# Open Quantum Random Walks and Quantum Markov chains on Trees II: The recurrence 

Farrukh Mukhamedov*<br>Department of Mathematical Sciences, College of Science, United Arab Emirates University 15551, Al-Ain, United Arab Emirates and<br>Institute of Mathematics named after V.I.Romanovski, 4, University str., 100125, Tashkent, Uzbekistan<br>e-mail: far75m@gmail.com; farrukh.m@uaeu.ac.ae<br>Abdessatar Souissi*<br>Department of Accounting, College of Business Management<br>Qassim University, Ar Rass, Saudi Arabia and<br>Preparatory institute for scientific and technical studies, Carthage University, Amilcar 1054, Tunisia<br>e-mail: a.souaissi@qu.edu.sa; abdessattar.souissi@ipest.rnu.tn<br>Tarek Hamdi<br>Department of Management Information Systems, College of Business Management<br>Qassim University, Ar Rass, Saudi Arabia and<br>Laboratoire d'Analyse Mathématiques et applications LR11ES11<br>Université de Tunis El-Manar, Tunisia<br>e-mail: t.hamdi@qu.edu.sa

Amen Allah Andolsi
Nuclear Physics and High Energy Physics Research Unit, Faculty of Sciences of Tunis, University of Tunis El Manar, 2092 Tunis, Tunisia
e-mail: amenallah.andolsi@fst.utm.tn

* Corresponding authors


#### Abstract

In the present paper, we construct QMC (Quantum Markov Chains) associated with Open Quantum Random Walks such that the transition operator of the chain is defined by OQRW and the restriction of QMC to the commutative subalgebra coincides with the distribution of OQRW. Furthermore, we first propose a new construction of QMC on trees, which is an extension of QMC considered in Ref. [9]. Using such a construction, we are able to construct QMCs on tress associated with OQRW. Our investigation leads to the detection of the phase transition phenomena within the proposed scheme. This kind of phenomena appears first time in this direction. Moreover, mean entropies of QMCs are calculated.


Mathematics Subject Classification: 46L53, 46L60, 82B10, 81Q10.
Key words: Open quantum random walks; Quantum Markov chain; Cayley tree; recurrence;

## 1 Introduction

Motivated largely by the prospect of superefficient algorithms, the theory of quantum Markov chains (QMC), especially in the guise of quantum walks, has generated a huge number of works, including many discoveries of fundamental importance [23, 26, 34, 38]. In [20] a novel approach has been proposed to investigate quantum cryptography problems by means of QMC, where quantum effects are entirely encoded into super-operators labelling transitions, and the nodes of its transition graph carry only classical information and thus they are discrete. In [10, 18] QMC have been applied to the investigations of so-called "open quantum random walks" (OQRW) [11, 14, 15, 25, 27]. OQRW are related to the study of asymptotic behavior of trace-preserving completely positive maps, which belong to fundamental topics of quantum information theory (see, for instance [13, 28, 36]). These quantum walks are possible noncommutative generalizations of classical Markov chains and have applications in quantum computing, quantum optics [24, 29]. We refer the reader to [40] for a recent survey on the subject.

Recently, in [33] we first have proposed a new construction of QMC on trees, which is an extension of QMC considered in [9]. Using such a construction, QMCs are defined on tress associated with OQRW. The investigation led to the detection of the phase transition phenomena within the proposed scheme. Such kind of phenomena appeared for the first time in this direction. In the present paper, we continue the proposed investigation to discuss the recurrence problem for the associated QMC. In one dimensional setting the recurrence problem had been paid attention by many authors (see for example $[6,7,18,21,27])$.

For the sake of clarity, let us recall some necessary information about OQRW. Let $\mathcal{K}$ denote a separable Hilbert space and let $\{|i\rangle\}_{i \in \Lambda}$ be its orthonormal basis indexed by the vertices of some graph $\Lambda$ (here the set $\Lambda$ of vertices might be finite or countable). Let $\mathcal{H}$ be another Hilbert space, which will describe the degrees of freedom given at each point of $\Lambda$. Then we will consider the space $\mathcal{H} \otimes \mathcal{K}$. For each pair $i, j$ one associates a bounded linear operator $B_{j}^{i}$ on $\mathcal{H}$. This operator describes the effect of passing from $|j\rangle$ to $|i\rangle$. We will assume that for each $j$, one has

$$
\begin{equation*}
\sum_{i} B_{j}^{i *} B_{j}^{i}=\mathbb{I}, \tag{1}
\end{equation*}
$$

where, if infinite, such series is strongly convergent. This constraint means: the sum of all the effects leaving site $j$ is $\mathbb{I}$. The operators $B_{j}^{i}$ act on $\mathcal{H}$ only, we dilate them as operators on $\mathcal{H} \otimes \mathcal{K}$ by putting

$$
M_{j}^{i}=B_{j}^{i} \otimes|i\rangle\langle j| .
$$

The operator $M_{j}^{i}$ encodes exactly the idea that while passing from $|j\rangle$ to $|i\rangle$ on the lattice, the effect is the operator $B_{j}^{i}$ on $\mathcal{H}$.

According to [11] one has

$$
\begin{equation*}
\sum_{i, j} M_{j}^{i^{*}} M_{j}^{i}=\mathbb{I} \tag{2}
\end{equation*}
$$

Therefore, the operators $\left(M_{j}^{i}\right)_{i, j}$ define a completely positive mapping

$$
\begin{equation*}
\mathcal{M}(\rho)=\sum_{i} \sum_{j} M_{j}^{i} \rho M_{j}^{i^{*}} \tag{3}
\end{equation*}
$$

on $\mathcal{H} \otimes \mathcal{K}$.
In what follows, we consider density matrices on $\mathcal{H} \otimes \mathcal{K}$ which take the form

$$
\begin{equation*}
\rho=\sum_{i} \rho_{i} \otimes|i\rangle\langle i|, \tag{4}
\end{equation*}
$$

assuming that $\sum_{i} \operatorname{Tr}\left(\rho_{i}\right)=1$.
For a given initial state of such form, the Open Quantum Random Walk (OQRW) is defined by the mapping $\mathcal{M}$, which has the following form

$$
\begin{equation*}
\mathcal{M}(\rho)=\sum_{i}\left(\sum_{j} B_{j}^{i} \rho_{j} B_{j}^{i *}\right) \otimes|i\rangle\langle i| . \tag{5}
\end{equation*}
$$

By means of the map $\mathcal{M}$ one defines a family of classical random process on $\varnothing=\Lambda^{\mathbb{Z}_{+}}$. Namely, for any density operator $\rho$ on $\mathcal{H} \otimes \mathcal{K}$ (see (4)) the probability distribution is defined by

$$
\begin{equation*}
\mathbb{P}_{\rho}\left(i_{0}, i_{1}, \ldots, i_{n}\right)=\operatorname{Tr}\left(B_{i_{n-1}}^{i_{n}} \cdots B_{i_{1}}^{i_{2}} B_{i_{0}}^{i_{1}} \rho_{i_{0}} B_{i_{0}}^{i_{1} *} B_{i_{1}}^{i_{2} *} \cdots B_{i_{n-1}}^{i_{n} *}\right) . \tag{6}
\end{equation*}
$$

We point out that this distribution is not a Markov measure [12].
On the other hand, it is well-known $[10,35]$ that to each classical random walk one can associate a certain Markov chain and some properties of the walk can be explored by the constructed chain. Recently, in [18, 19], we have found a quantum Markov chain ${ }^{1} \varphi$ on the algebra $\mathcal{A}=\otimes_{i \in \mathbb{Z}_{+}} \mathcal{A}_{i}$, where $\mathcal{A}_{i}$ is isomorphic to $B(\mathcal{H}) \otimes B(\mathcal{K}), i \in \mathbb{Z}_{+}$, such that the transition operator $P$ equals to the mapping $\mathcal{M}^{* 2}$ and the restriction of $\varphi$ to the commutative subalgebra of $\mathcal{A}$ coincides with the distribution $\mathbb{P}_{\rho}$, i.e.

$$
\begin{equation*}
\varphi\left(\left(\mathbb{I} \otimes\left|i_{0}><i_{0}\right|\right) \otimes \cdots \otimes\left(\mathbb{I} \otimes\left|i_{n}><i_{n}\right|\right)\right)=\mathbb{P}_{\rho}\left(i_{0}, i_{1}, \ldots, i_{n}\right) . \tag{7}
\end{equation*}
$$

Hence, this result allows us to interpret the distribution $\mathbb{P}_{\rho}$ as a QMC , and to study further properties of $\mathbb{P}_{\rho}$.

In [33], we have initiated to look at the probability distribution (6) as a Markov field over the Cayley tree $\Gamma^{k}$ Roughly speaking, $\left(i_{0}, i_{1}, \ldots, i_{n}\right)$ is considered as a configuration on $\Omega=\Lambda^{\Gamma^{k}}$. Such kind of consideration allows us to investigated a phase transition phenomena associated for OQRW within QMC scheme [30, 31].

We stress that, in physics, a spacial classes of QMC, called "Matrix Product States" (MPS) and more generally "Tensor Network States" [17, 37] were used to investigate quantum phase transitions for several lattice models. This method uses the density matrix renormalization group (DMRG) algorithm which opened a new way of performing the renormalization procedure in 1D systems and gave extraordinary precise results. This is done by keeping the states of subsystems which are relevant to describe the whole wave-function, and not those that minimize the energy on that subsystems [39].

In this paper, we propose to investigate the recurrence problem for QMC on trees, and apply it to the QMC associated with OQRW on trees. Notice that the mentioned problem has been investigated for discrete-time nearest-neighbor open quantum random walks on the integer line in [16]. However, in the present work, we focus on the recurrence problem associated with QMC, while in [16, 21, 22] the recurrence has been teated with respect to the probability distribution (6).

## 2 Preliminaries

Let $\Gamma_{+}^{k}=(V, E)$ be the semi-infinite Cayley tree of order $k$ with root $o$. The Cayley tree of order $k$ is characterized by being a tree for which every vertex has exactly $k+1$ nearest-neighbors. Recall that, two vertices $x$ and $y$ are nearest neighbors (denoted $x \sim y$ ) if they are joined through an edge (i.e. $<x, y>\in E)$. A list $x \sim x_{1} \sim \cdots \sim x_{d-1} \sim y$ of vertices is called a path from $x$ to $y$. The distance on

[^0]the tree between two vertices $x$ and $y$ (denoted $d(x, y))$ is the length of the shortest edge-path joining them.

Define

$$
\begin{aligned}
W_{n} & :=\{x \in V \\
\Lambda_{n} & :=\bigcup_{j \leq n} W_{j} ;
\end{aligned} \quad \Lambda_{[m, n]}=\bigcup_{j=m}^{n} W_{j} .
$$

Recall a coordinate structure in $\Gamma_{+}^{k}$ : every vertex $x$ (except for $x^{0}$ ) of $\Gamma_{+}^{k}$ has coordinates $\left(i_{1}, \ldots, i_{n}\right)$, here $i_{m} \in\{1, \ldots, k\}, 1 \leq m \leq n$ and for the vertex $x^{0}$ we put (0). Namely, the symbol (0) constitutes level 0 , and the sites $\left(i_{1}, \ldots, i_{n}\right)$ form level $n$ (i.e. $d\left(x^{0}, x\right)=n$ ) of the lattice. Using this structure, vertices $x_{W_{n}}^{(1)}, x_{W_{n}}^{(2)}, \cdots, x_{W_{n}}^{\left(\left|W_{n}\right|\right)}$ of $W_{n}$ can be represented as follows:

$$
\begin{gather*}
x_{W_{n}}^{(1)}=(1,1, \cdots, 1,1), \quad x_{W_{n}}^{(2)}=(1,1, \cdots, 1,2), \cdots \quad x_{W_{n}}^{(k)}=(1,1, \cdots, 1, k,),  \tag{8}\\
x_{W_{n}}^{(k+1)}=(1,1, \cdots, 2,1), \quad x_{W_{n}}^{(2)}=(1,1, \cdots, 2,2), \cdots \\
\vdots \\
x_{W_{n}}^{(2 k)}=(1,1, \cdots, 2, k), \\
x_{W_{n}}^{\left(\left|W_{n}\right|-k+1\right)}=(k, k, \cdots, k, 1), x_{W_{n}}^{\left(\left|W_{n}\right|-k+2\right)}=(k, k, \cdots, k, 2), \cdots x_{W_{n}}^{\left|W_{n}\right|}=(k, k, \cdots, k, k) .
\end{gather*}
$$

In the above notations, we write

$$
W_{n}=\left\{\left(i_{1}, i_{2}, \cdots, i_{n}\right) ; \quad i_{j}=1,2, \cdots, k\right\}
$$

So one can see that $\left|W_{n}\right|=k^{n}$. The set of direct successors for a given vertex $x \in V$ is defined by

$$
\begin{equation*}
S(x):=\{y \in V: x \sim y \text { and } d(y, o)>d(x, o)\} . \tag{9}
\end{equation*}
$$

The vertex $x$ has exactly $k$ direct successors denoted $(x, i), i=1,2, \cdots, k$

$$
S(x)=\{(x, 1),(x, 2), \cdots,(x, k)\} .
$$

To each vertex $x$, we associate a $\mathrm{C}^{*}$-algebra of observable $\mathcal{A}_{x}$ with identity $\mathbb{1}_{x}$. For a given bounded region $V^{\prime} \subset V$, we consider the algebra $\mathcal{A}_{V^{\prime}}=\otimes_{x \in V^{\prime}} \mathcal{A}_{x}$. We have the the following natural embedding

$$
\mathcal{A}_{\Lambda_{n}} \equiv \mathcal{A}_{\Lambda_{n}} \otimes \mathbb{1}_{W_{n+1}} \subset \mathcal{A}_{\Lambda_{n+1}}
$$

The algebra $\mathcal{A}_{\Lambda_{n}}$ is then a subalgebra of $\mathcal{A}_{\Lambda_{n+1}}$. It follows the local algebra

$$
\begin{equation*}
\mathcal{A}_{V ; l o c}:=\bigcup_{n \in \mathbb{N}} \mathcal{A}_{\Lambda_{n}} \tag{10}
\end{equation*}
$$

and the quasi-local algebra

$$
\mathcal{A}_{V}:={\overline{\mathcal{A}_{V ; l o c}}}^{C}
$$

The set of states on a $\mathrm{C}^{*}$-algebra $\mathcal{A}$ will be denoted $\mathcal{S}(\mathcal{A})$.
There are $k$ natural shifts on the Cayley tree of order $k$ : for each $x=\left(i_{1}, i_{2}, \cdots, i_{n}\right) \in \Lambda_{n}$ and $j \in\{1, \ldots, k\}$

$$
\begin{equation*}
\alpha_{j}(x)=(j, x)=\left(j, i_{1}, i_{2}, \cdots, i_{n}\right) \in \Lambda_{n+1} \tag{11}
\end{equation*}
$$

Let $g=\left(j_{1}, j_{2}, \cdots, j_{N}\right) \in V$ one defines

$$
\alpha_{g}(x):=\alpha_{j_{1}} \circ \alpha_{j_{2}} \circ \cdots \circ \alpha_{j_{N}}(x)=\left(j_{1}, j_{2}, \cdots, j_{N}, i_{1}, i_{2}, \cdots, i_{n}\right) .
$$

The $\alpha_{j}$ 's action on the algebra $\mathcal{A}_{V}$ is given as follows:

$$
\begin{equation*}
\alpha_{j}\left(\bigotimes_{x \in \Lambda_{\leq n}} a_{x}\right):=\mathbb{I}^{(o)} \otimes \bigotimes_{x \in \Lambda_{\leq n}} a_{x}^{(j, x)} . \tag{12}
\end{equation*}
$$

The shift $\alpha_{j}$ induces a $*$-isomorphism from $\mathcal{A}_{V}$ into $\mathcal{A}_{V_{(o, j)}}$. Let $\alpha_{j}^{-1}$ its inverse isomorphism. For $g \in V$, the map $\alpha_{g}$ defines a $*$-isomporphism from $\mathcal{A}_{V}$ into $\mathcal{A}_{V_{g}}$ and its inverse isomorphism will be denoted by $\alpha_{g}^{-1}$.

Consider a triplet $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$ of $\mathrm{C}^{*}$-algebras. A quasi-conditional expectation [3] is a completely positive identity preserving linear map $E: \mathcal{A} \rightarrow \mathcal{B}$ such that $E(c a)=c E(a)$, for all $a \in \mathcal{A}, c \in \mathcal{C}$.

Definition 2.1. [3] Let $\mathcal{B} \subseteq \mathcal{A}$ be two unitary $C^{*}$-algebra $\mathbb{I}$. A Markov transition expectation from $\mathcal{A}$ into $\mathcal{B}$ is a completely positive identity preserving map.

Definition 2.2. [5, 8] A (backward) quantum Markov chain on $\mathcal{A}_{V}$ is a triplet $\left(\phi_{o},\left(E_{\Lambda_{n}}\right)_{n \geq 0},\left(h_{n}\right)_{n}\right)$ of initial state $\phi_{o} \in \mathcal{S}\left(\mathcal{A}_{o}\right)$, a sequence of quasi-conditional expectations $\left(E_{\Lambda_{n}}\right)_{n}$ w.r.t. the triple $\mathcal{A}_{\Lambda_{n-1}} \subseteq \mathcal{A}_{\Lambda_{n}} \subseteq \mathcal{A}_{\Lambda_{n+1}}$ and a sequence $h_{n} \in \mathcal{A}_{W_{n},+}$ of boundary conditions such that for each $a \in \mathcal{A}_{V}$ the limit

$$
\begin{equation*}
\varphi(a):=\lim _{n \rightarrow \infty} \phi_{0} \circ E_{\Lambda_{0}} \circ E_{\Lambda_{1}} \circ \cdots \circ E_{\Lambda_{n}}\left(h_{n+1}^{1 / 2} a h_{n+1}^{1 / 2}\right) \tag{13}
\end{equation*}
$$

exists in the weak-*-topology and defines a state. In this case the state $\varphi$ defined by (13) is also called quantum Markov chain (QMC).

A QMC $\varphi$ on $\mathcal{A}_{V}$ is said to be tree-homogeneous if

$$
\begin{equation*}
\varphi \circ \alpha_{j}=\varphi \tag{14}
\end{equation*}
$$

for every $j \in\{1,2, \cdots, k\}$.
In the sequel, we restrict ourselves to the case of trivial boundary condition $h=\mathbb{I}$ and the associated tree-homogeneous quantum Markov chain $\varphi$ is determined by the pair $\varphi \equiv\left(\phi_{o}, \mathcal{E}\right) \equiv$ $\left(\phi_{o}, \mathcal{E}, h=\mathbb{I}\right) .{ }^{3}$ Here, $\mathcal{E}$ is a Markov transition expectation from $\mathcal{A}_{(o)} \otimes \mathcal{A}_{(1)} \otimes \cdots \otimes \mathcal{A}_{(k)}$ into $\mathcal{A}_{(o)}$. For each $u$ by $\mathcal{E}_{u}$ we denote the $\alpha_{u}$-shift of $\mathcal{E}$ given by

$$
\begin{equation*}
\mathcal{E}_{u}=\alpha_{u} \circ \mathcal{E} \circ \alpha_{u}^{-1} \tag{15}
\end{equation*}
$$

Clearly, $\mathcal{E}_{u}$ is a transition expectation from $\mathcal{A}_{u} \otimes \mathcal{A}_{(u, 1)} \otimes \cdots \otimes \mathcal{A}_{(u, k)}$ into $\mathcal{A}_{u}$. For each $n \in \mathbb{N}$, we consider

$$
\mathcal{E}_{W_{n}}:=\bigotimes_{u \in W_{n}} \mathcal{E}_{u}
$$

One can see that $\mathcal{E}_{W_{n}}$ is a Markov transition expectation from $\mathcal{A}_{\Lambda_{[n, n+1]}}$ into $\mathcal{A}_{W_{n}}$. Following [4, 32], we have the next result.

Theorem 2.3. Let $\varphi=\left(\phi_{o}, \mathcal{E}\right)$ be a tree-homogeneous quantum Markov chain. There exists a unique conditional expectation $E_{o]}$ from $\mathcal{A}_{V}$ into $\mathcal{A}_{o}$ characterized by

$$
\begin{equation*}
E_{o]}(a)=\mathcal{E}_{o}\left(a_{o} \otimes \mathcal{E}_{W_{1}}\left(a_{W_{1}} \cdots \otimes \mathcal{E}_{W_{n}}\left(a_{W_{n}} \otimes h_{n+1}\right)\right)\right) \tag{16}
\end{equation*}
$$

for all $a=a_{o} \otimes a_{W_{1}} \otimes \cdots \otimes a_{W_{n}}$. Moreover, one has

$$
\begin{equation*}
\varphi(\cdot)=\phi_{o} \circ E_{o]}(\cdot) \tag{17}
\end{equation*}
$$

[^1]The forward Markov operator associated with $\mathcal{E}_{u}$ is given by:

$$
\begin{equation*}
T_{u}(a)=\mathcal{E}_{u}\left(a \otimes \mathbb{1}_{S(u)}\right), \quad a \in \mathcal{A}_{u} \tag{18}
\end{equation*}
$$

While, there are $k$ backward Markov operators corresponding to the successors $(u, \ell), j=1, \ldots, k$ of $u$,

$$
\begin{equation*}
P_{u}^{(u, \ell)}(a)=\mathcal{E}_{u}\left(\mathbb{I}^{(u)} \otimes a \otimes \mathbb{1}_{S(u) \backslash\{(u, \ell)\}}\right), \quad \forall a \in \mathcal{A}_{(u, \ell)} \tag{19}
\end{equation*}
$$

For any ray $r=\left(u_{n}\right)_{n}$, one defines

$$
\begin{equation*}
P_{u_{n}}^{u_{n+m}}=P_{u_{n}}^{u_{n+1}} \circ \cdots \circ P_{u_{n+m-1}}^{u_{n+m}} ; \quad m, n \in \mathbb{N} \tag{20}
\end{equation*}
$$

The map $P_{u_{n}}^{u_{m}}$ defines a Markov operator from $\mathcal{A}_{u_{m}}$ into $\mathcal{A}_{u_{n}}$.

## 3 Recurrence of quantum Markov chains on trees

This section is devoted to the notions of recurrence and weak recurrence for quantum Markov chains on trees.

Following [6,41] a given projection $e \in \operatorname{Proj}(\mathcal{A})$ and a ray $r=\left(u_{n}\right)_{n} \in \operatorname{Paths}(o, \infty)$, a stopping time $\tau_{e ; r}=\left(\tau_{u_{n}}\right)_{n}$ on the algebra $\mathcal{A}_{V}$, is defined as follows:

$$
\begin{align*}
\tau_{e ; o} & =e^{(o)} \otimes \mathbb{1}_{V \backslash\{o\}} \\
\tau_{e ; u_{1}} & =e^{\perp(o)} \otimes e^{\left(u_{1}\right)} \otimes \mathbb{1}_{\left.V \backslash\left\{u_{1}\right]\right\}} \\
\vdots &  \tag{21}\\
\tau_{e ; u_{n}} & \left.=e^{\perp(o)} \otimes \cdots \otimes e^{\perp\left(u_{n-1}\right)} \otimes e^{\left(u_{n}\right)} \otimes \mathbb{1}_{V \backslash\left\{x_{n}\right\}}\right\}  \tag{22}\\
\tau_{e ; u_{n} ; \infty}:= & e^{\perp(o)} \otimes e^{\perp\left(u_{1}\right)} \otimes \cdots \otimes e^{\perp\left(u_{n-1}\right)} \otimes e^{\perp\left(u_{n}\right)} \otimes \mathbb{I}_{V \backslash\left\{u_{n}\right]},
\end{align*}
$$

where for each $a \in \mathcal{A}$ one has $a^{(u)}=\alpha_{u}(a)$. Put

$$
\tau_{e ; r ; \infty}=\lim _{n \rightarrow \infty} \tau_{e ; u_{n} ; \infty}=\bigotimes_{n \in \mathbb{N}} e^{\perp\left(u_{n}\right)}
$$

Definition 3.1. Let $\varphi=\left(\phi_{o}, \mathcal{E}\right)$ be a tree-homogeneous quantum Markov chain. A projection $e \in$ $\operatorname{Proj}(\mathcal{A})$ is said to be
(i) $\mathcal{E}$-completely accessible if

$$
\begin{equation*}
E_{o]}\left(\tau_{e ; r ; \infty}\right):=\lim _{n \rightarrow \infty} E_{o]}\left(\tau_{e ; x_{n} ; \infty}\right)=0 \tag{23}
\end{equation*}
$$

for every ray $r=\left(x_{n}\right)_{n}$.
(ii) $\varphi$-completely accessible if $\varphi\left(\tau_{e ; r ; \infty}\right)=0$, for every ray $r=\left(x_{n}\right)_{n}$.
(iii) $\mathcal{E}$-recurrent if $0<\operatorname{Tr}(\mathcal{E}(e \otimes \mathbb{I}))<\infty$ and one has

$$
\begin{equation*}
\frac{1}{\operatorname{Tr}(\mathcal{E}(e \otimes \mathbb{I}))} \operatorname{Tr}\left(E_{o]}\left(\sum_{n \geq 0} e \otimes \tau_{e ; x_{n}}\right)=1\right. \tag{24}
\end{equation*}
$$

for every ray $r=\left(x_{n}\right)_{n}$.
(iv) $\varphi$-recurrent if $\varphi\left(\alpha_{o}(e)\right) \neq 0$ and

$$
\begin{equation*}
\frac{1}{\varphi\left(\alpha_{o}(e)\right)} \varphi\left(\sum_{n} e \otimes \tau_{e ; x_{n}}\right)=1 \tag{25}
\end{equation*}
$$

for every ray $r=\left(x_{n}\right)_{n}$.
Definition 3.2. Let $\varphi=\left(\phi_{o}, \mathcal{E}\right)$ be a tree-homogeneous quantum Markov chain. Let e, $f \in \operatorname{Proj}(\mathcal{A}), e, f \neq$ 0 . The projection $f$ is
(i) $\mathcal{E}$-accessible from $e$ (and we write $e \rightarrow^{\mathcal{E}} f$ ) if for any ray $r=\left(x_{n}\right)_{n}$ there exists $m \in \mathbb{N}$ such that

$$
E_{o]}\left(\alpha_{0}(e) \alpha_{x_{m}}(f)\right) \neq 0
$$

(ii) $\varphi$-accessible from $e$ (we denote it as $e \rightarrow^{\varphi} f$ if for any ray $r=\left(x_{n}\right)_{n}$ there exists $m \in \mathbb{N}$ such that

$$
\varphi\left(\alpha_{0}(e) \alpha_{x_{m}}(f)\right) \neq 0
$$

Lemma 3.3. In the above notations:

$$
\begin{equation*}
\sum_{n \geq 0} \tau_{e ; x_{n}}=\mathbb{I}_{\mathcal{A}_{V}}-\tau_{e ; r ; \infty} \tag{26}
\end{equation*}
$$

Proof. see [41]
Theorem 3.4. Let $\varphi \equiv\left(\phi_{o}, \mathcal{E}\right)$ be a tree-homogeneous quantum Markov chain on $\mathcal{A}_{V}$. Let $e \in$ $\operatorname{Proj}\left(\mathcal{A}_{V}\right)$ be a projection
(i) $e$ is $\mathcal{E}$-recurrent if and only if for any ray $r=\left(x_{n}\right)_{n}$ one has

$$
\begin{equation*}
\mathcal{E}\left(e \otimes E_{o]}\left(\tau_{e ; r ; \infty}\right)\right)=0 \tag{27}
\end{equation*}
$$

(ii) $e$ is $\varphi$-recurrent if and only if for any ray $r=\left(x_{n}\right)_{n}$ one has

$$
\begin{equation*}
\varphi\left(e \otimes \tau_{e ; r ; \infty}\right)=0 \tag{28}
\end{equation*}
$$

(iii) $e$ is $\mathcal{E}$-accessible from $f$ if and only if for any ray $r=\left(x_{n}\right)_{n}$ there exists $m \in \mathbb{N}$ such that

$$
\begin{equation*}
\mathcal{E}\left(e \otimes P_{x_{1}}^{x_{m}} T_{x_{m}} f\right) \neq 0 \tag{29}
\end{equation*}
$$

(iv) $e$ is $\varphi$-accessible from $f$ if and only if for any ray $r=\left(x_{n}\right)_{n}$ there exists $m \in \mathbb{N}$ such that

$$
\begin{equation*}
\varphi\left(e \otimes P_{x_{1}}^{x_{m}} T_{x_{m}} f\right) \neq 0 \tag{30}
\end{equation*}
$$

Proof. From Lemma 3.3 one has

$$
\sum_{n \geq 0} e \otimes \tau_{x_{n}}=e \otimes \mathbb{I}-e \otimes \tau_{e ; n ; \infty}
$$

This leads to (i) and (ii).
One has

$$
\mathcal{E}_{W_{n}}\left(f^{\left(x_{m}\right)} \otimes \mathbb{I}\right)=\mathcal{E}_{x_{m}}\left(f^{\left(x_{m}\right)} \otimes \mathbb{I}\right)=T_{x_{m}} f
$$

and

$$
\begin{aligned}
& \mathcal{E}_{W_{m}}\left(\mathbb{I}_{W_{m-1}} \otimes\left(T_{x_{n}} f\right)^{\left(x_{m-1}\right)}\right)=P_{x_{m-1}}^{x_{m}} T_{x_{m}} f \\
& E_{o]}\left(\alpha_{0}(e) \alpha_{x_{m}}(f)\right)= \mathcal{E}_{W_{0}}\left(e \otimes \mathcal { E } _ { W _ { 1 } } \left(\mathbb { I } _ { W _ { 1 } } \otimes \cdots \otimes \mathcal { E } _ { W _ { m } } \left(\mathbb{I}_{W_{m-1}} \otimes \mathcal{E}_{W_{m}}\left(f^{\left(x_{m}\right)} \otimes \mathbb{I}_{W_{m}+1}\right)\right.\right.\right. \\
&= \mathcal{E}_{W_{0}}\left(e \otimes \mathcal { E } _ { W _ { 1 } } \left(\mathbb { I } _ { W _ { 1 } } \otimes \cdots \otimes \mathcal { E } _ { W _ { m - 2 } } \left(\mathbb{I}_{W_{m-2}} \otimes\left(P_{x_{m-1}}^{x_{m}} T_{x_{m}} f\right)^{\left(x_{m-1}\right)}\right.\right.\right. \\
& \vdots \\
&= \mathcal{E}\left(e \otimes P_{x_{1}}^{x_{m}} T_{x_{m}} f\right)
\end{aligned}
$$

This proves (iii) and using (17) one gets (iv).
Corollary 3.5. Let $\varphi \equiv\left(\phi_{o}, \mathcal{E}\right)$ be a tree-homogeneous quantum Markov chain. Any $\mathcal{E}$-recurrence projection is $\varphi$-recurrent. Conversely, if the initial state $\phi_{o}$ is faithful then Any $\varphi$-recurrence projection is $\mathcal{E}$-recurrent.
$\operatorname{Proof}$. Let $e \in \operatorname{Proj}(\mathcal{A})$ be a projection. For each $\ell \in\{1, \ldots, k\}$, one has

$$
E_{o]}\left(a_{o} \otimes \tau_{\ell}(a)\right)=\mathcal{E}\left(a_{o} \otimes E_{o]}(a)\right) ; \quad \forall a_{o} \in \mathcal{A}_{o}, \forall a \in \mathcal{A}_{V}
$$

Then

$$
\begin{aligned}
\left.\varphi\left(e \otimes \tau_{e ; r ; \infty}\right)\right) & =\varphi\left(\alpha_{o}(e) \otimes \alpha_{\left(x_{1}\right)}\left(\tau_{e ; r ; \infty}\right)\right) \\
& \stackrel{(17)}{=} \phi_{o}\left(E_{o]}\left(\left(\alpha_{o}(e) \otimes \alpha_{\left(x_{1}\right)}\left(\tau_{e ; r ; \infty}\right)\right)\right)\right. \\
& =\phi_{o}\left(\mathcal{E}\left(e \otimes E_{o]}\left(\tau_{e ; r ; \infty}\right)\right)\right)
\end{aligned}
$$

Therefore, if $\mathcal{E}\left(e \otimes E_{o]}\left(\tau_{e ; r ; \infty}\right)\right)=0$ then $\left.\varphi\left(e \otimes \tau_{e ; r ; \infty}\right)\right)=0$. This shows the first implication.
If the initial state $\phi_{o}$ is faithful, since $\mathcal{E}\left(e \otimes E_{o]}\left(\tau_{e ; r ; \infty}\right)\right) \geq 0$ then from the above computation, we have

$$
\left.\varphi\left(e \otimes \tau_{e ; r ; \infty}\right)\right)=0 \Rightarrow \mathcal{E}\left(e \otimes E_{o]}\left(\tau_{e ; r ; \infty}\right)\right)=0
$$

This shows the converse direction, and finishes the proof.

## 4 Recurrence of QMC associated with OQRW

Let $\mathcal{H}$ and $\mathcal{K}$ be given two separable Hilbert spaces over the complex field $\mathbb{C}$. Let $\{|i\rangle\}_{i \in \Lambda}$ be an orthonormal basis of $\mathcal{K}$ indexed by a graph $\Lambda$ with almost-countable vertex set. The algebra of observable at a site $x \in V$ is considered to be $\mathcal{A}_{x}=\mathcal{A}:=\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$.

Let $\mathcal{M}$ be a OQRW given by (5). In the language of OQRW [11] the Hilbert space $\mathcal{H}$ describes the internal degree of freedom of the quantum walker, while $\mathcal{K}$ describes the state space of the dynamics where the walk is dome through the oriented graph $\Lambda$. The transition of the walker from a site $j$ to site $i$ is described by a bounded operator $B_{j}^{i} \in \mathcal{B}(\mathcal{H})$ such that

$$
\begin{equation*}
\sum_{i \in \Lambda} B_{j}^{i *} B_{j}^{i}=\mathbb{1}_{\mathcal{B}(\mathcal{H})} . \tag{31}
\end{equation*}
$$

The initial density matrix of the dynamics is $\rho \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$, of the form

$$
\rho=\sum_{i \in \Lambda} \rho_{i} \otimes|i\rangle\langle i| ; \quad \rho_{i} \in \mathcal{B}(\mathcal{H})^{+} .
$$

In what follows, for the sake of simplicity of calculations, we assume that $\rho_{i} \neq 0$ for all $i \in \Lambda$ (see [19, Remark 4.5] for other kind of initial states).

Let

$$
\begin{equation*}
M_{j}^{i}=B_{j}^{i} \otimes|i\rangle\langle j| \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K}) . \tag{32}
\end{equation*}
$$

and

$$
\begin{equation*}
A_{j}^{i}:=\frac{1}{\operatorname{Tr}\left(\rho_{j}\right)^{1 / 2}} \rho_{j}^{1 / 2} \otimes|i\rangle\langle j|, \quad i, j \in \Lambda \tag{33}
\end{equation*}
$$

For each $u \in V$, we define

$$
\begin{equation*}
K_{j}^{i(u, S(u))}:=M_{j}^{i *(u)} \otimes \bigotimes_{v \in S(u)} A_{j}^{i(v)} \in \mathcal{A}_{\{u\} \cup S(u)} . \tag{34}
\end{equation*}
$$

The interaction of a vertex $u \in V$ with its set of direct successors it describled by

$$
K^{(u, S(u))}=\sum_{i, j} K_{j}^{i(u, S(u))} \in \mathcal{A}_{\{u\} \cup S(u)}
$$

Put

$$
\begin{equation*}
\mathcal{E}_{u}(a):=\operatorname{Tr}_{u]}\left(K^{(u, S(u))} a K^{(u, S(u)) *}\right) ; \quad a \in \mathcal{A}_{\{u\} \cup S(u)} . \tag{35}
\end{equation*}
$$

For each $j, j^{\prime} \in \Lambda$ we set

$$
\begin{equation*}
\varphi_{j j^{\prime}}(b):=\frac{1}{\operatorname{Tr}\left(\rho_{j}\right)^{1 / 2} \operatorname{Tr}\left(\rho_{j^{\prime}}\right)^{1 / 2}} \operatorname{Tr}\left(\rho_{j}^{1 / 2} \rho_{j^{\prime}}^{1 / 2} \otimes\left|j^{\prime}\right\rangle\langle j| b\right) ; \quad \forall a \in \mathcal{A} \tag{36}
\end{equation*}
$$

One can see that $\varphi_{j j^{\prime}}$ is a linear functional on $\mathcal{A}$. If $j=j^{\prime}$, we denote it simply denote $\varphi_{j}$ instead of $\varphi_{j j}$ one has

$$
\begin{equation*}
\varphi_{j}(a)=\frac{1}{\operatorname{Tr}\left(\rho_{j}\right)} \operatorname{Tr}\left(\rho_{j} \otimes|j\rangle\langle j| a\right) \tag{37}
\end{equation*}
$$

The functional $\varphi_{j}$ is then, a state on $\mathcal{A}$.
Theorem 4.1. In the above notations, the map $\mathcal{E}_{u}$ defines a Markov transition expectation from $\mathcal{A}_{\{u\} \cup S(u)}$ into $\mathcal{A}_{u}$ and

$$
\begin{equation*}
\mathcal{E}_{u}\left(a_{u} \otimes a_{(u, 1)} \otimes \cdots \otimes a_{(u, k)}\right)=\sum_{\left(i, j, j j^{\prime}\right) \in \Lambda^{3}} M_{j}^{i *} a_{(u)} M_{j^{\prime}}^{i} \prod_{\ell=1}^{k} \varphi_{j j^{\prime}}\left(a_{(u, \ell)}\right) \tag{38}
\end{equation*}
$$

Moreover, the backward Markov operators associated with $\mathcal{E}_{u}$ are given by

$$
\begin{equation*}
P_{u}^{(u, \ell)}\left(a_{(u, \ell)}\right)=\sum_{j}\left(\mathbb{I}_{\mathcal{B}(\mathcal{H})} \otimes|j\rangle\langle j|\right) \varphi_{j}\left(a_{(u, \ell)}\right) \tag{39}
\end{equation*}
$$

The forward Markov operator associated with $\mathcal{E}_{u}$ is given by

$$
\begin{equation*}
T_{u}\left(a_{u}\right)=\sum_{i j} M_{j}^{i, *} a_{u} M_{j}^{i} \tag{40}
\end{equation*}
$$

where $a_{u} \in \mathcal{A}$ and $a_{(u, \ell)} \in \mathcal{A}_{(u, \ell)}$ for each $\ell \in\{1, \cdots, k\}$.

Proof. The map $\mathcal{E}_{u}$ (35), is clearly completely positive.
Let $a=a_{u} \otimes a_{(u, 1)} \otimes \cdots \otimes a_{(u, k)}$. Taking into account (34) and (32) one gets

$$
\begin{aligned}
\mathcal{E}_{u}(a) & =\operatorname{Tr}_{u]}\left(\left(\sum_{(i, j) \in \Lambda^{2}} K_{j}^{i}\right) a\left(\sum_{(i, j) \in \Lambda^{2}} K_{j}^{i}\right)^{*}\right) \\
& =\sum_{(i, j),\left(i^{\prime}, j^{\prime}\right) \in \Lambda^{2}} \operatorname{Tr}_{u]}\left(K_{j}^{i(u, S(u))} a_{u} \otimes a_{(u, 1)} \cdots \otimes a_{(u, k)} K_{j^{\prime}}^{i^{\prime}(u, S(u)) *}\right) \\
& =\sum_{(i, j),\left(i^{\prime}, j^{\prime}\right) \in \Lambda^{2}} \operatorname{Tr}_{u]}\left(M_{j}^{i(u) *} a_{u} M_{j^{\prime}}^{i^{\prime}(u)} \otimes \bigotimes_{\ell=1}^{k}\left(A_{j}^{i} a_{(u, \ell)} A_{j^{\prime}}^{i^{\prime} *}\right)^{(u, \ell)}\right) \\
& =\sum_{(i, j),\left(i^{\prime}, j^{\prime}\right) \in \Lambda^{2}} M_{j}^{i *} a_{0}^{(u, 0)} M_{j^{\prime}}^{i^{\prime}} \prod_{\ell=1}^{k} \operatorname{Tr}\left(A_{j}^{i} a_{\ell}^{(u, \ell)} A_{j^{\prime}}^{i^{\prime} *}\right)
\end{aligned}
$$

For each $\ell \in\{1, \ldots, k\}$, one has

$$
\begin{aligned}
\operatorname{Tr}\left(A_{j}^{i} a_{(u, \ell)} A_{j^{\prime}}^{i^{\prime} *}\right) & \stackrel{(33)}{=} \operatorname{Tr}_{B} i g\left(A_{j^{\prime}}^{i^{\prime} *} A_{j}^{i} a_{(u, \ell)}\right) \\
& =\frac{1}{\operatorname{Tr}\left(\rho_{j}\right)^{1 / 2} \operatorname{Tr}\left(\rho_{j^{\prime}}\right)^{1 / 2}} \operatorname{Tr}\left(\rho_{j^{\prime}}^{1 / 2} \rho_{j}^{1 / 2} \otimes\left|j^{\prime}\right\rangle\langle j| a_{(u, \ell)}\right) \delta_{i, i^{\prime}} \\
& \stackrel{(36)}{=} \varphi_{j j^{\prime}}\left(a_{(u, \ell)}\right) \delta_{i, i^{\prime}}
\end{aligned}
$$

where $\delta_{i, i^{\prime}}$ denotes the Kronecker symbol. This leads to (38). One has

$$
\mathcal{E}_{u}\left(\mathbb{I}_{(u, S(u))}\right)=\sum_{i, j, j^{\prime}} M_{j}^{i *} M_{j^{\prime}}^{i} \prod_{\ell=1}^{k} \varphi_{j j^{\prime}}\left(\mathbb{I}_{(u, \ell)}\right) \stackrel{(36)}{=} \sum_{i, j} M_{j}^{i *} M_{j}^{i}=\mathbb{1}_{u}
$$

Then $\mathcal{E}_{u}$ is a Markov transition expectation.
From (19) one has

$$
\begin{aligned}
P_{u}^{(u, \ell)}\left(a_{(u, \ell)}\right. & =\sum_{i, j, j^{\prime}} M_{j}^{i *} M_{j^{\prime}}^{i} \varphi_{j j^{\prime}}\left(a_{(u, \ell)}\right) \prod_{\substack{\ell^{\prime} \prime \\
\ell^{\prime} \neq \ell}}^{k} \varphi_{j j^{\prime}}\left(\mathbb{I}_{\left(u, \ell^{\prime}\right)}\right) \\
& =\sum_{i, j} M_{j}^{i *} M_{j}^{i} \varphi_{j}\left(a_{(u, \ell)}\right) \\
& =\sum_{j}\left(\sum_{i} B_{j}^{i *} B_{j}^{i}\right) \varphi_{j}\left(a_{(u, \ell)}\right) \\
& \stackrel{(31)}{=} \sum_{j}\left(\mathbb{I}_{\mathcal{H}} \otimes|j\rangle\langle j|\right) \varphi_{j}\left(a_{(u, \ell)}\right)
\end{aligned}
$$

The forward Markov operator (18) associated with $\mathcal{E}_{u}$ satisfies

$$
T_{u}\left(a_{u}\right)=\sum_{i, j, j^{\prime}} M_{j}^{i *} a_{u} M_{j^{\prime}}^{i} \prod_{\ell=1}^{k} \varphi_{j j^{\prime}}\left(\mathbb{I}_{(u, \ell)}\right)=\sum_{i, j} M_{j}^{i *} a_{u} M_{j}^{i}
$$

This finishes the proof.

Now we are ready to Build the conditional expectation $E_{o]}$ in the case of open quantum random walks using the transition expectations of the form (35) and the quantum Markov chain $\varphi \equiv\left(\phi_{o}, \mathcal{E}\right)$, where

$$
\begin{equation*}
\mathcal{E}(a):=\mathcal{E}_{o}(a)=\sum_{i, j} M_{j}^{i *} a_{o} M_{j}^{i} \prod_{\ell=1}^{k} \varphi_{j}\left(a_{(o, \ell)}\right) \tag{41}
\end{equation*}
$$

for each $a=a_{o} \otimes a_{(o, 1)} \otimes \cdots \otimes a_{(o, k)}$.
It is clear that for each $u \in V$ the transition expectation $\mathcal{E}_{u}$ is a copy of $\mathcal{E}$ in the sense of (15).
Theorem 4.2. In the above notations, the conditional expectation associated $E_{o]}$ associated with $\mathcal{E}$ through (16) has the following expression

$$
\begin{equation*}
E_{o]}(a)=\sum_{j} \mathcal{M}_{j}\left(a_{o}\right) \prod_{u \in \Lambda_{[1, n]}} \psi_{j}\left(a_{u}\right) \tag{42}
\end{equation*}
$$

where

$$
\begin{equation*}
\psi_{j}(b)=\frac{1}{\operatorname{Tr}\left(\rho_{j}\right)} \sum_{i \in \Lambda} \operatorname{Tr}\left(B_{j}^{i} \rho_{j} B_{j}^{i^{*}} \otimes|i\rangle\langle i| b\right), \quad \forall b \in \mathcal{A} . \tag{43}
\end{equation*}
$$

and $a=\bigotimes_{u \in \Lambda_{n}} a_{u} \in \mathcal{A}_{\Lambda_{n}}$. Moreover, for any initial state $\phi_{o}=\operatorname{Tr}\left(\omega_{o} \cdot\right)$ the tree-homogeneous quantum Markov chain $\varphi \equiv\left(\phi_{o}, \mathcal{E}\right)$ is given by

$$
\begin{equation*}
\left.\varphi(a)=\sum_{j} \operatorname{Tr}\left(\omega_{o}\right) \mathcal{M}_{j}\left(a_{o}\right)\right) \prod_{u \in \Lambda_{[1, n]}} \psi_{j}\left(a_{u}\right) \tag{44}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{M}_{j}(\cdot)=\sum_{i \in \Lambda} M_{j}^{i *} \cdot M_{j}^{i} \tag{45}
\end{equation*}
$$

Remark 4.3. We notice that in our previous work [33] the expression (44) defines the QMC associated with the disordered phase of the system that deals with phase transitions for QMC on trees associated with $O Q R W$.
Theorem 4.4. In the notations of Theorem 4.2, if e is a projection in $\mathcal{A}$ such that

$$
\begin{equation*}
p:=\sup _{j \in \Lambda} \psi_{j}\left(e^{\perp}\right)<1 \tag{46}
\end{equation*}
$$

then $e$ is $\mathcal{E}$-recurrent.
Proof. Let $r=\left(x_{n}\right)_{n}$ be a ray one the semi-infinite Cayley tree. One has

$$
\begin{aligned}
E_{o]}\left(\tau_{e ; x_{n} ; \infty}\right) & \stackrel{(42)}{=} \sum_{j \in \Lambda} M_{j}^{i *} \alpha_{o}\left(e^{\perp}\right) M_{j}^{i} \prod_{m=1}^{n} \psi_{j}\left(\alpha_{x_{m}}\left(e^{\perp}\right)\right) \\
& \stackrel{(43)}{=} \sum_{j \in \Lambda} M_{j}^{i *} e^{\perp} M_{j}^{i}\left(\psi_{j}\left(e^{\perp}\right)\right)^{n} \\
& \leq \sum_{j} M_{j}^{i *} M_{j}^{i} p^{n} \\
& =p^{n}
\end{aligned}
$$

From (4.4) one gets

$$
0 \leq E_{o]}\left(\tau_{e ; r ; \infty}\right)=\lim _{n \rightarrow \infty} E_{o]}\left(\tau_{e ; x_{n} ; \infty}\right)=0
$$

Thus $E_{o]}\left(\tau_{e ; r ; \infty}\right)=0$ and by (27) the projection $e$ is $\mathcal{E}$-recurrent.

## 5 Examples

In this section, we are going to illustrate the obtained results on recurrence for quantum Markov chains associated with OQRW.

Let $\mathcal{H}=\mathcal{K}=\mathbb{C}^{2}$. The algebra of observable at a site $u$ is then $\mathcal{A}_{u}=\mathcal{B}(\mathcal{H} \otimes \mathcal{H}) \equiv M_{4}(\mathbb{C})$. Let $\Lambda=\{1,2\}$. The interactions are given by

$$
B_{1}^{1}=\left(\begin{array}{cc}
a & 0  \tag{47}\\
0 & b
\end{array}\right), \quad B_{2}^{1}=\left(\begin{array}{cc}
0 & 1 \\
0 & 0
\end{array}\right), \quad B_{1}^{2}=\left(\begin{array}{cc}
c & 0 \\
0 & d
\end{array}\right), \quad B_{2}^{2}=\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right)
$$

where $|a|^{2}+|c|^{2}=|b|^{2}+|d|^{2}=1, a c \neq 0$. Put

$$
p=\left(\begin{array}{cc}
1 & 0  \tag{48}\\
0 & 0
\end{array}\right), \quad q=\left(\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right)
$$

and

$$
|1\rangle=\left[\begin{array}{l}
1 \\
0
\end{array}\right],|2\rangle=\left[\begin{array}{l}
0 \\
1
\end{array}\right]
$$

Notice that $(|1\rangle,|2\rangle)$ is an ortho-normal basis of $\mathcal{K} \equiv \mathbb{C}^{2}$. In the sequel elements of $\mathcal{B}(\mathcal{H})$ will be denoted by means of $2 \times 2$ complex matrices, while elements of $\mathcal{B}(\mathcal{K})$ will be written using Dirac notation $|i\rangle\langle j|$.

Recall that (c.f. []) any rank-1 projection in $\mathbb{M}_{2}(\mathbb{C})$ has the form

$$
p(\varepsilon, z)=\left(\begin{array}{cc}
\varepsilon & z \sqrt{\varepsilon(1-\varepsilon)}  \tag{49}\\
\bar{z} \sqrt{\varepsilon(1-\varepsilon)} & 1-\varepsilon
\end{array}\right)
$$

where $\varepsilon \in[0,1], z \in \mathbb{C}$ with $|z|=1$. Then we consider the projection on $\mathcal{A}$ having the following form

$$
e(\varepsilon, z, \xi)=p(\varepsilon, z) \otimes|\xi\rangle\langle\xi|
$$

where

$$
|\xi\rangle:=\sum_{i \in \Lambda} \xi_{i}|i\rangle \in \mathcal{K}
$$

being a unit vector. i.e. $\sum_{i \in \Lambda}\left|\xi_{i}\right|^{2}=1$.
Example 5.1 ( $\mathcal{E}$-recurrence). Using (43) one compute

$$
\psi_{j}(e(\varepsilon, z, \xi))=\frac{1}{\operatorname{Tr}\left(\rho_{j}\right)} \sum_{i \in \Lambda} \operatorname{Tr}\left(B_{j}^{i} \rho_{j} B_{j}^{i *} p(\varepsilon, z)\right)\left|\xi_{i}\right|^{2}
$$

Then, for $\rho_{j}=\left(\begin{array}{ll}1 & 0 \\ 0 & 0\end{array}\right)$, one gets

$$
\begin{gathered}
\operatorname{Tr}\left(B_{1}^{1} \rho_{j} B_{1}^{1 *} p(\varepsilon, z)\right)=\varepsilon|a|^{2} \\
\operatorname{Tr}\left(B_{2}^{1} \rho_{2} B_{2}^{1 *} p(\varepsilon, z)\right)=0 \\
\operatorname{Tr}\left(B_{1}^{2} \rho_{j} B_{1}^{2 *} p(\varepsilon, z)\right)=\varepsilon|c|^{2} \\
\quad \operatorname{Tr}\left(B_{2}^{2} \rho_{j} B_{2}^{2 *} p(\varepsilon, z)\right)=\varepsilon
\end{gathered}
$$

Hence,

$$
\begin{equation*}
\psi_{1}(e(\varepsilon, z, \xi))=\sum_{i \in \Lambda} \operatorname{Tr}\left(B_{1}^{i} \rho_{j} B_{1}^{i *} p(\varepsilon, z)\right)\left|\xi_{i}\right|^{2}=\quad \varepsilon|a|^{2}\left|\xi_{1}\right|^{2} \tag{50}
\end{equation*}
$$

and

$$
\begin{equation*}
\psi_{2}(e(\varepsilon, z, \xi))=\sum_{i \in \Lambda} \operatorname{Tr}\left(B_{2}^{i} \rho_{j} B_{2}^{i *} p(\varepsilon, z)\right)\left|\xi_{i}\right|^{2}=\quad \varepsilon|c|^{2}\left|\xi_{1}\right|^{2}+\varepsilon\left|\xi_{2}\right|^{2} \tag{51}
\end{equation*}
$$

Thus, Theorem 4.4 implies that $e(\varepsilon, z, \xi)^{\perp}$ is $\mathcal{E}$-recurrent whenever $\varepsilon<1$. If $\varepsilon=|a|=\left|\xi_{1}\right|=1$, the projection $e(\varepsilon, z, \xi)$ becomes

$$
e(1, z, \xi)=\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right) \otimes|1\rangle\langle 1| .
$$

Put

$$
e:=e(1, z, \xi)^{\perp}=\mathbb{1}_{M_{2}} \otimes|2\rangle\langle 2|+\left(\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right) \otimes|1\rangle\langle 1|
$$

From (50) and (51) one has $\psi_{1}\left(e^{\perp}\right)=1$ and $\psi_{2}\left(e^{\perp}\right)=0$. Then, from (42) one gets

$$
\begin{aligned}
E_{o]}\left(\tau_{e ; x_{n} ; \infty}\right)=\mathcal{M}_{1}\left(e^{\perp}\right) & =\sum_{i=1}^{2} M_{1}^{i *} e^{\perp} M_{1}^{i} \\
& =B_{1}^{1^{*}}\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right) B_{1}^{1} \otimes|1\rangle\langle 1| \\
& =e^{\perp}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathcal{E}\left(e \otimes E_{o]}\left(\tau_{e ; r ; \infty}\right)\right) & =\mathcal{E}\left(e \otimes e^{\perp}\right) \\
& =\mathcal{M}_{1}(e) \psi_{1}\left(e^{\perp}\right) \\
& =\mathcal{M}_{1}(e) \\
& =\left(\begin{array}{cc}
0 & 0 \\
0 & |b|^{2}
\end{array}\right) \otimes|1\rangle\langle 1|+\left(\begin{array}{cc}
0 & 0 \\
0 & |d|^{2}
\end{array}\right) \otimes|2\rangle\langle 2| \neq 0
\end{aligned}
$$

Thus, from (29) the projection $e$ is not $\mathcal{E}$-recurrent. This means that the inequality (4.4) is optimal.
Example 5.2 ( $\mathcal{E}$-accessibility). Recall that for $\ell=1,2$, the backward Markov operator is given by

$$
P_{u}^{(u, \ell)}\left(a_{(u, \ell)}\right)=\sum_{j=1}^{2}\left(\mathbb{I}_{\mathcal{B}(\mathcal{H})} \otimes|j\rangle\langle j|\right) \varphi_{j}\left(a_{(u, \ell)}\right) .
$$

Recall also that the forward Markov operator is given by

$$
T_{u}\left(a_{u}\right)=\sum_{i j} M_{j}^{i, *} a_{u} M_{j}^{i}
$$

Then,

$$
\mathcal{E}\left(e \otimes P_{x_{0}}^{x_{m}} T_{x_{m}} f\right)=\sum_{j} \psi_{j}(f) \mathcal{E}(e \otimes \mathbb{I} \otimes|j\rangle\langle j|)
$$

- Take $e \in \operatorname{Proj}(\mathcal{A})$ and $f=e(\varepsilon, z, \xi)$, then using (50) and (51), we obtain

$$
\begin{aligned}
\mathcal{E}\left(e \otimes P_{x_{0}}^{x_{m}} T_{x_{m}} e(\varepsilon, z, \xi)\right) & =\sum_{j} \psi_{j}(e(\varepsilon, z, \xi)) \mathcal{E}(e \otimes \mathbb{I} \otimes|j\rangle\langle j|) \\
& =\varepsilon|a|^{2}\left|\xi_{1}\right|^{2} \mathcal{E}(e \otimes \mathbb{I} \otimes|1\rangle\langle 1|)+\varepsilon\left(|c|^{2}\left|\xi_{1}\right|^{2}+\left|\xi_{2}\right|^{2}\right) \mathcal{E}(e \otimes \mathbb{I} \otimes|2\rangle\langle 2|) \\
& =\varepsilon\left[|a|^{2}\left|\xi_{1}\right|^{2} \mathcal{M}_{1}(e)+\left(|c|^{2}\left|\xi_{1}\right|^{2}+\left|\xi_{2}\right|^{2}\right) \mathcal{M}_{2}(e)\right]
\end{aligned}
$$

for any projection e. In particular, one easily can see that there is no projection $e$ which is $\mathcal{E}$-accessible from

$$
e(0, z, \xi)=\left(\begin{array}{cc}
0 & 0 \\
0 & 1
\end{array}\right) \otimes|\xi\rangle\langle\xi| .
$$

- Now, take

$$
f=\sigma^{x_{W_{n}}(1)}=\mathbb{1}_{M_{2}} \otimes|1\rangle\langle 1|
$$

where $x_{W_{n}}(1)$ is defined by (8). Then, we have

$$
\psi_{1}\left(\sigma^{x_{W_{n}}(1)}\right)=\operatorname{Tr}\left(B_{1}^{1} p B_{1}^{1^{*}}\right)=|a|^{2} \quad \text { and } \psi_{2}\left(\sigma^{x_{W_{n}}(1)}\right)=\operatorname{Tr}\left(B_{2}^{1} p B_{2}^{1^{*}}\right)=0
$$

Hence,

$$
\begin{aligned}
\mathcal{E}\left(e \otimes P_{x_{1}}^{x_{m}} T_{x_{m}} \sigma^{x_{W_{n}}(1)}\right) & =|a|^{2} \mathcal{E}\left(e \otimes(\mathbb{I} \otimes|1\rangle\langle 1|)^{\left(x_{1}\right)}\right) \\
& =|a|^{2} \sum_{i} M_{1}^{i^{*}} e M_{1}^{i} \\
& =|a|^{2} \mathcal{M}_{1}(e) .
\end{aligned}
$$

In particular, if $|a|>0$, we deduce that

$$
e_{1}=\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right) \otimes|1\rangle\langle 1|
$$

is $\mathcal{E}$-accessible from $\sigma^{x_{W_{n}}(1)}$, since $\mathcal{M}_{1}\left(e_{1}\right)=e_{1}$.
Example 5.3 ( $\varphi$-accessibility). We notice that,

$$
\varphi\left(e \otimes P_{x_{0}}^{x_{m}} T_{x_{m}} f\right)=\sum_{j} \psi_{j}(f) \varphi(e \otimes \mathbb{I} \otimes|j\rangle\langle j|)
$$

where

$$
\begin{aligned}
\varphi(e \otimes \mathbb{I} \otimes|j\rangle\langle j|) & =\sum_{k} \operatorname{Tr}\left(\omega_{o} \mathcal{M}_{k}(e)\right) \psi_{k}(\mathbb{I} \otimes|j\rangle\langle j|) \\
& =\sum_{k} \operatorname{Tr}\left(\omega_{o} \mathcal{M}_{k}(e)\right) \psi_{k}(\mathbb{I} \otimes|j\rangle\langle j|) \\
& =\sum_{k} \operatorname{Tr}\left(\omega_{o} \mathcal{M}_{k}(e)\right) \frac{\operatorname{Tr}\left(B_{k}^{j} \rho_{k} B_{k}^{j^{*}}\right)}{\operatorname{Tr}\left(\rho_{k}\right)} .
\end{aligned}
$$

Hence, for $|a|>0$ and

$$
\omega_{0}=\left(\begin{array}{c|c}
(0) & (0) \\
\hline(0) & (*)
\end{array}\right)
$$

we deduce that

$$
e_{1}=\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right) \otimes|1\rangle\langle 1|
$$

is not $\varphi$-accessible from $\sigma^{x_{W_{n}}(1)}$, since

$$
\operatorname{Tr}\left(\omega_{o} \mathcal{M}_{1}\left(e_{1}\right)\right)=\operatorname{Tr}\left(\omega_{o} e_{1}\right)=0 \quad \text { and } \quad \mathcal{M}_{2}\left(e_{1}\right)=0
$$

## Declaration of Competing Interest

The authors confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

## Data availability

The paper does not use any data.

## Acknowledgments

The authors gratefully acknowledge Qassim University, represented by the Deanship of Scientific Research, on the financial support for this research under the number (10173-cba-2020-1-3-I) during the academic year $1442 \mathrm{AH} / 2020 \mathrm{AD}$.

## References

[1] L. Accardi, On noncommutative Markov property, Funct. Anal. Appl. 8 (1975), 1-8.
[2] L. Accardi, A. Frigerio, Markovian cocycles, Proc. Royal Irish Acad. 83A (1983) 251-263.
[3] L. Accardi, C. Cecchini, Conditional expectations in von Neumann algebras and a Theorem of Takesaki, J. Funct. Anal. 45, 245-273 (1982).
[4] Accardi L., Fidaleo F., Non homogeneous quantum Markov states and quantum Markov fields, J. Funct. Anal. 200 (2003), 324--347.
[5] L. Accardi, F. Fidaleo, F. Mukhamedov, Markov states and chains on the CAR algebra, Inf. Dim. Analysis, Quantum Probab. Related Topics 10 (2007), 165-183.
[6] L. Accardi, D. Koroliuk, Stopping times for quantum Markov chains, J. Theor. Probab. 5(1992), 521-535.
[7] L. Accardi, D. Koroliuk, Quantum Markov chains: The recurrence problem. In book: Quantum Prob. and Related Topics VII, 63-73 (1991).
[8] L. Accardi, A. Souissi, E. Soueidy, Quantum Markov chains: A unification approach, Inf. Dim. Analysis, Quantum Probab. Related Topics 23(2020), 2050016.
[9] L. Accardi, H. Ohno, F. Mukhamedov, Quantum Markov fields on graphs, Inf. Dim. Analysis, Quantum Probab. Related Topics 13(2010), 165-189.
[10] L. Accardi, G.S. Watson, Quantum random walks, in book: L. Accardi, W. von Waldenfels (eds) Quantum Probability and Applications IV, Proc. of the year of Quantum Probability, Univ. of Rome Tor Vergata, Italy, 1987, LNM, 1396(1987), 73-88.
[11] S. Attal, F. Petruccione, C. Sabot, I. Sinayskiy. Open Quantum Random Walks. J. Stat. Phys. 147(2012), 832-852.
[12] I. Bardet, D. Bernard, Y. Pautrat, Passage times, exit times and Dirichlet problems for open quantum walks, J. Stat. Phys. 167(2017), 173-204.
[13] D. Burgarth, V. Giovannetti, The generalized Lyapunov theorem and its application to quantum channels. New J. Phys. 9 (2007) 150.
[14] R. Carbone, Y. Pautrat. Homogeneous open quantum random walks on a lattice. J. Stat. Phys. 160 (2015), 1125-1152.
[15] R. Carbone, Y. Pautrat. Open quantum random walks: reducibility, period, ergodic properties. Ann. Henri Poincaré 17(2016), 99-135.
[16] S.L. Carvalho, L.F. Guidi, C.F. Lardizabal, Site recurrence of open and unitary quantum walks on the line, Quantum Infor. Proc.16(2017), Article 17.
[17] J.I. Cirac, F. Verstraete, Renormalization and tensor product states in spin chains and lattices, J. Phys. A. Math. Theor. 42 (2009), 504004.
[18] A. Dhahri. C.K. Ko, H.J. Yoo, Quantum Markov chains associated with open quantum random walks, J. Stat. Phys. 176(2019), 1272-1295
[19] A. Dhahri, F. Mukhamedov, Open quantum random walks, quantum Markov chains and recurrence. Rev. Math. Phys. 31(2019), 1950020.
[20] Y. Feng, N. Yu and M. Ying, Model checking quantum Markov chains, J. Computer Sys. Sci. 79, 1181-1198 (2013).
[21] F.A. Grünbaum, C.F. Lardizabal, L.Velázquez, Quantum Markov chains: recurrence, Schur functions and splitting rules, Ann. Henri Poincare, 21(2020), 189-239.
[22] T. S. Jacq, C. F. Lardizabal, Homogeneous open quantum walks on the line: criteria for site recurrence and absorption, Quantum Inf. Comput., 21(2021), 37-58.
[23] J. Kempe, Quantum random walks-an introductory overview, Contemporary Physics, 44(2003), 307-327.
[24] C. K. Ko, N. Konno, E. Segawa, H. J. Yoo. Central limit theorems for open quantum random walks on the crystal lattices, J. Stat. Phys. 176(2019), 710-735.
[25] N. Konno, H. J. Yoo. Limit theorems for open quantum random walks. J. Stat. Phys. 150 (2013), 299-319.
[26] B. Kümmerer, Quantum Markov processes and applications in physics. In book: Quantum independent increment processes. II, 259-330, Lecture Notes in Math., 1866, Springer, Berlin, 2006.
[27] C. F. Lardizabal, R. R. Souza. On a class of quantum channels, open random walks and recurrence. J. Stat. Phys. 159(2015), 772-796.
[28] C. Liu, N. Petulante. On Limiting distributions of quantum Markov chains. Int. J. Math. and Math. Sciences. 2011(2011), ID 740816.
[29] A. Marais, I. Sinayskiy, A. Kay, F. Petruccione, A. Ekert, Decoherence-assisted transport in quantum networks, New J. Phys., 15(2013), 013038.
[30] F. Mukhamedov, A. Barhoumi, A. Souissi, Phase transitions for quantum Markov chains associated with Ising type models on a Cayley tree, J. Stat. Phys. 163, 544-567 (2016).
[31] F. Mukhamedov, A. Barhoumi, A. Souissi, S. El Gheteb, A quantum Markov chain approach to phase transitions for quantum Ising model with competing XY-interactions on a Cayley tree, J. Math. Phys. 61, 093505 (2020).
[32] F. Mukhamedov, A. Souissi, Quantum Markov States on Cayley trees, J. Math. Anal. Appl. 473(2019), 313-333.
[33] F. Mukhamedov, A. Souissi, T. Hamdi, Open Quantum Random Walks and Quantum Markov chains on Trees I: Phase transitions, Preprint
[34] M. A. Nielsen, I. L. Chuang. Quantum computation and quantum information. Cambridge Univ. Press, 2000.
[35] J. R. Norris. Markov chains. Cambridge Univ. Press, 1997.
[36] J. Novotný, G. Alber, I. Jex. Asymptotic evolution of random unitary operations. Cent. Eur. J. Phys. 8(2010), 1001-1014.
[37] R. Orus, A practical introduction of tensor networks: matrix product states and projected entangled pair states, Ann of Physics 349 (2014) 117-158.
[38] R. Portugal. Quantum walks and search algorithms. Springer, 2013.
[39] S. Rommer, S. Ostlund, A class of ansatz wave functions for 1D spin systems and their relation to DMRG, Phys. Rev. B 55 (1997) 2164.
[40] I. Sinayskiy , F. Petruccione, Open quantum walks, Eur. Phys. J. Spec. Top. 227(2019), 1869-1883.
[41] Souissi A., On Stopping Rules for Tree-indexed Quantum Markov chains, preprint (2022)


[^0]:    ${ }^{1}$ We note that a Quantum Markov Chain is a quantum generalization of a Classical Markov Chain where the state space is a Hilbert space, and the transition probability matrix of a Markov chain is replaced by a transition amplitude matrix, which describes the mathematical formalism of the discrete time evolution of open quantum systems, see $[1,2]$ for more details.
    ${ }^{2}$ The dual of $\mathcal{M}$ is defined by the equality $\operatorname{Tr}(\mathcal{M}(\rho) x)=\operatorname{Tr}\left(\rho \mathcal{M}^{*}(x)\right)$ for all density operators $\rho$ and observables $x$.

[^1]:    ${ }^{3}$ The existence of other boundary conditions leads to the problem of a phase transition within QMC scheme which was considered in [33, 30].

