Multipartite concurrence of W-class states based on sub-partite quantum systems

Wei Chen¹, Yanmin Yang²^{*}, Shao-Ming Fei^{3,4}, Zhu-Jun Zheng⁵, Yan-Ling Wang¹

¹ School of Computer Science and Technology,

Dongguan University of Technology, Dongguan, 523808, P.R. China

² Faculty of Science,

Kunning University of Science and Technology, Kunning, 650500, P.R. China

³ School of Mathematical Sciences,

Capital Normal University, Beijing 100048, P.R. China

⁴ Max-Planck-Institute for Mathematics in the Sciences, Leipzig 04103, Germany

⁵ School of Mathematics,

South China University of Technology, Guangzhou 510641, P.R. China

Abstract

We study the concurrence for arbitrary N-partite W-class states based on the (N-1)-partite partitions of subsystems by taking account to the structures of W-class states. By using the method of permutation and combination we give analytical formula of concurrence and some elegant relations between the multipartite concurrence and the (N-1)-partite concurrence for arbitrary multipartite W-class states. Applying these relations we present better lower bounds of concurrence for multipartite mixed states. An example is given to demonstrate that our lower bounds can detect more entanglements.

Keywords Multipartite concurrence \cdot W-class states \cdot (N-1)-partite partitions \cdot Lower bound of concurrence

1 Introduction

Quantum entanglement is a striking feature of quantum physics [1–4] and an essential resource in quantum information processing varying from quantum teleportation [5] and quantum cryptography [6] to dense coding [7]. Due to its variety of usages, quantum entanglement has attracted much attention in recent years [8–14].

^{*}e-mail: ym.yang@kust.edu.cn

To quantify the entanglement of a state, the concept of entanglement measure has been naturally introduced, such as the entanglement of formation [15] for bipartite quantum systems and concurrence [16] for any multipartite quantum systems. For the two-qubit case, the entanglement of formation is proven to be a monotonically increasing function of the concurrence and an elegant formula for concurrence was derived analytically by Wootters [17]. However, except for bipartite qubit systems and some special symmetric states [18], there have been no explicit analytic formulas of concurrence for arbitrary high-dimensional mixed states, due to the extremizations involved in the computation.

Instead of analytic formulas, some progress has been made toward the analytical lower bounds of concurrence. In [19, 20], the authors presented a lower bound of concurrence by decomposing the joint Hilbert space into many $2 \otimes 2$ and $s \otimes t$ -dimensional subspaces, which improve all the known lower bounds of concurrence. Similar nice algorithms and progress have been made towards the lower bounds of concurrence for tripartite quantum systems [21,22] and other multipartite quantum systems [23,24] based on bipartite partitions of the whole quantum system. The authors in [25] improve the lower bound of concurrence by using tripartite and *M*-partite concurrences of an *N*-partite ($2 \leq M < N$) systems.

As a particular kind of quantum states, the well-known W-class states [26] have been widely studied. In this paper we first study the multipartite concurrence for W-class states and derive an analytical formula for pure W-class states. Then we investigate the N-partite concurrence of W-class states based on the (N-1)-partite quantum systems and present an elegant relation between among them. Based on the results for the W-class states, we derive better lower bounds of concurrence for a class of multipartite mixed states. An example is given to illustrate that our lower bound may detect more entanglements.

2 Multipartite concurrence of *W*-class states

We first recall the definition of the multipartite concurrence. Let H_i , $i = 1, \dots, N$, be d_i dimensional Hilbert spaces. The concurrence of an N-partite pure state $|\psi\rangle \in H_1 \otimes H_2 \otimes \cdots \otimes H_N$ is defined by [27],

$$C_N(|\psi\rangle) = 2^{1-\frac{N}{2}} \sqrt{(2^N - 2) - \sum_{\alpha} Tr(\rho_{\alpha}^2)},$$
(1)

where the index α labels all $2^N - 2$ non-trivial subsystems of the *N*-partite quantum systems and ρ_{α} are the corresponding reduced density matrices. For a mixed multipartite quantum state $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i | \in H_1 \otimes H_2 \otimes \cdots \otimes H_N, p_i \ge 0, \sum_i p_i = 1$, the concurrence is given by the convex roof:

$$\mathcal{C}_N(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \mathcal{C}_N(|\psi_i\rangle), \qquad (2)$$

where the minimum is taken over all possible pure state decompositions of ρ .

In [23] the authors obtained the lower bounds of multipartite concurrence in terms of the concurrences of bipartite partitioned states of the whole quantum system. For an Npartite quantum pure state $|\psi\rangle \in H_1 \otimes H_2 \otimes \cdots \otimes H_N$, the concurrence of bipartite partition between the subsystems $12 \cdots M$ and $M + 1 \cdots N$ is defined by

$$\mathcal{C}_2(|\psi\rangle\langle\psi|) = \sqrt{2(1 - Tr(\rho_{12\cdots M}^2))},\tag{3}$$

where $\rho_{12\cdots M} = Tr_{M+1\cdots N}\{|\psi\rangle\langle\psi|\}$ is the reduced density matrix of $\rho = |\psi\rangle\langle\psi|$ by tracing over the subsystems $M+1\cdots N$. For a mixed multipartite quantum state $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \in$ $H_1 \otimes H_2 \otimes \cdots \otimes H_N$, the corresponding concurrence $\mathcal{C}_2(\rho)$ is given by the convex roof:

$$C_2(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_2(|\psi_i\rangle\langle\psi_i|).$$
(4)

A relation between the concurrence (2) and the bipartite concurrence (4) has been presented in [23]. For a multipartite quantum state $\rho \in H_1 \otimes H_2 \otimes \cdots \otimes H_N$ with $N \geq 3$, the following inequality holds,

$$\mathcal{C}_N(\rho) \ge \max 2^{\frac{3-N}{2}} \mathcal{C}_2(\rho), \tag{5}$$

where the maximum is taken over all kinds of bipartite concurrences.

In terms of the lower bounds of bipartite concurrence, in [24] further relations between the concurrence (2) and the bipartite concurrence (4) have been derived:

$$C_N(\rho) \ge \max_{M=1,2,\cdots,N-1} \{ 2^{\frac{1-N}{2}} \sqrt{2^{N-M} + 2^M - 2} C_2(\rho_M) \}$$
(6)

for $N \geq 3$, where the maximum is taken over all kinds of bipartite concurrences for given M. In particularly, if N = 3, one has $C_3(\rho) \geq \max\{C_2(\rho_1), C_2(\rho_2)\}$. If N = 4, one gets $C_4(\rho) \geq \max\{C_2(\rho_1), \frac{\sqrt{3}}{2}C_2(\rho_2), C_2(\rho_3)\}$.

In order to improve the lower bounds of concurrence, instead of the bipartite concurrence $C_2(\rho)$, the authors in [22] consider tripartite concurrence $C_3(\rho)$. In [25] the authors improve the lower bound of concurrence by using tripartite and *M*-partite concurrences of an *N*-partite (M < N) system. For an *N*-partite quantum pure state $|\psi\rangle \in H_1 \otimes H_2 \otimes \cdots \otimes H_N$ $(N \ge 3)$, denote $\{i^1\}, \{i^2\}, \cdots, \{i^{M_1}\}, \{k_1^1, k_2^1\}, \{k_1^2, k_2^2\}, \cdots, \{k_1^{M_2}, k_2^{M_2}\}, \cdots, \{q_1^1, \cdots, q_j^1\},$ $\{q_1^2, \cdots, q_j^2\}, \cdots, \{q_1^{M_j}, \cdots, q_j^{M_j}\}$ as the *M* decompositions among the subsystems, where $\{i^1, i^2, \cdots, i^{M_1}, k_1^1, k_2^1, k_2^2, \cdots, k_1^{M_2}, k_2^{M_2}, \cdots, q_1^1, \cdots, q_j^1, \cdots, q_1^{M_j}, \cdots, q_j^{M_j}\} = \{1, 2, \cdots, N\}$ and $\sum_{k=1}^j M_k = M, \sum_{k=1}^j k M_k = N$. The concurrence of the *M*-partite decompositions among the above subsystems is given by

$$\mathcal{C}_M(|\psi\rangle\langle\psi|) = 2^{1-\frac{M}{2}} \sqrt{(2^M - 2) - \sum_{\alpha} Tr(\rho_{\alpha}^2)},\tag{7}$$

where $\emptyset \neq \alpha \subsetneq \{\{i^1\}, \{i^2\}, \cdots, \{i^{M_1}\}, \{k_1^1, k_2^1\}, \{k_1^2, k_2^2\}, \cdots, \{k_1^{M_2}, k_2^{M_2}\}, \cdots, \{q_1^1, \cdots, q_j^1\}, \cdots, \{q_1^{M_j}, \cdots, q_j^{M_j}\}\}$ and ρ_{α} are the corresponding reduced density matrices. The rearrangement of the subsystems are implied naturally. Taking N = 4 and M = 3, one has six different partitions of the four-partite system: 1|2|34, 1|3|24, 1|4|23, 12|3|4, 13|2|4 and 14|2|3. In terms of the lower bounds of tripartite concurrence, in [25] the authors derived a relation between the concurrence (2) and the bipartite concurrence (7), $C_4^2(\rho) \geq \widetilde{C_3}^2(\rho)$, where $\widetilde{C_3}^2(\rho) = \frac{1}{6}(C_3^2(\rho_{1|2|34}) + C_3^2(\rho_{1|3|24}) + C_3^2(\rho_{1|4|23}) + C_3^2(\rho_{12|3|4}) + C_3^2(\rho_{13|2|4}) + C_3^2(\rho_{14|2|3})).$

In order to improve the above lower bounds, we first consider the N-qubit W-class states [26],

$$|W\rangle_{A_1A_2\cdots A_N} = a_1|10\cdots 0\rangle_{A_1A_2\cdots A_N} + a_2|01\cdots 0\rangle_{A_1A_2\cdots A_N} + \dots + a_N|00\cdots 1\rangle_{A_1A_2\cdots A_N}, \quad (8)$$

where $\sum_{i=1}^{N} |a_i|^2 = 1$. Let $\rho \triangleq \rho_{A_1 A_2 \cdots A_N} = |W\rangle_{A_1 A_2 \cdots A_N} \langle W|$ and

$$\rho_{i_1 i_2 \cdots i_k} \triangleq \rho_{A_{i_1} A_{i_2} \cdots A_{i_k}} = tr_{A_1 \cdots \widehat{A_{i_1}} \cdots \widehat{A_{i_2}} \cdots \widehat{A_{i_k}} \cdots A_N} (|W\rangle_{A_1 A_2 \cdots A_N} \langle W|)$$

for any $1 \leq i_1 < i_2 < \cdots < i_k \leq N$. We have

$$\rho_{i_1 i_2 \cdots i_k} = (a_{i_1} | 10 \cdots 0 \rangle + \dots + a_{i_k} | 00 \cdots 1 \rangle)_{A_{i_1} A_{i_2} \cdots A_{i_k}} (a_{i_1}^* \langle 10 \cdots 0 | + a_{i_k}^* \langle 00 \cdots 1 |) \\ + \sum_{j \neq i_1, \cdots, i_k} |a_j|^2 | 00 \cdots 0 \rangle_{A_1 \cdots \widehat{A_{i_1}} \cdots \widehat{A_{i_2}} \cdots \widehat{A_{i_k}} \cdots A_N} \langle 00 \cdots 0 |, \qquad (9)$$

and

$$1 - Tr(\rho_{i_1 i_2 \cdots i_k}^2) = 2(|a_{i_1}|^2 + \dots + |a_{i_k}|^2) \sum_{j \neq i_1, \cdots, i_k} |a_j|^2,$$
(10)

where $A_1 A_2 \cdots \widehat{A_i} \cdots A_N = A_1 A_2 \cdots A_{i-1} A_{i+1} \cdots A_N$. For simplicity, we denote by $\rho_{i_1 i_2 \cdots i_k}$ the reduced density operator $\rho_{A_{i_1} A_{i_2} \cdots A_{i_k}}$. We have the following lemmas for the *N*-qubit *W*-class states.

Lemma 1 When N (N > 1) is even, we have

$$2C_N^0 + 2C_N^1 + \dots + 2C_N^{\frac{N}{2}-1} + C_N^{\frac{N}{2}} = 2^N.$$
 (11)

Proof. As $C_N^i = C_N^{N-i}$ for all integer $0 \le i < N$, we have

$$2C_N^0 + 2C_N^1 + \dots + 2C_N^{\frac{N}{2}-1} + C_N^{\frac{N}{2}}$$
$$= C_N^0 + C_N^1 + \dots + C_N^{\frac{N}{2}-1} + C_N^{\frac{N}{2}} + C_N^{\frac{N}{2}+1} + \dots + C_N^{N}$$
$$= (1+1)^N = 2^N.$$

Hence, $2C_N^0 + 2C_N^1 + \dots + 2C_N^{\frac{N}{2}-1} + C_{N-2}^{\frac{N}{2}} = 2^N$. \Box

Lemma 2 When $N (N \ge 1)$ is odd, we have

$$2C_N^0 + 2C_N^1 + \dots + 2C_N^{\frac{N-1}{2}} = 2^N.$$
 (12)

Proof. As $C_N^i = C_N^{N-i}$ for all integer $0 \le i < N$, we have

$$2C_N^0 + 2C_N^1 + \dots + 2C_N^{\frac{N-1}{2}}$$

= $C_N^0 + C_N^1 + \dots + C_N^{\frac{N-1}{2}} + C_N^{\frac{N+1}{2}} + C_{N-1}^N + C_N^N$
= $(1+1)^N = 2^N$.

Hence, $2C_N^0 + 2C_N^1 + \dots + 2C_N^{\frac{N-1}{2}} = 2^N$. \Box

Theorem 1 The squared multipartite concurrence of the N-qubit W-class state $|W\rangle_{A_1A_2\cdots A_N}$ is given by

$$\mathcal{C}_{N}^{2}(|W\rangle_{A_{1}A_{2}\cdots A_{N}}) = 4\sum_{1 \le i < j \le N} |a_{i}|^{2} |a_{j}|^{2}$$
(13)

for $N \geq 2$.

 $1 \le i_1 < i_2 < \dots < i_{d-1} \le N$

Proof. For N = 2, we have $|W\rangle_{A_1A_2} = a_1|10\rangle_{A_1A_2} + a_2|01\rangle_{A_1A_2}$ and $C_2(|W\rangle_{A_1A_2}) = \sqrt{(2^2 - 2) - Tr(\rho_1^2) - Tr(\rho_2^2)}$. Then $C_2^2(|W\rangle_{A_1A_2}) = 2 - Tr(\rho_1^2) - Tr(\rho_2^2) = 4|a_1|^2|a_2|^2$. i) N > 3 and N is even. Denote $d = \frac{N}{2}$. For pure states one has $1 - Tr(\rho_{i_1 \ i_2 \cdots i_k}^2) = 1 - Tr(\rho_{1 \ \cdots \widehat{i_1} \cdots \widehat{i_2} \cdots \widehat{i_k} \cdots N})$. By (1) we have

$$\mathcal{C}_{N}^{2}(|W\rangle_{A_{1}A_{2}\cdots A_{N}}) = \frac{1}{2^{N-3}} \left[\sum_{i=1}^{N} (1 - tr\rho_{i}^{2}) + \sum_{1 \le i_{1} < i_{2} \le N} (1 - tr\rho_{i_{1}i_{2}}^{2}) + \cdots + \sum_{i_{1} < i_{2} \le N} (1 - tr\rho_{i_{1}i_{2}\cdots i_{d-1}}^{2}) + \sum_{i_{1} < i_{2} \le N} (1 - tr\rho_{i_{1}i_{2}\cdots i_{d-1}}^{2})\right].$$
(14)

 $1 < i_1 < i_2 < \dots < i_{d-1} \le N$

Concerning the numbers of the terms in the summations of (14), we only need to consider the following non-trivial index α , see **Table 1**,

Type of α	Details of α	Number of α
with one subsystem	$1,2,\cdots,N$	C_N^1
with two subsystems	$12, 13, \cdots, (N-1)N$	C_N^2
• • •		• • •
with d subsystems	$12\cdots d, 13\cdots (d+1), \cdots, 1(d+2)\cdots N$	$C_{N-1}^{d-1} = \frac{1}{2}C_N^d$

Table 1: Non-trivial index α in (14).

By equality (10), we have that each item of $1 - Tr(\rho_{i_1i_2\cdots i_{d-1}}^2)$ has the form $2|a_i|^2|a_j|^2$. To compute $\mathcal{C}_N^2(|W\rangle_{A_1A_2\cdots A_N})$ we just need to determine the total coefficients of the term $|a_i|^2|a_j|^2$ for every $1 - Tr(\rho_{i_1i_2\cdots i_{d-1}}^2)$ in (14). Taking the coefficient of the term $|a_1|^2|a_2|^2$ as an example, see **Table 2**, the total coefficient of the term $|a_1|^2|a_2|^2$ is $\frac{1}{2^{N-3}}(2 \cdot 2C_{N-2}^0 + 2 \cdot 2C_{N-2}^1 + \cdots + 2 \cdot C_{N-2}^{d-1})$, which is equal to 4 by **Lemma 1**. Similarly, we can prove that any item $|a_i|^2|a_j|^2$ has the coefficient 4. Hence we have $\mathcal{C}_N^2(|W\rangle_{A_1A_2\cdots A_N}) = 4\sum_{1\leq i< j\leq N} |a_i|^2|a_j|^2$.

ii) $N \ge 3$ and N is odd. Denote $d = \frac{N-1}{2}$. Similarly we have

$$\mathcal{C}_{N}^{2}(|W\rangle_{A_{1}A_{2}\cdots A_{N}}) = \frac{1}{2^{N-3}} \left[\sum_{i=1}^{N} (1 - tr\rho_{i}^{2}) + \sum_{1 \le i_{1} < i_{2} \le N} (1 - tr\rho_{i_{1}i_{2}}^{2}) + \cdots \right]$$

$$+\sum_{1\leq i_1< i_2<\dots< i_{d-1}\leq N} (1-tr\rho_{i_1i_2\cdots i_{d-1}}^2) + \sum_{1\leq i_1< i_2<\dots< i_d\leq N} (1-tr\rho_{1i_1i_2\cdots i_{d-1}}^2)].$$
 (15)

Type of $1 - Tr(\rho_{i_1 i_2 \cdots i_k}^2)$	$\frac{1 - Tr(\rho_{i_1 i_2 \cdots i_k}^2)}{\text{which has the item } a_1 ^2 a_2 ^2}$	The coefficient of $ a_1 ^2 a_2 ^2$
k = 1	$1 - tr(\rho_1^2), \ 1 - tr(\rho_2^2)$	$2 \cdot 2C_{N-2}^0$
k = 2	$1 - tr(\rho_{13}^2), 1 - tr(\rho_{14}^2), \cdots, 1 - tr(\rho_{1N}^2); 1 - tr(\rho_{23}^2), 1 - tr(\rho_{24}^2), \cdots, 1 - tr(\rho_{2N}^2)$	$2 \cdot 2C_{N-2}^1$
k = d - 1	$1 - tr(\rho_{134\cdots d}^2), \cdots, 1 - tr(\rho_{1(d+3)\cdots N}^2);$ $1 - tr(\rho_{234\cdots d}^2), \cdots, 1 - tr(\rho_{2(d+3)\cdots N}^2)$	$2 \cdot 2C_{N-2}^{d-2}$
k = d	$1 - tr(\rho_{134\cdots(d+1)}^2), \cdots, 1 - tr(\rho_{1(d+2)\cdots N}^2)$	$2 \cdot 1C_{N-2}^{d-1}$

Table 2: The coefficients of the term $|a_1|^2 |a_2|^2$ for even N).

Type of α	Details of α	Number of α
with one subsystem	$1, 2, \cdots, N$	C_N^1
with two subsystems	$12,13,\cdots,(N-1)N$	C_N^2
		•••
with d subsystems	$12 \cdots d, 13 \cdots (d+1), \cdots, 1(d+3) \cdots N, \cdots, (d+2)(d+3) \cdots N$	C_N^d

Table 3: Non-trivial index α in (15).

The non-trivial index α we need to consider is shown in **Table 3**.

In order to compute $C_N^2(|W\rangle_{A_1A_2\cdots A_N})$ we need to determine the total coefficient of the terms $|a_i|^2|a_j|^2$ for every $1 - Tr(\rho_{i_1i_2\cdots i_{d-1}}^2)$ in (15). Still taking the coefficient of $|a_1|^2|a_2|^2$ as an example, from **Table 4** we have that the total coefficient of the term $|a_1|^2|a_2|^2$ is $\frac{1}{2^{N-3}}(2 \cdot 2C_{N-2}^0 + 2 \cdot 2C_{N-2}^1 + \cdots + 2 \cdot 2C_{N-2}^{d-1})$ which is equal to 4 by **Lemma 2**. Similarly,

Type of $1 - Tr(\rho_{i_1 i_2 \cdots i_k}^2)$	$\frac{1 - Tr(\rho_{i_1 i_2 \cdots i_k}^2)}{\text{which has the item } a_1 ^2 a_2 ^2}$	The coefficient of $ a_1 ^2 a_2 ^2$
k = 1	$1 - tr(\rho_1^2), \ 1 - tr(\rho_2^2)$	$2 \cdot 2C_{N-2}^0$
k = 2	$1 - tr(\rho_{13}^2), 1 - tr(\rho_{14}^2), \cdots, 1 - tr(\rho_{1N}^2); 1 - tr(\rho_{23}^2), 1 - tr(\rho_{24}^2), \cdots, 1 - tr(\rho_{2N}^2)$	$2 \cdot 2C_{N-2}^1$
•••		
k = d	$1 - tr(\rho_{134\cdots d}^2), \cdots, 1 - tr(\rho_{1(d+3)\cdots N}^2);$ $1 - tr(\rho_{234\cdots (d+1)}^2), \cdots, 1 - tr(\rho_{2(d+3)\cdots N}^2)$	$2 \cdot 2C_{N-2}^{d-1}$

Table 4: The coefficient of the term $|a_1|^2 |a_2|^2$ (N odd).

we can prove that any item $|a_i|^2 |a_j|^2$ has the coefficient 4. Therefore, $C_N^2(|W\rangle_{A_1A_2\cdots A_N}) = 4\sum_{1\leq i< j\leq N} |a_i|^2 |a_j|^2$. \Box

3 N-partite concurrence of W-Class States based on (N-1)-partite quantum systems

If M = N - 1, under the rearrangement of the sub-systems there are C_N^2 different partitions of an N-partite system: $ij|1|\cdots |\hat{i}|\cdots |\hat{j}|\cdots |N, 1 \leq i < j \leq N$. Denote by $\widetilde{C}_{N-1}^2(|W_{A_1A_2\cdots A_N}\rangle) = \sum_{\mathcal{P}} \mathcal{C}_{N-1}^2(\rho_{\mathcal{P}})$, where the index \mathcal{P} labels all C_N^2 different (N-1)-partite partitions of the N-partite systems, and $\mathcal{C}_{N-1}^2(\rho_{\mathcal{P}})$ is the (N-1)-partite concurrence with respect to the partition \mathcal{P} .

Theorem 2 For the N-qubit W-class states $\rho_{12\dots N} = |W\rangle_{A_1A_2\dots A_N} \langle W|$, under the partition $12|3|\dots|N$ we have

$$C_{N-1}^2(\rho_{12|3|\cdots|N}) = 4 \sum_{\substack{1 \le i < j \le N\\(i,j) \ne (1,2)}} |a_i|^2 |a_j|^2$$
(16)

for $N \geq 3$.

Proof. i) N > 3 and N is even. From equality (7) we have

$$\mathcal{C}_{N-1}^2(\rho_{12|3|\cdots|N}) = 2^{2-(N-1)}[(2^{N-1}-2) - \sum_{\alpha} Tr(\rho_{\alpha}^2)],$$
(17)

where the index α labels all $2^{N-1} - 2$ non-trivial subsystems of the (N-1)-partite quantum systems $12|3|\cdots|N$, and ρ_{α} are the corresponding reduced density matrices. According to the relation $1 - Tr(\rho_{i_1 \ i_2 \cdots i_k}^2) = 1 - Tr(\rho_{1 \ \cdots \ i_1 \ \cdots \ i_2 \ \cdots \ i_k \ \cdots \ N})$, we have

$$\mathcal{C}_{N-1}^{2}(\rho_{12|3|\cdots|N}) = 2^{3-(N-1)}[(1-tr\rho_{12}^{2}) + \sum_{i=3}^{N}(1-tr\rho_{i}^{2}) + \sum_{i=3}^{N}(1-tr\rho_{12i}^{2}) + \sum_{i=3}^{N}(1-tr\rho_{12i_{1}i_{2}}^{2}) + \sum_{i=3}^{N}(1-tr\rho_{12i_{1}i_{2}i_{2}}^{2}$$

Concerning the number of the terms in the summations of (18), we only need to consider the following non-trivial index α , see **Table 5**. From equality (10), we have that each item of

Details of α	Number of α
12; 3, 4, \cdots , N	$C_{N-1}^1 = C_{N-2}^0 + C_{N-2}^1$
$123, 124, \cdots, 12N; \ 34, 35, \cdots, (N-1)N$	$C_{N-1}^2 = C_{N-2}^1 + C_{N-2}^2$
$123\cdots \frac{N}{2}, \cdots, 12(\frac{N}{2}+2)\cdots N$	$C^{\frac{N}{2}-1} - C^{\frac{N}{2}-2} + C^{\frac{N}{2}-1}$
$34\cdots(\frac{N}{2}+1),\cdots,(\frac{N}{2}+2)(\frac{N}{2}+3)\cdots N$	$O_{N-1} = O_{N-2} + O_{N-2}$

Table 5: Non-trivial index α in the equality (18).

 $1 - tr(\rho_{i_1 i_2 \cdots i_k}^2)$ has the form $2|a_i|^2 |a_j|^2$. To compute $\mathcal{C}_{N-1}^2(\rho_{12|3|\cdots|N})$ we just need to determine the coefficients c_{ij} of the term $|a_i|^2 |a_j|^2$ for each $1 - tr(\rho_{i_1 i_2 \cdots i_k}^2)$ in equality (18).

As $c_{12} = 0$, we next calculate the coefficients c_{ij} for $1 \le i < j \le N$ and $(i, j) \ne (1, 2)$. Taking the coefficient c_{13} as an example, we can see from **Table 6** that the total coefficient c_{13} of the term $|a_1|^2 |a_3|^2$ is $2^{3-(N-1)}(2 \cdot 2C_{N-3}^0 + 2 \cdot 2C_{N-3}^1 + \cdots + 2 \cdot C_{N-3}^{\frac{N-2}{2}-1})$, which is equal to 4 by **Lemma 2**. Other coefficients c_{ij} , $1 \le i < j \le N$, $(i, j) \ne (1, 2)$, can be calculated in

$1 - Tr(\rho_{\alpha}^2)$ containing the item $ a_1 ^2 a_3 ^2$	The coefficient c_{13}
$1 - Tr(\rho_{12}^2); 1 - Tr(\rho_3^2)$	$2 \cdot 2C_{N-3}^0$
$1 - Tr(\rho_{124}^2), \cdots, 1 - Tr(\rho_{12N}^2)$ $1 - Tr(\rho_{34}^2), \cdots, 1 - Tr(\rho_{3N}^2)$	$2 \cdot 2C^1_{N-3}$
$1 - Tr(\rho_{1245}^2), \cdots, 1 - Tr(\rho_{12(N-1)N}^2)$ $1 - Tr(\rho_{345}^2), \cdots, 1 - Tr(\rho_{3(N-1)N}^2)$	$2 \cdot 2C_{N-3}^2$
	• • •
$ 1 - Tr(\rho_{1245\cdots(\frac{N}{2}+1)}^2), \cdots, 1 - Tr(\rho_{12(\frac{N}{2}+2)\cdots N}^2) $ $1 - Tr(\rho_{345\cdots(\frac{N}{2}+1)}^2), \cdots, 1 - Tr(\rho_{3(\frac{N}{2}+2)\cdots N}^2) $	$2 \cdot 2C_{N-3}^{\frac{N-2}{2}-1}$

Table 6: The coefficients of $|a_1|^2 |a_3|^2$ for even N).

a similar way, which are all equal to 4. Hence we have $C_{N-1}^2(\rho_{12|3|\cdots|N}) = 4 \sum_{\substack{1 \le i < j \le N \\ (i,j) \ne (1,2)}} |a_i|^2 |a_j|^2.$

ii) $N \geq 3$ and N is odd. Similarly we have

$$\mathcal{C}_{N-1}^{2}(\rho_{12|3|\cdots|N}) = 2^{3-(N-1)} [(1 - tr\rho_{12}^{2}) + \sum_{i=3}^{N} (1 - tr\rho_{i}^{2}) + \sum_{i=3}^{N} (1 - tr\rho_{12i}^{2}) + \sum_{i=3}^{N} (1 - tr\rho_{12i_{1}i_{2}}^{2}) + \sum_{i=3}^{N} (1 - tr\rho_{12i_{1}i_{2}}^{2}) + \sum_{i=3}^{N} (1 - tr\rho_{12i_{1}i_{2}\cdots i_{\frac{N-3}{2}}}^{2}) + \sum_{i=3}^{N} (1 - tr\rho_{12i_{2$$

The non-trivial index α we need to consider is shown in **Table 7**.

To compute $C_{N-1}^2(\rho_{12|3|\cdots|N})$ we need to determine the coefficients c_{ij} of the term $|a_i|^2 |a_j|^2$ for each $1 - tr(\rho_{i_1i_2\cdots i_k}^2)$ in equality (19). Obviously, $c_{12} = 0$. We still take the coefficient c_{13} as an example. From **Table 8** we have that the total coefficient c_{13} of the term $|a_1|^2 |a_3|^2$ is $2^{3-(N-1)}(2 \cdot 2C_{N-3}^0 + 2 \cdot 2C_{N-3}^1 + \cdots + 2 \cdot 2C_{N-3}^{\frac{N-1}{2}-2} + 2 \cdot C_{N-3}^{\frac{N-1}{2}-1})$, which is equal to 4 by **Lemma 1**. Similarly, we can prove that other coefficients c_{ij} , $1 \le i < j \le N$, $(i, j) \ne (1, 2)$ are all equal to 4. Therefore, $C_{N-1}^2(\rho_{12|3|\cdots|N}) = 4 \sum_{\substack{1 \le i < j \le N \\ (i,j) \ne (1,2)}} |a_i|^2 |a_j|^2$. \Box

Corollary 1 For the N-qubit W-class states $\rho_{12\cdots N} = |W\rangle_{A_1A_2\cdots A_N} \langle W|$, we have

$$\mathcal{C}_{N-1}^{2}(\rho_{ij|1|\cdots|\hat{i}|\cdots|\hat{j}|\cdots|N}) = 4\sum_{\substack{1 \le k < l \le N\\(k,l) \ne (i,j)}} |a_{k}|^{2} |a_{l}|^{2}$$
(20)

Details of α	Number of α
12; $3, 4, \cdots, N$	$C_{N-1}^1 = C_{N-2}^0 + C_{N-2}^1$
$123, 124, \cdots, 12N; 34, 35, \cdots, (N-1)N$	$C_{N-1}^2 = C_{N-2}^1 + C_{N-2}^2$
$123\cdots \frac{N-1}{2}, \cdots, 12(\frac{N-1}{2}+4)\cdots N$	$C^{\frac{N-1}{2}-1} - C^{\frac{N-1}{2}-2} + C^{\frac{N-1}{2}-1}$
$34\cdots(\frac{N-1}{2}+1),\cdots,(\frac{N-1}{2}+2)(\frac{N-1}{2}+3)\cdots N$	$C_{N-1} = C_{N-2} + C_{N-2}$
$123\cdots(\frac{N-1}{2}+1),\cdots,12(\frac{N-1}{2}+3)\cdots N$	$\frac{\frac{1}{2}C_{N-1}^{\frac{N-1}{2}} = C_{N-2}^{\frac{N-1}{2}-1}$

Table 7: Non-trivial index α in the equality (19).

$1 - Tr(\rho_{\alpha}^2)$ which contains the item $ a_1 ^2 a_3 ^2$	The coefficients c_{13}	
$1 - Tr(\rho_{12}^2); 1 - Tr(\rho_3^2)$	$2 \cdot 2C_{N-3}^0$	
$1 - Tr(\rho_{124}^2), \cdots, 1 - Tr(\rho_{12N}^2)$	$2 \cdot 2C^{1}_{N-3}$	
$1 - Tr(\rho_{34}^2), \cdots 1 - Tr(\rho_{3N}^2)$	11 0	
$1 - Tr(\rho_{1245}^2), \dots, 1 - Tr(\rho_{12(N-1)N}^2)$	$2 \cdot 2C_{X}^2$	
$1 - Tr(\rho_{345}^2), \cdots 1 - Tr(\rho_{3(N-1)N}^2)$	2 2 N-3	
	• • •	
$1 - Tr(\rho_{124\cdots(\frac{N-1}{2}+1)}^2), \cdots, 1 - Tr(\rho_{12(\frac{N-1}{2}+4)\cdots N}^2)$	$2 \cdot 2C_{n^2}^{\frac{N-1}{2}-2}$	
$1 - Tr(\rho_{34\cdots(\frac{N-1}{2}+1)}^2), \cdots \ 1 - Tr(\rho_{3(\frac{N-1}{2}+4)\cdots N}^2)$	2 20 N-3	
$1 - Tr(\rho_{124\cdots(\frac{N-1}{2}+2)}^2), \cdots, 1 - Tr(\rho_{12(\frac{N-1}{2}+3)\cdots N}^2)$	$2 \cdot 1C_{N-3}^{\frac{N-1}{2}-1}$	

Table 8: The coefficients of $|a_1|^2 |a_3|^2$ (N odd).

for $N \ge 3$ and $1 \le i < j \le N$,

Based on the above conclusions, the relation between the N-partite concurrence and the (N-1)-partite concurrence for W-class states are given by the following theorem.

Theorem 3 For the N-qubit W-class states $|W\rangle_{A_1A_2\cdots A_N}$, we have

$$C_N^2(|W_{A_1A_2\cdots A_N}\rangle) = \frac{1}{C_N^2 - 1} \widetilde{C}_{N-1}^2(|W_{A_1A_2\cdots A_N}\rangle)$$
(21)

for $N \geq 3$.

Proof. From Corollary 1, we have

$$\widetilde{\mathcal{C}}_{N-1}^{2}(|W_{A_{1}A_{2}\cdots A_{N}}\rangle) = \sum_{\mathcal{P}} \mathcal{C}_{N-1}^{2}(\rho_{\mathcal{P}}) \\ = 4 \cdot (C_{N}^{2} - 1) \sum_{1 \le i < j \le N} |a_{i}|^{2} |a_{j}|^{2},$$

where the index \mathcal{P} labels all C_N^2 different (N-1)-partite partitions of N-partite systems. By **Theorem 1**, we have $\mathcal{C}_N^2(|W_{A_1A_2\cdots A_N}\rangle) = \frac{1}{C_N^2-1}\widetilde{\mathcal{C}}_{N-1}^2(|W_{A_1A_2\cdots A_N}\rangle)$. \Box

4 Generalized results to get lower bound of multipartite concurrence

Theorem 3 gives an explicit expression of the concurrence for pure multipartite Wclass states. Based on **Theorem 3** we can also derive tighter lower bounds of concurrence for mixed multipartite states.

Let us consider the case of N = 4. We first take a look at the 4-qubit W-class states $|W\rangle_{A_1A_2A_3A_4}$ with density matrix $\rho_{1234} = |W\rangle_{A_1A_2A_3A_4}\langle W|$. From equality (14) we have

$$\mathcal{C}_{4}^{2}(|W\rangle_{A_{1}A_{2}A_{3}A_{4}}) = \frac{1}{2} \left[\sum_{i=1}^{4} (1 - tr\rho_{i}^{2}) + \sum_{i=2}^{4} (1 - tr\rho_{1i}^{2})\right].$$
(22)

From equality (18) we get $C_3^2(\rho_{12|3|4}) = (1 - Tr\rho_{12}^2) + (1 - Tr\rho_3^2) + (1 - Tr\rho_4^2)$. Therefore,

$$\widetilde{\mathcal{C}}_{3}^{2}(|W_{A_{1}A_{2}A_{3}A_{4}}\rangle) = 3\sum_{i=1}^{4} (1 - tr\rho_{i}^{2}) + 2\sum_{i=2}^{4} (1 - tr\rho_{1i}^{2}).$$
(23)

Moreover, from **Theorem 3**, we obtain $C_4^2(|W\rangle_{A_1A_2A_3A_4}) = \frac{1}{5}\widetilde{C}_3^2(|W_{A_1A_2A_3A_4}\rangle)$. Hence, for the 4-qubit W-class states $|W\rangle_{A_1A_2A_3A_4}$ we have $\sum_{i=1}^4 (1 - tr\rho_i^2) = \sum_{i=2}^4 (1 - tr\rho_{1i}^2)$. Under such particular properties, we have the following lower bounds for some multipartite mixed states.

Theorem 4 For any 4-partite quantum state $\rho \in H_1 \otimes H_2 \otimes H_3 \otimes H_4$, if $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$ attains the minimal partition of the multipartite concurrence and $\sum_{i=1}^4 (1-tr\rho_i^2) = \sum_{i=2}^4 (1-tr\rho_{1i}^2)$ for any $|\psi_i\rangle$ in above partition, then

$$C_4^2(\rho) \ge \frac{1}{5} \widetilde{C_3}^2(\rho),$$
 (24)

where $\widetilde{C_3}^2(\rho) = C_3^2(\rho_{1|2|34}) + C_3^2(\rho_{1|3|24}) + C_3^2(\rho_{1|4|23}) + C_3^2(\rho_{12|3|4}) + C_3^2(\rho_{13|2|4}) + C_3^2(\rho_{14|2|3}).$

Proof. First consider the pure state $|\psi\rangle \in H_1 \otimes H_2 \otimes H_3 \otimes H_4$ with $\rho = |\psi\rangle\langle\psi|$. From equality (1) we have

$$C_4^2(\rho) = \frac{1}{2} \left(\sum_{i=1}^4 (1 - tr\rho_i^2) + \sum_{i=2}^4 (1 - tr\rho_{1i}^2) \right)$$
(25)

and

$$C_3^2(\rho_{i|j|kl}) = (1 - tr\rho_i^2) + (1 - tr\rho_j^2) + (1 - tr\rho_{kl}^2),$$
(26)

where $\rho_i = Tr_{jkl}(\rho)$, $\rho_j = Tr_{ikl}(\rho)$ and $\rho_{kl} = Tr_{ij}(\rho)$. Since $\sum_{i=1}^4 (1 - tr\rho_i^2) = \sum_{i=2}^4 (1 - tr\rho_{1i}^2)$ we have $C_4^2(\rho) = \sum_{i=1}^4 (1 - tr\rho_i^2)$ and $\widetilde{C_3}^2(\rho) = 5(\sum_{i=1}^4 (1 - tr\rho_i^2))$, hence we get $C_4^2(\rho) = \frac{1}{5}\widetilde{C_3}^2(\rho)$, where $\widetilde{C_3}^2(\rho) = C_3^2(\rho_{1|2|34}) + C_3^2(\rho_{1|3|24}) + C_3^2(\rho_{1|4|23}) + C_3^2(\rho_{12|3|4}) + C_3^2(\rho_{13|2|4}) + C_3^2(\rho_{14|2|3})$.

As a mixed state $\rho = \sum_{i} p_i |\psi_i\rangle \langle \psi_i|$ attains the minimal partition of the multipartite concurrence, and $\sum_{i=1}^{4} (1 - tr\rho_i^2) = \sum_{i=2}^{4} (1 - tr\rho_{1i}^2)$ for any $|\psi_i\rangle$ in above partition, we have

$$\begin{split} C_4^2(\rho) &= (\sum_i p_i C_4(|\psi_i\rangle\langle\psi_i|))^2 \\ &= (\sum_i p_i \sqrt{\frac{1}{5}} (C_3^2((|\psi_i\rangle)_{1|2|34}) + C_3^2((|\psi_i\rangle)_{1|3|24}) + \dots + C_3^2((|\psi_i\rangle)_{14|2|3}))))^2 \\ &\geq (\sum_i p_i \frac{1}{\sqrt{6}} C_3((|\psi_i\rangle)_{1|2|34}))^2 + (\sum_i p_i \frac{1}{\sqrt{5}} C_3((|\psi_i\rangle)_{1|3|24}))^2 + \dots + (\sum_i p_i \frac{1}{\sqrt{5}} C_3((|\psi_i\rangle)_{14|2|3}))^2 \\ &\geq \frac{1}{5} (C_3^2(\rho_{1|2|34}) + C_3^2(\rho_{1|3|24}) + C_3^2(\rho_{1|4|23}) + C_3^2(\rho_{12|3|4}) + C_3^2(\rho_{13|2|4}) + C_3^2(\rho_{14|2|3})), \end{split}$$

where the relation $(\sum_{j} (\sum_{i} x_{ij})^2)^{\frac{1}{2}} \leq \sum_{i} (\sum_{j} x_{ij}^2)^{\frac{1}{2}}$ has been used in first inequality. Therefore, we get (24).

To illustrate our lower bound (24), let us consider the following example.

 $\begin{aligned} & \textbf{Example 1} \quad Consider the 4-qubit W-class state |W\rangle_{A_1A_2A_3A_4} = a_1 |1000\rangle + a_2 |0100\rangle + a_3 |0010\rangle + a_4 |0001\rangle, where \sum_{i=1}^{4} |a_i|^2 = 1. \quad Denote \ \rho = |W\rangle_{A_1A_2A_3A_4} \langle W|. \quad We \ have \ \mathcal{C}_4^2(\rho) = \frac{1}{2}[(1-tr(\rho_1^2)) + (1-tr(\rho_2^2)) + (1-tr(\rho_3^2)) + (1-tr(\rho_4^2)) + (1-tr(\rho_{12}^2)) + (1-tr(\rho_{13}^2)) + (1-tr(\rho_{14}^2))] \ and \ 1-tr(\rho_1^2) = 2|a_1|^2(|a_2|^2 + |a_3|^2 + |a_4|^2). \quad Similarly \ we \ have \ 1-tr(\rho_2^2) = 2|a_2|^2(|a_1|^2 + |a_3|^2 + |a_4|^2), \ 1-tr(\rho_3^2) = 2|a_3|^2(|a_1|^2 + |a_2|^2 + |a_4|^2), \ 1-tr(\rho_4^2) = 2|a_4|^2(|a_1|^2 + |a_2|^2 + |a_3|^2), \ 1-tr(\rho_{12}^2) = 2(|a_1|^2 + |a_2|^2)(|a_3|^2 + |a_4|^2), \ 1-tr(\rho_{13}^2) = 2(|a_1|^2 + |a_3|^2)(|a_2|^2 + |a_4|^2), \ 1-tr(\rho_{13}^2) = 2(|a_1|^2 + |a_3|^2)(|a_2|^2 + |a_4|^2), \ 1-tr(\rho_{14}^2) = 2(|a_1|^2 + |a_4|^2)(|a_2|^2 + |a_3|^2). \end{aligned}$ $We \ easily \ verify \ that \ \sum_{i=1}^{4} (1-tr\rho_i^2) = \ \sum_{i=2}^{4} (1-tr\rho_{1i}^2). \quad Hence \ by \ Theorem \ 4 \ we \ can \ get \ \mathcal{C}_4^2(|W_{A_1A_2A_3A_4}\rangle) = \ \frac{1}{5}\widetilde{\mathcal{C}}_3^2(|W_{A_1A_2A_3A_4}\rangle), \ which \ is \ better \ than \ the \ lower \ bound \ given \ in \ [25]. \end{aligned}$

 $= 4 ((1, A_1A_2A_3A_4)) = 5 \cdot 3 ((1, A_1A_2A_3A_4)), \quad \text{where we could use to we could be all the set of all$

5 Conclusions

The multipartite concurrence plays important roles in quantifying the entanglement of multipartite quantum systems. By taking into account the structures of W-class states we have studied the multipartite concurrence for arbitrary multipartite W-class states in terms of the (N - 1)-partitions of subsystems. We have used the method of permutation and combination to present explicit expressions of multipartite concurrence for arbitrary W-class states. We have shown the relations between the multipartite concurrence and the (N-1)-partite concurrence for W-class states. At last, motivated by the explicit expressions of concurrence for W-class states, we have presented a lower bound of concurrence for a class of four-partite mixed states. Similarly, our approach may be also applied to study the multipartite concurrence for arbitrary N-partite W-class states based on arbitrary Mpartite partition of subsystems, and to give relations between the multipartite concurrence and arbitrary M-partite ($2 \le M \le N-2$) concurrence for W-class states, as well as the corresponding lower bounds of concurrence for more general multipartite mixed states.

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