Charlie; the storage center Charlie has the ability to store quantum states and is responsible for distributing the graph states prepared by Bob to the client and selecting the third-party verifier, and is ensured honesty. When the protocol is executed between Alice₁ and Bob, any other client Alice_t in A where $t \in \{2, \dots, l\}$ can perform public verification.



Fig. 4 The tripartite relationship in the proposed protocol.

The graph states used in our protocol all correspond to 2-colorable graphs. Taking client Alice₁, who has a computing request, as an example, the specific steps are as follows (also shown in Figure 5):

- Step 1 Alice₁ sends preparation request to Charlie, then Charlie forwards it to Bob, where the graph state $|G\rangle$ requested is an *n*-qubit state corresponding to a 2-colorable graph and $n \ge 6$.
- Step 2 Bob prepares a 5Kn-qubit state $|G\rangle^{5K}$, where $K = \lceil n^2 \log n \rceil, \lceil \cdot \rceil$ is ceiling function. Then he send it to Charlie one by one qubit.
- Step 3 Charlie divides the state sent from Bob into 5K *n*-qubit states $|G\rangle$ in turn and stores them in *n*-qubit registers respectively. He selects 2K registers independently, evenly and randomly from these 5K registers and keeps them, and then sends the rest 3K to Alice₁ in turn.
- Step 4 Alice₁ divides the states sent from Charlie into 3K *n*-qubit registers and selects 2K registers independently, evenly and randomly from them, then executes Verification Algorithm (see Algorithm 1) where $C = \frac{K}{2n}$.
- Step 5 If it accepts, Alice₁ considers Bob honest and proceeds to the next step, and otherwise considers Bob dishonest and refuses to pay for services.
- Step 6 Alice₁ randomly selects one register from the remaining K registers and discards the others, then uses this register to perform MBQC, i.e., measures particles on the basis determined by her algorithm.
- Step 7 If Alice₁ claims that Bob is dishonest, Bob can ask Charlie for public verification, and then Charlie randomly selects a third party Alice_t from Alice₂, Alice₃, \cdots , Alice_l to send verification request.
- Step 8 If Alice_t accepts the request, Charlie sends the 2K copies in his hand to Alice_t in turn. According to the graph state type, Alice_t executes Verification Algorithm, where $C = \frac{3K}{4n}$. If it accepts, Alice_t claims that Alice₁ is dishonest, otherwise Bob is dishonest.

4 Performance analysis

4.1 Completeness analysis

Completeness means that when Bob faithfully prepares the required graph state, it must be accepted by Alice₁ or Alice_t with high probability. In Step 3 of Algorithm 1, the random variable $M_j^{\varrho} = \prod_{i \in S_j} \frac{x_i \prod_{k \in N(i)} z_k + 1}{2} \in \{0, 1\}$ to be calculated is the measurement result of $\prod_{i \in S_j} \frac{g_i + I}{2}$. According to quantum measurement theory, we have

$$\overline{M}_{j}^{\varrho} = Tr\left(\prod_{i\in S_{j}}\frac{g_{i}+I}{2}\varrho\right),\tag{2}$$



Fig. 5 The process of public verifiable protocol. Solid lines represent graph states transmission and dotted lines represent requests transmission.

where $\overline{M}_{j}^{\varrho}$ is the mathematical expectation of M_{j}^{ϱ} . Assume Bob prepares the correct state $\varrho = |G\rangle \langle G|$, then we have

$$\overline{M}_{j}^{\varrho} = Tr\left(\prod_{i \in S_{j}} \frac{g_{i} + I}{2} \left| G \right\rangle \left\langle G \right|\right) = Tr\left(\left| G \right\rangle \left\langle G \right|\right) = 1.$$
(3)

We have $M_j^{\varrho} \in \{0, 1\}$, which means for each register $k \in \Pi^{(j)}$, $M_j^{\varrho_k} = 1$, i.e., $K_j = 0$. Therefore $\forall 0 \leq C \leq 2K, K_1 + K_2 \leq C$, i.e., it must be accepted, leading to the completeness.

4.2 Soundness analysis

Soundness means that if Alice₁ or Alice_t accepts the state ρ prepared by Bob, it must close to the graph state required by Alice₁ with high probability. Fidelity $F = \langle G | \rho | G \rangle$ is generally used to measure the closeness. Fidelity estimation is based on the following inequality [15]:

$$F \ge \frac{1}{2} - \frac{1}{2} Tr\left(W^{(2)}\varrho\right). \tag{4}$$

In our protocol, the expectation $Tr(W^{(2)}\varrho)$ is obtained by measuring the entanglement witness $W^{(2)}$ detecting $|G\rangle$ to determine whether the state ϱ is close to $|G\rangle$, so as to verifies the behavior of Bob. We have the following theorem about the soundness of our protocol:

Theorem 1 In our protocol,

(1) When Alicet hasn't involved in arbitration, if Alice₁ measures $K_1 + K_2 \leq \frac{K}{2n}$, then we have

$$F \ge 1 - \frac{4\sqrt{\lambda_1} + 1}{n} \tag{5}$$

with a probability

$$P \ge 1 - 4n^{-\frac{\lambda_1}{2}},\tag{6}$$

where λ_1 is arbitrary constant satisfying $\log_n 16 \leq \lambda_1 \leq \frac{(n-1)^2}{16}$.

(2) When Alicet involves in arbitration, if Alicet measures $K_1 + K_2 \leq \frac{3K}{4n}$, then we have

$$F \ge 1 - \frac{3\sqrt{\lambda_2} + 1}{n} \tag{7}$$

with a probability

$$P \ge 1 - 4n^{-\lambda_2},\tag{8}$$

where λ_2 is arbitrary constant satisfying $\log_n 4 \leq \lambda_2 \leq \frac{(n-1)^2}{9}$.

Proof In the theorem, (1) has been proved [15] and we only need to prove (2), using some existing probability inequalities. If we perform the *j*-th measurement setting on the rest 3K registers, then for each group of K registers selected by Charlie, we can obtain an upper bound of the number of registers satisfying $M_j^{\varrho} = 0$ in the rest 3Kregisters measured and the relevant confidence probability, Hence, the lower bound of $\sum_{i=1}^{3K} M_{i}^{\varrho_k}$ is directly given. We then obtain a lower bound of \overline{M}_i^{ϱ} and the relevant

of $\sum_{k=1}^{3K} M_1^{\varrho_k}$ is directly given. We then obtain a lower bound of \overline{M}_1^{ϱ} and the relevant

^{$\kappa=1$} confidence probability. By Eq. 2 we can obtain a lower bound of $Tr\left(\prod_{i\in S_j} \frac{g_i+I}{2}\varrho\right)$.

By Eq. 1 and Eq. 4, we finally prove that the fidelity F satisfies a lower bound with a certain confidence probability P. See Appendix A for details.

Therefore, if the verification is passed, then the state prepared by Bob is close to the required graph state with high probability, leading to the soundness.

In the protocol, $K = O(n^2 \log n)$ so that the probability in the theorem is $1 - O(n^{-\lambda})$ for a constant λ , which is high enough; we set $n \ge 6$ so that the above λ exists. See Appendix A for details.

4.3 Efficiency analysis

Efficiency refers to the number of copies of the resource state prepared by the protocol and the number of local measurements required. As mentioned above, we set $K = O(n^2 \log n)$ so that the probability is high enough. The detail parameters of our and Sato et al.'s protocol[17] are shown in Table 1.

Protocol	Copies numbers	Parameters ranges	Observables measured
Sato et al.'s	$ G\rangle^{2k+m+1}$	$k \ge 4n^2 - 1, \ m \ge (2\ln 2) \ K^n n^5$	$\prod_{i=1}^{n} g_i^{k_i}$
Our	$ G\rangle^{5K}$	$K \ge \left\lceil n^2 \log n \right\rceil$	$\prod_{i\in S_j}^n g_i$

Table 1 The detail parameters of public verifiable MBQC protocols

We first consider the number of copies. For the same *n*-qubit graph state $|G\rangle$, in Sato et al.'s protocol, the number of copies is

$$2k + m + 1 \ge 8n^2 - 2 + (2\ln 2) k^n n + 1$$

= 8n^2 - 2 + (2 ln 2) k^n n + 1, (9)
= $\Theta(n^{2n+5})$

i.e., $O\left(n^{2n+5}\right)$ at least; in our protocol it is $5K \ge 5 \left\lceil n^2 \log n \right\rceil = \Theta\left(n^2 \log n\right)$, i.e., $O\left(n^2 \log n\right)$ at least. Thus our protocol has advantages.

Then, the number of required local measurement is taken into account. In Sato et al.'s protocol, they measured stabilizers $\prod_{i=1}^{n} g_i^{k_i}$ on k or 2k n-qubit graph states, where $k = (k_1k_2...k_n)$ is a string randomly selected. Considering that the local measurement decomposition of $\prod_{i=1}^{n} g_i^{k_i}$ may be complex, and for a specific graph structure it may be $O(2^n)[22]$, thus the number of local measurements is $O(kn) = O(n^{2n+5}2^n)$; in our protocol, we measure observables $\prod_{i\in S_j}^{n} g_i$ on 2K n-qubit graph states and for each graph state the number of the local measurements is O(n) as mentioned above, thus the total number is $O(Kn) = O(n^3 \log n)$. Therefore, our protocol still has advantages.

5 Error propagation and mitigation

It is worth noting that all the above analyses are based on the fact that the quantum channel does not contain noise. However, the actual channel has a certain amount of noise, so there must be certain errors in the graph state's propagation. Since Bob is not assumed to be honest in our protocol (i.e., the sent graph state is not guaranteed to be correct), it is impossible to determine whether the graph state is disturbed by noise by comparing it with the target state. To mitigate the impact of noise, we have the following two methods.

(1) Use channel noise detection For each n-qubit graph state $|G\rangle$, the sender (Bob or Charlie) insert some additional qubits that are not entangled with the graph state into the n qubits. These qubits' initial states are agreed in advance by both the sender and the receiver (Charlie or Alice), and they will be sent together with the graph state as a whole. In this way, if some noise

is encountered during one transmission, the state of these extra qubits will be changed with a certain probability, and thus the noise can be detected. If the noise is considered to be too high, the receiver may reject the communication and request retransmission.

For example, assuming that k additional qubits are in the initial state $|0\rangle$, bit-flip noise occurs in the quantum channel with probability p, and r qubits are flipped to $|1\rangle$ after transmission. Since r has a binomial distribution, its expectation is kp. By Azuma-Hoeffding bound[23] (see Appendix A for details), $\forall t > 0$, we have

$$\Pr\left(\left(1-\frac{r}{k}\right)-(1-p)\leq t\right)=\Pr\left(p\leq\frac{r}{k}+t\right)\geq 1-\exp\left(-2kt^2\right).$$
 (10)

If p_{th} is the noise threshold, then

$$\Pr\left(p \le p_{th}\right) \ge 1 - \exp\left(-2k\left(p_{th} - \frac{r}{k}\right)^2\right). \tag{11}$$

Let $1 - \exp\left(-2k\left(p_{th} - \frac{r}{k}\right)^2\right) \ge 99\%$, then $r \le kp_{th} - p\sqrt{k \ln 10} = r_{th}$. For instance, let k = 5, p = 1%, then $r_{th} = 3.44$. If $r \le 3$, then we can say $\Pr\left(p \le p_{th}\right) \ge 99\%$. The larger the k, the smaller the $\frac{r_{th} - \lfloor r_{th} \rfloor}{r_{th}} \le \frac{1}{r_{th}}$, and the tighter the upper bound. Other noise types can be detected similarly, and only the corresponding initial states and measurement bases need to be agreed.

(2) Use fault-tolerant quantum computing (FTQC) As mentioned by Morimae et al.[7], using a computational model that can handle particle losses can effectively mitigate noise. An [n, k, d] quantum error-correcting codes (QECC) encodes n physical qubits into k logical qubits, and a QECC with distance d can correct up to $\frac{d-1}{2}$ errors on arbitrary qubits[24]. The entanglement of the graph state will not be destroyed in a qubit stabilizer QECC scheme, because it is not necessary to really know the initial state of the target qubit, but only to measure and compare the relative change between the physical qubits. On the other hand, only the measurements and quantum gates of single-qubit Pauli operators X, Y, Z are required, which means that even receiver with weak quantum ability can implement it. In existing fault-tolerant quantum computing, the noise threshold can even reach 24.9%[25].

Note that the above two methods can be used in combination because they are independent of each other. First, the channel noise detection can ensure that the noise factor is lower than a certain threshold, and then the fault tolerance mechanism can correct small errors. By using the two methods, the error caused by channel noise can be mitigated to a certain extent. Of course, when the noise reaches a certain level, even retransmission will fail.

6 Conclusion

In this paper, we proposes a public verifiable measurement-only blind quantum computation protocol. By introducing a storage center, it allows the thirdparty verifier to be any other client randomly selected. Compared with the previous protocol, our protocol is public verifiable in the true sense. In the protocol, the fidelity estimation between arbitrary states and graph states are realized by measuring the entanglement witnesses detecting the graph states. Without loss of completeness and soundness, the nature of 2-colorable graph states reduce the number of local measurements $(O(n^3 \log n))$ and the number of copies of the graph states resources $(O(n^2 \log n))$. Compared with the arbitrable protocol of Sato et al.[17] (the number of local measurements is $O(n^{2n+5})$ and of copies of the resource states is $O(n^{2n+5}2^n)$), our protocol has obvious advantages. We also consider the communication error and give some error mitigation schemes.

On the other hand, since we have only considered 2-colorable graphs, the proposed protocol is not applicable to arbitrary graph states. For more general graph states, more research is needed to further improve the efficiency and performance of existing schemes.

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Appendix A Proof of Theorem 1

In the theorem, (1) has been proved [15] with a condition $n \ge 6$. Now we prove (2). At first we introduce the following two probability bounds which will be used in the analysis, where $Pr(\cdot)$ represents the event probability and $E(\cdot)$ represents the mathematical expectation:

(1) Serfling's bound[26] Given a set $Y = (Y_1, Y_2, ..., Y_T)$ of T binary random variables with $Y_k \in \{0, 1\}$ and two arbitrary positive integers N and K that satisfy T = N + K, select K samples that are distinguished from each other independently, evenly and randomly from Y, and let Π be the set of these samples, $\overline{\Pi} = Y - \Pi$, then $\forall 0 < v < 1$, we have

$$\Pr\left(\sum_{k\in\overline{\Pi}}Y_k \le \frac{N}{K}\sum_{k\in\Pi}Y_k + Nv\right)$$

$$\ge 1 - \exp\left(-\frac{2v^2NK^2}{(N+K)(K+1)}\right).$$
(A1)

(2) Azuma-Hoeffding bound[23] Given independent random variables $\xi_1, \xi_2, \dots, \xi_n$ where $\xi_i \in [a_i, b_i], i = 1, 2, \dots, n$, then $\forall t > 0$, we have

$$\Pr\left(\frac{\xi_1 + \xi_2 + \dots + \xi_n}{n} - \operatorname{E}\left(\frac{\xi_1 + \xi_2 + \dots + \xi_n}{n}\right) \le t\right)$$

$$\ge 1 - \exp\left(-\frac{2n^2t^2}{\sum\limits_{i=1}^n (b_i - a_i)}\right).$$
(A2)

For the first K registers selected, we denote them as $\Pi^{(1)}$ and the rest 4K as $\overline{\Pi}^{(1)}$. Let $T = 5K, N = 4K, Y_k = \begin{cases} 0, M_1^{\varrho'_k} = 1\\ 1, M_1^{\varrho'_k} = 0 \end{cases}$, where ϱ'_k is the state in the k-th register in $\Pi^{(1)}$ or $\overline{\Pi}^{(1)}$, then we have

$$\Pr\left(\sum_{k\in\overline{\Pi}^{(1)}}Y_k \le \frac{4K}{K}\sum_{k\in\Pi^{(1)}}Y_k + 4Kv\right) \ge 1 - \exp\left(-\frac{2v^24KK^2}{(4K+K)(K+1)}\right)$$
(A3)

by Eq A1, which means if we perform the *j*-th measurement on the rest 4K registers, then the upper bound of the number of the registers satisfying $M_1^{\varrho} = 0$ (i.e., $Y_k = 1$) in $\overline{\Pi}^{(1)}$ is $4 \sum_{k \in \Pi^{(1)}} Y_k + 4Kv$, with the probability on the right-side of Eq A3. Similarly, for the second K registers selected, we denote them as $\Pi^{(2)}$ and the rest 3K as $\overline{\Pi}^{(2)}$. Let $T = 4K, N = 3K, Y_k = \begin{cases} 0, M_2^{\varrho'_k} = 1 \\ 1, M_2^{\varrho'_k} = 0 \end{cases}$, where ϱ'_k is the state in the k-th register in $\Pi^{(2)}$ or $\overline{\Pi}^{(2)}$, then we have

$$\Pr\left(\sum_{k\in\overline{\Pi}^2}Y_k \le \frac{3K}{K}\sum_{k\in\Pi^2}Y_k + 3Kv\right) \ge 1 - \exp\left(-\frac{2v^2 3KK^2}{(3K+K)\left(K+1\right)}\right),\tag{A4}$$

which means if we perform the *j*-th measurement on the rest 3K registers, then the upper bound of the number of the registers satisfying $M_2^{\varrho} = 0$ in $\overline{\Pi}^{(2)}$ is $4 \sum_{k \in \Pi^{(2)}} Y_k + 4Kv$, with the probability on the right-side of Eq A4. In the protocol, any two clients do not trust each other, thus it can be considered that the rest 3K registers haven't been measured. If we perform the first measurement on the rest 3K registers, there will be $3K - \left(4\sum_{k \in \Pi^{(1)}} Y_k + 4Kv\right)$

registers satisfying $M_1^{\varrho} = 1$ at least, i.e.,

$$\sum_{k=1}^{3K} M_1^{\varrho_k} \ge 3K - \left(4\sum_{k\in\Pi^{(1)}} Y_k + 4Kv\right).$$
 (A5)

Similarly, we have

$$\sum_{k=1}^{3K} M_2^{\varrho_k} \ge 3K - \left(3\sum_{k\in\Pi^{(2)}} Y_k + 3Kv\right).$$
(A6)

Let $n = 3K, \xi_k = M_1^{\varrho}$ or M_2^{ϱ} , then by Eq A2 we have

$$\Pr\left(\frac{1}{3K}\sum_{k=1}^{3K}M_1^{\varrho_k} - \overline{M}_1^{\varrho} \le t\right) \ge 1 - \exp\left(-2 \cdot 3Kt^2\right). \tag{A7}$$

By $Tr\left(\prod_{i\in S_1} \frac{g_i+I}{2}\varrho\right) = \overline{M}_1^{\varrho}$ and Eq A5 we have

$$\Pr\left(Tr\left(\prod_{i\in S_1}\frac{g_i+I}{2}\varrho\right) > 1 - \frac{1}{3K}\left(4\sum_{k\in\Pi^{(1)}}Y_k + 4Kv\right) - t\right), \quad (A8)$$
$$\geq 1 - \exp\left(-6Kt^2\right)$$

and similarly we have

$$\Pr\left(Tr\left(\prod_{i\in S_2} \frac{g_i+I}{2}\varrho\right) > 1 - \frac{1}{3K}\left(3\sum_{k\in\Pi^{(2)}} Y_k + 3Kv\right) - t\right).$$
(A9)
$$\geq 1 - \exp\left(-6Kt^2\right)$$

Therefore, we have

$$\begin{split} F &\geq \frac{1}{2} - \frac{1}{2} Tr\left(W^{(2)}\varrho\right) \\ &= \frac{1}{2} - \frac{1}{2} Tr\left(3I\varrho\right) + \frac{1}{2} Tr\left(2\prod_{i\in S_{1}} \frac{g_{i}+I}{2}\varrho\right) + Tr\left(2\prod_{i\in S_{2}} \frac{g_{i}+I}{2}\varrho\right) \\ &= -1 + 1 - \frac{1}{AK} \left(4\sum_{k\in\Pi^{(1)}} Y_{k} + 4Kv\right) - t + 1 - \frac{1}{3K} \left(3\sum_{k\in\Pi^{(2)}} Y_{k} + 3Kv\right) - t \\ &= 1 - \left(2 + \frac{1}{3}\right)v - 2t - \frac{1}{3K} \left(4\sum_{k\in\Pi^{(1)}} Y_{k} + 3\sum_{k\in\Pi^{(2)}} Y_{k}\right) \\ &\geq 1 - \left(2 + \frac{1}{3}\right)v - 2t - \frac{4}{3K} \left(\sum_{k\in\Pi^{(1)}} Y_{k} + \sum_{k\in\Pi^{(2)}} Y_{k}\right) \\ &= 1 - \left(2 + \frac{1}{3}\right)v - 2t - \frac{4}{3K} \left(K_{1} + K_{2}\right). \end{split}$$
(A10)

with a probability

$$P \ge \left[1 - \exp\left(-\frac{8v^2K}{5\left(1 + \frac{1}{K}\right)}\right)\right] \left[1 - \exp\left(-\frac{3v^2K}{2\left(1 + \frac{1}{K}\right)}\right)\right] \left[1 - \exp\left(-6Kt^2\right)\right]^2,$$

$$\ge \left[1 - \exp\left(-Kv^2\right)\right]^2 \left[1 - \exp\left(-6Kt^2\right)\right]^2$$
(A11)

where the second inequality in Eq A11 holds as long as $K \ge 2$. Obviously $\sum_{k\in\Pi^{(1)}} Y_k = K_1$ and $\sum_{k\in\Pi^{(2)}} Y_k = K_2$ in Eq A10. To make $F = \langle G | \rho | G \rangle$ is $1 - O\left(\frac{1}{n}\right)$, which is high enough, we need that $v = O\left(\frac{1}{n}\right), t = O\left(\frac{1}{n}\right), \frac{4}{3K}(K_1 + K_2) \le \frac{1}{n}$ which leads to $F = 1 - O\left(\frac{1}{n}\right)$. Therefore, we set $v = \frac{\sqrt{\lambda_2}}{n}, t = \frac{\sqrt{\lambda_2}}{\sqrt{6n}}$, then consider the acceptance condition $K_1 + K_2 \le \frac{3K}{4n}$ in Algorithm 1, we have

$$F \ge 1 - \frac{7}{3}\frac{\sqrt{\lambda_2}}{n} - 2\frac{\sqrt{\lambda_2}}{\sqrt{6}n} - \frac{1}{n} = 1 - \frac{\left(\frac{7}{3} + \frac{2}{\sqrt{6}}\right)\sqrt{\lambda_2} + 1}{n} \ge 1 - \frac{3.15\sqrt{\lambda_2} + 1}{n}$$
(A12)

with a probability

$$P \ge \left[1 - \exp\left(-\frac{\lambda_2}{n^2}K\right)\right]^4 \ge 1 - 4\exp\left(-\frac{\lambda_2}{n^2}K\right) \ge 1 - 4\exp\left(-\frac{\lambda_2}{n^2}n^2\log n\right).$$
(A13)

To make the probability $P = 1 - O(n^{-\lambda})$ for a constant λ , we set $K = \lfloor n^2 \log n \rfloor$, then

$$P \ge 1 - 4 \exp\left(-\frac{\lambda_2}{n^2}n^2\log n\right) = 1 - 4n^{-\lambda_2},$$
 (A14)

which is high enough. The condition for above F, P both to be positive is $\log_n 4 \leq \lambda_2 \leq \frac{(n-1)^2}{10}$, where $n \geq 5$. When $n \geq 5$, we have $n^2 > \frac{(n-1)^2}{10}$, thus $\lambda_2 < n^2$, then $v = \frac{\sqrt{\lambda_2}}{n} < 1$. Consider the condition of (1), we have $n \geq 6$.

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