Sedentariness in quantum walks

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Abstract

We formalize the notion of a sedentary vertex and present a relaxation of the concept of a sedentary family of graphs introduced by Godsil [*Linear Algebra Appl.* 614:356-375, 2021]. We provide sufficient conditions for a given vertex in a graph to exhibit sedentariness. We also show that a vertex with at least two twins (vertices that share the same neighbours) is sedentary. We prove that there are infinitely many graphs containing strongly cospectral vertices that are sedentary, which reveals that, even though strong cospectrality is a necessary condition for pretty good state transfer, there are strongly cospectral vertices which resist high probability state transfer to other vertices. Moreover, we derive results about sedentariness in products of graphs which allow us to construct new sedentary families, such as Cartesian powers of complete graphs and stars.

Keywords: quantum walks, sedentary walks, twin vertices, strongly cospectral vertices, adjacency matrix, Laplacian matrix

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1 Introduction

A *(continuous-time) quantum walk* (CTQW) on X describes the propagation of quantum states across a quantum spin network modelled by the graph X, where the qubits in the spin network and the interactions between them are represented by the vertices and edges of X, respectively. If H is a real symmetric matrix that encodes the adjacencies in X, i.e., $H_{u,v} = 0$ if and only if $[u,v]$ is not an edge in X, then the CTQW on X with respect to H is determined by the complex symmetric unitary matrix

$$U_H(t) = e^{itH}. \tag{1}$$

We call $U_H(t)$ and H resp. the *transition matrix* and the *Hamiltonian* of the quantum walk. If H in (1) is clear from the context, then we write $U_H(t)$ as $U(t)$. As $U(t)$ is unitary, $\sum_{j\in V(X)}|U(t)_{u,j}|^2 = 1$ for all t, and so $|U(t)_{u,v}|^2$ is interpreted as the probability of quantum state transfer between u and v at time t.

The concept of a sedentary family of graphs was introduced by Godsil [God21], which was mainly motivated by the behaviour of quantum walks on complete graphs. Godsil defined a family $\mathscr{F}$ of graphs to be *sedentary* if there is a constant $a > 0$ such that for each $X \in \mathscr{F}$ and each vertex u of X, we have $|U(t)_{u,u}| \geqslant 1 - \frac{a}{|V(X)|}$ for all t. Consequently, $|U(t)_{u,u}|$ tends to 1 for each $u \in V(X)$ as $|V(X)|$ increases. Godsil showed that large classes of strongly regular graphs are sedentary at any vertex. While the main focus in [God21] was to study sedentary families of graphs, Godsil also investigated sedentariness of a single vertex in a graph by showing that cones over k-regular graphs exhibit varying degrees of sedentariness at their apexes with respect to the adjacency matrix depending on the value k. Frigerio and Paris showed that cones are also sedentary at the apex with respect to the Laplacian matrix [FP23]. To the best of our knowledge, no other families of finite graphs are known to exhibit sedentariness.

In order to better understand the notion of single vertex sedentariness and obtain more sedentary families of graphs, we formalize the definition of a sedentary vertex and propose a relaxation of the notion of a sedentary family of graphs. Let $0 < C \leqslant 1$ be a constant. We say that a vertex u of X is C-*sedentary* if $|U(t)_{u,u}| \geqslant C$ for all t. We say that a family $\mathscr{F}$ of graphs is C-*sedentary* if there exists a real-valued function f satisfying $0 < f(s) \leqslant 1$ for all $s > 0$ such that (i) for each $X \in \mathscr{F}$, some vertex u of X is $f(|V(X)|)$-sedentary and (ii) $f(s) \to C$ as s increases. If $f(s) = 1 - \frac{a}{s}$ for some $a > 0$, then $C = 1$, and if we add that each vertex of each $X \in \mathscr{F}$ is $f(|V(X)|)$-sedentary, then the concept of a C-sedentary family coincides with Godsil's notion of a sedentary family. If $C = 0$ and $\inf_{t>0}|U(t)_{u,u}| = f(|V(X)|)$ for each $X \in \mathscr{F}$, then the family is said to be *quasi-sedentary*, a concept first introduced in this paper. We emphasize that these properties depend on the matrix H, which we later choose to be a generalized adjacency matrix or a generalized normalized adjacency matrix of X.

The main goal of this paper is to provide sufficient conditions for C-sedentariness of a vertex and construct families of graphs that are C-sedentary. We prove our main result, which states that by an appropriate choice of a subset S of the eigenvalue support of a vertex u, one may be able to show that u is sedentary. We then use this result to establish that for any vertex u in a set of twins T, $|U(t)_{u,u}| \geqslant 1 - \frac{2}{|T|}$ for all t. Consequently, vertices with at least two twins are sedentary, which allows us to construct new families of graphs that are C-sedentary. This includes graphs built from joins, graphs with tails and blow-ups of graphs. We also show that sedentariness is preserved under Cartesian products, which provides another way to construct C-sedentary families. Another result, which is rather unexpected, is that there are infinitely many graphs containing strongly cospectral vertices that are sedentary. This reveals that some strongly cospectral vertices resist high probability transfer to other vertices. We also discuss the connection of sedentariness to other types of quantum state transfer. Even though sedentary vertices do not exhibit pretty good state transfer, we show that there are C-sedentary and quasi-sedentary families whose each member graph exhibits proper fractional revival at the sedentary vertices. For local uniform mixing, we show that this is only possible for quasi-sedentary families.

Throughout this paper, we assume that X is a connected weighted undirected graph with possible loops but no multiple edges. We denote the vertex and edge sets of X resp. by $V(X)$ and $E(X)$, and we allow the edges of X to have nonzero real weights (i.e., an edge can have either positive or negative weight). We denote an edge between vertices u and v by $[u,v]$. We say that X is *simple* if X has no loops, and X is *unweighted* if all edges of X have weight one. For $u \in V(X)$, we denote the set of neighbours of u in X as $N_X(u)$, and the characteristic vector of u as $\mathbf{e}_u$, which is a vector with a 1 on the entry indexed by u and 0's elsewhere. The all-ones vector of order n, the zero vector of order n, the $m \times n$ all-ones matrix, and the $n \times n$ identity matrix are denoted resp. by $\mathbf{1}_n$, $\mathbf{0}_n$, $\mathbf{J}_{m,n}$ and I_n. If $m = n$, then we write $\mathbf{J}_{m,n}$ as $\mathbf{J}_n$, and if the context is clear, then we simply write these matrices resp. as $\mathbf{1}$, $\mathbf{0}$, $\mathbf{J}$ and I. We also represent the transpose of M by M^T. We denote the simple unweighted empty, cycle, complete, and path graphs on n vertices resp. as O_n, C_n, K_n, and P_n. We also denote the simple unweighted complete bipartite graph with partite sets of sizes $n_1, \dots, n_k$ as $K_{n_1,\dots,n_k}$.

For two graphs X and Y, the *join* $X \vee Y$ is the resulting graph after adding all edges $[u,v]$ of weight one, where $u \in V(X)$ and $v \in V(Y)$, while the *union* $X \cup Y$ is the resulting graph with $V(X \cup Y) = V(X) \cup V(X)$ and $E(X \cup Y) = E(X) \cup E(Y)$. The *Cartesian product* $X \square Y$ is a graph with vertex set $V(X) \times V(Y)$ where (u,x) and (v,y) are adjacent in $X \square Y$ if either $u = v$ and $[x,y]$ is an edge in Y or $x = y$ and $[u,v]$ is an edge in X. The weight of the edge between (u,x) and (v,y) is equal to the weight of $[u,v]$ if $x = y$ and $[x,y]$ if $u = v$. The *direct product* $X \times Y$ is the graph with vertex set $V(X) \times V(Y)$ where (u,x) and (v,y) are adjacent in $X \times Y$ if $[u,v]$ and $[x,y]$ are edges resp. in X and Y. The weight of the edge between (u,x) and (v,y) is equal to the product of the weights of the edges $[u,v]$ and $[x,y]$.

We define the adjacency matrix $A(X)$ of X entrywise as

$$A(X)_{u,v} = \begin{cases} \omega_{u,v}, & \text{if } u \text{ and } v \text{ are adjacent} \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

where $0 \neq \omega_{u,v} \in \mathbb{R}$ is the weight of $[u,v]$. The degree matrix $D(X)$ of X is the diagonal matrix of vertex degrees of X, where $\deg(u) = 2\omega_{u,u} + \sum_{j \neq u} \omega_{u,j}$ for each $u \in V(X)$. We say that X is weighted-regular if $D(X)$ is a scalar multiple of the identity matrix. As X is weighted, it is possible that $\deg(u) = 0$ without u being isolated. Assuming that $\deg(u) \geqslant 0$ for all $u \in V(X)$, we define $D(X)^{-\frac{1}{2}}$ as the diagonal matrix whose (u,u) entry is $1/\sqrt{\deg(u)}$ if $\deg(u) > 0$ and 0 otherwise.

Let $a, b, c \in \mathbb{R}$ with $a \neq 0$. A matrix of the form $\mathbf{A}(X) = cI + bD(X) + aA(X)$ is called a *generalized adjacency matrix* $\mathbf{A}(X)$ of X, and a matrix of the form $\mathcal{A}(X) = bI + aD(X)^{-\frac{1}{2}}A(X)D(X)^{-\frac{1}{2}}$ is called a *generalized normalized adjacency matrix* $\mathcal{A}(X)$ of X. These two matrices were first studied in [Mon22] in the context of quantum state transfer (in particular, in relation to the concept of strong cospectrality). We consider these two types of matrices, which are generalizations of well-known matrices associated to graphs. Indeed, if $c = 0$, $b = 1$ and $a = -1$, then $\mathbf{A}$ becomes the *Laplacian matrix* $L(X)$ of X, while $\mathcal{A}(X)$ becomes the *normalized Laplacian* matrix $\mathcal{L}(X)$ of X. Since the quantum walks determined by $bI + aH$ and H are equivalent, we simplify the discussion by considering the matrices

$$\mathbf{A}(X) = \alpha D(X) + A(X) \quad \text{and} \quad \mathcal{A}(X) = D(X)^{-\frac{1}{2}}A(X)D(X)^{-\frac{1}{2}}, \tag{3}$$

and note that the quantum walks determined by $\mathbf{A}$ are equivalent for all $\alpha \in \mathbb{R}$ whenever the graph is weighted-regular. We use $M(X)$ to denote $\mathbf{A}(X)$ or $\mathcal{A}(X)$, and use $H = M(X)$ in (1). If the context is clear, then we write $M(X)$, $A(X)$, $L(X)$, $\mathcal{L}(X)$ and $D(X)$ resp. as M, A, L, $\mathcal{L}$ and D. Finally, if $U_{X\square Y}(t)$ is the transition matrix of $X \square Y$ with respect to $\mathbf{A}$, then it is known that

$$U_{X\square Y}(t) = U_X(t) \otimes U_Y(t), \tag{4}$$

while if $U_{X\times Y}(t)$ is the transition matrix of $X \times Y$ with respect to $\mathcal{A}$, then $U_{X\times Y}(t) = U_X(t) \otimes U_Y(t)$ whenever X and Y are simple. Here, $A \otimes B$ denotes the Kronecker product of matrices A and B.

2 Sedentariness

We begin with the definition of a sedentary vertex.

Definition 1. *We say that vertex u of X is C-sedentary if for some constant $0 < C \leqslant 1$,*

$$\inf_{t>0} |U_M(t)_{u,u}| \geqslant C. \tag{5}$$

If equality holds in (5), then we say that u is sharply C-sedentary, while if the infimum in (5) is attained for some $t > 0$, then we say that u is tightly C-sedentary.

We also say that u is *not sedentary* if $\inf_{t>0}|U_M(t)_{u,u}| = 0$. Note that for a sharply C-sedentary vertex, C is the best lower bound one can get for $|U_M(t)_{u,u}|$ for all t. It is also clear that a tightly sedentary vertex is sharply sedentary, but the converse is not true. If C is not important, then we resp. say sedentary, sharply sedentary, and tightly sedentary. Sedentariness of X at u implies that $|U_M(t)_{u,u}|$ is bounded away from 0, and as a result, the quantum state initially at vertex u tends to stay at u.

As M is real symmetric, it admits a spectral decomposition $M = \sum_j \lambda_j E_j$, and so we can write (1) as

$$U_M(t) = \sum_j e^{it\lambda_j} E_j,$$

where the λ_j's are the distinct eigenvalues of M and E_j is the orthogonal projection matrix associated with λ_j. The *eigenvalue support* of vertex u with respect to M is the set $\sigma_u(M) = \{\lambda_j : E_j\mathbf{e}_u \neq \mathbf{0}\}$. We say that two vertices u and v are *cospectral* if $(E_j)_{u,u} = (E_j)_{v,v}$ for each j. It is immediate that if X has an automorphism mapping u to v, then they are cospectral.

Let S_1 and S_2 be two non-empty disjoint proper subsets of $V(X)$. Order the vertices of X in a way that S_1 comes first, followed by S_2 and then $V(X)\backslash(S_1 \cup S_2)$. We say that there is *pretty good state transfer* (PGST) from S_1 and S_2 if for each $\epsilon > 0$, there exists a time t_ϵ such that $U_M(t_\epsilon)$ has the block form

$$U_M(t_\epsilon) = \begin{bmatrix} * & U_\epsilon & * \\ U_\epsilon^T & * & * \\ * & * & * \end{bmatrix},$$

where U_ϵ is an $|S_1|$-by-$|S_2|$ matrix satisfying $\|U_\epsilon\| > 1-\epsilon$. Clearly, if $|S_1| = |S_2| = 1$, then we get PGST between two vertices. If $\|U_\epsilon\| = 1$, then we say that *perfect state transfer* (PST) occurs from S_1 and S_2, a notion that is equivalent to *group state transfer* (GST) introduced by Brown et al. [BMW21]. As $U_M(t_1)$ is non-singular, if PST occurs from S_1 and S_2, then $|S_1| \leqslant |S_2|$, and the case $|S_1| = |S_2| = 1$ yields PST between two vertices. PST, PGST and GST (and later on, uniform mixing and fractional revival) fall under the general notion of quantum state transfer, which is an important physical concept.

The following basic properties of C-sedentary vertices are immediate from the fact that $U(t)$ is unitary.

Proposition 2. *Let X be a graph with vertex u.*

1. *If X is (sharply or tightly) C_1-sedentary at u, where $0 < C_1 \leqslant 1$, then the following hold.*
   - *(a) X is also C_2-sedentary at u whenever $0 < C_2 \leqslant C_1$.*
   - *(b) If u and v are cospectral vertices, then X is also (sharply or tightly) C_1-sedentary at v.*
   - *(c) Any subset S of $V(X)$ containing u cannot be involved in pretty good state transfer in X.*
   - *(d) For any vertex $v \neq u$, $\sup_{t>0}|U_M(t)_{u,v}| \leqslant \sqrt{1 - C_1^2}$.*

2. *If for each vertex $v \neq u$ of X, there is a constant $C_v < 1$ such that $\sup_{t>0}|U_M(t)_{u,v}| = C_v$, then u is sharply C-sedentary if and only if $1 - \sum_{v\neq u} C_v^2 > 0$, in which case $C = 1 - \sqrt{\sum_{v\neq u} C_v^2}$.*

By Proposition 2(1a), it is desirable to find the least $C < 1$ such that a vertex is C-sedentary, i.e., the C such that u is sharply C-sedentary. Proposition 2(1c) implies that PGST and sedentariness are mutually exclusive. Thus, our investigation of sedentariness is motivated in the same way as the study of PGST, in a sense that identifying sedentary vertices rules out the existence of PGST. Proposition 2(1d) tells us that a necessary condition for sedentariness of u is that $|U(t)_{u,v}|$ is bounded away from 1 for any $v \neq u$, while Proposition 2(2) provides a sufficient condition for sedentariness. But since not much is known about pairs of vertices such that $|U(t)_{u,v}|$ bounded away from 1, Proposition 2(2) will not be very useful to us. Instead, we present a sufficient condition for sedentariness in Section 4 that only depends on the diagonal entries of the E_j's. Next, we define what it means for a family of graphs to be C-sedentary.

Definition 3. *Let $0 \leqslant C \leqslant 1$ and $\mathscr{F}$ be a (countable) family of graphs. We call $\mathscr{F}$ is C-sedentary if there is a function $f : \mathbb{R}^+ \to (0,1]$ such that (i) for each $X \in \mathscr{F}$ and some $u \in V(X)$, the graph X is $f(|V(X)|)$-sedentary at u and (ii) $f(s) \to C$ as $s \to \infty$. Further, if $C = 1$, then we call $\mathscr{F}$ is sedentary; if each $X \in \mathscr{F}$ is sharply (resp., tightly) $f(|V(X)|)$-sedentary at u, then call $\mathscr{F}$ is sharply (resp., tightly) C-sedentary; and if $C = 0$ and $\mathscr{F}$ is sharply C-sedentary, then call $\mathscr{F}$ is quasi-sedentary.*

Note that if $C = 1$, then the above notion coincides with Godsil's definition of sedentary quantum walks. For example, if $\mathscr{K}$ is the family of complete graphs on $n \geqslant 3$ vertices, then for all t,

$$|U(t)_{u,u}| = \frac{|n - 1 + e^{itn}|}{n} \geqslant 1 - \frac{2}{n} \tag{6}$$

with equality if and only if $t = \frac{j\pi}{n}$ for odd j. Thus, $\mathscr{K}$ is a sedentary family. The case $0 \leqslant C < 1$ is a more relaxed version of C-sedentariness than the case $C = 1$. In [God21], Godsil showed that cones on d-regular graphs on n vertices are C-sedentary at the apex with respect to A, where $C = \frac{d^2}{d^2+4n}$. Thus, the concept of C-sedentariness for $0 < C \leqslant 1$ is not entirely new, although this concept is first formalized in this paper. Quasi-sedentariness, on the other hand, is a new concept introduced in this paper, and may be regarded as the weakest form of sedentariness for families of graphs.

3 Products

Consider $\mathbf{A}$ and $\mathcal{A}$ in (3). In this section, we derive results about sedentariness in products of graphs.

Theorem 4. *Let $X_1, \dots, X_n$ be weighted graphs with possible loops, and $Z = \square_{j=1}^n X_j$.*

1. *If each X_j is C_j-sedentary at u_j, then Z is $\prod_{j=1}^n C_j$-sedentary at $(u_1, \dots, u_n)$. In particular, if each X_j is sharply C_j-sedentary at u_j, then Z is sharply C'-sedentary at $(u_1, \dots, u_n)$ with $C' \geqslant \prod_{j=1}^n C_j$.*
2. *If Z is C-sedentary at $(u_1, \dots, u_n)$, then each X_j is sharply C_j-sedentary at u_j for some $0 < C_j \leqslant 1$.*
3. *If each X_j is tightly C_j-sedentary at u_j and there exists a time t_1 such that $|U_{X_j}(t_1)_{u_j,u_j}| = C_j$ for each j, then Z is tightly C-sedentary at $(u_1, \dots, u_n)$ with $|U_Z(t_1)_{(u_1,\dots,u_n),(u_1,\dots,u_n)}| = \prod_{j=1}^n C_j$.*

Proof. From (4), we have $|U_{X_1 \square X_2}(t)_{(u_1,u_2),(u_1,u_2)}| = |U_{X_1}(t)_{u_1,u_1}| \cdot |U_{X_2}(t)_{u_2,u_2}|$. Using the fact that $\inf_{t>0} f(t)g(t) \geqslant \inf_{t>0} f(t) \inf_{t>0} g(t)$ for all nonnegative functions f and g yields (1-3). □

By Theorem 4, Cartesian products of graphs with sedentary vertices also contain sedentary vertices. Consequently, Cartesian products of sedentary families also yield a sedentary family.

Corollary 5. *Let $\mathscr{F}_1, \dots, \mathscr{F}_n$ be families of weighted graphs with possible loops. If each $\mathscr{F}_j$ is C_j-sedentary, then $\mathscr{F} = \left\{\square_{j=1}^n X_j : X_j \in \mathscr{F}_j\right\}$ is $\prod_{j=1}^n C_j$-sedentary.*

If the graphs involved are simple, then Theorem 4 and Corollary 5 also hold for the direct product. We also note that if X and Y are simple and weighted-regular, then so are $X \square Y$ and $X \times Y$, and so the quantum walks determined by $\mathbf{A}$ and $\mathcal{A}$ are equivalent. In this case, Theorem 4 and Corollary 5 apply to $\mathcal{A}$, and their analogs for the direct product also apply to $\mathbf{A}$. However, if X is not weighted-regular, then it is not clear how to obtain simple expressions for $e^{it\mathcal{A}(X\square Y)}$ and $e^{it\mathbf{A}(X\times Y)}$.

Next, we examine Cartesian products of complete graphs. Since these are regular, our results apply to $\mathbf{A}$ and $\mathcal{A}$. We use $\nu_2(b)$ to denote the largest power of two that divides an integer b.

Theorem 6. *Let $n_1, \dots, n_m \geqslant 2$ and $X = \square_{j=1}^m K_{n_j}$. The following hold.*

1. *If $n_j = 2$ for some j, then X is not sedentary at any vertex.*
2. *If each $n_j \geqslant 3$, then X is C-sedentary at any vertex, where $C = \prod_{j=1}^m (1 - \frac{2}{n_j})$. In particular, if the $\nu_2(n_j)$'s are all equal, then $|U_X(t_1)_{w,w}| \geqslant C$ for any vertex w with equality at $t_1 = \pi/2^{\nu_2(n_1)}$.*

Proof. Let $n_j = 2$. If X is sedentary at some vertex $(u_1, \dots, u_m)$, where $u_j \in V(K_2)$, then Theorem 4(2) implies that K_2 is sedentary at u_j, which is a contradiction because K_2 exhibits PST. This proves (1). Now, if each $n_j \geqslant 3$, then (6) and Theorem 4 imply that X is C-sedentary. If we add that the $\nu_2(n_j)$'s are all equal, then each $n_j' = n_j/2^{\nu_2(n_1)}$ is odd, and so (6) implies that each K_{n_j} is tightly $(1 - \frac{2}{n_j})$-sedentary at any vertex at time $t_1 = \pi/2^{\nu_2(n_1)}$. Invoking Theorem 4(3) completes the proof of (2). □

The following corollary is immediate from Theorem 6.

Corollary 7. *Fix k and let $\mathscr{F}$ be a family of graphs of the form $\square_{j=1}^k K_{n_j}$, where each $n_j \geqslant 3$.*

1. *If n_j is fixed for some j, then $\mathscr{F}$ is $(1 - \frac{2}{n_j})$-sedentary at any vertex.*
2. *If each n_j increases as $\prod_{j=1}^k n_j \to \infty$, then $\mathscr{F}$ is sedentary at any vertex.*

If we add that the $\nu_2(n_j)$'s are equal for all $\square_{j=1}^k K_{n_j} \in \mathscr{F}$, then the sedentariness in (1) and (2) is tight.

The Hamming graph $H(k, n)$ is obtained by taking the Cartesian product of $k \geqslant 1$ copies of K_n. Combining Theorem 6(2) and Corollary 7(2) yields the following result about Hamming graphs.

Corollary 8. *Let u be a vertex of $H(k, n)$ and $\mathscr{F}$ be a family of Hamming graphs $H(k, n)$. If $n \geqslant 3$, then u is tightly sedentary in $H(k, n)$ and $\mathscr{F}$ is a sedentary family of graphs.*

4 A sufficient condition

We say that vertex u is *periodic* in X with respect to M if $|U_M(t_1)_{u,u}| = 1$ for some time t_1, and the minimum such $t_1 > 0$ is called the minimum period of u, denote by ρ. If u is periodic, then $|U_M(t)_{u,u}|$ is a periodic function because $|U_M(t+\rho)_{u,u}| = |(U_M(t)U(\rho))_{u,u}| = |U_M(t)_{u,u}| \cdot |U_M(\rho)_{u,u}| = |U_M(t)_{u,u}|$. In this case, $\inf_{t>0}|U_M(t)_{u,u}| = \min_{t\in[0,\rho]}|U_M(t)_{u,u}|$, and so the following is immediate.

Lemma 9. *If u is periodic, then u is tightly sedentary if and only if $U_M(t)_{u,u} \neq 0$ for all $t \in [0, \rho]$.*

From Lemma 9, a periodic sedentary vertex is tightly sedentary. Since a rook graph has all integer eigenvalues, it is periodic. By Theorem 6(2), it follows that each vertex in a rook graph is tightly sedentary.

Example 10. *By Theorem 6(2), the rook graphs $X = K_3 \square K_4$ and $Y = K_3 \square K_5$ resp. are $\frac{1}{6}$- and $\frac{1}{5}$-sedentary at any vertex. Since both are periodic, Lemma 9 implies that both are tightly sedentary. Invoking Theorem 6(2), we get $\min_{t>0}|U_Y(t)_{w,w}| = \frac{1}{5}$ is attained at $t_1 = \pi$, and so Y is tightly $\frac{1}{5}$-sedentary at any vertex. But since $\nu_2(3) \neq \nu_2(4)$, we cannot say that X is tightly $\frac{1}{6}$-sedentary. Indeed, by computing $U_{K_3}(t)$ and $U_{K_4}(t)$, and using the fact that $|U_X(t)_{w,w}| = |U_{K_3}(t)_{u,u}| \cdot |U_{K_4}(t)_{v,v}|$, where $w = (u, v)$, one checks that $\min_{t>0}|U_X(t)_{w,w}| \approx 0.2064$ is attained at $t_1 \approx 0.9556$. Thus, X is tightly C-sedentary at any vertex, where $C \approx 0.7936$. Moreover, since $\min_{t>0}|U_{K_3}(t)_{u,u}| = \frac{1}{3}$ and $\min_{t>0}|U_{K_4}(t)_{v,v}| = \frac{1}{2}$, which are attained at $t_1 = \frac{\pi}{3}$ and $t_1 = \frac{\pi}{4}$ resp., we conclude that the converse of Theorem 4(3) does not hold.*

We now prove the main result in this section which could be used to prove that a vertex is sedentary.

Theorem 11. *Let u be a vertex of X with $\sigma_u(M) = \{\lambda_1, \ldots, \lambda_r\}$, where E_j is the orthogonal projection matrix corresponding to λ_j. If S is a non-empty proper subset of $\sigma_u(M)$, say $S = \{\lambda_1, \ldots, \lambda_s\}$, such that*

$$\sum_{j=1}^{s} (E_j)_{u,u} = a \tag{7}$$

for some $\frac{1}{2} \leqslant a < 1$, then

$$|U_M(t)_{u,u}| \geqslant \left| \sum_{j=1}^{s} e^{it\lambda_j} (E_j)_{u,u} \right| - (1 - a) \quad \text{for all } t. \tag{8}$$

If there exists a time $t_1 > 0$ such that

$$\left| \sum_{j=1}^{s} e^{it_1\lambda_j} (E_j)_{u,u} \right| \geqslant 1 - a, \tag{9}$$

and for all $j \in \{1, \ldots, s\}$ and $k \in \{s + 1, \ldots, r\}$,

$$e^{it_1(\lambda_1 - \lambda_j)} = 1 \quad \text{and} \quad e^{it_1(\lambda_1 - \lambda_k)} = -1, \tag{10}$$

then equality holds in (8), in which case $|U_M(t_1)_{u,u}| = 2a - 1$ and u is periodic at time $2t_1$.

Proof. For brevity, let $\alpha_j = (E_j)_{u,u}$ for each $j = 1, \ldots, r$. We know that $U_M(t)_{u,u} = \sum_{j=1}^{r} \alpha_j e^{it\lambda_j}$. Suppose (7) holds, where $1 \leqslant s < r$ and $\frac{1}{2} \leqslant a < 1$. Then $\sum_{j=s+1}^{r} \alpha_j = 1 - a$, and because $\alpha_j > 0$ for each j, we obtain $\left|\sum_{k=s+1}^{r} \alpha_k e^{it\lambda_k}\right| \leqslant \sum_{k=s+1}^{r} \alpha_k = 1 - a$ by triangle inequality. Hence, for all t, we have

$$|U_M(t)_{u,u}| \overset{(*)}{\geqslant} \left|\sum_{j=1}^{s} \alpha_j e^{it\lambda_j}\right| - \left|\sum_{k=s+1}^{r} \alpha_k e^{it\lambda_k}\right| \overset{(**)}{\geqslant} \left|\sum_{j=1}^{s} \alpha_j e^{it\lambda_j}\right| - (1 - a).$$

This proves (8). Equality holds in $(**)$ if and only if for some t_1 and $\gamma \in \mathbb{C}$, $e^{it_1\lambda_k} = -\gamma$ for each $k \in \{s + 1, \ldots, r\}$. This reduces $(*)$ to $\left|\sum_{j=1}^{s} \alpha_j e^{it_1(\lambda_j - \lambda_{s+1} - \pi)} - (1 - a)\right| \geqslant \left|\sum_{j=1}^{s} \alpha_j e^{it_1\lambda_j}\right| - (1 - a)$, which is an equality if and only if $\left|\sum_{j=1}^{s} \alpha_j e^{it_1\lambda_j}\right| \geqslant 1 - a$ and $e^{it_1(\lambda_j - \lambda_{s+1})} = -1$ for each j. The latter yields $e^{it_1\lambda_j} = \gamma$ for $j = 1, \ldots, r$. This proves (9) and (10). If equality holds in $(*)$ and $(**)$, then $\left|\sum_{j=1}^{s} \alpha_j e^{it_1\lambda_j}\right| = a$, and so $|U_M(t_1)_{u,u}| = 2a - 1$. The statement about periodicity is straightforward. □

The following lemma helps us identify sharply sedentary vertices which are not tightly sedentary.

Lemma 12. *Suppose the premise of Theorem 11 holds. If ℓ_j and m_j are integers such that*

$$\sum_{j=1}^{s} m_j\lambda_j + \sum_{j=s+1}^{r} \ell_j\lambda_j = 0 \quad \text{and} \quad \sum_{j=1}^{s} m_j + \sum_{j=s+1}^{r} \ell_j = 0$$

implies that $\sum_{j=1}^{s} m_j$ is even, then there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty}|U_M(t_k)_{u,u}| = 2a - 1$.

The proof of Lemma 12 is similar to the proof of a characterization of PGST between two vertices [KLY17, Lemma 2.2], except that we replace the sets $\sigma_{uv}^{+}(M)$ and $\sigma_{uv}^{-}(M)$ resp. by $\sigma_u(M)\backslash S$ and S.

Using Theorem 11 and Lemma 12, we obtain the following sufficient conditions for sedentariness.

Corollary 13. *Let u be a vertex of X and suppose $\varnothing \neq S \subseteq \sigma_u(M)$.*

1. *Let $S = \{\lambda_1\}$. If $(E_1)_{u,u} = a$, then $|U(t)_{u,u}| \geqslant 2a - 1$ for all t. The following also hold.*
   (a) *If $a > \frac{1}{2}$, then u is $(2a-1)$-sedentary. This is tight (resp., sharp) whenever (10) (resp., Lemma 12) holds. Moreover, if u is periodic, then u is tightly C-sedentary for some $C \geqslant 2a - 1$.*
   (b) *Suppose (9) and (10) hold, or Lemma 12 holds. If $a = \frac{1}{2}$, then u is not sedentary.*
2. *Let $|S| \geqslant 2$, $b > 0$ and $F(t) = \left|\sum_{j=1}^{s} e^{it\lambda_j}(E_j)_{u,u}\right|$. If $a > \frac{1}{2}$, then u is sedentary whenever (i) $F(t) - (1-a) > b$ for all t or (ii) $F(t) \geqslant 1 - a$ for all t, u is periodic and $U(t_1)_{u,u} \neq 0$ for all t_1 with $F(t_1) = 1 - a$.*

Proof. The statement in (1) follows from Theorem 11. To prove (1a), let $a < \frac{1}{2}$. Then u is clearly $2a$-sedentary. Since (9) holds by default, the sedentariness is tight by Theorem 11 whenever (10) holds. If the premise of Lemma 12 holds, then $\inf_{t>0}|U_M(t)_{u,u}| = 2a - 1$, and so u is tightly sedentary. If we add that u is periodic, then $\min_{t>0}|U_M(t)_{u,u}| = C \geqslant 2a - 1$, where $\min_{t>0}|U_M(t)_{u,u}|$ is attained at some $t_1 \in (o, \rho)$. Thus, u is C-sedentary. For (1b), if $a = \frac{1}{2}$, then $|U_M(t)_{u,u}| \geqslant 0$. If (9) and (10) hold, then $|U_M(t_1)_{u,u}| = 0$ at some $t_1 \in (o, \rho)$, while if Lemma 12 holds, then $\inf_{t>0}|U_M(t)_{u,u}| = 0$. This proves (1b). Finally, let $|S| \geqslant 2$ and $a > \frac{1}{2}$. If (2i) holds, then u is sedentary by (8). If (2ii) holds, then $|U_M(t)_{u,u}| > 0$ for all t, and so Lemma 9 implies that u is tightly sedentary. This proves (2). □

As we will see, Corollary 13(1) will be useful in the later sections. We end this section with an example that illustrates Corollary 13.

Example 14. *Consider the path P_3 with end vertex u. Then $\sigma_u(A) = \{\pm\sqrt{2}, 0\}$ with associated eigenvectors $(1, \pm\sqrt{2}, 1)$ and $(1, 0, -1)$, while $\sigma_u(L) = \{3, 1, 0\}$ with associated eigenvectors $(1, -2, 1)$, $(1, 0, -1)$ and $\mathbf{1}$. Note that u is periodic in both cases. Moreover,*

$$U_A(t)_{u,u} = \tfrac{1}{4}e^{i\sqrt{2}t} + \tfrac{1}{4}e^{-i\sqrt{2}t} + \tfrac{1}{2} \quad \text{and} \quad U_L(t)_{u,u} = \tfrac{1}{6}e^{i3t} + \tfrac{1}{2}e^{it} + \tfrac{1}{3}.$$

For A, let $S = \{0\}$. Then $(E_0)_{u,u} = 1/2$ and one checks that (9) and (10) hold at $t_1 = \pi/\sqrt{2}$. By Corollary 13(1b), u is not sedentary, which is consistent with the fact that adjacency PST occurs between end vertices of P_3 at t_1. For L, take $S = \{3, 1\}$ so that $(E_3)_{u,u} + (E_1)_{u,u} = 2/3$. Applying Theorem 11 with $a = 2/3$, we get that $F(t) = \left|\frac{1}{6}e^{i3t} + \frac{1}{2}e^{it}\right| - \frac{1}{3}$. Now, $F(t_1) = 0$ if and only if $t_1 = j\pi/2$ for any odd j. Since $U_L(t_1)_{u,u} \neq 0$ and u is periodic, Corollary 13(2ii) implies that u is tightly sedentary.

5 Twin vertices

In this section, we show that a vertex with at least two twins is sedentary. Unless otherwise stated, all results in this section apply to both $\mathbf{A}$ and $\mathcal{A}$.

Two vertices u and v of X are *twins* if (i) $N_X(u)\backslash\{u,v\} = N_X(v)\backslash\{u,v\}$, (ii) the edges (u,w) and (v,w) have the same weight for each $w \in N_X(u)\backslash\{u,v\}$, and (iii) the loops on u and v have the same weight if they exist. We say that a subset $T = T(\omega,\eta)$ of $V(X)$ with at least two vertices is a *set of twins* in X if each pair of vertices in T are twins, where each vertex in T has a loop of weight ω whenever $\omega \neq 0$ and every pair of vertices in T are connected by an edge with weight η whenever $\eta \neq 0$. Since there exists an automorphism that switches any pair of twins, it follows that all vertices in T are pairwise cospectral. For a more extensive treatment of the role of twin vertices in quantum state transfer, see [Mon21].

We now restate a spectral characterization of twin vertices [Mon22, Lemma 2.9].

Lemma 15. *Let $T = T(\omega,\eta)$ be a set of twins in X. Then $u,v \in T$ if and only if $\mathbf{e}_u - \mathbf{e}_v$ is an eigenvector of M corresponding to the eigenvalues θ given by*

$$\theta = \begin{cases} \alpha \deg(u) + \omega - \eta, & \text{if } M = \mathbf{A} \\ \frac{\omega-\eta}{\deg(u)}, & \text{if } M = \mathcal{A}. \end{cases} \tag{11}$$

If $u \in T$, then $\theta \in \sigma_u(M)$ by Lemma 15. We use this to prove our main result.

Theorem 16. *Let T be a set of twins in X. If $u \in T$ with $\sigma_u(M) = \{\theta, \lambda_2, \dots, \lambda_r\}$, then*

$$|U_M(t)_{u,u}| \geqslant 1 - \frac{2}{|T|} \quad \text{for all } t, \tag{12}$$

with equality whenever (10) holds with $S = \{\theta\}$. Further, if $|T| \geqslant 3$, then u is $(1 - \frac{2}{|T|})$-sedentary.

Proof. Let T be a set of twins in X. If we index the first $|T|$ rows of M by the elements of T, then for a fixed $u \in T$, Lemma 15 implies that $\mathbf{e}_u - \mathbf{e}_v$ is an eigenvector for M for all $v \in T\backslash\{u\}$ corresponding to the eigenvalue θ in (11). Assuming u is the first row of M, we get $E_\theta = \left(I_{|T|} - \frac{1}{|T|}\mathbf{J}_{|T|}\right) + F$ for some matrix F. Taking $S = \{\theta\}$, we get $1 - a = 1 - \frac{1}{|T|} \geqslant \frac{1}{2}$. Applying Corollary 13(1) yields the desired result. ☐

Remark 17. *If X is simple and unweighted, and T is a set of twins in X, then T is also a set of twins in the complement X^c of X. Thus, if X^c is connected, then Theorem 16 also holds for X^c.*

Theorem 16 reveals that twin vertices in quantum walks behave like vertices in a complete graph, which is an interesting observation because the underlying graph induced by a set of twins is either complete or empty. But unlike complete graphs, equality in (12) may not be attained for other graphs.

Joins

Since the property of being twins is preserved under joins, Theorem 16 yields the following results.

Corollary 18. *Let T be a set of twins in Y. If $|T| \geqslant 3$, then the vertices in T are $(1 - \frac{2}{|T|})$-sedentary in $Y \vee X$ for any weighted graph X with possible loops.*

Corollary 19. *Let X be a weighted graph with possible loops. For each $m \geqslant 3$, the vertices of K_m and O_m resp. are $(1 - \frac{2}{m})$-sedentary in $K_m \vee X$ and $O_m \vee X$.*

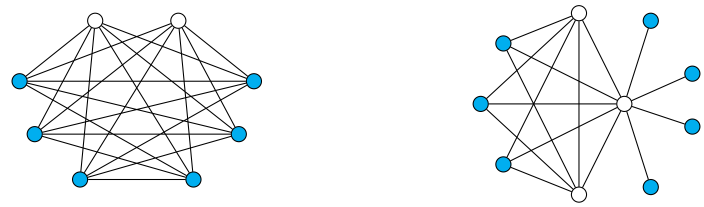

Figure 1: The complete multipartite graph $K_{2,3,3}$ (left) and the threshold graph $((O_3 \vee K_2) \cup O_4) \vee K_1$ (right) with sedentary vertices marked blue

By Corollary 19, a degree $m-1$ vertex of $K_m\backslash e = K_{m-2} \vee O_2$ is $(1-\frac{2}{m-1})$-sedentary for all $m \geqslant 5$. We now examine sedentariness in two well known classes of graphs obtained using the join operation.

Corollary 20. *Let $n_1, \ldots, n_{2k}$ be integers such that $n_j \geqslant 3$ for some $j \in \{1, \ldots, k\}$.*

1. *Each vertex of $K_{n_1,n_2,\ldots,n_k}$ in the partite set of size n_j is $(1-\frac{2}{n_j})$-sedentary. Moreover, if $|\{\ell : n_\ell = 1\}| = p \geqslant 3$, then each vertex in a singleton partite set of $K_{n_1,n_2,\ldots,n_k}$ is $(1-\frac{2}{p})$-sedentary.*
2. *Each vertex of $Z \in \{K_{n_j}, O_{n_j}\}$ is $(1-\frac{2}{n_j})$-sedentary in the threshold graph*

$$((((O_{n_1} \vee K_{n_2}) \cup O_{n_3}) \vee K_{n_4}) \cdots) \vee K_{n_{2k}} \quad \text{or} \quad ((((K_{n_1} \cup O_{n_2}) \vee K_{n_3}) \cup O_{n_4}) \cdots) \vee K_{n_{2k+1}}. \tag{13}$$

Proof. If $n_j \geqslant 3$, then each partite set of $K_{n_1,n_2,\ldots,n_k}$ and those vertices in each K_{n_j} and O_{n_j} in (13) form a set of twins. If $|\{\ell : n_\ell = 1\}| = p \geqslant 3$, then the singleton partite sets also form a set of twins size p. Applying Theorem 16 yields the desired result. □

Threshold graphs with form given in (13), where $n_1 \geqslant 2$ and $n_j \geqslant 1$ for $j \geqslant 2$, are precisely all the connected threshold graphs as characterized by Kirkland and Severini (see [KS11, Lemma 1]).

Next, we have the following immediate consequence of Theorem 16.

Corollary 21. *Let $\mathscr{F}$ be a family of graphs with a set of twins T with $|T| \geqslant 3$.*

1. *If $|T|$ is fixed for all $X \in \mathscr{F}$, then $\mathscr{F}$ is $(1-\frac{2}{|T|})$-sedentary at every vertex in T.*
2. *If $|V(X)\backslash T|$ is fixed for all $X \in \mathscr{F}$, then $\mathscr{F}$ is sedentary at every vertex of T.*

For the family $\mathscr{K}_1$ of complete graphs on $m \geqslant 5$ vertices minus an edge, all vertices in $K_m\backslash e$, except for the non-adjacent pair, form a set of twins T with $|T| = m-2$. Thus, $|V(X)\backslash T| = 2$ is fixed, and so by Corollary 21(2), $\mathscr{K}_1$ is a family that is sedentary at all vertices except for the non-adjacent pair.

The next result follows immediately from Corollaries 19, 20 and 21.

Corollary 22. *The following hold.*

1. *Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be families of graphs resp. of the form $O_m \vee X$ and $K_m \vee X$. Let $Z \in \{O_m, K_m\}$. If X has fixed number of vertices, then each $\mathscr{F}_i$ is sedentary at every vertex of Z. If $m \geqslant 3$ is fixed, then each $\mathscr{F}_i$ is $(1-\frac{2}{m})$-sedentary at every vertex of Z.*
2. *Let $\mathscr{F}_1$ and $\mathscr{F}_2$ resp. be families of complete multipartite graphs $K_{n_1,\ldots,n_k} = \bigvee_{\ell=1}^{k} O_{n_\ell}$ and threshold graphs in (13). Let $Z \in \{K_{n_j}, O_{n_j}\}$. If $n_j \geqslant 3$ is fixed, then each $\mathscr{F}_i$ is $(1-\frac{2}{n_j})$-sedentary at every vertex in Z. If $k \geqslant 1$ is fixed and $n_j \to \infty$, then each $\mathscr{F}_i$ is sedentary at every vertex of Z.*

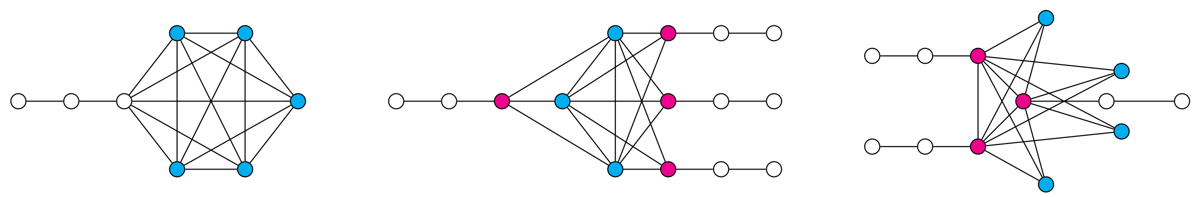

Figure 2: The lollipop graph $L_{4,2}$ (left), the graph $X_{3,4,2}$ (center) and the graph $Y_{3,4,2}$ (right) with sedentary vertices marked blue

Graphs with tails

For $n \geqslant 4$ and $k \geqslant 1$, let $L_{n,k}$ be a lollipop graph, which is a graph obtained after attaching a path P_k to a vertex u of K_n. The vertices $v \neq u$ of K_n in $L_{n,k}$ form a set of twins of size $n-1 \geqslant 3$, and so each of them is $(1-\frac{2}{n-1})$-sedentary by Theorem 16. More generally, if $n \geqslant k+3$, then attaching k paths (possibly with different lengths) to k vertices of K_n leaves the remaining $n-k$ vertices of K_n $(1-\frac{2}{n-k})$-sedentary in the resulting graph. The same holds in the complement of $L_{n,k}$.

For $n, m \geqslant 3$ and $k \geqslant 0$, let $X_{n,m,k}$ and $Y_{n,m,k}$ be graphs obtained from $K_n \vee O_m$ by attaching copies of P_k resp. to the vertices of O_m and K_n. The vertices of K_n form a set of twins in $X_{n,m,k}$ of size n, while those of O_m form a set of twins in $Y_{n,m,k}$ of size m. Thus, the vertices of K_n and O_m resp. are $(1-\frac{2}{n})$- and $(1-\frac{2}{m})$-sedentary in $X_{n,m,k}$ and $Y_{n,m,k}$. This remains true even if we vary the lengths of paths attached to O_m and K_n. This also holds in the complements of $X_{n,m,k}$ and $Y_{n,m,k}$.

The above considerations combined with Corollary 21 yield the following result.

Corollary 23. *Let $\mathscr{F}$ be a family of simple unweighted lollipop graphs $L_{n,k}$, $\mathscr{F}_1$ be a family of graphs $X_{n,m,k}$, and $\mathscr{F}_2$ be a family of graphs $Y_{n,m,k}$.*

1. *Let $n \geqslant 4$ and $k \geqslant 1$. If k is fixed, then $\mathscr{F}$ is sedentary at every vertex of $v \neq u$ of K_n. If n is fixed, then $\mathscr{F}$ is $(1-\frac{2}{n-1})$-sedentary at every vertex $v \neq u$ of K_n*

2. *Let $n, m \geqslant 3$ and $k \geqslant 0$. If m and k are fixed, then $\mathscr{F}_1$ (resp., $\mathscr{F}_2$) is sedentary at every vertex of K_n (resp., O_n). If n is fixed, then $\mathscr{F}_1$ is $(1-\frac{2}{n})$-sedentary at every vertex of K_n, while if m is fixed, then $\mathscr{F}_2$ is $(1-\frac{2}{m})$-sedentary at every vertex of O_m.*

For $n \geqslant 4$, let L_n be an infinite lollipop, which is the resulting graph after attaching an infinite path to a vertex u of a complete graph K_n. In [BTVX22, Proposition 3], Bernard et al. showed that the family of infinite lollipops is sedentary at each vertex $v \neq u$ of K_n. This complements Corollary 23(1) which states that the family of lollipop graphs $L_{n,k}$ with k fixed is sedentary at every vertex $v \neq u$ of K_n. They also showed that attaching infinite paths to the vertices of O_m in $K_n \vee O_m$ yields a family that is sedentary at every vertex of K_n [BTVX22, Theorem 4], which again, complements our result in Corollary 23(2a), which states that the family of graphs $X_{n,m,k}$ is sedentary at every vertex of K_n whenever m and k are fixed.

Similar to lollipop graphs, one may construct barbell-type graphs with sedentary vertices. Barbell-type graphs are obtained by joining corresponding vertices of two copies of complete graphs with a path. For instance, if $m, n \geqslant 4$ and $k \geqslant 1$ then the barbell-type graph $L_{n,k,m}$ formed by joining vertices u of K_n and w of K_m by a path P_k is $(1-\frac{2}{n-1})$- and $(1-\frac{2}{m-1})$-sedentary resp. at any vertex $v \neq u$ of K_n and $v \neq w$ of K_m. One can then derive results about sedentary families of barbell-type graphs similar to Corollary 23.

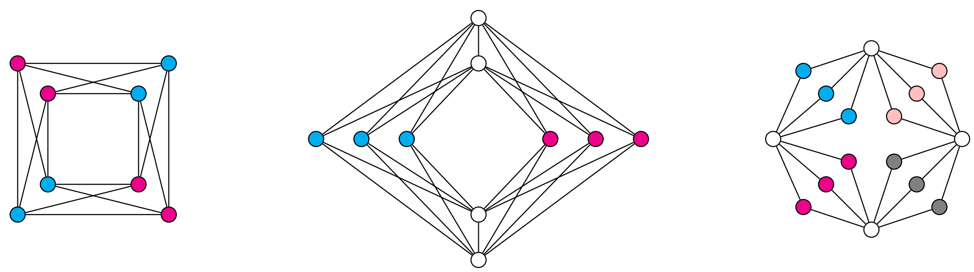

Figure 3: Blow-ups of C_4: $C_4^2(V)$ (left), $C_4(2,3,2,3)(V)$ (center), and $C_4^3(E)$ (right) with sets of twins filled with the same color, all members of which are sedentary

Blow-ups

Let X be a weighted graph with possible loops with vertices $v_1,\dots,v_n$ and edges with distinct endpoints (i.e., non-loops) $e_1,\dots,e_m$. Let $(k_1,\dots,k_n)$ and $(k_1,\dots,k_m)$ be n- and m-tuples of positive integers.

A $(k_1,\dots,k_n)$-*vertex blow-up* of X, denoted $X(k_1,\dots,k_n)(V)$, is the graph obtained by replacing every vertex v_j of X by the graph $X_j \in \{O_{k_j}, K_{k_j}\}$ such that a vertex in X_j is adjacent to a vertex in X_ℓ in the resulting graph if and only if v_j and v_ℓ are adjacent in X, and the weight of each edge between X_j and X_ℓ is the same as the weight of the edge $[v_j, v_\ell]$ in X. If $k_1 = \cdots = k_m = k$, then we call the resulting graph a *k-vertex blow-up* of X, denoted $X^k(V)$. Vertex blow-ups in the literature typically mean replacing each vertex by an empty graph, but in our definition, we have the freedom to choose between an empty or a complete graph. For example, $K_{m,n}$ and K_{m+n} are (m,n)-vertex blow-ups of K_2, where each vertex of K_2 was replaced by an empty graph for the former, and by a complete graph for the latter.

A $(k_1,\dots,k_m)$-*edge blow-up* of X, denoted $X(k_1,\dots,k_n)(E)$, is a graph obtained by replacing every edge $e_j = [u_j, v_j]$ of X by $X_j \in \{O_{k_j}, K_{k_j}\}$ and adding edges $[u_j, w]$ and $[v_j, w]$ for all vertices w of X_j, each with weight equal to that of e_j. If $k_1 = \cdots = k_m = k$, then we call the resulting graph a *k-edge blow-up* of X, denoted $X^k(E)$. A 1-edge blow-up of X is obtained by subdividing every edge of X.

Theorem 24. *Let X be a weighted graph with possible loops with vertices $v_1,\dots,v_n$ and edges $e_1,\dots,e_m$ with distinct endpoints. Let $(k_1,\dots,k_n)$ and $(k_1,\dots,k_m)$ be n and m-tuples of positive integers.*

1. *If $k_j \geqslant 3$ for some j, then the vertices of $X_j \in \{O_{k_j}, K_{k_j}\}$ added in place of v_j (resp., e_j) are $(1-\frac{2}{k_j})$-sedentary in $X(k_1,\dots,k_n)(V)$ (resp., $X(k_1,\dots,k_m)(E)$). If $k \geqslant 3$, then each vertex in $X^k(V)$ is $(1-\frac{2}{k})$-sedentary, while each vertex in $\bigcup_{j=1}^m X_j$ is $(1-\frac{2}{k})$-sedentary in $X^k(E)$.*

2. *Let T be a set of twins in X with $k_j \geqslant 2$ for some $v_j \in T$. Suppose $W_1 = \bigcup_{v_j \in T, X_j = K_{K_j}} X_j$ and $W_2 = \bigcup_{v_j \in T, X_j = O_{k_j}} X_j$. If the vertices in T are pairwise adjacent, then W_1 is a set of twins in $X(k_1,\dots,k_n)(V)$. Otherwise, W_2 is a set of twins in $X(k_1,\dots,k_n)(V)$. If $|W_i| \geqslant 3$ for some $i \in \{1,2\}$, then each vertex in W_i is $(1-\frac{2}{|W_i|})$-sedentary.*

Proof. Since the vertices of X_j form a set of twins of size $k_j \geqslant 3$, (1) follows directly from Theorem 16. Now, let T be a set of twins in X such that $k_j \geqslant 2$ for some $v_j \in T$. Note that the vertices in T are either all pairwise adjacent, or all pairwise non-adjacent. Suppose the former holds. If $v_1, v_2 \in T$ are distinct, and we replace v_1 by O_{k_1} with $k_1 \geqslant 2$ and v_2 by $Z_2 \in \{O_{k_2}, K_{k_2}\}$, then a vertex u_1 in O_{k_1} and a vertex u_2 in Z are not twins in $X(k_1,\dots,k_n)(V)$, because u_1 is not adjacent to least one vertex $w \neq u_1$ in O_{k_1} while u_2 is adjacent to this w. The same holds if reverse the roles of v_1 and v_2. Thus, we are left with the case when v_1 and v_2 are replaced by K_{k_1} and K_{k_2}. In this case, any two vertices in $K_{k_1} \cup K_{k_2}$ are adjacent twins, and so the first statement in (2) holds. The second follows by using the same argument, and the third is a direct consequence of Theorem 16. ☐

Theorem 24(2) tells us that if T is a set of non-adjacent (resp., adjacent) twins in X and each vertex in T is replaced by an empty (resp., complete) graph, one of which has size at least two, then $W = \bigcup_{v_j \in T} V(X_j)$ is a set of twins in $X(k_1, \ldots, k_n)(V)$ and each vertex in W is $(1 - \frac{2}{|W|})$-sedentary.

Example 25. *Figure 3 depicts three blow-ups of C_4: $C_4^2(V)$ is obtained by replacing each vertex of C_4 by O_2, $C_4(2,3,2,3)(V)$ by replacing two vertices of C_4 by two copies of K_2 and the rest by O_3, and $C_4^3(E)$ by replacing all edges of C_4 by copies of O_3. By Theorem 24(1), the vertices in the two copies of O_3 are $\frac{1}{3}$-sedentary in $C_4(2,3,2,3)(V)$, while the 12 coloured vertices are $\frac{1}{3}$-sedentary in $C_4^3(E)$. By Theorem 24(2), a set of two twins in C_4 becomes a set of four in $C_4^2(V)$, all members of each set are $\frac{1}{2}$-sedentary.*

6 Cones

A graph of the form $K_1 \vee X$ is called a *cone* on X with *apex* u, where $V(K_1) = \{u\}$. A graph of the form $Z \vee X$, where $Z \in \{K_2, O_2\}$ is called a *double cone* on X, and any vertex of Z is called an *apex*. In particular, $K_2 \vee X$ and $O_2 \vee X$ are resp. called *connected* and *disconnected* double cones.

For cones over d-regular graphs on n vertices, Godsil showed that $|U_A(t)_{u,u}| \geqslant \frac{d^2}{d^2+4n}$, with equality if and only if $t = \frac{\pi}{\sqrt{d^2+4n}}$ [God21]. This yields the following.

Proposition 26. *Let $d > 0$ and $\mathscr{C}$ be a family of cones over weighted d-regular graphs on n vertices.*

1. *If $d^2/n \to \infty$ as n increases, then $\mathscr{C}$ is tightly sedentary at the apex.*
2. *If γ is a constant such that $d^2/n \to \gamma$ as n increases, then $\mathscr{F}$ is $\frac{\gamma}{\gamma+4}$-sedentary at the apex. In particular, if d is fixed, then $\mathscr{C}$ is quasi-sedentary at the apex.*

Remark 27. *If $d = 0$, then $|U(\frac{\pi}{2\sqrt{n}})_{u,u}| = 0$, and so the apex in this case is not sedentary.*

Theorem 28. *For each $0 \leqslant C \leqslant 1$, there exists a C-sedentary family with respect to the adjacency matrix.*

Proof. If $0 \leqslant C < 1$, then $\mathscr{C}$ is C-sedentary at the apex by Proposition 26(2) whenever $d^2/n \to \frac{4C}{1-C}$. If $d^2/n \to \infty$, then $\mathscr{C}$ is sedentary at the apex by Proposition 26(1). ☐

For the Laplacian case, we prove a more general result for cones.

Theorem 29. *Let $m \geqslant 1$ and X be a simple positively weighted graph on $n \geqslant 2$ vertices. For any vertex u of K_m in $K_m \vee X$, $|U_L(t)_{u,u}| \geqslant 1 - \frac{2}{m+n}$ for all t with equality if and only if $t = \frac{j\pi}{m+n}$ for some odd j. Thus, the family of joins $K_m \vee X$ is tightly sedentary at any vertex of K_m.*

Proof. Let u be a vertex of K_m in $K_m \vee X$. By [ADL+16, Equation 31], $U_L(t)_{u,u} = \frac{1}{m+n} + \frac{m+n-1}{m+n} e^{it(m+n)}$. Thus, $|U_L(t)_{u,u}|^2 \geqslant \frac{(m+n-2)^2}{(m+n)^2}$ for all t and the result is immediate. ☐

By Corollary 22(1), if $m \geqslant 3$ is fixed, then $K_m \vee X$ is $(1 - \frac{2}{m})$-sedentary at every vertex of K_m with respect to M. But since $|U_L(t)_{u,u}| \geqslant 1 - \frac{2}{m+n} > 1 - \frac{2}{m}$, this family of joins is, in fact, sedentary with respect to L. This also implies that Theorem 29 yields a sharper bound than Theorem 16, which suggests that the bound obtained in Theorem 16 can be improved if we take a more specific Hamiltonian.

Taking $m \in \{1, 2\}$ in Theorem 29 yields the following result.

Corollary 30. *The families of cones and connected double cones on simple positively weighted graphs are tightly sedentary at the apexes with respect to the Laplacian matrix.*

7 Strongly cospectral vertices

We say that two vertices u and v are *strongly cospectral* if $E_j\mathbf{e}_u = \pm E_j\mathbf{e}_v$ for all $\lambda_j \in \sigma_u(M)$. In this case,

$$\sigma_{uv}^{+}(M) = \{E_j\mathbf{e}_u = E_j\mathbf{e}_u \neq \mathbf{0}\} \quad \text{and} \quad \sigma_{uv}^{-}(M) = \{E_j\mathbf{e}_u = -E_j\mathbf{e}_u \neq \mathbf{0}\}.$$

partition $\sigma_u(M)$. The interest in the study of strongly cospectrality is motivated by the fact that it is a requirement for two vertices to exhibit PGST [God12, Lemma 13.1]. In this section, we show that there are infinitely many graphs with strongly cospectral vertices that are sedentary. But as the next result shows, the machinery that we have developed in Section 4 has limitations for strongly cospectral vertices.

Proposition 31. *Let u and v are strongly cospectral. If $S = \sigma_{uv}^{\pm}(M)$, then $a = \frac{1}{2}$ in (7).*

Proof. Assume $\sigma_{uv}^{+}(M) = \{\lambda_1, \dots, \lambda_s\}$ and $\sigma_{uv}^{-}(M) = \{\lambda_{s+1}, \dots, \lambda_r\}$. Then we have $(E_j)_{u,u} = (E_j)_{u,v}$ for $j = 1, \dots, s$, while $(E_j)_{u,u} = -(E_j)_{u,v}$ for $j = s+1, \dots, r$. As the E_j's sum to identity, we get $\sum_{j=1}^{s}(E_j)_{u,u} = \sum_{k=s+1}^{r}(E_k)_{u,u} = \frac{1}{2}$. $\square$

Let u and v be strongly cospectral. If we take $S \in \{\sigma_{uv}^{+}(M), \sigma_{uv}^{-}(M)\}$, then $a = \frac{1}{2}$ from Proposition 31. In this case, Theorem 11 is not very useful, as (8) yields the trivial statement $|U(t)_{u,u}| \geqslant 0$ for all t. If we add that either (9) and (10) hold or Corollary 12(2) holds, then Corollary 13(1b) implies that u is not sedentary. Indeed, this holds because strong cospectrality together with either (9) and (10) or Corollary 12(2) resp. yield PST or PGST between u and v. Hence, in order for Theorem 11 to work for strongly cospectral vertices, one may avoid taking $S \in \{\sigma_{uv}^{+}(M), \sigma_{uv}^{-}(M)\}$. For the case of strongly cospectral twin vertices, it is known that $|\sigma_{uv}^{-}(M)| = 1$ [Mon22, Theorem 3.4]), and so the only viable option is to choose S such that $\sigma_{uv}^{-}(M)$ is a proper subset of S, in which case, $|S| \geqslant 2$ and $\theta \in S$, where θ is given in (11). However, we shall see in Remark 33 of the next subsection that, unlike the case $S = \{\theta\}$ which yields Theorem 16, the case $|S| \geqslant 2$ with $\theta \in S$ requires more work in order to establish sedentariness of u.

To achieve our goal of showing that there are infinitely many graphs with strongly cospectral vertices that are sedentary, we consider disconnected double cones. Indeed, the apexes of such graphs are strongly cospectral with respect to A and L by [Mon22, Corollary 6.9]. Our main motivation for considering these graphs is that their apexes form a set of twins of size two, and our results in Corollary 19(1) prompt us to investigate whether the apexes of $O_2 \vee X$ are also sedentary. Results in the literature indicate that the apexes of disconnected double cones are excellent sources of PST and PGST (see for instance [ADL+16] for the Laplacian case and [KMP22] for the adjacency and signless Laplacian case), and so one might be inclined to speculate that these apexes are not sedentary. But as it turns out, the apexes of disconnected double cones are sedentary whenever they do not exhibit PST or PGST.

Laplacian case

Let X be a simple positively weighted graph on n vertices. Then

$$U_L(t)_{u,u} = \frac{1}{m+n} + \frac{(m-1)e^{itn}}{m} + \frac{ne^{it(m+n)}}{m(m+n)} \tag{14}$$

for each vertex u of O_m in $O_m \vee X$ (see [ADL+16, Equation 33]).

Theorem 32. *Let X be a simple positively weighted graph on n vertices, and let u be an apex of $O_2 \vee X$.*

1. *If $n \equiv 2 \pmod 4$, then $O_2 \vee X$ is not sedentary at u.*

2. *If $n \equiv 0 \pmod 4$, then $|U_L(t)_{u,u}| \geqslant 1 - \frac{n}{n+2}$ with equality if and only if $t = \frac{j\pi}{2}$ for any odd j.*

3. *Let n be odd. If $n = 1$, then $|U_L(t)_{u,u}| \geqslant \frac{1}{3}$ with equality if and only if $t = \ell\pi$ for any odd ℓ, while $n \geqslant 3$, then $|U_L(t)_{u,u}| \geqslant 1 - \frac{n+2-\sqrt{2}}{n+2}$ with equality if and only if $t = \frac{j\pi}{2}$ for any odd j.*

Proof. Let u be an apex of $O_2 \vee X$. From (14), $U_L(t)_{u,u} = \frac{1}{n+2} + \frac{e^{itn}}{2} + \frac{ne^{it(n+2)}}{2(n+2)}$, and so

$$|U_L(t)_{u,u}|^2 = \frac{n^2 + 2n + 4 + h(t)}{2(n+2)^2}, \tag{15}$$

where $h(t) = n(n+2)\cos(2t) + 2(n+2)\cos(tn) + 2n\cos(t(n+2))$. Note that $|U_L(t)_{u,u}|^2$ is maximum (resp., minimum) if and only if $h(t)$ is maximum (resp., minimum). One can then verify that

$$h'(t) \overset{(*)}{=} -2n(n+2)\left[\sin(2t) + \sin(tn) + \sin(t(n+2))\right] = -8n(n+2)\cos(t)\cos(tn/2)\sin(t(n+2)/2).$$

Thus, $h'(t) = 0$ if and only if either (i) $t = j\pi/2$ for some odd j, (ii) $t = \ell\pi/n$ for some odd ℓ, or (iii) $t = \frac{k\pi}{n+2}$ for some even k. We now differentiate $h'(t)$ in $(*)$ to get

$$h''(t) = -4n(n+2)\left[2\cos(t(n+4)/2)\cos(tn/2) + n\cos(t(n+1))\cos(t)\right] \tag{16}$$

If $t = \frac{j\pi}{2}$ for some odd j, then $\cos(t) = 0$ and $\cos(t(n+4)/2) = -\cos(tn/2)$ because j is odd. While if $t = \ell\pi/n$ for some odd ℓ, then $\cos(tn/2) = 0$ and $\cos(t(n+1)) = -\cos(t)$ because ℓ is odd. In both cases, (16) yields $h''(t) > 0$, and so $|U_L(t)_{u,u}|^2$ has a relative minimum. Now, if $t = \frac{k\pi}{n+2}$ for some even k, then $\cos(t(n+4)/2) = \cos(tn/2)$ and $\cos(t(n+1)) = \cos(t)$. Using (16), one checks that $h''(t) < 0$, and so $|U_L(t)_{u,u}|^2$ has a relative maximum. From these three cases, it suffices to compare the values of $|U_L(t)_{u,u}|^2$ at $t = j\pi/2$ for odd j and $t = \ell\pi/n$ for odd ℓ to get the absolute minimum. We begin with $t = j\pi/2$ for some odd j. In this case, $\cos(2t) = -1$ and $\cos(t(n+2)) = -\cos(tn)$, and so (15) yields

$$|U_L(t)_{u,u}|^2 = \frac{4\left[1 + \cos(tn)\right]}{2(n+2)^2}. \tag{17}$$

If $n \equiv 2 \pmod 4$, then u exhibits PST [ADL+16, Corollary 4], and so it is not sedentary. This proves (a). Thus, we have two remaining cases.

- If $n = 4m$, then $\cos(tn) = \cos(2jm\pi) = 1$, and so (17) yields $|U_L(t)_{u,u}|^2 = \frac{4}{(n+2)^2}$.

- If n is odd, then $\cos(tn) = 0$, and so (17) gives us $|U_L(t)_{u,u}|^2 = \frac{2}{(n+2)^2}$.

Next, let $t = \frac{\ell\pi}{n}$ for odd ℓ. Then $\cos(tn) = -1$ and $\cos(t(n+2)) = -\cos(2t)$. From (15), we obtain

$$|U_L(t)_{u,u}|^2 = \frac{n^2(1 + \cos(2t))}{2(n+2)^2}. \tag{18}$$

- Let $n = 4m$. Then $2t = \frac{\ell\pi}{2m}$ cannot be an odd multiple of π, and so $\cos(2t) > -1$. The closest that $2t$ will be from an odd multiple of π is when $\ell = 2ms \pm 1$ for some odd s, in which case, $2t = \left(s \pm \frac{1}{2m}\right)\pi$. From (18), we get $|U_L(t)_{u,u}|^2 \geqslant \frac{n^2(1-\cos(2\pi/n))}{2(n+2)^2} = \frac{n^2\sin^2(\pi/n)}{(n+2)^2} \geqslant \frac{8}{(n+2)^2}$.

- Let n be odd. If $n = 1$, then $2t = 2\ell\pi$ and so $|U_L(t)_{u,u}|^2 = \frac{1}{9}$. If $n > 1$, then using (18) and the same argument in the case $n = 4m$ yields $|U_L(t)_{u,u}|^2 \geqslant \frac{n^2(1-\cos(\pi/n))}{2(n+2)^2} = \frac{n^2\sin^2(\pi/2n)}{(n+2)^2} \geqslant \frac{2.25}{(n+2)^2}$.

Finally, comparing the above subcases yields the desired result. $\square$

Remark 33. *In the above proof, if we take $S = \{0, n\}$, then $\sigma_u(L)\backslash S = \{n+2\}$ and $(E_n)_{u,u} + (E_0)_{u,u} = a$, where $a = 1 - \frac{n}{2(n+2)} > \frac{1}{2}$. If $F(t) := |\frac{1}{n+2} + \frac{1}{2}e^{itn}| - \frac{n}{2(n+2)}$, then Theorem 11 yields $|U(t)_{u,u}| \geqslant F(t) \geqslant 0$ for all t. If $n \equiv 0$ (mod 4), then one checks that (10) holds at $t_1 = j\pi/2$ for odd j, i.e., the first inequality is an equality, in which case $U(t_1) = 2a - 1 = 1 - \frac{n}{n+2}$. As $F(t)$ is not a constant function, we are not guaranteed that $|U(t)_{u,u}| \geqslant 1 - \frac{n}{n+2}$ for all t. To show that this is indeed the case, we need to establish that $|U(t)_{u,u}|$ is minimum at t_1, which was the approach taken in the proof of Theorem 32.*

The following is an immediate consequence of Theorem 32.

Corollary 34. *The family of disconnected double cones on simple positively weighted graphs on n vertices, where $n \not\equiv 2$ (mod 4), is quasi-sedentary at the apexes with respect to the Laplacian matrix.*

In keeping with Theorem 29, we show that the bound in Theorem 16 is tight for the vertices of O_m in $O_m \vee X$ for some values of m and n.

Theorem 35. *Let $m \geqslant 3$ and X be a simple positively weighted graph on $n \geqslant 1$ vertices. If $\nu_2(m) = \nu_2(n)$, then for any vertex u of O_m in $O_m \vee X$, $|U_L(t)_{u,u}| = 1 - \frac{2}{m}$ whenever $t_1 = \frac{j\pi}{\gcd(m,n)}$ for any odd j.*

Proof. By Theorem 16, $|U_L(t)_{u,u}| \geqslant 1 - \frac{2}{m}$ for all t. Letting $t_1 = \frac{j\pi}{g}$, where j is odd and $g = \gcd(m, n)$, one can check that using (14) that $|U_L(t)_{u,u}| = 1 - \frac{2}{m}$. $\square$

Let m be fixed. By Theorem 35, the family of joins $O_m \vee X$ such that $\nu_2(m) = \nu_2(n)$ is tightly $(1 - \frac{2}{m})$-sedentary at the vertices of O_m. Our numerical observations indicate that if $\nu_2(m) \neq \nu_2(n)$ with m fixed, then any vertex of O_m is $C(n)$-sedentary in $O_m \vee X$, where $C(n)$ satisfies $C(n) < 1 - \frac{2}{m}$ for all n and $C(n) \to 1 - \frac{2}{m}$ as n increases. This suggests that in general, the family of graphs of the form $O_m \vee X$ with fixed m is sharply $(1 - \frac{2}{m})$-sedentary at the vertices of O_m.

Adjacency case

Next, we examine the adjacency case for disconnected double cones.

Theorem 36. *Let X be a simple uweighted d-regular graph on n vertices, and let u be an apex of $O_2 \vee X$.*

1. *If either (i) $d^2 + 8n$ is not a perfect square, (ii) $d = 0$, or (iii) $d^2 + 8n$ is a perfect square and $\nu_2(d + \sqrt{d^2 + 8n}) = \nu_2(d - \sqrt{d^2 + 8n})$, then $O_2 \vee X$ is not sedentary at u.*
2. *Let $d > 0$ and $n = \frac{1}{2}s(d + s)$ for some integer $s > 0$ such that $s(d + s)$ is even and $\nu_2(d + s) \neq \nu_2(s)$ (i.e., $d^2 + 8n$ is a perfect square and $\nu_2(d + \sqrt{d^2 + 8n}) \neq \nu_2(d - \sqrt{d^2 + 8n})$). Suppose $d_1 = d/g$ and $s_1 = s/g$, where $g = \gcd(d, s)$. The following hold.*
   (a) *If $s_1 = 1$, then $|U_A(t)_{u,u}| \geqslant \frac{1}{d_1+2}$ with equality if and only if $t = \frac{j\pi}{g}$ for any odd j.*
   (b) *If $s_1 \geqslant 2$, then $|U_A(t)_{u,u}| \geqslant \frac{\sqrt{2}}{d_1+2s_1}$ with equality if and only if $t = \frac{j\pi}{g}$ for any odd j.*

Proof. Let Y be a double cone on X with apexes u and v. From [KMP22, Lemma 3(2)], we know that $\sigma_u(A) = \{0, \lambda^{\pm}\}$, where $\lambda^{\pm} = \frac{1}{2}\left(d \pm \sqrt{d^2 + 8n}\right)$. Applying [CG21, Lemma 12.3.1], we obtain

$$U_A(t)_{u,u} = \frac{1}{2} + \frac{n}{2n + (\lambda^+)^2} e^{it\lambda^+} + \frac{n}{2n + (\lambda^-)^2} e^{it\lambda^-} \tag{19}$$

If d^2+8n is not a perfect square, then PGST occurs between u and v [KMP22, Theorem 11(1)], while if $d=0$, or d^2+8n is a perfect square and $\nu_2(d+\sqrt{d^2+8n})=\nu_2(d-\sqrt{d^2+8n})$, then PST occurs between u and v [KMP22, Theorem 14(2a)]. Invoking Proposition 2(1c) yields (1a). To prove (1b), suppose d^2+8n is a perfect square, $d>0$, and $\nu_2(d+\sqrt{d^2+8n})\neq\nu_2(d-\sqrt{d^2+8n})$. This is equivalent to the existence of an integer $s>0$ such that $n=\frac{1}{2}s(d+s)$ with $s(d+s)$ is even and

$$\nu_2(s)\geqslant\nu_2(d). \tag{20}$$

One also checks that $2n+(\lambda^+)^2=(d+s)(d+2s)$ and $2n+(\lambda^-)^2=s(d+2s)$. Thus, we obtain $\frac{n}{2n+(\lambda^+)^2}=\frac{s(d+s)}{2(d+s)(d+2s)}=\frac{s}{2(d+2s)}$ and $\frac{n}{2n+(\lambda^-)^2}=\frac{s(d+s)}{2s(d+2s)}=\frac{d+s}{2(d+2s)}$. Combining this with (19), we get that $U_A(t)_{u,u}=\frac{1}{2(d+2s)}\left((d+2s)+se^{it(d+s)}+(d+s)e^{-its}\right)$, and so

$$|U_A(t)_{u,u}|^2=\frac{d^2+3ds+3s^2+h(t)}{2(d+2s)^2} \tag{21}$$

where $h(t)=s(d+s)\cos(t(d+2s))+s(d+2s)\cos(t(d+s))+(d+s)(d+2s)\cos(ts)$. Following the Laplacian case, $|U_A(t)_{u,u}|^2$ is maximum (resp., minimum) if and only if $h(t)$ is, and we have

$$h'(t)=-4s(d+s)(d+2s)\cos\left(ts/2\right)\cos\left(t(d+s)/2\right)\sin\left(t(d+2s)/2\right). \tag{22}$$

and

$$h''(t)=-2s(d+s)(d+2s)\left[(d+s)\cos\left(t(2d+3s)/2\right)\cos\left(ts/2\right)+s\cos\left(t(d+3s)/2\right)\cos\left(t(d+s)/2\right)\right]. \tag{23}$$

From (22), $h'(t)=0$ if and only if either $t=j\pi/s$ for some odd j, $t=\frac{\ell\pi}{d+s}$ for some odd ℓ, and $t=\frac{k\pi}{d+2s}$ for even $k\neq 0$. Among these three values, one can use (23) to show that $h''(t)>0$ if and only if $t=j\pi/s$ for odd j and $t=\frac{\ell\pi}{d+s}$ for odd ℓ. Thus, it suffices to compare the values of $|U_A(t)_{u,u}|^2$ at the points $t=j\pi/s$ for some odd j and $t=\frac{\ell\pi}{d+s}$ for some odd ℓ to obtain the absolute minimum. Let's start with $t=j\pi/s$ for some odd j. In this case, $\cos(ts)=-1$, $\cos(t(d+2s))=\cos\left(jd\pi/s\right)$ and $\cos(t(d+s))=-\cos\left(jd\pi/s\right)$. Thus, $h(t)=-s^2\cos\left(jd\pi/s\right)-(d+s)(d+2s)$, and (21) gives us

$$|U_A(t)_{u,u}|^2=\frac{s^2\left[1-\cos\left(jd\pi/s\right)\right]}{2(d+2s)^2} \tag{24}$$

Let $d=gd_1$ and $s=gs_1$, where $g=\gcd(d,s)$. Then we can write $jd\pi/s=jd_1\pi/s_1$, where d_1 is odd by (20). We proceed with two subcases.

- Let $s_1=1$. Then (24) yields $|U_A(t)_{u,u}|^2=\frac{1-\cos(jd_1\pi)}{2(d_1+2)^2}=\frac{1}{(d_1+2)^2}$.
- Let $s_1\geqslant 2$. As j is odd, $jd_1\pi/s_1$ is not an even multiple of π. Hence, $\cos\left(jd_1\pi/s_1\right)\leqslant\cos\left(\pi/s_1\right)$, and making use of (24) then yields $|U_A(t)_{u,u}|^2\geqslant\frac{s_1^2[1-\cos(\pi/s_1)]}{2(d_1+2s_1)^2}=\frac{s_1^2\sin^2(\pi/2s_1)}{(d_1+2s_1)^2}\geqslant\frac{2}{(d_1+2s_1)^2}$.

Next, let $t=\frac{\ell\pi}{d+s}$ for some odd ℓ. In this case, $\cos(t(d+s))=-1$ and $\cos(t(d+2s))=\cos\left(\frac{\ell s\pi}{d+s}\right)$. Thus, $h(t)=-s(d+2s)+(d+s)^2\cos(ts)$, and making use of (21) gives us

$$|U_A(t)_{u,u}|^2=\frac{(d+s)^2\left[1+\cos\left(\frac{\ell s\pi}{d+s}\right)\right]}{2(d+2s)^2} \tag{25}$$

Note that we can write $\frac{\ell s\pi}{d+s}=\frac{\ell s_1\pi}{d_1+s_1}$. We proceed with two subcases.

- Let $s_1 = 1$. The same argument in the proof of Theorem 32 for the case $t = \frac{\ell\pi}{n}$ for odd ℓ and $n = 4m$ yields $|U_A(t)_{u,u}|^2 \geqslant \frac{(d_1+1)^2\left[1-\cos\left(\frac{\pi}{d_1+1}\right)\right]}{2(d_1+2)^2} = \frac{(d_1+1)^2\sin^2\left(\frac{\pi}{2(d_1+1)}\right)}{(d_1+2)^2} \geqslant \frac{2}{(d_1+2)^2}$.

- Let $s_1 \geqslant 2$. Since $d_1 + s_1$ and s_1 must have opposite parities, $\frac{\ell s_1\pi}{d_1+s_1}$ is not an odd multiple of 2π. Thus, (25) yields $|U_A(t)_{u,u}|^2 \geqslant \frac{(d_1+s_1)^2\left[1-\cos\left(\frac{\pi}{d_1+s_1}\right)\right]}{2(d_1+2s_1)^2} = \frac{(d_1+s_1)\sin^2\left(\frac{\pi}{2(d_1+s_1)}\right)}{(d_1+2s_1)^2} \geqslant \frac{2.25}{(d_1+2s_1)^2}$.

Comparing the minima for each subcases above yields the desired conclusion. $\square$

The following is a straightforward consequence of Theorem 36(2).

Corollary 37. *With the assumption in Theorem 36(2), let $\mathscr{F}$ be a family of disconnected double cones on simple unweighted d-regular graphs on n vertices. If s_1 and d_1 are fixed, then $\mathscr{F}$ is $\frac{1}{d_1+2}$- and $\frac{\sqrt{2}}{d_1+2s_1}$-sedentary at the apexes resp. whenever $s_1 = 1$ and $s_1 \geqslant 2$. On the other hand, if either $d_1 \to \infty$ or $s_1 \to \infty$ as n increases, then $\mathscr{F}$ is quasi-sedentary at the apexes.*

We end this section with the following remark.

Remark 38. *Let u and v be the apexes of $O_2 \vee X$, where X is regular whenever $M = A$.*

1. *Suppose the assumption Theorem 32 holds. If $n \equiv 0 \pmod 4$, then $|U_L(t)_{u,v}| \leqslant 1 - \delta$ for all t, where $\delta = \frac{2}{n+2}$, while if $n \geqslant 3$ is odd, then $|U_L(t)_{u,v}| \leqslant 1 - \delta$ for all t, where $\delta = \frac{\sqrt{2}}{n+2}$.*

2. *Suppose the assumption in Theorem 36(2) holds. If $s_1 = 1$, then $|U_A(t)_{u,v}| \leqslant 1 - \delta$ for all t, where $\delta = \frac{1}{d_1+2}$, while if $s_1 \geqslant 2$, then $|U_A(t)_{u,v}| \leqslant 1 - \delta$ for all t, where $\delta = \frac{\sqrt{2}}{d_1+2s_1}$.*

In [GS17]*, Godsil and Smith asked to find examples of strongly cospectral vertices u and v such that for some constant $\delta > 0$, $|U(t)_{u,v}| \leqslant 1 - \delta$ for all t. Mirror symmetric vertices in paths without PGST and antipodal vertices in even cycles without PGST are infinite families that answer this question. However, we do not know whether paths and cycles are sedentary. Thus, the families in (1) and (2) are the first examples that answer Godsil and Smith's question, whereby the vertices involved are sedentary.*

8 Trees

Our first result in this section is a direct consequence of Theorem 16.

Proposition 39. *Let T be a set of leaves of a tree X that share a common neighbour. Then T is a set of twins in X, and for each $u \in T$, we have $|U_M(t)_{u,u}| \geqslant 1 - \frac{2}{|T|}$ for all t.*

Next, we examine whether the central vertex of a star is sedentary.

Proposition 40. *Let T be the set of leaves of $K_{1,n}$ and $u \in T$. Then $|U_M(t)_{u,u}| \geqslant 1 - \frac{2}{n}$ for all t. Hence, the family $\mathscr{S}$ of stars $K_{1,n}$ on $n \geqslant 3$ vertices is sedentary at every leaf vertex. The following also hold.*

1. *For all $n \geqslant 2$, $|U_A(t)_{u,u}| = 1 - \frac{2}{n}$ if and only if $t = \frac{j\pi}{\sqrt{n}}$ for any odd j. Moreover, the central vertex of $K_{1,n}$ is not sedentary with respect to A.*

2. *If n is odd, then $|U_L(t)_{u,u}| = 1 - \frac{2}{n}$ whenever $t = j\pi$ for any odd j. For all $n \geqslant 2$, the central vertex w of $K_{1,n}$ satisfies $|U_L(t)_{w,w}| \geqslant 1 - \frac{2}{n+1}$ with equality if and only if $t = \frac{j\pi}{n+1}$ for any odd j.*

Proof. The first statement follows from Proposition 39, while the second one is obtained by by noting that $|V(X)\backslash T| = 1$ and applying Corollary 21(2). To prove (1a), one can use the fact that $\sigma_u(A) = \{0, \pm\sqrt{n}\}$ to show that (9) and (10) hold if and only if $t_1 = \frac{j\pi}{\sqrt{n}}$ for odd j. Thus, equality holds in (12) in Theorem 16, which yields the first statement of (1a). As $K_{1,n}$ is a cone on a 0-regular graph, the second statement follows from Remark 27. Finally, since $K_{1,n} = O_n \vee K_1$ with $T = V(O_n)$, Theorem 35 yields the first statement of (2), while the second follows by noting that $K_{1,n} = O_n \vee K_1$ and applying Theorem 29. □

Proposition 40(2) implies that the apex of a cone on any simple positively weighted graph X on n vertices is Laplacian tightly $(1 - \frac{2}{n+1})$- sedentary. Next, we use Proposition 40 to create more sedentary families using Cartesian products.

Example 41. *Let $k \geqslant 1$, $n \geqslant 3$ and $Z_{k,n}$ be the Cartesian product of $K_{1,n}$ with itself k times. Let u be a leaf of $K_{1,n}$. By Proposition 40, $|U_{Z_{k,n}}(t)_{(u,\dots,u),(u,\dots,u)}| \geqslant (1 - \frac{2}{n})^k$ for all t with respect to M. Consider the families $\mathscr{F}_1 = \{Z_{k,n} : k \text{ fixed}\}$, $\mathscr{F}_2 = \{Z_{k,n} : n = \lfloor mk \rfloor \text{ for some } m > 0\}$ and $\mathscr{F}_3 = \{Z_{k,n} : n \text{ fixed}\}$.*

1. *As n increases, $(1 - \frac{2}{n})^k \to 1$ in $\mathscr{F}_1$ and $(1 - \frac{2}{n})^k \to 1/\sqrt[m]{e^2}$ in $\mathscr{F}_2$. Thus, $\mathscr{F}_1$ is sedentary while $\mathscr{F}_2$ is $1/\sqrt[m]{e^2}$-sedentary at $(u, \dots, u)$. If $M = A$, then the sedentariness of $\mathscr{F}_1$ and $\mathscr{F}_2$ is tight by Proposition 40(1). If $M = L$, then the sedentariness of the subfamilies of $\mathscr{F}_1$ and $\mathscr{F}_2$ is tight whenever n in each $Z_{k,n}$ is odd by virtue of Proposition 40(2).*
2. *Since $(1 - \frac{2}{n})^k \to 0$ as $k \to \infty$, $\mathscr{F}_3$ is quasi-sedentary by Proposition 40.*

If u in Example 41 is instead the degree n vertex of $K_{1,n}$, then $|U_{Z_{k,n}}(t)_{(u,\dots,u),(u,\dots,u)}| \geqslant (1 - \frac{2}{n+1})^k$ for all t with respect to L by Proposition 40(2). Thus, $\mathscr{F}_1$ is sedentary, $\mathscr{F}_2$ is $1/\sqrt[m]{e^2}$-sedentary, and $\mathscr{F}_3$ is quasi-sedentary at $(u, \dots, u)$ with respect to L. Now, this does not hold for A by Proposition 40, and so we get a family that is sedentary with respect to L but not to A. By Theorems 32 and 36, one may construct a family that is sedentary with respect to A but not to L by taking the family of disconnected double cones on d-regular graphs on $n \equiv 2 \pmod 4$ vertices satisfying condition (2) of Theorem 36.

We also note that if we replace the $Z_{k,n}$'s in the above example by the Hamming graphs $H(k, n)$, then $\mathscr{F}_1$ is tightly sedentary (which we already know by Corollary 8), $\mathscr{F}_2$ is tightly $1/\sqrt[m]{e^2}$-sedentary at $(u, \dots, u)$, and $\mathscr{F}_3$ is quasi-sedentary. Moreover, this applies to any M because each $H(k, n)$ is regular.

A *double star* $S_{k,\ell}$ is a tree resulting from attaching k and ℓ pendent vertices to the vertices of K_2. Like the central vertex of $K_{1,n}$, we show that an internal vertex of $S_{k,k}$ is not sedentary whenever $M = A$.

Theorem 42. *Let $k \geqslant 1$, and consider a double star $S_{k,k}$ with internal vertices u and v.*

1. *Let $4k + 1$ be a perfect square, and let w be a leaf of $S_{k,k}$. If $k = 2$, then $|U_A(t)_{w,w}| \geqslant \frac{1}{4}$, with equality whenever $t = \frac{j\pi}{3}$, where $j \equiv 2, 4 \pmod 6$. If $k > 2$, then $|U_A(t)_{w,w}| \geqslant 1 - \frac{2}{k}$, with equality whenever $t = j\pi$ for an integer j such that $j\sqrt{4k + 1} \equiv 3 \pmod 4$.*
2. *For all $k \geqslant 1$, u and v are not sedentary in $S_{k,k}$ with respect to the adjacency matrix.*

Proof. Suppose we index the vertices of $A(S_{k,\ell})$ starting with the k leaves attached to u, followed by u and v, and then the k leaves attached to v. Then we can write $A(S_{k,k}) = \begin{bmatrix} A(K_{1,k}) & Y \\ Y^T & A(K_{1,k}) \end{bmatrix}$, where $A(K_{1,k}) = \begin{bmatrix} \mathbf{0}_k & \mathbf{1} \\ \mathbf{1}^T & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} \mathbf{0} & \mathbf{0}_k \\ 1 & \mathbf{0} \end{bmatrix}$. Thus, $\mathbf{e}_1 - \mathbf{e}_j$ for $j = 1, \dots, k$ and $\mathbf{e}_{k+2} - \mathbf{e}_j$ for $j = k + 3, \dots, 2k + 2$ are eigenvectors for $A(S_{k,k})$ corresponding to the eigenvalue 0. Moreover,

$$\lambda_1 = -\frac{1}{2}(1 + \sqrt{4k + 1}), \lambda_2 = \frac{1}{2}(-1 + \sqrt{4k + 1}), \lambda_3 = \frac{1}{2}(1 - \sqrt{4k + 1}) \text{ and } \lambda_4 = \frac{1}{2}(1 + \sqrt{4k + 1}) \quad (26)$$

are simple eigenvalues of $A(S_{k,k})$ resp. with eigenvectors $\mathbf{v}_1 = \left[-\mathbf{1}, \frac{1+\sqrt{4k+1}}{2}, -\frac{1+\sqrt{4k+1}}{2}, \mathbf{1}\right]$, $\mathbf{v}_2 = \left[-\mathbf{1}, \frac{1-\sqrt{4k+1}}{2}, \frac{-1+\sqrt{4k+1}}{2}, \mathbf{1}\right]$, $\mathbf{v}_3 = \left[\mathbf{1}, \frac{1-\sqrt{4k+1}}{2}, \frac{1-\sqrt{4k+1}}{2}, \mathbf{1}\right]$ and $\mathbf{v}_4 = \left[\mathbf{1}, \frac{1+\sqrt{4k+1}}{2}, \frac{1+\sqrt{4k+1}}{2}, \mathbf{1}\right]$. Thus, $E_0 = I_k - \frac{1}{k}\mathbf{J}_k \oplus O_2 \oplus I_k - \frac{1}{k}\mathbf{J}_k$, where $A \oplus B$ is the direct sum of matrices A and B, and $E_{\lambda_j} = \frac{1}{\|\mathbf{v}_j\|^2}\mathbf{v}_j^T\mathbf{v}_j$ for $j \in \{1,2,3,4\}$. Thus, $\sigma_u(A) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and $\sigma_w(A) = \{0\} \cup \sigma_u(A)$ for any leaf w of $S_{k,k}$. Using spectral decomposition and the fact that $\lambda_1 = -\lambda_4$ and $\lambda_2 = -\lambda_3$ yields

$$U_A(t)_{u,u} = \frac{(1+\sqrt{4k+1})^2\cos(t\lambda_1)}{2(4k+1+\sqrt{4k+1})} + \frac{(1-\sqrt{4k+1})^2\cos(t\lambda_2)}{2(4k+1-\sqrt{4k+1})} \tag{27}$$

and

$$U_A(t)_{w,w} = \frac{k-1}{k} + \frac{2\cos(t\lambda_1)}{4k+1+\sqrt{4k+1}} + \frac{2\cos(t\lambda_2)}{4k+1-\sqrt{4k+1}}. \tag{28}$$

Let $4k+1$ be a perfect square. Note that the first statement of (1) can be easily verified using (28). Since Theorem 16 yields $|U_A(t)_{w,w}| \geqslant 1 - \frac{2}{k}$ for all t, one can check using (28) that indeed, $|U_A(t_1)_{w,w}| = 1 - \frac{2}{k}$ whenever $t_1 = j\pi$, where j is an integer such that $j\ell \equiv 3 \pmod 4$. This proves (1). To prove (2), it suffices to check the case when $4k+1$ is a perfect square by Proposition 2(1c) and [FG13, Theorem 5.3]. Observe that $U_A(t)_{u,u}$ in (27) is a real valued continuous function that has positive and negative values as t ranges across $[0, 2\pi]$. By IVT, there exists a $t \in [0, 2\pi]$ such that $U_A(t)_{u,u} = 0$, i.e., u is not sedentary. □

Corollary 43. *Let u be a vertex of $S_{k,\ell}$ with $\deg(u) = k+1$. The following hold for $M = A$.*

1. *If $k = 2$, then the leaves attached to u in $S_{2,\ell}$ are sedentary if and only if $\ell = 2$. In particular, if $k = \ell = 2$, then all leaves in $S_{2,2}$ are tightly $\frac{1}{4}$-sedentary.*

2. *If $k \geqslant 3$ and the nonzero eigenvalues of $A(S_{k,\ell})$ are linearly independent over $\mathbb{Q}$, then the leaves attached to u are sharply $(1 - \frac{2}{k})$-sedentary. In particular, if $4k+1$ is not a perfect square, then all leaves of $S_{k,k}$ are sharply $(1 - \frac{2}{k})$-sedentary.*

Proof. If $k = 2$, then PGST occurs between the leaves attached to u if and only if $\ell = 2$ [FG13, Theorem 5.2]. By Proposition 2(1c), we only need to check $S_{2,2}$. By Theorem 42(1), a leaf attached to u is tightly $\frac{1}{4}$-sedentary. As all leaves in $S_{2,2}$ are cospectral, Proposition 2(1b) implies they are tightly $\frac{1}{4}$-sedentary. This proves (1). Now, assume the premise of (2). Take $S = \{0\}$ in Theorem 11 so that $(E_0)_{u,u} = a \geqslant \frac{1}{2}$, where $a = 1 - \frac{1}{k}$. If m_j and ℓ_j are integers such that $\sum_{\lambda_j \neq 0} \ell_j\lambda_j = 0$ and $m_j + \sum_{\lambda_j \neq 0} \ell_j = 0$, then $m_j = 0$. Invoking Lemma 12 yields sharp $(1 - \frac{2}{k})$-sedentariness at u. The last statement follows from the linear independence of the nonzero eigenvalues of $A(S_{k,k})$ when $4k+1$ is not a perfect square. □

It is natural to ask whether the internal vertices of $S_{k,\ell}$ are in general sedentary. We leave this as an open question.

9 Other types of state transfer

By Proposition 2(1c), a sedentary vertex cannot be involved in PGST. Hence, we ask, which types of state transfer can a sedentary vertex exhibit? Here, we show that there are sedentary families where each member graph exhibits proper fractional revival and local uniform mixing at a sedentary vertex.

Proper (α, β)-*fractional revival* (FR) occurs between u and v at time t_1 if $\alpha^2 + \beta^2 = 1$, where $\alpha = |U(t_1)_{u,u}|$ and $\beta = |U(t_1)_{u,v}| \neq 0$. In [CJL+21, Theorem 11], Chan et al. showed that proper

Laplacian FR occurs between the apexes of $O_2 \vee X$, where X is a simple unweighted graph on n vertices. Meanwhile, in [CCT+19, Example 6.3], Chan et al. showed that $O_2 \vee X$ exhibits proper Laplacian FR between its apexes, where X is a simple unweighted d-regular graph on n vertices. If X is a simple positively weighted graph, it is shown in [Mon23] that the apexes of $K_2 \vee X$ do not admit proper Laplacian FR. Combining these facts with Corollaries 30, 34 and 37, we obtain families where each member graph exhibits (resp., does not exhibit) proper FR involving a sedentary vertex. This tells us that, unlike PGST, FR and sedentariness can occur together, although they do not always happen together.

Example 44. *The following hold.*

1. *Each graph in the quasi-sedentary family of disconnected double cones in Corollary 34 exhibits proper Laplacian FR between apexes. Moreover, each graph in the C-sedentary families of disconnected double cones in Corollary 37 (1a-c) exhibits proper adjacency FR between apexes.*

2. *Each graph in the sedentary family of complete graphs on $n \geqslant 3$ vertices does not exhibit proper FR between any two vertices with respect to A and $\mathcal{A}$. Moreover, each graph in the sedentary family of connected double cones in Corollary 30 does not exhibit proper Laplacian FR between apexes.*

We say that u admits *(instantaneous) local uniform mixing* in X at time t_1 if $|U(t_1)_{u,v}| = 1/\sqrt{|V(X)|}$ for each vertex v in X. We say that X admits *(instantaneous) uniform mixing* in X at time t_1 if each vertex in X admits local uniform mixing at time t_1.

Proposition 45. *Let $0 < C \leqslant 1$ and $\mathscr{F}$ be a C-sedentary family of graphs.*

1. *Almost all graphs in $\mathscr{F}$ do not exhibit local uniform mixing.*

2. *If the function f in Definition 3 satisfies $f(|V(X)|) > \frac{1}{\sqrt{|V(X)|}}$ for all $X \in \mathscr{F}$, then each $X \in \mathscr{F}$ does not exhibit local uniform mixing at a sedentary vertex.*

Proof. By assumption, for each $X \in \mathscr{F}$ and some vertex u of X, we have $|U_M(t)_{u,u}| \geqslant f(|V(X)|)$ for all t, where $f(s) \to C > 0$ as s increases. Now, if $X \in \mathscr{F}$ admits local uniform mixing, then $|U_M(t_1)_{u,u}| = 1/\sqrt{|V(X)|}$ for some time t_1. But since $C > 0$ and $1/\sqrt{s} \to 0$ as s increases, only finitely many graphs in $\mathscr{F}$ can exhibit local uniform mixing. This proves (1), and (2) is straightforward. □

If $\mathscr{K}'$ is the family of complete graphs on $n \geqslant 5$ vertices, then from (6), we may take f such that $f(n) = 1 - \frac{2}{n}$. Since $f(n) > \frac{1}{\sqrt{n}}$ for all $n \geqslant 5$, no member of $\mathscr{K}'$ exhibits local uniform mixing by Proposition 45(2). Proposition 45(1), on the other hand, implies that only quasi-sedentary families exhibit local uniform mixing at a sedentary vertex as illustrated by our next examples.

Example 46. *Let $\mathscr{F}$ be a family of cones on weighted d-regular graphs, where $0 < d \leqslant 2$. Combining Proposition 26(2) and* [God21, Lemma 7.1]*, we conclude that $\mathscr{F}$ is quasi-sedentary at the apex and each $X \in \mathscr{F}$ admits local uniform mixing at the apex with respect to A.*

Example 47. *Consider $Z_{k,3}$ in Example 41, which is a Cartesian power of $K_{1,3}$. This graph admits uniform mixing at $t_1 = \frac{\pi}{3\sqrt{3}}$* [GZ15, Section 11]*, and so Example 41(2) implies that each graph in the quasi-sedentary family $\mathscr{F} = \{Z_{k,3} : k \geqslant 1\}$ admits adjacency uniform mixing.*

Since Cartesian powers of K_3 admit uniform mixing at time $t_1 = \frac{\pi}{9}$, the same result holds if we replace $Z_{k,3}$ in the previous example by $H(k,3)$. Moreover, this result applies to any M because $H(k,3)$ is regular.

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