# Multipartite concurrence of $W$-class states based on sub-partite quantum systems 

Wei Chen ${ }^{1}$, Yanmin Yang ${ }^{2}$, Shao-Ming Fei ${ }^{3,4}$, Zhu-Jun Zheng ${ }^{5}$, Yan-Ling Wang ${ }^{1}$<br>${ }^{1}$ School of Computer Science and Technology, Dongguan University of Technology, Dongguan, 523808, P.R. China<br>${ }^{2}$ Faculty of Science, Kunming University of Science and Technology, Kunming, 650500, P.R. China<br>${ }^{3}$ School of Mathematical Sciences, Capital Normal University, Beijing 100048, P.R. China<br>${ }^{4}$ Max-Planck-Institute for Mathematics in the Sciences, Leipzig 04103, Germany<br>${ }^{5}$ School of Mathematics,<br>South China University of Technology, Guangzhou 510641, P.R. China


#### Abstract

We study the concurrence for arbitrary $N$-partite $W$-class states based on the ( $N-1$ )-partite partitions of subsystems by taking account to the structures of $W$-class states. By using the method of permutation and combination we give analytical formula of concurrence and some elegant relations between the multipartite concurrence and the ( $N-1$ )-partite concurrence for arbitrary multipartite $W$-class states. Applying these relations we present better lower bounds of concurrence for multipartite mixed states. An example is given to demonstrate that our lower bounds can detect more entanglements.


Keywords Multipartite concurrence • W-class states • $(N-1)$-partite partitions - Lower bound of concurrence

## 1 Introduction

Quantum entanglement is a striking feature of quantum physics [1-4] and an essential resource in quantum information processing varying from quantum teleportation [5] and quantum cryptography [6] to dense coding [7]. Due to its variety of usages, quantum entanglement has attracted much attention in recent years [8-14].

[^0]To quantify the entanglement of a state, the concept of entanglement measure has been naturally introduced, such as the entanglement of formation 15 for bipartite quantum systems and concurrence [16] for any multipartite quantum systems. For the two-qubit case, the entanglement of formation is proven to be a monotonically increasing function of the concurrence and an elegant formula for concurrence was derived analytically by Wootters [17]. However, except for bipartite qubit systems and some special symmetric states [18], there have been no explicit analytic formulas of concurrence for arbitrary high-dimensional mixed states, due to the extremizations involved in the computation.

Instead of analytic formulas, some progress has been made toward the analytical lower bounds of concurrence. In [19, 20], the authors presented a lower bound of concurrence by decomposing the joint Hilbert space into many $2 \otimes 2$ and $s \otimes t$-dimensional subspaces, which improve all the known lower bounds of concurrence. Similar nice algorithms and progress have been made towards the lower bounds of concurrence for tripartite quantum systems [21,22] and other multipartite quantum systems [23, 24] based on bipartite partitions of the whole quantum system. The authors in [25] improve the lower bound of concurrence by using tripartite and $M$-partite concurrences of an $N$-partite $(2 \leq M<N)$ systems.

As a particular kind of quantum states, the well-known $W$-class states [26] have been widely studied. In this paper we first study the multipartite concurrence for $W$-class states and derive an analytical formula for pure $W$-class states. Then we investigate the $N$-partite concurrence of $W$-class states based on the $(N-1)$-partite quantum systems and present an elegant relation between among them. Based on the results for the $W$-class states, we derive better lower bounds of concurrence for a class of multipartite mixed states. An example is given to illustrate that our lower bound may detect more entanglements.

## 2 Multipartite concurrence of $W$-class states

We first recall the definition of the multipartite concurrence. Let $H_{i}, i=1, \cdots, N$, be $d_{i}$ dimensional Hilbert spaces. The concurrence of an $N$-partite pure state $|\psi\rangle \in H_{1} \otimes H_{2} \otimes$ $\cdots \otimes H_{N}$ is defined by [27],

$$
\begin{equation*}
\mathcal{C}_{N}(|\psi\rangle)=2^{1-\frac{N}{2}} \sqrt{\left(2^{N}-2\right)-\sum_{\alpha} \operatorname{Tr}\left(\rho_{\alpha}^{2}\right)} \tag{1}
\end{equation*}
$$

where the index $\alpha$ labels all $2^{N}-2$ non-trivial subsystems of the $N$-partite quantum systems and $\rho_{\alpha}$ are the corresponding reduced density matrices. For a mixed multipartite quantum state $\rho=\sum_{i} p_{i}\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right| \in H_{1} \otimes H_{2} \otimes \cdots \otimes H_{N}, p_{i} \geq 0, \sum_{i} p_{i}=1$, the concurrence is given by the convex roof:

$$
\begin{equation*}
\mathcal{C}_{N}(\rho)=\min _{\left\{p_{i}, \mid \psi_{i}>\right\}} \sum_{i} p_{i} \mathcal{C}_{N}\left(\left|\psi_{i}\right\rangle\right) \tag{2}
\end{equation*}
$$

where the minimum is taken over all possible pure state decompositions of $\rho$.
In [23] the authors obtained the lower bounds of multipartite concurrence in terms of the concurrences of bipartite partitioned states of the whole quantum system. For an $N$ partite quantum pure state $|\psi\rangle \in H_{1} \otimes H_{2} \otimes \cdots \otimes H_{N}$, the concurrence of bipartite partition
between the subsystems $12 \cdots M$ and $M+1 \cdots N$ is defined by

$$
\begin{equation*}
\mathcal{C}_{2}(|\psi\rangle\langle\psi|)=\sqrt{2\left(1-\operatorname{Tr}\left(\rho_{12 \cdots M}^{2}\right)\right)} \tag{3}
\end{equation*}
$$

where $\rho_{12 \cdots M}=\operatorname{Tr}_{M+1 \cdots N}\{|\psi\rangle\langle\psi|\}$ is the reduced density matrix of $\rho=|\psi\rangle\langle\psi|$ by tracing over the subsystems $M+1 \cdots N$. For a mixed multipartite quantum state $\rho=\sum_{i} p_{i}\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right| \in$ $H_{1} \otimes H_{2} \otimes \cdots \otimes H_{N}$, the corresponding concurrence $\mathcal{C}_{2}(\rho)$ is given by the convex roof:

$$
\begin{equation*}
\mathcal{C}_{2}(\rho)=\min _{\left\{p_{i},\left|\psi_{i}\right\rangle\right\}} \sum_{i} p_{i} \mathcal{C}_{2}\left(\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right|\right) \tag{4}
\end{equation*}
$$

A relation between the concurrence (2) and the bipartite concurrence (4) has been presented in [23]. For a multipartite quantum state $\rho \in H_{1} \otimes H_{2} \otimes \cdots \otimes H_{N}$ with $N \geq 3$, the following inequality holds,

$$
\begin{equation*}
\mathcal{C}_{N}(\rho) \geq \max 2^{\frac{3-N}{2}} \mathcal{C}_{2}(\rho) \tag{5}
\end{equation*}
$$

where the maximum is taken over all kinds of bipartite concurrences.
In terms of the lower bounds of bipartite concurrence, in [24] further relations between the concurrence (2) and the bipartite concurrence (4) have been derived:

$$
\begin{equation*}
\mathcal{C}_{N}(\rho) \geq \max _{M=1,2, \cdots, N-1}\left\{2^{\frac{1-N}{2}} \sqrt{2^{N-M}+2^{M}-2} \mathcal{C}_{2}\left(\rho_{M}\right)\right\} \tag{6}
\end{equation*}
$$

for $N \geq 3$, where the maximum is taken over all kinds of bipartite concurrences for given M. In particularly, if $N=3$, one has $\mathcal{C}_{3}(\rho) \geq \max \left\{\mathcal{C}_{2}\left(\rho_{1}\right), \mathcal{C}_{2}\left(\rho_{2}\right)\right\}$. If $N=4$, one gets $\mathcal{C}_{4}(\rho) \geq \max \left\{\mathcal{C}_{2}\left(\rho_{1}\right), \frac{\sqrt{3}}{2} \mathcal{C}_{2}\left(\rho_{2}\right), \mathcal{C}_{2}\left(\rho_{3}\right)\right\}$.

In order to improve the lower bounds of concurrence, instead of the bipartite concurrence $\mathcal{C}_{2}(\rho)$, the authors in [22] consider tripartite concurrence $\mathcal{C}_{3}(\rho)$. In [25] the authors improve the lower bound of concurrence by using tripartite and $M$-partite concurrences of an $N$-partite $(M<N)$ system. For an $N$-partite quantum pure state $|\psi\rangle \in H_{1} \otimes H_{2} \otimes \cdots \otimes H_{N}$ $(N \geq 3)$, denote $\left\{i^{1}\right\},\left\{i^{2}\right\}, \cdots,\left\{i^{M_{1}}\right\},\left\{k_{1}^{1}, k_{2}^{1}\right\},\left\{k_{1}^{2}, k_{2}^{2}\right\}, \cdots,\left\{k_{1}^{M_{2}}, k_{2}^{M_{2}}\right\}, \cdots,\left\{q_{1}^{1}, \cdots, q_{j}^{1}\right\}$, $\left\{q_{1}^{2}, \cdots, q_{j}^{2}\right\}, \cdots,\left\{q_{1}^{M_{j}}, \cdots, q_{j}^{M_{j}}\right\}$ as the $M$ decompositions among the subsystems, where $\left\{i^{1}, i^{2}, \cdots, i^{M_{1}}, k_{1}^{1}, k_{2}^{1}, k_{1}^{2}, k_{2}^{2}, \cdots, k_{1}^{M_{2}}, k_{2}^{M_{2}}, \cdots, q_{1}^{1}, \cdots, q_{j}^{1}, \cdots, q_{1}^{M_{j}}, \cdots, q_{j}^{M_{j}}\right\}=\{1,2, \cdots, N\}$ and $\sum_{k=1}^{j} M_{k}=M, \sum_{k=1}^{j} k M_{k}=N$. The concurrence of the $M$-partite decompositions among the above subsysytems is given by

$$
\begin{equation*}
\mathcal{C}_{M}(|\psi\rangle\langle\psi|)=2^{1-\frac{M}{2}} \sqrt{\left(2^{M}-2\right)-\sum_{\alpha} \operatorname{Tr}\left(\rho_{\alpha}^{2}\right)} \tag{7}
\end{equation*}
$$

where $\emptyset \neq \alpha \subsetneq\left\{\left\{i^{1}\right\},\left\{i^{2}\right\}, \cdots,\left\{i^{M_{1}}\right\},\left\{k_{1}^{1}, k_{2}^{1}\right\},\left\{k_{1}^{2}, k_{2}^{2}\right\}, \cdots,\left\{k_{1}^{M_{2}}, k_{2}^{M_{2}}\right\}, \cdots,\left\{q_{1}^{1}, \cdots, q_{j}^{1}\right\}, \cdots\right.$, $\left.\left\{q_{1}^{M_{j}}, \cdots, q_{j}^{M_{j}}\right\}\right\}$ and $\rho_{\alpha}$ are the corresponding reduced density matrices. The rearrangement of the subsystems are implied naturally. Taking $N=4$ and $M=3$, one has six different partitions of the four-partite system: $1|2| 34,1|3| 24,1|4| 23,12|3| 4,13|2| 4$ and $14|2| 3$. In terms of the lower bounds of tripartite concurrence, in [25] the authors derived a relation between the concurrence (2) and the bipartite concurrence (7), $C_{4}^{2}(\rho) \geq{\widetilde{C_{3}}}^{2}(\rho)$, where $\widetilde{C}_{3}^{2}(\rho)=\frac{1}{6}\left(C_{3}^{2}\left(\rho_{1|2| 34}\right)+C_{3}^{2}\left(\rho_{1|3| 24}\right)+C_{3}^{2}\left(\rho_{1|4| 23}\right)+C_{3}^{2}\left(\rho_{12|3| 4}\right)+C_{3}^{2}\left(\rho_{13|2| 4}\right)+C_{3}^{2}\left(\rho_{14|2| 3}\right)\right)$.

In order to improve the above lower bounds, we first consider the $N$-qubit $W$-class states [26],

$$
\begin{equation*}
|W\rangle_{A_{1} A_{2} \cdots A_{N}}=a_{1}|10 \cdots 0\rangle_{A_{1} A_{2} \cdots A_{N}}+a_{2}|01 \cdots 0\rangle_{A_{1} A_{2} \cdots A_{N}}+\cdots+a_{N}|00 \cdots 1\rangle_{A_{1} A_{2} \cdots A_{N}} \tag{8}
\end{equation*}
$$

where $\sum_{i=1}^{N}\left|a_{i}\right|^{2}=1$. Let $\rho \triangleq \rho_{A_{1} A_{2} \cdots A_{N}}=|W\rangle_{A_{1} A_{2} \cdots A_{N}}\langle W|$ and

$$
\rho_{i_{1} i_{2} \cdots i_{k}} \triangleq \rho_{A_{i_{1}} A_{i_{2}} \cdots A_{i_{k}}}=\operatorname{tr}_{A_{1} \cdots \widehat{A_{i_{1}}} \cdots \widehat{A_{i_{2}}} \cdots \widehat{A_{i_{k}}} \cdots A_{N}}\left(|W\rangle_{A_{1} A_{2} \cdots A_{N}}\langle W|\right)
$$

for any $1 \leq i_{1}<i_{2}<\cdots<i_{k} \leq N$. We have

$$
\begin{align*}
\rho_{i_{1} i_{2} \cdots i_{k}}=\left(a_{i_{1}}|10 \cdots 0\rangle+\cdots\right. & \left.+a_{i_{k}}|00 \cdots 1\rangle\right)_{A_{i_{1}} A_{i_{2}} \cdots A_{i_{k}}}\left(a_{i_{1}}^{*}\langle 10 \cdots 0|+a_{i_{k}}^{*}\langle 00 \cdots 1|\right) \\
& +\sum_{j \neq i_{1}, \cdots, i_{k}}\left|a_{j}\right|^{2}|00 \cdots 0\rangle_{A_{1} \cdots \widehat{A_{i_{1}}} \cdots \widehat{A_{i_{2}}} \cdots \widehat{A_{i_{k}}} \cdots A_{N}}\langle 00 \cdots 0|, \tag{9}
\end{align*}
$$

and

$$
\begin{equation*}
1-\operatorname{Tr}\left(\rho_{i_{1} i_{2} \cdots i_{k}}^{2}\right)=2\left(\left|a_{i_{1}}\right|^{2}+\cdots+\left|a_{i_{k}}\right|^{2}\right) \sum_{j \neq i_{1}, \cdots, i_{k}}\left|a_{j}\right|^{2}, \tag{10}
\end{equation*}
$$

where $A_{1} A_{2} \cdots \widehat{A_{i}} \cdots A_{N}=A_{1} A_{2} \cdots A_{i-1} A_{i+1} \cdots A_{N}$. For simplicity, we denote by $\rho_{i_{1} i_{2} \cdots i_{k}}$ the reduced density operator $\rho_{A_{i_{1}} A_{i_{2}} \cdots A_{i_{k}}}$. We have the following lemmas for the $N$-qubit $W$-class states.

Lemma 1 When $N(N>1)$ is even, we have

$$
\begin{equation*}
2 C_{N}^{0}+2 C_{N}^{1}+\cdots+2 C_{N}^{\frac{N}{2}-1}+C_{N}^{\frac{N}{2}}=2^{N} \tag{11}
\end{equation*}
$$

Proof. As $C_{N}^{i}=C_{N}^{N-i}$ for all integer $0 \leq i<N$, we have

$$
\begin{aligned}
& 2 C_{N}^{0}+2 C_{N}^{1}+\cdots+2 C_{N}^{\frac{N}{2}-1}+C_{N}^{\frac{N}{2}} \\
& =C_{N}^{0}+C_{N}^{1}+\cdots+C_{N}^{\frac{N}{2}-1}+C_{N}^{\frac{N}{2}}+C_{N}^{\frac{N}{2}+1}+\cdots+C_{N}^{N} \\
& =(1+1)^{N}=2^{N}
\end{aligned}
$$

Hence, $2 C_{N}^{0}+2 C_{N}^{1}+\cdots+2 C_{N}^{\frac{N}{2}-1}+C_{N-2}^{\frac{N}{2}}=2^{N}$.
Lemma 2 When $N(N \geq 1)$ is odd, we have

$$
\begin{equation*}
2 C_{N}^{0}+2 C_{N}^{1}+\cdots+2 C_{N}^{\frac{N-1}{2}}=2^{N} \tag{12}
\end{equation*}
$$

Proof. As $C_{N}^{i}=C_{N}^{N-i}$ for all integer $0 \leq i<N$, we have

$$
\begin{aligned}
& 2 C_{N}^{0}+2 C_{N}^{1}+\cdots+2 C_{N}^{\frac{N-1}{2}} \\
& =C_{N}^{0}+C_{N}^{1}+\cdots+C_{N}^{\frac{N-1}{2}}+C_{N}^{\frac{N+1}{2}}+C_{N-1}^{N}+C_{N}^{N} \\
& =(1+1)^{N}=2^{N}
\end{aligned}
$$

Hence, $2 C_{N}^{0}+2 C_{N}^{1}+\cdots+2 C_{N}^{\frac{N-1}{2}}=2^{N}$.

Theorem 1 The squared multipartite concurrence of the $N$-qubit $W$-class state $|W\rangle_{A_{1} A_{2} \cdots A_{N}}$ is given by

$$
\begin{equation*}
\mathcal{C}_{N}^{2}\left(|W\rangle_{A_{1} A_{2} \cdots A_{N}}\right)=4 \sum_{1 \leq i<j \leq N}\left|a_{i}\right|^{2}\left|a_{j}\right|^{2} \tag{13}
\end{equation*}
$$

for $N \geq 2$.
Proof. For $N=2$, we have $|W\rangle_{A_{1} A_{2}}=a_{1}|10\rangle_{A_{1} A_{2}}+a_{2}|01\rangle_{A_{1} A_{2}}$ and $\mathcal{C}_{2}\left(|W\rangle_{A_{1} A_{2}}\right)=$ $\sqrt{\left(2^{2}-2\right)-\operatorname{Tr}\left(\rho_{1}^{2}\right)-\operatorname{Tr}\left(\rho_{2}^{2}\right)}$. Then $\mathcal{C}_{2}^{2}\left(|W\rangle_{A_{1} A_{2}}\right)=2-\operatorname{Tr}\left(\rho_{1}^{2}\right)-\operatorname{Tr}\left(\rho_{2}^{2}\right)=4\left|a_{1}\right|^{2}\left|a_{2}\right|^{2}$.
i) $N>3$ and $N$ is even. Denote $d=\frac{N}{2}$. For pure states one has $1-\operatorname{Tr}\left(\rho_{i_{1}}^{2} i_{2} \cdots i_{k}\right)=$


$$
\begin{array}{r}
\quad \mathcal{C}_{N}^{2}\left(|W\rangle_{A_{1} A_{2} \cdots A_{N}}\right)=\frac{1}{2^{N-3}}\left[\sum_{i=1}^{N}\left(1-\operatorname{tr} \rho_{i}^{2}\right)+\sum_{1 \leq i_{1}<i_{2} \leq N}\left(1-\operatorname{tr} \rho_{i_{1} i_{2}}^{2}\right)+\cdots\right. \\
\left.+\sum_{1 \leq i_{1}<i_{2}<\cdots<i_{d-1} \leq N}\left(1-\operatorname{tr} \rho_{i_{1} i_{2} \cdots i_{d-1}}^{2}\right)+\sum_{1<i_{1}<i_{2}<\cdots<i_{d-1} \leq N}\left(1-\operatorname{tr} \rho_{1 i_{1} i_{2} \cdots i_{d-1}}^{2}\right)\right] . \tag{14}
\end{array}
$$

Concerning the numbers of the terms in the summations of (14), we only need to consider the following non-trivial index $\alpha$, see Table [1,

| Type of $\alpha$ | Details of $\alpha$ | Number of $\alpha$ |
| :---: | :---: | :---: |
| with one subsystem | $1,2, \cdots, N$ | $C_{N}^{1}$ |
| with two subsystems | $12,13, \cdots,(N-1) N$ | $C_{N}^{2}$ |
| $\cdots$ | $\cdots$ | $\cdots$ |
| with $d$ subsystems | $12 \cdots d, 13 \cdots(d+1), \cdots, 1(d+2) \cdots N$ | $C_{N-1}^{d-1}=\frac{1}{2} C_{N}^{d}$ |

Table 1: Non-trivial index $\alpha$ in (14).
By equality (10), we have that each item of $1-\operatorname{Tr}\left(\rho_{i_{1} i_{2} \cdots i_{d-1}}^{2}\right)$ has the form $2\left|a_{i}\right|^{2}\left|a_{j}\right|^{2}$. To compute $\mathcal{C}_{N}^{2}\left(|W\rangle_{A_{1} A_{2} \cdots A_{N}}\right)$ we just need to determine the total coefficients of the term $\left|a_{i}\right|^{2}\left|a_{j}\right|^{2}$ for every $1-\operatorname{Tr}\left(\rho_{i_{1} i_{2} \cdots i_{d-1}}^{2}\right)$ in (14). Taking the coefficient of the term $\left|a_{1}\right|^{2}\left|a_{2}\right|^{2}$ as an example, see Table 2, the total coefficient of the term $\left|a_{1}\right|^{2}\left|a_{2}\right|^{2}$ is $\frac{1}{2^{N-3}}\left(2 \cdot 2 C_{N-2}^{0}+2\right.$. $2 C_{N-2}^{1}+\cdots+2 \cdot C_{N-2}^{d-1}$ ), which is equal to 4 by Lemma 1. Similarly, we can prove that any item $\left|a_{i}\right|^{2}\left|a_{j}\right|^{2}$ has the coefficient 4. Hence we have $\mathcal{C}_{N}^{2}\left(|W\rangle_{A_{1} A_{2} \cdots A_{N}}\right)=4 \sum_{1 \leq i<j \leq N}\left|a_{i}\right|^{2}\left|a_{j}\right|^{2}$.
ii) $N \geq 3$ and $N$ is odd. Denote $d=\frac{N-1}{2}$. Similarly we have

$$
\begin{align*}
& \quad \mathcal{C}_{N}^{2}\left(|W\rangle_{A_{1} A_{2} \cdots A_{N}}\right)=\frac{1}{2^{N-3}}\left[\sum_{i=1}^{N}\left(1-\operatorname{tr} \rho_{i}^{2}\right)+\sum_{1 \leq i_{1}<i_{2} \leq N}\left(1-\operatorname{tr} \rho_{i_{1} i_{2}}^{2}\right)+\cdots\right. \\
& \left.+\sum_{1 \leq i_{1}<i_{2}<\cdots<i_{d-1} \leq N}\left(1-\operatorname{tr} \rho_{i_{1} i_{2} \cdots i_{d-1}}^{2}\right)+\sum_{1 \leq i_{1}<i_{2}<\cdots<i_{d} \leq N}\left(1-\operatorname{tr} \rho_{1 i_{1} i_{2} \cdots i_{d-1}}^{2}\right)\right] . \tag{15}
\end{align*}
$$

| Type of $1-\operatorname{Tr}\left(\rho_{i_{1} i_{2} \cdots i_{k}}^{2}\right)$ | $1-\operatorname{Tr}\left(\rho_{i_{1} i_{2} \cdots i_{k}}^{2}\right)$ <br> which has the item $\left\|a_{1}\right\|^{2}\left\|a_{2}\right\|^{2}$ | The coefficient of $\left\|a_{1}\right\|^{2}\left\|a_{2}\right\|^{2}$ |
| :---: | :---: | :---: |
| $k=1$ | $1-\operatorname{tr}\left(\rho_{1}^{2}\right), 1-\operatorname{tr}\left(\rho_{2}^{2}\right)$ | $2 \cdot 2 C_{N-2}^{0}$ |
| $k=2$ | $1-\operatorname{tr}\left(\rho_{13}^{2}\right), 1-\operatorname{tr}\left(\rho_{14}^{2}\right), \cdots, 1-\operatorname{tr}\left(\rho_{1 N}^{2}\right) ;$ <br> $1-\operatorname{tr}\left(\rho_{23}^{2}\right), 1-\operatorname{tr}\left(\rho_{24}^{2}\right), \cdots, 1-\operatorname{tr}\left(\rho_{2 N}^{2}\right)$ | $2 \cdot 2 C_{N-2}^{1}$ |
| $\cdots$ | $\cdots$ | $\cdots$ |
| $k=d-1$ | $1-\operatorname{tr}\left(\rho_{134 \cdots d}^{2}\right), \cdots, 1-\operatorname{tr}\left(\rho_{1(d+3) \cdots N}^{2}\right) ;$ <br> $1-\operatorname{tr}\left(\rho_{234 \cdots d}^{2}\right), \cdots, 1-\operatorname{tr}\left(\rho_{2(d+3) \cdots N}^{2}\right)$ | $2 \cdot 2 C_{N-2}^{d-2}$ |
| $k=d$ | $1-\operatorname{tr}\left(\rho_{134 \cdots(d+1)}^{2}\right), \cdots, 1-\operatorname{tr}\left(\rho_{1(d+2) \cdots N}^{2}\right)$ | $2 \cdot 1 C_{N-2}^{d-1}$ |

Table 2: The coefficients of the term $\left|a_{1}\right|^{2}\left|a_{2}\right|^{2}$ for even N).

| Type of $\alpha$ | Details of $\alpha$ | Number of $\alpha$ |
| :---: | :---: | :---: |
| with one subsystem | $1,2, \cdots, N$ | $C_{N}^{1}$ |
| with two subsystems | $12,13, \cdots,(N-1) N$ | $C_{N}^{2}$ |
| $\cdots$ | $\cdots$ | $\cdots$ |
| with $d$ subsystems | $12 \cdots d, 13 \cdots(d+1), \cdots, 1(d+3) \cdots N, \cdots,(d+2)(d+3) \cdots N$ | $C_{N}^{d}$ |

Table 3: Non-trivial index $\alpha$ in (15).

The non-trivial index $\alpha$ we need to consider is shown in Table 3,
In order to compute $\mathcal{C}_{N}^{2}\left(|W\rangle_{A_{1} A_{2} \cdots A_{N}}\right)$ we need to determine the total coefficient of the terms $\left|a_{i}\right|^{2}\left|a_{j}\right|^{2}$ for every $1-\operatorname{Tr}\left(\rho_{i_{1} i_{2} \cdots i_{d-1}}^{2}\right)$ in (15). Still taking the coefficient of $\left|a_{1}\right|^{2}\left|a_{2}\right|^{2}$ as an example, from Table 4 we have that the total coefficient of the term $\left|a_{1}\right|^{2}\left|a_{2}\right|^{2}$ is $\frac{1}{2^{N-3}}\left(2 \cdot 2 C_{N-2}^{0}+2 \cdot 2 C_{N-2}^{1}+\cdots+2 \cdot 2 C_{N-2}^{d-1}\right)$ which is equal to 4 by Lemma 2. Similarly,

| Type of $1-\operatorname{Tr}\left(\rho_{i_{1} i_{2} \cdots i_{k}}^{2}\right)$ | $1-\operatorname{Tr}\left(\rho_{i_{1} i_{2} \cdots i_{k}}^{2}\right)$ <br> which has the item $\left\|a_{1}\right\|^{2}\left\|a_{2}\right\|^{2}$ | The coefficient of $\left\|a_{1}\right\|^{2}\left\|a_{2}\right\|^{2}$ |
| :---: | :---: | :---: |
| $k=1$ | $1-\operatorname{tr}\left(\rho_{1}^{2}\right), 1-\operatorname{tr}\left(\rho_{2}^{2}\right)$ | $2 \cdot 2 C_{N-2}^{0}$ |
| $k=2$ | $1-\operatorname{tr}\left(\rho_{13}^{2}\right), 1-\operatorname{tr}\left(\rho_{14}^{2}\right), \cdots, 1-\operatorname{tr}\left(\rho_{1 N}^{2}\right) ;$ <br> $1-\operatorname{tr}\left(\rho_{23}^{2}\right), 1-\operatorname{tr}\left(\rho_{24}^{2}\right), \cdots, 1-\operatorname{tr}\left(\rho_{2 N}^{2}\right)$ | $2 \cdot 2 C_{N-2}^{1}$ |
| $\cdots$ | $\cdots$ | $\cdots$ |
| $k=d$ | $1-\operatorname{tr}\left(\rho_{134 \cdots d}^{2}\right), \cdots, 1-\operatorname{tr}\left(\rho_{1(d+3) \cdots N}^{2}\right) ;$ <br> $1-\operatorname{tr}\left(\rho_{234 \cdots(d+1)}^{2}\right), \cdots, 1-\operatorname{tr}\left(\rho_{2(d+3) \cdots N}^{2}\right)$ | $2 \cdot 2 C_{N-2}^{d-1}$ |

Table 4: The coefficient of the term $\left|a_{1}\right|^{2}\left|a_{2}\right|^{2}$ (N odd).
we can prove that any item $\left|a_{i}\right|^{2}\left|a_{j}\right|^{2}$ has the coefficient 4. Therefore, $\mathcal{C}_{N}^{2}\left(|W\rangle_{A_{1} A_{2} \cdots A_{N}}\right)=$ $4 \sum_{1 \leq i<j \leq N}\left|a_{i}\right|^{2}\left|a_{j}\right|^{2}$.

## 3 N-partite concurrence of $W$-Class States based on ( $N-1$ )-partite quantum systems

If $M=N-1$, under the rearrangement of the sub-systems there are $C_{N}^{2}$ different partitions of an $N$-partite system: $i j|1| \cdots|\widehat{i}| \cdots|\widehat{j}| \cdots \mid N, 1 \leq i<j \leq N$. Denote by $\widetilde{\mathcal{C}}_{N-1}^{2}\left(\left|W_{A_{1} A_{2} \cdots A_{N}}\right\rangle\right)=\sum_{\mathcal{P}} \mathcal{C}_{N-1}^{2}\left(\rho_{\mathcal{P}}\right)$, where the index $\mathcal{P}$ labels all $C_{N}^{2}$ different $(N-1)$-partite partitions of the $N$-partite systems, and $\mathcal{C}_{N-1}^{2}\left(\rho_{\mathcal{P}}\right)$ is the $(N-1)$-partite concurrence with respect to the partition $\mathcal{P}$.

Theorem 2 For the $N$-qubit $W$-class states $\rho_{12 \cdots N}=|W\rangle_{A_{1} A_{2} \cdots A_{N}}\langle W|$, under the partition $12|3| \cdots \mid N$ we have

$$
\begin{equation*}
\mathcal{C}_{N-1}^{2}\left(\rho_{12|3| \cdots \mid N}\right)=4 \sum_{\substack{1 \leq i<j \leq N \\(i, j) \neq(1,2)}}\left|a_{i}\right|^{2}\left|a_{j}\right|^{2} \tag{16}
\end{equation*}
$$

for $N \geq 3$.
Proof. i) $N>3$ and $N$ is even. From equality (7) we have

$$
\begin{equation*}
\mathcal{C}_{N-1}^{2}\left(\rho_{12|3| \cdots \mid N}\right)=2^{2-(N-1)}\left[\left(2^{N-1}-2\right)-\sum_{\alpha} \operatorname{Tr}\left(\rho_{\alpha}^{2}\right)\right], \tag{17}
\end{equation*}
$$

where the index $\alpha$ labels all $2^{N-1}-2$ non-trivial subsystems of the $(N-1)$-partite quantum systems $12|3| \cdots \mid N$, and $\rho_{\alpha}$ are the corresponding reduced density matrices. According to the relation $1-\operatorname{Tr}\left(\rho_{i_{1}}^{2} i_{2} \cdots i_{k}\right)=1-\operatorname{Tr}\left(\rho_{1 \cdots \widehat{i_{1}} \cdots \hat{i_{2}} \cdots \widehat{i_{k} \cdots N}}^{2}\right)$, we have

$$
\begin{align*}
\mathcal{C}_{N-1}^{2}\left(\rho_{12|3| \cdots \mid N}\right)= & 2^{3-(N-1)}\left[\left(1-\operatorname{tr} \rho_{12}^{2}\right)+\sum_{i=3}^{N}\left(1-\operatorname{tr} \rho_{i}^{2}\right)+\sum_{i=3}^{N}\left(1-\operatorname{tr} \rho_{12 i}^{2}\right)+\right. \\
& \sum_{3 \leq i_{1}<i_{2} \leq N}\left(1-\operatorname{tr} \rho_{i_{1} i_{2}}^{2}\right)+\cdots+\sum_{3 \leq i_{1}<i_{2}<\cdots<i_{\frac{N}{2}-2} \leq N}\left(1-\operatorname{tr} \rho_{12 i_{1} i_{2} \cdots i_{\frac{N}{2}-2}}^{2}\right) \\
& \left.+\sum_{3 \leq i_{1}<i_{2}<\cdots<i_{\frac{N}{2}-1} \leq N}\left(1-\operatorname{tr} \rho_{i_{1} i_{2} \cdots i_{\frac{N}{2}-1}}^{2}\right)\right] . \tag{18}
\end{align*}
$$

Concerning the number of the terms in the summations of (18), we only need to consider the following non-trivial index $\alpha$, see Table 5. From equality (10), we have that each item of

| Details of $\alpha$ | Number of $\alpha$ |
| :---: | :---: |
| $12 ; 3,4, \cdots, N$ | $C_{N-1}^{1}=C_{N-2}^{0}+C_{N-2}^{1}$ |
| $123,124, \cdots, 12 N ; 34,35, \cdots,(N-1) N$ | $C_{N-1}^{2}=C_{N-2}^{1}+C_{N-2}^{2}$ |
| $\cdots$ | $\cdots$ |
| $123 \cdots \frac{N}{2}, \cdots, 12\left(\frac{N}{2}+2\right) \cdots N$ | $C_{N-1}^{\frac{N}{2}-1}=C_{N-2}^{\frac{N}{2}-2}+C_{N-2}^{\frac{N}{2}-1}$ |
| $34 \cdots\left(\frac{N}{2}+1\right), \cdots,\left(\frac{N}{2}+2\right)\left(\frac{N}{2}+3\right) \cdots N$ |  |

Table 5: Non-trivial index $\alpha$ in the equality (18).
$1-\operatorname{tr}\left(\rho_{i_{1} i_{2} \cdots i_{k}}^{2}\right)$ has the form $2\left|a_{i}\right|^{2}\left|a_{j}\right|^{2}$. To compute $\mathcal{C}_{N-1}^{2}\left(\rho_{12|3| \cdots \mid N}\right)$ we just need to determine the coefficients $c_{i j}$ of the term $\left|a_{i}\right|^{2}\left|a_{j}\right|^{2}$ for each $1-\operatorname{tr}\left(\rho_{i_{1} i_{2} \cdots i_{k}}^{2}\right)$ in equality (18).

As $c_{12}=0$, we next calculate the coefficients $c_{i j}$ for $1 \leq i<j \leq N$ and $(i, j) \neq(1,2)$. Taking the coefficient $c_{13}$ as an example, we can see from Table 6 that the total coefficient $c_{13}$ of the term $\left|a_{1}\right|^{2}\left|a_{3}\right|^{2}$ is $2^{3-(N-1)}\left(2 \cdot 2 C_{N-3}^{0}+2 \cdot 2 C_{N-3}^{1}+\cdots+2 \cdot C_{N-3}^{\frac{N-2}{2}-1}\right)$, which is equal to 4 by Lemma 2. Other coefficients $c_{i j}, 1 \leq i<j \leq N,(i, j) \neq(1,2)$, can be calculated in

| $1-\operatorname{Tr}\left(\rho_{\alpha}^{2}\right)$ containing the item $\left\|a_{1}\right\|^{2}\left\|a_{3}\right\|^{2}$ | The coefficient $c_{13}$ |
| :---: | :---: |
| $1-\operatorname{Tr}\left(\rho_{12}^{2}\right) ; 1-\operatorname{Tr}\left(\rho_{3}^{2}\right)$ | $2 \cdot 2 C_{N-3}^{0}$ |
| $\begin{gathered} 1-\operatorname{Tr}\left(\rho_{124}^{2}\right), \cdots, 1-\operatorname{Tr}\left(\rho_{12 N}^{2}\right) \\ 1-\operatorname{Tr}\left(\rho_{34}^{2}\right), \cdots 1-\operatorname{Tr}\left(\rho_{3 N}^{2}\right) \end{gathered}$ | $2 \cdot 2 C_{N-3}^{1}$ |
| $\begin{gathered} 1-\operatorname{Tr}\left(\rho_{1245}^{2}\right), \cdots, 1-\operatorname{Tr}\left(\rho_{12(N-1) N}^{2}\right) \\ 1-\operatorname{Tr}\left(\rho_{345}^{2}\right), \cdots 1-\operatorname{Tr}\left(\rho_{3(N-1) N}^{2}\right) \end{gathered}$ | $2 \cdot 2 C_{N-3}^{2}$ |
| $\ldots$ | $\ldots$ |
| $\begin{gathered} 1-\operatorname{Tr}\left(\rho_{1245 \cdots\left(\frac{N}{2}+1\right)}^{2}\right), \cdots, 1-\operatorname{Tr}\left(\rho_{12\left(\frac{N}{2}+2\right) \cdots N}^{2}\right) \\ 1-\operatorname{Tr}\left(\rho_{345 \cdots\left(\frac{N}{2}+1\right)}^{2}\right), \cdots 1-\operatorname{Tr}\left(\rho_{3\left(\frac{N}{2}+2\right) \cdots N}^{2}\right) \end{gathered}$ | $2 \cdot 2 C_{N-3}^{\frac{N-2}{2}-1}$ |

Table 6: The coefficients of $\left|a_{1}\right|^{2}\left|a_{3}\right|^{2}$ for even $\left.N\right)$.
a similar way, which are all equal to 4 . Hence we have $\mathcal{C}_{N-1}^{2}\left(\rho_{12|3| \cdots \mid N}\right)=4 \sum_{\substack{1 \leq i<j \leq N \\(i, j) \neq(1,2)}}\left|a_{i}\right|^{2}\left|a_{j}\right|^{2}$.
ii) $N \geq 3$ and $N$ is odd. Similarly we have

$$
\begin{align*}
\mathcal{C}_{N-1}^{2}\left(\rho_{12|3| \cdots \mid N}\right)= & 2^{3-(N-1)}\left[\left(1-\operatorname{tr} \rho_{12}^{2}\right)+\sum_{i=3}^{N}\left(1-\operatorname{tr} \rho_{i}^{2}\right)+\sum_{i=3}^{N}\left(1-\operatorname{tr} \rho_{12 i}^{2}\right)+\right. \\
& \sum_{3 \leq i_{1}<i_{2} \leq N}\left(1-\operatorname{tr} \rho_{i_{1} i_{2}}^{2}\right)+\cdots+\sum_{3 \leq i_{1}<i_{2}<\cdots<i_{\frac{N-3}{2}} \leq N}\left(1-\operatorname{tr} \rho_{12 i_{1} i_{2} \cdots i_{\frac{N-3}{2}}^{2}}\right) \\
& \left.+\sum_{3 \leq i_{1}<i_{2}<\cdots<i_{\frac{N-1}{2}} \leq N}\left(1-\operatorname{tr} \rho_{i_{1} i_{2} \cdots i_{\frac{N-1}{2}}^{2}}^{2}\right)\right] . \tag{19}
\end{align*}
$$

The non-trivial index $\alpha$ we need to consider is shown in Table 7.
To compute $\mathcal{C}_{N-1}^{2}\left(\rho_{12|3| \cdots \mid N}\right)$ we need to determine the coefficients $c_{i j}$ of the term $\left|a_{i}\right|^{2}\left|a_{j}\right|^{2}$ for each $1-\operatorname{tr}\left(\rho_{i_{1} i_{2} \cdots i_{k}}^{2}\right)$ in equality (19). Obviously, $c_{12}=0$. We still take the coefficient $c_{13}$ as an example. From Table 8 we have that the total coefficient $c_{13}$ of the term $\left|a_{1}\right|^{2}\left|a_{3}\right|^{2}$ is $2^{3-(N-1)}\left(2 \cdot 2 C_{N-3}^{0}+2 \cdot 2 C_{N-3}^{1}+\cdots+2 \cdot 2 C_{N-3}^{\frac{N-1}{2}-2}+2 \cdot C_{N-3}^{\frac{N-1}{2}-1}\right)$, which is equal to 4 by Lemma 1 . Similarly, we can prove that other coefficients $c_{i j}, 1 \leq i<j \leq N,(i, j) \neq(1,2)$ are all equal to 4. Therefore, $\mathcal{C}_{N-1}^{2}\left(\rho_{12|3| \cdots \mid N}\right)=4 \sum_{\substack{1 \leq i<j \leq N \\(i, j) \neq(1,2)}}\left|a_{i}\right|^{2}\left|a_{j}\right|^{2}$.
Corollary 1 For the $N$-qubit $W$-class states $\rho_{12 \cdots N}=|W\rangle_{A_{1} A_{2} \cdots A_{N}}\langle W|$, we have

$$
\begin{equation*}
\mathcal{C}_{N-1}^{2}\left(\rho_{i j|1| \cdots|\hat{i}| \cdots|\hat{j}| \cdots \mid N}\right)=4 \sum_{\substack{1 \leq k<l \leq N \\(k, l) \neq(i, j)}}\left|a_{k}\right|^{2}\left|a_{l}\right|^{2} \tag{20}
\end{equation*}
$$

| Details of $\alpha$ | Number of $\alpha$ |
| :---: | :---: |
| $12 ; 3,4, \cdots, N$ | $C_{N-1}^{1}=C_{N-2}^{0}+C_{N-2}^{1}$ |
| $123,124, \cdots, 12 N ; 34,35, \cdots,(N-1) N$ | $C_{N-1}^{2}=C_{N-2}^{1}+C_{N-2}^{2}$ |
| $\cdots$ | $\cdots$ |
| $123 \cdots \frac{N-1}{2}, \cdots, 12\left(\frac{N-1}{2}+4\right) \cdots N$ | $C_{N-1}^{\frac{N-1}{2}-1}=C_{N-2}^{\frac{N-1}{2}-2}+C_{N-2}^{\frac{N-1}{2}-1}$ |
| $34 \cdots\left(\frac{N-1}{2}+1\right), \cdots,\left(\frac{N-1}{2}+2\right)\left(\frac{N-1}{2}+3\right) \cdots N$ |  |
| $123 \cdots\left(\frac{N-1}{2}+1\right), \cdots, 12\left(\frac{N-1}{2}+3\right) \cdots N$ | $\frac{1}{2} C_{N-1}^{\frac{N-1}{2}}=C_{N-2}^{\frac{N-1}{2}-1}$ |

Table 7: Non-trivial index $\alpha$ in the equality (19).

| $1-\operatorname{Tr}\left(\rho_{\alpha}^{2}\right)$ which contains the item $\left\|a_{1}\right\|^{2}\left\|a_{3}\right\|^{2}$ | The coefficients $c_{13}$ |
| :---: | :---: |
| $1-\operatorname{Tr}\left(\rho_{12}^{2}\right) ; 1-\operatorname{Tr}\left(\rho_{3}^{2}\right)$ | $2 \cdot 2 C_{N-3}^{0}$ |
| $\begin{gathered} 1-\operatorname{Tr}\left(\rho_{124}^{2}\right), \cdots, 1-\operatorname{Tr}\left(\rho_{12 N}^{2}\right) \\ 1-\operatorname{Tr}\left(\rho_{34}^{2}\right), \cdots 1-\operatorname{Tr}\left(\rho_{3 N}^{2}\right) \end{gathered}$ | $2 \cdot 2 C_{N-3}^{1}$ |
| $\begin{gathered} 1-\operatorname{Tr}\left(\rho_{1245}^{2}\right), \cdots, 1-\operatorname{Tr}\left(\rho_{12(N-1) N}^{2}\right) \\ 1-\operatorname{Tr}\left(\rho_{345}^{2}\right), \cdots 1-\operatorname{Tr}\left(\rho_{3(N-1) N}^{2}\right) \end{gathered}$ | $2 \cdot 2 C_{N-3}^{2}$ |
| $\ldots$ | $\ldots$ |
| $\begin{gathered} 1-\operatorname{Tr}\left(\rho_{124 \cdots\left(\frac{N-1}{2}+1\right)}^{2}\right), \cdots, 1-\operatorname{Tr}\left(\rho_{12\left(\frac{N-1}{2}+4\right) \cdots N}^{2}\right) \\ 1-\operatorname{Tr}\left(\rho_{34 \cdots\left(\frac{N-1}{2}+1\right)}^{2}\right), \cdots 1-\operatorname{Tr}\left(\rho_{3\left(\frac{N-1}{2}+4\right) \cdots N}^{2}\right) \end{gathered}$ | $2 \cdot 2 C_{N-3}^{\frac{N-1}{2}-2}$ |
| $1-\operatorname{Tr}\left(\rho_{124 \cdots\left(\frac{N-1}{2}+2\right)}^{2}\right), \cdots, 1-\operatorname{Tr}\left(\rho_{12\left(\frac{N-1}{2}+3\right) \cdots N}^{2}\right)$ | $2 \cdot 1 C_{N-3}^{\frac{N-1}{2}-1}$ |

Table 8: The coefficients of $\left|a_{1}\right|^{2}\left|a_{3}\right|^{2}$ ( $N$ odd).
for $N \geq 3$ and $1 \leq i<j \leq N$,
Based on the above conclusions, the relation between the $N$-partite concurrence and the $(N-1)$-partite concurrence for $W$-class states are given by the following theorem.

Theorem 3 For the $N$-qubit $W$-class states $|W\rangle_{A_{1} A_{2} \cdots A_{N}}$, we have

$$
\begin{equation*}
\mathcal{C}_{N}^{2}\left(\left|W_{A_{1} A_{2} \cdots A_{N}}\right\rangle\right)=\frac{1}{C_{N}^{2}-1} \widetilde{\mathcal{C}}_{N-1}^{2}\left(\left|W_{A_{1} A_{2} \cdots A_{N}}\right\rangle\right) \tag{21}
\end{equation*}
$$

for $N \geq 3$.
Proof. From Corollary 1, we have

$$
\begin{aligned}
\widetilde{\mathcal{C}}_{N-1}^{2}\left(\left|W_{A_{1} A_{2} \cdots A_{N}}\right\rangle\right) & =\sum_{\mathcal{P}} \mathcal{C}_{N-1}^{2}\left(\rho_{\mathcal{P}}\right) \\
& =4 \cdot\left(C_{N}^{2}-1\right) \sum_{1 \leq i<j \leq N}\left|a_{i}\right|^{2}\left|a_{j}\right|^{2}
\end{aligned}
$$

where the index $\mathcal{P}$ labels all $C_{N}^{2}$ different $(N-1)$-partite partitions of $N$-partite systems. By Theorem 1, we have $\mathcal{C}_{N}^{2}\left(\left|W_{A_{1} A_{2} \cdots A_{N}}\right\rangle\right)=\frac{1}{C_{N}^{2}-1} \widetilde{\mathcal{C}}_{N-1}^{2}\left(\left|W_{A_{1} A_{2} \cdots A_{N}}\right\rangle\right)$.

## 4 Generalized results to get lower bound of multipartite concurrence

Theorem 3 gives an explicit expression of the concurrence for pure multipartite $W$ class states. Based on Theorem 3 we can also derive tighter lower bounds of concurrence for mixed multipartite states.

Let us consider the case of $N=4$. We first take a look at the 4 -qubit $W$-class states $|W\rangle_{A_{1} A_{2} A_{3} A_{4}}$ with density matrix $\rho_{1234}=|W\rangle_{A_{1} A_{2} A_{3} A_{4}}\langle W|$. From equality (14) we have

$$
\begin{equation*}
\mathcal{C}_{4}^{2}\left(|W\rangle_{A_{1} A_{2} A_{3} A_{4}}\right)=\frac{1}{2}\left[\sum_{i=1}^{4}\left(1-\operatorname{tr} \rho_{i}^{2}\right)+\sum_{i=2}^{4}\left(1-\operatorname{tr} \rho_{1 i}^{2}\right)\right] . \tag{22}
\end{equation*}
$$

From equality (18) we get $\mathcal{C}_{3}^{2}\left(\rho_{12|3| 4}\right)=\left(1-\operatorname{Tr} \rho_{12}^{2}\right)+\left(1-\operatorname{Tr} \rho_{3}^{2}\right)+\left(1-\operatorname{Tr} \rho_{4}^{2}\right)$. Therefore,

$$
\begin{equation*}
\widetilde{\mathcal{C}}_{3}^{2}\left(\left|W_{A_{1} A_{2} A_{3} A_{4}}\right\rangle\right)=3 \sum_{i=1}^{4}\left(1-\operatorname{tr} \rho_{i}^{2}\right)+2 \sum_{i=2}^{4}\left(1-\operatorname{tr} \rho_{1 i}^{2}\right) . \tag{23}
\end{equation*}
$$

Moreover, from Theorem 3, we obtain $\mathcal{C}_{4}^{2}\left(|W\rangle_{A_{1} A_{2} A_{3} A_{4}}\right)=\frac{1}{5} \widetilde{\mathcal{C}}_{3}^{2}\left(\left|W_{A_{1} A_{2} A_{3} A_{4}}\right\rangle\right)$. Hence, for the 4-qubit $W$-class states $|W\rangle_{A_{1} A_{2} A_{3} A_{4}}$ we have $\sum_{i=1}^{4}\left(1-\operatorname{tr} \rho_{i}^{2}\right)=\sum_{i=2}^{4}\left(1-\operatorname{tr} \rho_{1 i}^{2}\right)$. Under such particular properties, we have the following lower bounds for some multipartite mixed states. Theorem 4 For any 4-partite quantum state $\rho \in H_{1} \otimes H_{2} \otimes H_{3} \otimes H_{4}$, if $\rho=\sum_{i} p_{i}\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right|$ attains the minimal partition of the multipartite concurrence and $\sum_{i=1}^{4}\left(1-\operatorname{tr} \rho_{i}^{2}\right)=\sum_{i=2}^{4}\left(1-\operatorname{tr} \rho_{1 i}^{2}\right)$ for any $\left|\psi_{i}\right\rangle$ in above partition, then

$$
\begin{equation*}
C_{4}^{2}(\rho) \geq \frac{1}{5} \widetilde{C}_{3}^{2}(\rho) \tag{24}
\end{equation*}
$$

where ${\widetilde{C_{3}}}^{2}(\rho)=C_{3}^{2}\left(\rho_{1|2| 34}\right)+C_{3}^{2}\left(\rho_{1|3| 24}\right)+C_{3}^{2}\left(\rho_{1|4| 23}\right)+C_{3}^{2}\left(\rho_{12|3| 4}\right)+C_{3}^{2}\left(\rho_{13|2| 4}\right)+C_{3}^{2}\left(\rho_{14|2| 3}\right)$.
Proof. First consider the pure state $|\psi\rangle \in H_{1} \otimes H_{2} \otimes H_{3} \otimes H_{4}$ with $\rho=|\psi\rangle\langle\psi|$. From equality (1) we have

$$
\begin{equation*}
C_{4}^{2}(\rho)=\frac{1}{2}\left(\sum_{i=1}^{4}\left(1-\operatorname{tr} \rho_{i}^{2}\right)+\sum_{i=2}^{4}\left(1-\operatorname{tr} \rho_{1 i}^{2}\right)\right) \tag{25}
\end{equation*}
$$

and

$$
\begin{equation*}
C_{3}^{2}\left(\rho_{i|j| k l}\right)=\left(1-\operatorname{tr} \rho_{i}^{2}\right)+\left(1-\operatorname{tr} \rho_{j}^{2}\right)+\left(1-\operatorname{tr} \rho_{k l}^{2}\right), \tag{26}
\end{equation*}
$$

where $\rho_{i}=\operatorname{Tr}_{j k l}(\rho), \rho_{j}=\operatorname{Tr}_{i k l}(\rho)$ and $\rho_{k l}=\operatorname{Tr} r_{i j}(\rho)$. Since $\sum_{i=1}^{4}\left(1-\operatorname{tr} \rho_{i}^{2}\right)=\sum_{i=2}^{4}\left(1-\operatorname{tr} \rho_{1 i}^{2}\right)$ we have $C_{4}^{2}(\rho)=\sum_{i=1}^{4}\left(1-\operatorname{tr} \rho_{i}^{2}\right)$ and ${\widetilde{C_{3}}}^{2}(\rho)=5\left(\sum_{i=1}^{4}\left(1-\operatorname{tr} \rho_{i}^{2}\right)\right)$, hence we get $C_{4}^{2}(\rho)=\frac{1}{5}{\widetilde{C_{3}}}^{2}(\rho)$, where $\widetilde{C}_{3}^{2}(\rho)=C_{3}^{2}\left(\rho_{1|2| 34}\right)+C_{3}^{2}\left(\rho_{1|3| 24}\right)+C_{3}^{2}\left(\rho_{1|4| 23}\right)+C_{3}^{2}\left(\rho_{12|3| 4}\right)+C_{3}^{2}\left(\rho_{13|2| 4}\right)+C_{3}^{2}\left(\rho_{14|2| 3}\right)$.

As a mixed state $\rho=\sum_{i} p_{i}\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right|$ attains the minimal partition of the multipartite concurrence, and $\sum_{i=1}^{4}\left(1-\operatorname{tr} \rho_{i}^{2}\right)=\sum_{i=2}^{4}\left(1-\operatorname{tr} \rho_{1 i}^{2}\right)$ for any $\left|\psi_{i}\right\rangle$ in above partition, we have

$$
\begin{aligned}
& C_{4}^{2}(\rho)=\left(\sum_{i} p_{i} C_{4}\left(\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right|\right)\right)^{2} \\
= & \left(\sum_{i} p_{i} \sqrt{\frac{1}{5}\left(C_{3}^{2}\left(\left(\left|\psi_{i}\right\rangle\right)_{1|2| 34}\right)+C_{3}^{2}\left(\left(\left|\psi_{i}\right\rangle\right)_{1|3| 24}\right)+\cdots+C_{3}^{2}\left(\left(\left|\psi_{i}\right\rangle\right)_{14|2| 3}\right)\right)}\right)^{2} \\
\geq & \left(\sum_{i} p_{i} \frac{1}{\sqrt{6}} C_{3}\left(\left(\left|\psi_{i}\right\rangle\right)_{1|2| 34}\right)\right)^{2}+\left(\sum_{i} p_{i} \frac{1}{\sqrt{5}} C_{3}\left(\left(\left|\psi_{i}\right\rangle\right)_{1|3| 24}\right)\right)^{2}+\cdots+\left(\sum_{i} p_{i} \frac{1}{\sqrt{5}} C_{3}\left(\left(\left|\psi_{i}\right\rangle\right)_{14|2| 3}\right)\right)^{2} \\
\geq & \frac{1}{5}\left(C_{3}^{2}\left(\rho_{1|2| 34}\right)+C_{3}^{2}\left(\rho_{1|3| 24}\right)+C_{3}^{2}\left(\rho_{1|4| 23}\right)+C_{3}^{2}\left(\rho_{12|3| 4}\right)+C_{3}^{2}\left(\rho_{13|2| 4}\right)+C_{3}^{2}\left(\rho_{14|2| 3}\right)\right)
\end{aligned}
$$

where the relation $\left(\sum_{j}\left(\sum_{i} x_{i j}\right)^{2}\right)^{\frac{1}{2}} \leq \sum_{i}\left(\sum_{j} x_{i j}^{2}\right)^{\frac{1}{2}}$ has been used in first inequality. Therefore, we get (24).

To illustrate our lower bound (24), let us consider the following example.
Example 1 Consider the 4-qubit $W$-class state $|W\rangle_{A_{1} A_{2} A_{3} A_{4}}=a_{1}|1000\rangle+a_{2}|0100\rangle+a_{3}|0010\rangle+$ $a_{4}|0001\rangle$, where $\sum_{i=1}^{4}\left|a_{i}\right|^{2}=1$. Denote $\rho=|W\rangle_{A_{1} A_{2} A_{3} A_{4}}\langle W|$. We have $\mathcal{C}_{4}^{2}(\rho)=\frac{1}{2}\left[\left(1-\operatorname{tr}\left(\rho_{1}^{2}\right)\right)+\right.$ $\left.\left(1-\operatorname{tr}\left(\rho_{2}^{2}\right)\right)+\left(1-\operatorname{tr}\left(\rho_{3}^{2}\right)\right)+\left(1-\operatorname{tr}\left(\rho_{4}^{2}\right)\right)+\left(1-\operatorname{tr}\left(\rho_{12}^{2}\right)\right)+\left(1-\operatorname{tr}\left(\rho_{13}^{2}\right)\right)+\left(1-\operatorname{tr}\left(\rho_{14}^{2}\right)\right)\right]$ and $1-\operatorname{tr}\left(\rho_{1}^{2}\right)=2\left|a_{1}\right|^{2}\left(\left|a_{2}\right|^{2}+\left|a_{3}\right|^{2}+\left|a_{4}\right|^{2}\right)$. Similarly we have

$$
\begin{gathered}
1-\operatorname{tr}\left(\rho_{2}^{2}\right)=2\left|a_{2}\right|^{2}\left(\left|a_{1}\right|^{2}+\left|a_{3}\right|^{2}+\left|a_{4}\right|^{2}\right), 1-\operatorname{tr}\left(\rho_{3}^{2}\right)=2\left|a_{3}\right|^{2}\left(\left|a_{1}\right|^{2}+\left|a_{2}\right|^{2}+\left|a_{4}\right|^{2}\right) \\
1-\operatorname{tr}\left(\rho_{4}^{2}\right)=2\left|a_{4}\right|^{2}\left(\left|a_{1}\right|^{2}+\left|a_{2}\right|^{2}+\left|a_{3}\right|^{2}\right), 1-\operatorname{tr}\left(\rho_{12}^{2}\right)=2\left(\left|a_{1}\right|^{2}+\left|a_{2}\right|^{2}\right)\left(\left|a_{3}\right|^{2}+\left|a_{4}\right|^{2}\right) \\
1-\operatorname{tr}\left(\rho_{13}^{2}\right)=2\left(\left|a_{1}\right|^{2}+\left|a_{3}\right|^{2}\right)\left(\left|a_{2}\right|^{2}+\left|a_{4}\right|^{2}\right), 1-\operatorname{tr}\left(\rho_{14}^{2}\right)=2\left(\left|a_{1}\right|^{2}+\left|a_{4}\right|^{2}\right)\left(\left|a_{2}\right|^{2}+\left|a_{3}\right|^{2}\right)
\end{gathered}
$$

We easily verify that $\sum_{i=1}^{4}\left(1-\operatorname{tr} \rho_{i}^{2}\right)=\sum_{i=2}^{4}\left(1-\operatorname{tr} \rho_{1 i}^{2}\right)$. Hence by Theorem 4 we can get $\mathcal{C}_{4}^{2}\left(\left|W_{A_{1} A_{2} A_{3} A_{4}}\right\rangle\right)=\frac{1}{5} \widetilde{\mathcal{C}}_{3}^{2}\left(\left|W_{A_{1} A_{2} A_{3} A_{4}}\right\rangle\right)$, which is better than the lower bound given in [25].

## 5 Conclusions

The multipartite concurrence plays important roles in quantifying the entanglement of multipartite quantum systems. By taking into account the structures of $W$-class states we have studied the multipartite concurrence for arbitrary multipartite $W$-class states in terms of the $(N-1)$-partitions of subsystems. We have used the method of permutation and combination to present explicit expressions of multipartite concurrence for arbitrary $W$-class states. We have shown the relations between the multipartite concurrence and the ( $N-1$ )-partite concurrence for $W$-class states. At last, motivated by the explicit expressions of concurrence for $W$-class states, we have presented a lower bound of concurrence for a class of four-partite mixed states. Similarly, our approach may be also applied to study
the multipartite concurrence for arbitrary $N$-partite $W$-class states based on arbitrary $M$ partite partition of subsystems, and to give relations between the multipartite concurrence and arbitrary $M$-partite $(2 \leq M \leq N-2)$ concurrence for $W$-class states, as well as the corresponding lower bounds of concurrence for more general multipartite mixed states.

Acknowledgments This work was supported by the Basic and Applied Basic Research Funding Program of Guangdong Province (Grant No. 2019A1515111097), Yunnan Provincial Research Foundation for Basic Research, China (Grant No. 202001AU070041) and Guangdong Universities' Special Projects in Key Fields of Natural Science (No.2019KZDZX1005), National Natural Science Foundation of China (NSFC) under Grants 12075159 and 12171044, Beijing Natural Science Foundation (Grant No. Z190005), and the Academician Innovation Platform of Hainan Province.

## References

[1] Nielsen, M.A., Chuang, I.L.: Quantum Computation and Quantum Information(Cambridge University Press, Cambridge, 2000).
[2] Einstein, A., Podolsky, B., Rosen, N.: Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?. Phys. Rev. 47, 777(1935).
[3] Osterloh, A., Amico, L., Falci, G., Fazio, R.: Scaling of entanglement close to a quantum phase transition. Nature(London)416, 608(2002).
[4] Werner, R.F.: Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model. Phys. Rev. A 40, 4277(1989).
[5] Bennett, C.H., Brassard, G., Crepeau, C., Jozsa, R., Peres, A.,Wootters,W.K.: Teleporting an unknown quantum state via dual classical and EinsteinCPodolskyCRosen channels. Phys. Rev. Lett. 70, 1895 (1993).
[6] Ekert, A.K.: Quantum cryptography based on Bells theorem. Phys. Rev. Lett. 67, 661 (1991).
[7] Bennett, C.H., Wiesner, S.J.: Communication via one- and two-particle operators on EinsteinC PodolskyCRosen states. Phys. Rev. Lett. 69, 2881 (1992)
[8] Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K.: Quantum entanglement. Rev. Mod. Phys. 81, 865(2009).
[9] Mintert, F., Kuś, M., Buchleitner, A.: Concurrence of mixed bipartite quantum states in arbitrary dimensions. Phys. Rev. Lett. 92, 167902(2004);
Mintert, F.: Measures and dynamics of entangled states, Ph.D. thesis, Munich University, Munich, 2004.
[10] Chen, K., Albeverio, S., Fei, S.M.: Concurrence of arbitrary dimensional bipartite quantum states. Phys. Rev. Lett. 95, 040504(2005).
[11] Breuer, H.P.: Separability criteria and bounds for entanglement measures. J. Phys. A: Math. Gen. 39, 11847 (2006).
[12] Breuer, H.P.: Optimal entanglement criterion for mixed quantum states. Phys. Rev. Lett. 97, 080501(2006).
[13] de Vicente, J.I.: Lower bounds on concurrence and separability conditions. Phys. Rev. A $75,052320(2007)$.
[14] Zhang, C.J., Zhang, Y.S., Zhang, S., Guo, G.C.: Optimal entanglement witness based on local orthogonal observables. Phys. Rev. 76, 012334(2007).

Fuchs, C.A., Gisin, N., Griffiths, R.B., Niu, C.S., Peres, A.: Optimal eavesdropping in quantum cryptography. I. Information bound and optimal strategy. Phys. Rev. A 56, 1163(1997).
[15] Bennett, C.H., DiVincenzo, D.P., Smolin, J.A., Wootters, W.K.: Mixed-state entanglement and quantum error correction. Phys. Rev. A 54,3824(1996);
Plenio M.B., Virmani, S.: An introduction to entanglement measures. Quant. Inf. Comput. 7, 1(2007).
[16] Uhlmann, A.: Fidelity and concurrence of conjugated states. Phys. Rev. A 62, 032307(2000);
Rungta, P., Bužek, V., Caves, C.M., Hillery, M., Milburn, G.J.: Universal state inversion and concurrence in arbitrary dimensions. Phys. Rev. A 64, 042315(2001);
Albeverio S., Fei, S.M.: A note on invariants and entanglements. J. Opt. B: Quantum Semiclassical Opt. 3, 223(2001).
[17] Wootters, W.K.: Entanglement of formation of an arbitrary state of two qubits. Phys. Rev. Lett. 80, 2245(1998).
[18] Terhal, B.M., Vollbrecht, K.G.H.: Entanglement of formation for isotropic states. Phys. Rev. Lett. 85, 2625(2000)
Fei, S.M., Jost, J., Li-Jost, X.Q., Wang, G.F.: Entanglement of formation for a class of quantum states. Phys. Lett. A 310, 333(2003);
Rungta P., Caves, C.M.: Concurrence-based entanglement measures for isotropic states. Phys. Rev. A 67, 012307(2003);
Fei, S.M., Li-Jost, X.Q.: A class of special matrices and quantum entanglement. Rep. Math. Phys. 53, 195(2004);
Fei, S.M., Wang, Z.X., Zhao, H.: A note on entanglement of formation and generalized concurrence. Phys. Lett. A 329, 414(2004).
[19] Ou, Y.C., Fan, H., Fei, S.M.: Proper monogamy inequality for arbitrary pure quantum states. Phys. Rev. A 78, 012311(2008).
[20] Zhao, M.J., Zhu, X.N., Fei, S.M., Li-Jost, X.Q.: Lower bound on concurrence and distillation for arbitrary-dimensional bipartite quantum states. Phys. Rev. A 84, 062322(2011).
[21] Zhu, X.N.,Zhao, M.J., Fei, S.M.: Lower bound of multipartite concurrence based on subquantum state decomposition. Phys. Rev. A 86, 022307(2012).
[22] Chen, W., Fei, S.M., Zheng, Z.J.: Lower bound on concurrence for arbitrary-dimensional tripartite quantum states. Quantum Inform. Processing 15, 3761-3771(2016).
[23] Li, M., Fei, S.M., Wang, Z.X.: Bounds for multipartite concurrence. Rep. Math. Phys. 65, 289-296(2010).
[24] Zhu, X.N., Li, M., Fei, S.M.: Lower bounds of concurrence for multipartite states. Aip Conf. Proc. Advances in Quan. Theory, 1424(2012).
[25] Chen, W., Zhu, X.N., Fei, S.M., Zheng, Z.J.: Lower bound of multipartite concurrence based on sub-partite quantum systems. Quantum Inform. Processing 16, 288(2017).
[26] J. S. Kim, and B. C. Sanders, J. Phys. A: Math. Theor. 41, 495301(2008).
[27] Aolita L., Mintert, F.: Measuring Multipartite Concurrence with a Single Factorizable Observable. Phys. Rev. Lett. 97, 050501(2006);
Carvalho, A.R.R., Mintert, F., Buchleitner, A.: Decoherence and Multipartite Entanglement. Phys. Rev. Lett. 93, 230501(2004).
[28] Zhu, X.N., Fei, S.M.: Lower bound of concurrence for qubit systems. Quantum Inform. Processing 13, 815-823(2014).
[29] Qi, X.F., Gao, T., Yan, F.l.: Lower bounds of concurrence for $N$-qubit systems and the detection of $k$-nonseparability of multipartite quantum systems. arXiv:1605.05000(2016).
[30] Kiesel, N., Schmid, C., Tóth, G., Solano, E., Weinfurter, H.: Experimental Observation of Four-Photon Entangled Dicke State with High Fidelity. Phys. Rev. Lett. 98, 063604(2007).


[^0]:    *e-mail: ym.yang@kust.edu.cn

