# How Much Entanglement Does a Quantum Code Need? 

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In the setting of entanglement-assisted quantum error-correcting codes (EAQECCs), the sender and the receiver have access to preshared entanglement. Such codes promise better information rates or improved error handling properties. Entanglement incurs costs and must be judiciously calibrated in designing quantum codes with good performance, relative to their deployment parameters.

Revisiting known constructions, we devise tools from classical coding theory to better understand how the amount of entanglement can be varied. We present three new propagation rules and discuss how each of them affects the error handling. Tables listing the parameters of the best performing qubit and qutrit EAQECCs that we can explicitly construct are supplied for reference and comparison.

## 1 Introduction

In quantum communication, the goal is to send as much quantum information as possible over a noisy quantum channel using a fixed number of quantum bits (qubits) or higher-dimensional systems (qudits). One aims at optimizing the transmission rate so that it approaches the channel capacity asymptotically. The communicating parties are assumed to be physically separated, but they might have access to additional resources, which may include access to classical communication channels, pre-shared randomness, and pre-shared entanglement. We focus on the use of entanglement in the design of quantum error-correcting codes (QECCs) to boost either their communication rates or error-control capabilities.

### 1.1 Quantum Codes

A general quantum error-correcting code for qubits that does not use additional resources is a $K$ -

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dimensional subspace of the complex Hilbert space of $n$ qubits, which has dimension $2^{n}$. We use the notation $\llbracket n, k \rrbracket_{2}$ for a code $\mathcal{Q}$ of dimension $K=2^{k}$ and say that $\mathcal{Q}$ encodes $k$ logical qubits into $n$ physical qubits. The encoding operation consists of two steps. First, the sender appends $n-k$ ancilla qubits in a fixed state, typically $|0\rangle^{\otimes(n-k)}$, to a state $|\varphi\rangle$ of $k$ qubits. Then an encoding unitary $U_{\text {enc }}$ is applied:

$$
\begin{align*}
|\varphi\rangle & \mapsto|\varphi\rangle \otimes|0\rangle^{\otimes(n-k)} \\
& \mapsto\left|\Psi_{L}\right\rangle:=U_{\mathrm{enc}}\left(|\varphi\rangle \otimes|0\rangle^{\otimes(n-k)}\right) \tag{1}
\end{align*}
$$

The state $\left|\Psi_{L}\right\rangle$ is called the encoded state or the logical state. The unitary $U_{\text {enc }}$ acts on the space formed by the input state $|\varphi\rangle$ and the ancillas $|0\rangle^{\otimes(n-k)}$.

In an entanglement-assisted setup [5], we replace $c$ of the ancillas by $c$ pairs of maximally entangled qubits (ebits in short), before applying $U_{\text {enc }}$. Let the state $\left|\Psi^{+}\right\rangle_{A B}$ be an EPR pair [14] shared between the sender Alice and the receiver Bob. The encoding operation performed on Alice's qubits is given by

$$
\begin{align*}
|\varphi\rangle & \mapsto|\varphi\rangle \otimes|0\rangle^{\otimes(n-k-c)} \otimes\left|\Psi^{+}\right\rangle_{A B}^{\otimes c} \\
& \mapsto\left(U_{\mathrm{enc}} \otimes I_{B}\right)\left(|\varphi\rangle \otimes|0\rangle^{\otimes(n-k-c)} \otimes\left|\Psi^{+}\right\rangle_{A B}^{\otimes c}\right) . \tag{2}
\end{align*}
$$

Here $I_{B}$ denotes the identity operator on the receiver's half of the $c$ maximally entangled pairs, i.e., the encoded state consists of $n+c$ qubits in total. The notation $\left|\Psi^{+}\right\rangle_{A B}^{\otimes c}$ should be understood as reordering the qubits such that the first $c$ qubits are with the sender Alice. We assume that the noise only affects the first $n$ qubits sent over the channel, while the half of the shared $c$ ebits that is with the receiver Bob is not affected.

The ebits are prepared ahead of time and are assumed to be error-free, using, e. g., entanglement distillation or a similar procedure. For more details on quantum entanglement, including its creation and delivery across quantum networks, one can consult the survey in [25], the experiment reported in [29], or a recent scheme to generate genuine multipartite entanglement of a large number of qubits in [37].

In teleportation, one perfect ebit is used in tandem with noiseless classical communication to perfectly


Figure 1: The basic structure of an entanglement-assisted quantum error-correcting code. Alice and Bob share the maximally entangled state $|\Phi\rangle_{A B}$ of $c$ maximally entangled qudits. Alice uses her half of the maximally entangled state and $n-k-c$ ancillas in a fixed state $|0\rangle$ in the encoding of her quantum information $|\varphi\rangle$ by the operator $U:=U_{\text {enc }}$. The resulting $q^{n}$-dimensional state passes through the noisy channel. Bob's half of the initial $c$ pairs of maximally entangled states is assumed to be error-free. They will be used in error diagnosis and recovery.
transmit one qubit. Entanglement-assisted quantum error-correcting codes (EAQECCs) use some noiseless ebits, without classical communication, to transmit quantum information over a noisy quantum channel.

In superdense coding, the sender can apply an operation to her half of a pair of maximally entangled states such that, after sending this qubit, the receiver can decode two classical bits of information. An ancilla in a standard QECC can be interpreted as a placeholder for one bit of classical information about any error that has occurred. Replacing an ancilla with one half of an ebit can, in theory, enable the receiver to extract two bits of classical information regarding the errors. This enhances the error-handling capabilities of EAQECCs over the standard counterparts.

One can execute some communication tasks with fewer total resources or better error control by using an EAQECC instead of a combination of a standard QECC and teleportation. An asymptotic analysis on the benefits of pre-shared entanglement in quantum communication is available in [13]. We view an EAQECC as a finite-length realization. It is in principle possible to approach the entanglement-assisted quantum capacity by building larger code blocks as shown in [4].

Shared entanglement does not come for free. The cost of sharing and purifying ebits means that EAQECCs do not automatically outperform standard quantum codes in all circumstances. One measure to assess the advantage is the net rate, which subtracts the number of ebits required from the number of qubits transmitted. In terms of construction via classical error-correcting codes, however, EAQECCs have fewer restrictions, allowing us to use larger families of classical codes.

The notation $\llbracket n, \kappa, \delta ; c \rrbracket_{q}$ signifies that the quantum code $\mathcal{Q}$ is a $q$-ary EAQECC that encodes $\kappa \log$ ical qudits (quantum systems of dimension $q$ ) into $n$
physical qudits, with the help of $n-\kappa-c$ ancillas and $c$ pairs of maximally entangled qudits. A quantum code with minimum distance $\delta$ can correct up to $\lfloor(\delta-1) / 2\rfloor$ single-qudit errors. As shown in Figure 1, Alice transmits the $n$ qudits to Bob. He then performs a syndrome measurement on them together with his half of the $c$ pairs of maximally entangled qudits to correct errors and to retrieve the $\kappa$ logical qudits. The rate $\rho$ and the net rate $\bar{\rho}$ of $\mathcal{Q}$ are, respectively,

$$
\begin{equation*}
\rho:=\frac{\kappa}{n} \quad \text { and } \quad \bar{\rho}:=\frac{\kappa-c}{n} . \tag{3}
\end{equation*}
$$

The abbreviated form $\llbracket n, \kappa, \delta \rrbracket_{q}$ is used when $c=0$.

### 1.2 Quantum Codes from Classical Codes

Let $p$ be a prime and let $s$ be a positive integer. Let $q$ be a prime power $q=p^{s}$ and let $\mathbb{F}_{q}$ be the finite field with $q$ elements. The multiplicative group of the nonzero elements of $\mathbb{F}_{q}$ is denoted by $\mathbb{F}_{q}^{*}$. For a positive integer $m$, we denote by $[m]$ the set $\{1,2, \ldots, m\}$. A code $\mathcal{C}$ of length $n$ is a nonempty subset of $\mathbb{F}_{q}^{n}$. Its codewords are vectors of length $n$ with entries from $\mathbb{F}_{q}$. The weight of a vector is the number of its nonzero entries. Given a nonempty $\mathcal{S} \subseteq \mathbb{F}_{q}^{n}$, we denote by $\mathrm{wt}(\mathcal{S})$ the number $\min \{\mathrm{wt}(\mathbf{v}): \mathbf{v} \in \mathcal{S}, \mathbf{v} \neq \mathbf{0}\}$.

A code $\mathcal{C}$ is linear with parameters $[n, k, d]_{q}$ if it is a $k$-dimensional subspace of $\mathbb{F}_{q}^{n}$ and its minimum distance, defined to be the smallest of the weights of its nonzero codewords, is $d$. A $k \times n$ matrix $G$ whose rows form a basis for $\mathcal{C}$ is a generator matrix of $\mathcal{C}$. If $G=\left(I_{k} A\right)$, where $I_{k}$ is the $k \times k$ identity matrix, we say that $G$ is in the standard form.

We equip $\mathbb{F}_{q}^{n}$ and $\mathbb{F}_{q^{2}}^{n}$ with the Euclidean and the Hermitian inner products, respectively. Given an arbitrary vector $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right)$ and a codeword $\mathbf{c}=\left(c_{1}, \ldots, c_{n}\right)$ in $\mathcal{C}$, the Euclidean dual $\mathcal{C}^{\perp}$ of $\mathcal{C}$ is

$$
\begin{equation*}
\mathcal{C}^{\perp}=\left\{\mathbf{x} \in \mathbb{F}_{q}^{n}: \sum_{i=1}^{n} x_{i} c_{i}=0, \text { for all } \mathbf{c} \in \mathcal{C}\right\} \tag{4}
\end{equation*}
$$

Analogously, the Hermitian dual $\mathcal{C}^{\perp_{\mathrm{H}}}$ of $\mathcal{C}$ is

$$
\begin{equation*}
\mathcal{C}^{\perp_{\mathrm{H}}}=\left\{\mathbf{x} \in \mathbb{F}_{q^{2}}^{n}: \sum_{i=1}^{n} x_{i} c_{i}^{q}=0, \text { for all } \mathbf{c} \in \mathcal{C}\right\} \tag{5}
\end{equation*}
$$

An $[n, k, d]_{q}$-code with $k \leq\left\lfloor\frac{n}{2}\right\rfloor$ is self-orthogonal if it is contained in its dual. If, moreover, $n=2 k$, then the code is self-dual. The notion of the hull of a code was introduced in [2] to define the intersection of the code with its dual. Hence, the Hermitian hull of $\mathcal{C}$ is the code $\operatorname{Hull}_{\mathrm{H}}(\mathcal{C})=\mathcal{C} \cap \mathcal{C}^{\perp_{\mathrm{H}}}$. A code whose hull is $\{\mathbf{0}\}$ intersects trivially with its dual and is called a linear complementary dual (LCD) code. Readers interested to know more about classical codes may consult [28].

Gottesman formulated the stabilizer formalism for QECCs in [19]. It was subsequently expressed in the language of classical coding theory in [8], triggering
fruitful cross-pollination of ideas and results between quantum error control and classical coding theory. A general treatment over any finite field followed in [1]. A survey can be found in [30]. The main ingredients are self-orthogonal classical (additive) codes under the (trace) Hermitian inner product. The orthogonality condition imposes constraints on the parameters of the corresponding quantum codes. Entanglementassisted schemes enlarge the pool of ingredients to include codes which are not self-orthogonal, but require maximally entangled states as an additional resource.

We recall a general construction route of EAQECCs via the non-commuting stabilizers as explained, with illustrations, in [6]. For the qubit case, a formal treatment is given in [7]. It links arbitrary classical codes over $\mathbb{F}_{4}$ as well as pairs of codes over $\mathbb{F}_{2}$ to qubit EAQECCs. Extensions to the qudit case, where $q>2$, are given in $[17,18]$. Using $\mathbb{F}_{q^{2}}$-linear codes, we have the following construction (see [17, Theorem 3]).
Proposition 1 (Hermitian construction). Let $\mathcal{C}$ be an $[n, k]_{q^{2}}$-code, and let $\mathcal{C}^{\perp_{\mathrm{H}}}$ denote its Hermitian dual. Then there exists an $\llbracket n, \kappa, \delta ; c \rrbracket_{q}$-code $\mathcal{Q}$ with

$$
\begin{aligned}
c & =k-\operatorname{dim}_{\mathbb{F}_{q^{2}}}\left(\mathcal{C} \cap \mathcal{C}^{\perp_{\mathrm{H}}}\right), \\
\kappa & =n-2 k+c, \\
\text { and } \quad \delta & = \begin{cases}\mathrm{wt}\left(\mathcal{C}^{\perp_{\mathrm{H}}}\right), & \text { if } \mathcal{C}^{\perp_{\mathrm{H}}} \subseteq \mathcal{C} ; \\
\mathrm{wt}\left(\mathcal{C}^{\perp_{\mathrm{H}}} \backslash\left(\mathcal{C} \cap \mathcal{C}^{\perp_{\mathrm{H}}}\right)\right), & \text { otherwise. } .\end{cases}
\end{aligned}
$$

We note that the construction includes the case that $\mathcal{C}$ is contained in its Hermitian dual $\mathcal{C}^{\perp_{\mathrm{H}}}$, which implies $c=0$, i.e., the quantum codes do not require entanglement assistance. The case $\mathcal{C}^{\perp_{H}} \subseteq \mathcal{C}$ has not been explicitly addressed in [17]. The resulting codes have $c=2 k-n$ and $\kappa=0$. For codes with $\kappa=0$ and minimum distance $\delta$, by definition the code has to be pure, i.e., there is no error of weight less than $\delta$ that acts trivially on the code.

Another construction uses a pair of $\mathbb{F}_{q}$-linear codes of equal length, yielding the so-called CSS-like family of EAQECCs (see [17, Theorem 4]).

Proposition 2 (CSS-like construction). If $\mathcal{C}_{i}$ is an $\left[n, k_{i}, d_{i}\right]_{q}$-code for $i=1,2$, then there is an $\llbracket n, \kappa, \delta ; c \rrbracket_{q}$-code $\mathcal{Q}$ with

$$
\begin{aligned}
c & =k_{1}-\operatorname{dim}\left(C_{1} \cap C_{2}^{\perp}\right), \\
\kappa & =n-\left(k_{1}+k_{2}\right)+c, \quad \text { and } \\
\delta & =\left\{\begin{aligned}
\min \left\{\operatorname{wt}\left(C_{1}^{\perp}\right), \operatorname{wt}\left(C_{2}^{\perp}\right)\right\}, & \text { if } C_{1}^{\perp} \subseteq C_{2} ; \\
\min \left\{\operatorname{wt}\left(C_{1}^{\perp} \backslash\left(C_{2} \cap C_{1}^{\perp}\right)\right),\right. & \\
\left.\operatorname{wt}\left(C_{2}^{\perp} \backslash\left(C_{1} \cap C_{2}^{\perp}\right)\right)\right\}, & \text { otherwise. }
\end{aligned}\right.
\end{aligned}
$$

Again, when $C_{2}^{\perp} \subseteq C_{1}$, we have $c=0$ and the resulting code does not require entanglement assistance. The case $C_{1}^{\perp} \subseteq C_{2}$, resulting in $c=k_{1}+k_{2}-n$ and $\kappa=0$, has not been explicitly addressed in [17] either.

The code $\mathcal{Q}$ in Proposition 1 is pure or nondegenerate if $\delta=\operatorname{wt}\left(\mathcal{C}^{\perp_{\mathrm{H}}}\right)=d\left(\mathcal{C}^{\perp_{\mathrm{H}}}\right)$. The code $\mathcal{Q}$ in Proposition 2 is pure whenever $\delta=\min \left\{d\left(C_{1}^{\perp}\right), d\left(C_{2}^{\perp}\right)\right\}$.

Otherwise, the code is said to be impure, and it is pure to distance wt $\left(\mathcal{C}^{\perp_{\mathrm{H}}}\right)$ or $\min \left\{d\left(C_{1}^{\perp}\right), d\left(C_{2}^{\perp}\right)\right\}$, respectively.

Another extremal case of Proposition 1 that has not been explicitly discussed in the literature arises when one considers the trivial code $\mathcal{C}=[n, n, 1]_{q^{2}}$. In this case, $\mathcal{C}^{\perp_{\mathrm{H}}}$ is the trivial code that contains only the zero codeword. We argue that the distance of the resulting EAQECC with parameters $c=n$ and $\kappa=0$ is $n+1$. As the code uses $c=n$ maximally entangled states, we are in a situation similar to superdense coding. Performing a joint measurement on $n$ qudits received from the channel and the $n$ qudits from the pre-shared entanglement, the receiver can distinguish $q^{2 n}$ different unitary operations applied by the channel, corresponding to all errors of weight at most $n$. For this, we do not require $q$ to be a prime power. In summary, we have the following proposition.
Proposition 3. For any $q \geq 2$, not necessarily $a$ prime power, there exists an EAQECC $\mathcal{Q}=\llbracket n, 0, n+$ $1 ; n \rrbracket_{q}$.

### 1.3 Our Contributions

1. We establish three propagation rules.

The first rule, given as Theorem 12, increases $c$, signifying that more entanglement is required. The derived quantum code can send more information without losing anything in terms of error handling.
Theorem 16 gives the second rule. It keeps $c$ fixed while lengthening the code, reducing its size. If some conditions are met, then the quantum distance may increase.
The third rule is in Theorem 18. It decreases $c$ while lengthening the code. There may be a price to pay in terms of smaller distances on some occasions.
For the last two rules, we have less theoretical control over the quantum distances and, therefore, searches are the next best option.
2. Our propagation rules are applicable to both stabilizer QECCs and EAQECCs. Most prior propagation rules were designed for stabilizer QECCs whereas our propagation rules works on nontrivial EAQECCs as well. It is in conducting searches for EAQECCs with excellent parameters that the main advantage of our propagation rules come to the fore. They allow us to control either the distance or the number of ebits.
3. In the realm of classical coding theory, Theorem 7 provides a simple proof on the equivalence of $\mathbb{F}_{q^{2}}$-linear codes with diverse Hermitian hull dimensions for $q>2$. For any $[n, k, d]_{q^{2}}$-code $\mathcal{C}$ with $\operatorname{dim}\left(\operatorname{Hull}_{\mathrm{H}}(\mathcal{C})\right)=\ell$, there exists an equivalent $[n, k, d]_{q^{2}}$-code $\mathcal{C}^{\prime}$ with $\operatorname{dim}\left(\operatorname{Hull}_{\mathrm{H}}\left(\mathcal{C}^{\prime}\right)\right)=\ell^{\prime}$
for each $\ell^{\prime} \in\{0,1, \ldots, \ell\}$. This generalizes the result for Hermitian LCD codes in [9, Section V] that considered only the case of $\ell^{\prime}=0$.

Given an $[n, k, d]_{q^{2}}$-code $\mathcal{C}$, Section 2 discusses three linear algebraic approaches that derive codes whose dimensions of Hermitian hulls vary. In the first two approaches, the derived codes have fixed parameters $[n, k, d]_{q^{2}}$, while the dimension of the hull decreases. In the third approach, $k$ is fixed, while both $n$ and the dimension of the hull increase by 1 , and the distance $d$ is either fixed or improved by 1 . Section 3 discusses upper bounds on the parameters of EAQECCs. They are subsequently used collectively in Section 4 as a measure of goodness to motivate our computational process and results. The parameters of the resulting qubit and qutrit EAQECCs are listed in the tables after the concluding remarks in Section 5.

## 2 Three New Propagation Rules

This section presents three new propagation rules based on their effects on $c$, which quantifies the amount of entanglement. We start by devising tools from the classical ingredients.

### 2.1 Tools from Classical Coding Theory

For any vector $\mathbf{v}:=\left(v_{1}, \ldots, v_{n}\right) \in \mathbb{F}_{q^{2}}^{n}$, we denote by $\mathbf{v}^{q}$ the vector $\left(v_{1}^{q}, \ldots, v_{n}^{q}\right)$. Let $\mathcal{C}$ be an $[n, k, d]_{q^{2}}$-code with generator matrix $G$, and let $\mathbf{v}_{1}, \ldots, \mathbf{v}_{k}$ be the rows of $G$. We use $G^{\dagger}$ to denote the $n \times k$ matrix whose columns are $\mathbf{v}_{1}^{q}, \ldots, \mathbf{v}_{k}^{q}$. We call $G^{\dagger}$ the Hermitian transpose of $G$. As usual, $\mathbf{x}^{\top}$ and $M^{\top}$ denote the respective transposes of a vector $\mathbf{x}$ and a matrix $M$.

We recall the relation between a code's generator matrix and its Hermitian hull.

Lemma 4. The dimension of the Hermitian hull is

$$
\begin{equation*}
\operatorname{dim}\left(\operatorname{Hull}_{\mathrm{H}}(\mathcal{C})\right)=k-\operatorname{rank}\left(G G^{\dagger}\right) . \tag{6}
\end{equation*}
$$

Proof. A vector $\mathbf{v}$ is an element of $\operatorname{Hull}_{\mathrm{H}}(\mathcal{C})$ if it is both a codeword of $\mathcal{C}$ and $\mathcal{C}^{\perp_{\mathrm{H}}}$. The first condition requires that $\mathbf{v}$ is in the row span of $G$, that is, $\mathbf{v}=\mathbf{u} G$ for some $\mathbf{u} \in \mathbb{F}_{q^{2}}^{k}$. The second condition requires that $\mathbf{v}$ is in the kernel of $G^{\dagger}$, i. e., $\mathbf{v} G^{\dagger}=\mathbf{0}$. In combination we have

$$
\begin{equation*}
\operatorname{Hull}_{\mathrm{H}}(\mathcal{C})=\left\{\mathbf{v}=\mathbf{u} G: \mathbf{u} \in \mathbb{F}_{q}^{k} \text { and } \mathbf{u} G G^{\dagger}=\mathbf{0}\right\} \tag{7}
\end{equation*}
$$

This implies (6).
A monomial matrix is a square matrix with exactly one nonzero entry in each row and each column and zeros elsewhere. The matrix is a permutation matrix if all of its nonzero entries are 1. Based on these two families of matrices, two equivalence relations among linear codes can be defined.

Definition 5. Let two linear codes $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$ with respective generator matrices $G_{1}$ and $G_{2}$ be given. Then the following statements hold.

1. The codes $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$ are permutation equivalent if there exists a permutation matrix $P$ such that $G_{1} P$ is a generator matrix of $\mathcal{C}_{2}$.
2. The codes $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$ are monomially equivalent if there exists a monomial matrix $M$ such that $G_{1} M$ is a generator matrix of $\mathcal{C}_{2}$.

Equivalent codes have the same length, dimension, and minimum distance. It can be shown (see, e. g., [27, Theorem 1.6.2]) that any linear code is permutation equivalent to a linear code whose generator matrix is in the standard form. The next result shows that the respective Hermitian hulls of two permutation equivalent codes have the same dimension.

Lemma 6. Any two permutation equivalent $\mathbb{F}_{q^{2}}$ linear codes $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$ have

$$
\operatorname{dim}\left(\operatorname{Hull}_{\mathrm{H}}\left(\mathcal{C}_{1}\right)\right)=\operatorname{dim}\left(\operatorname{Hull}_{\mathrm{H}}\left(\mathcal{C}_{2}\right)\right) .
$$

Proof. Let $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$ be permutation equivalent codes with parameters $[n, k, d]_{q^{2}}$. Let $G_{1}$ be a generator matrix of $\mathcal{C}_{1}$. Hence, there is a permutation matrix $P$ such that $G_{2}=G_{1} P$ is a generator matrix of $\mathcal{C}_{2}$. This implies that

$$
G_{2} G_{2}^{\dagger}=G_{1} P\left(G_{1} P\right)^{\dagger}=G_{1} P P^{\top} G_{1}^{\dagger}=G_{1} G_{1}^{\dagger}
$$

In combination with Lemma 4, the conclusion follows.

Regarding monomially equivalent codes, Carlet et al. in [9] demonstrated that Hermitian LCD codes over $\mathbb{F}_{q}$ exist for all possible parameters when $q>$ 2. The next result is a generalization of the results for Hermitian LCD codes in [9, Section V]. In our notation, only the case of $\ell^{\prime}=0$ was considered in the said reference.

Theorem 7. Let $q>2$ be a prime power and let $\mathcal{C}$ be an $[n, k, d]_{q^{2}}$-code with $\operatorname{dim}\left(\operatorname{Hull}_{\mathrm{H}}(\mathcal{C})\right)=\ell$. Then there exists an equivalent $[n, k, d]_{q^{2}}$-code $\mathcal{C}^{\prime}$ with $\operatorname{dim}\left(\operatorname{Hull}_{\mathrm{H}}\left(\mathcal{C}^{\prime}\right)\right)=\ell^{\prime}$ for each $\ell^{\prime} \in\{0,1, \ldots, \ell\}$.

Proof. Without loss of generality, we can assume that $\operatorname{Hull}_{\mathrm{H}}(C)$ is an $\left[n, \ell, d^{\prime}\right]_{q^{2}}$-code generated in the standard form by $G_{1}=\left(I_{\ell} A\right)$. Moreover, we can choose a generator matrix for $\mathcal{C}$ of the form

$$
G=\left(\begin{array}{cc}
I_{\ell} & A  \tag{8}\\
\mathbf{0} & B
\end{array}\right)
$$

and derive
$G G^{\dagger}=\left(\begin{array}{cc}I_{\ell} & A \\ \mathbf{0} & B\end{array}\right)\left(\begin{array}{cc}I_{\ell} & \mathbf{0} \\ A^{\dagger} & B^{\dagger}\end{array}\right)=\left(\begin{array}{cc}I_{\ell}+A A^{\dagger} & A B^{\dagger} \\ B A^{\dagger} & B B^{\dagger}\end{array}\right)$.

The submatrices $I_{\ell}+A A^{\dagger}$ and $A B^{\dagger}$ are zero since $G_{1}$ generates the Hermitian hull, which is contained in the Hermitian dual of $\mathcal{C}$. We have $\operatorname{rank}\left(B B^{\dagger}\right)=k-\ell$. We now consider the generator matrix

$$
\begin{equation*}
G^{\prime}=G \operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{\ell-\ell^{\prime}}, 1, \ldots, 1\right) \tag{10}
\end{equation*}
$$

with $a_{j} \in \mathbb{F}_{q^{2}}^{*}$ and $a_{j}^{q+1} \neq 1$ for $1 \leq j \leq \ell-\ell^{\prime}$. These conditions can always be met for $q>2$. Let

$$
T=\left(\begin{array}{cccc}
a_{1}^{q+1}-1 & 0 & \cdots & 0 \\
0 & a_{2}^{q+1}-1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{\ell-\ell^{\prime}}^{q+1}-1
\end{array}\right)
$$

We verify that $G^{\prime} G^{\prime \dagger}$ is the block-diagonal matrix

$$
\begin{align*}
& G^{\prime} G^{\prime \dagger}=\left(\begin{array}{ccc}
T & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & B B^{\dagger}
\end{array}\right)= \\
& \operatorname{diag}\left(a_{1}^{q+1}-1, \cdots, a_{\ell-\ell^{\prime}}^{q+1}-1,0, \cdots, 0, B B^{\dagger}\right) \tag{11}
\end{align*}
$$

and that $\operatorname{rank}\left(G^{\prime} G^{\prime \dagger}\right)=k-\ell^{\prime}$. This completes the proof.

Remark 8. Independently and coming from a different motivation, $H$. Chen derived a similar result to Theorem 7 in [10, Corollary 2.2]. The said corollary was added since version 2 of his pre-print following a private communication with the first author.

We will need the following lemma (see [26, Theorem 6.32]) later.

Lemma 9. Let $A$ be an $n \times n$ matrix with rank $s$ over $\mathbb{F}_{q^{2}}$ such that $A=A^{\dagger}$. Then $A$ is Hermitian congruent to

$$
\operatorname{diag}(\underbrace{1, \cdots, 1}_{s}, 0, \cdots, 0)
$$

Here two matrices $A$ and $B$ over $\mathbb{F}_{q^{2}}$ are Hermitian congruent if there exists a nonsingular matrix $D$ such that $B=D A D^{\dagger}$.

Using Lemma 9, we derive the following result. It enables an $[n, k]_{q^{2}}$-code with $\ell$-dimensional Hermitian hull to generate an $[n+1, k]_{q^{2}}$-code with $(\ell+1)$ dimensional Hermitian hull.

Proposition 10. Let $0 \leq \ell<\min \{k, n-k\}$. Given an $[n, k, d]_{q^{2}}$-code $\mathcal{C}$ with $\operatorname{dim}\left(\operatorname{Hull}_{\mathrm{H}}(\mathcal{C})\right)=\ell$ and generator matrix $G$, one can add one column to $G$ such that the Hermitian hull of the extended $\left[n+1, k, d^{\prime}\right]_{q^{2-}}$ code $\mathcal{C}^{\prime}$ has dimension $\operatorname{dim}\left(\operatorname{Hull}_{\mathrm{H}}\left(\mathcal{C}^{\prime}\right)\right)=\ell+1$ and minimum distance $d^{\prime}$, with $d \leq d^{\prime} \leq d+1$.

Proof. Let $q=p^{m}$. Let $G$ be a generator matrix of $\mathcal{C}$. Since $\operatorname{dim}\left(\operatorname{Hull}_{\mathrm{H}}(\mathcal{C})\right)=\ell$, by Lemma 4, $\operatorname{rank}\left(G G^{\dagger}\right)=$
$s$, where $s=k-\ell \geq 1$. By Lemma 9 , there exists a nonsingular $k \times k$ matrix $D$ over $\mathbb{F}_{q^{2}}$ such that

$$
\begin{equation*}
D G G^{\dagger} D^{\dagger}=\operatorname{diag}(\underbrace{1, \cdots, 1}_{s}, 0, \cdots, 0) \tag{12}
\end{equation*}
$$

Since $s \geq 1$, the first diagonal entry of the $k \times k$ diagonal matrix in (12) must be 1 . Because $D$ is nonsingular, $D G$ is also a generator matrix of $\mathcal{C}$. Let $\alpha \in \mathbb{F}_{q^{2}}$ be such that $\alpha^{q+1}=-1$. Such an $\alpha$ always exists since $\alpha^{q+1}$ runs through $\mathbb{F}_{q}$ when $\alpha$ runs through $\mathbb{F}_{q^{2}}$. Let $G^{\prime}$ be the $k \times(n+1)$ matrix defined by

$$
G^{\prime}=\left(\begin{array}{l|c} 
& \alpha  \tag{13}\\
D G & 0 \\
\vdots \\
& 0
\end{array}\right)
$$

Then $G^{\prime}$ generates an $\left[n+1, k, d^{\prime}\right]_{q}$-code $\mathcal{C}^{\prime}$ with $d \leq$ $d^{\prime} \leq d+1$. One then verifies that

$$
\begin{align*}
G^{\prime} G^{\prime \dagger} & =D G G^{\dagger} G^{\dagger}+\operatorname{diag}\left(\alpha^{q+1}, 0, \cdots, 0\right)  \tag{14}\\
& =\operatorname{diag}(0, \underbrace{1, \cdots, 1}_{s-1}, 0, \cdots, 0), \tag{15}
\end{align*}
$$

and, hence, $\operatorname{rank}\left(G^{\prime} G^{\prime \dagger}\right)=s-1$. The claim about the dimension of the hull follows from Lemma 4. As $D$ is invertible, the matrix

$$
D^{-1} G^{\prime}=\left(G \left\lvert\, D^{-1}\left(\begin{array}{c}
\alpha  \tag{16}\\
0 \\
\vdots \\
0
\end{array}\right)\right.\right)
$$

is a generator matrix for $\mathcal{C}^{\prime}$ in the desired form.
We note that the matrix $D$ in (12) is not unique. Moreover, there are $q+1$ choices for the element $\alpha$, which can be at any of the first $s$ positions. Hence, there are many choices for the additional column in Proposition 10.

In [32], Lisoněk and Singh proposed a modified construction of quantum codes by relaxing the selforthogonality requirement. From a linear code $\mathcal{C}$ that is not Hermitian self-orthogonal, one can obtain a new linear code which is Hermitian self-orthogonal by adding some rows and columns to a generator matrix of $\mathcal{C}$. Inspired by this result, we show that an $[n, k]_{q^{2}}$-code with $\ell$-dimensional Hermitian hull gives rise to an $[n+1, k+1]_{q^{2}}$-code with $(\ell+1)$-dimensional Hermitian hull.

Proposition 11. Let $\mathcal{C}$ be an $[n, k, d]_{q^{2}}$-code with $b a$ sis $\left\{\mathbf{a}_{1}, \cdots, \mathbf{a}_{k}\right\}$ and $\operatorname{dim}\left(\operatorname{Hull}_{\mathrm{H}}(\mathcal{C})\right)=\ell$, with $0 \leq$ $\ell<\min \{k, n-k\}$. Let $\mathbf{c}$ be a chosen codeword of $\mathcal{C}^{\perp_{\mathrm{H}}} \backslash \operatorname{Hull}_{\mathrm{H}}(\mathcal{C})$ such that $\mathbf{c c}^{\dagger} \neq 0$. Then there exists an $\left[n+1, k+1, d^{\prime}\right]_{q^{2}}$-code $\mathcal{C}^{\prime}$ with $\operatorname{dim}\left(\operatorname{Hull}_{\mathrm{H}}\left(\mathcal{C}^{\prime}\right)\right)=\ell+1$ and $d^{\prime}=\min \left\{d, d_{0}+1\right\}$, where $d_{0}$ is the minimum distance of the code generated by $\left\{\mathbf{a}_{1}, \cdots, \mathbf{a}_{k}, \mathbf{c}\right\}$.

Proof. Such a codeword $\mathbf{c} \in \mathcal{C}^{\perp_{H}} \backslash \operatorname{Hull}_{\mathrm{H}}(\mathcal{C})$ with $\mathbf{c c}^{\dagger} \neq 0$ always exists since $\operatorname{dim}\left(\operatorname{Hull}_{\mathrm{H}}(\mathcal{C})\right)=\ell$ with $0 \leq \ell<\min \{k, n-k\}$. Let $G$ be a generator matrix of $\mathcal{C}$ whose rows are $\left\{\mathbf{a}_{1}, \cdots, \mathbf{a}_{k}\right\}$. Let $\mathbf{c c}^{\dagger}=\alpha$. Since $\alpha \in \mathbb{F}_{q}^{*}$, there exists $\beta \in \mathbb{F}_{q^{2}}^{*}$ such that $\beta^{q+1}=-\alpha$. Let $G^{\prime}$ be the $(k+1) \times(n+1)$ matrix defined by

$$
G^{\prime}:=\left(\begin{array}{cc}
G & \mathbf{0}_{k \times 1}  \tag{17}\\
\mathbf{c} & \beta
\end{array}\right) .
$$

Then we obtain the code
$\mathcal{C}^{\prime}=\left\{\left(x_{1}, \cdots, x_{k+1}\right) \cdot G^{\prime}: x_{i} \in \mathbb{F}_{q^{2}}, i \in[1, \cdots, k+1]\right\}$.
Putting $x_{k+1}$ to be either zero or nonzero, we deduce that the minimum distance of $\mathcal{C}^{\prime}$ is $d^{\prime}=\min \left\{d, d_{0}+1\right\}$. It follows from (17) that

$$
G^{\prime} G^{\prime \dagger}=\left(\begin{array}{cc}
G G^{\dagger} & G \mathbf{c}^{\dagger}  \tag{18}\\
\mathbf{c} G^{\dagger} & 0
\end{array}\right)
$$

Since $\mathbf{c} \in \mathcal{C}^{\perp_{\mathrm{H}}}$, the column $G \mathbf{c}^{\dagger}$ is a zero vector. Thus, $\operatorname{dim}\left(\operatorname{Hull}_{\mathrm{H}}\left(\mathcal{C}^{\prime}\right)\right)=k+1-\operatorname{rank}\left(G G^{\dagger}\right)=\ell+1$.

### 2.2 Propagation Rules

The new propagation rules are presented based on how they affect the variable $c$.

Theorem 12 (More Entanglement). For $q>2$, the existence of a pure $\llbracket n, \kappa, \delta ; c \rrbracket_{q}$-code $\mathcal{Q}$, constructed by Proposition 1, implies the existence of an $\llbracket n, \kappa+$ $i, \delta ; c+i \rrbracket_{q}$-code $\mathcal{Q}^{\prime}$ that is pure to distance $\delta$ for each $i \in\{1, \ldots, \ell\}$, where $\ell$ is the dimension of the Hermitian hull of the $\mathbb{F}_{q^{2}}$-linear code $\mathcal{C}$ that corresponds to $\mathcal{Q}$.

Proof. To confirm the assertion, we apply the Hermitian construction in Proposition 1 on the codes from Theorem 7 and use Lemma 4 to cover the stated range of parameters.

Example 13. Let $\omega$ be a root of $x^{2}+2 x+2 \in \mathbb{F}_{3}[x]$ and let $\mathbb{F}_{9}=\mathbb{F}_{3}(\omega)$. The $[29,14,12]_{9}$-code $\mathcal{C}$ generated by $G=\left(I_{14} A\right)$, with $A$ being the matrix

$$
\left(\begin{array}{ccccccccccccccc}
2 & \omega & \omega^{5} & \omega^{7} & \omega^{7} & \omega^{2} & \omega & 0 & 0 & \omega^{5} & \omega^{6} & \omega^{3} & \omega^{3} & \omega & \omega^{5} \\
\omega^{5} & \omega & 2 & \omega^{3} & \omega & 0 & \omega^{6} & \omega & 0 & \omega^{6} & 2 & \omega^{3} & \omega^{5} & 2 & 2 \\
2 & 0 & 0 & \omega^{2} & 0 & \omega^{3} & \omega & \omega^{6} & \omega & \omega^{5} & \omega^{2} & \omega^{5} & \omega^{7} & 0 & \omega^{6} \\
\omega^{6} & \omega^{5} & \omega^{7} & \omega & \omega^{3} & 2 & \omega^{7} & \omega & \omega^{6} & \omega^{6} & \omega^{3} & 1 & \omega & 0 & \omega^{7} \\
\omega^{7} & \omega^{3} & \omega^{3} & \omega^{5} & \omega^{3} & \omega^{2} & 1 & \omega^{7} & \omega & \omega^{5} & 2 & \omega & \omega^{5} & \omega^{7} & 1 \\
1 & 2 & 1 & \omega^{7} & \omega^{2} & \omega & 1 & 1 & \omega^{7} & \omega^{5} & 1 & \omega^{2} & \omega^{6} & \omega & \omega^{6} \\
\omega^{6} & \omega^{6} & \omega^{2} & \omega^{2} & \omega^{6} & \omega & 1 & 1 & 1 & \omega^{3} & \omega^{3} & \omega^{3} & 1 & \omega & \omega^{6} \\
\omega^{6} & \omega & 1 & \omega^{3} & \omega^{3} & \omega^{3} & 1 & 1 & 1 & \omega & \omega^{6} & \omega^{2} & \omega^{2} & \omega^{6} & \omega^{6} \\
\omega^{6} & \omega & \omega^{6} & \omega^{2} & 1 & \omega^{5} & \omega^{7} & 1 & 1 & \omega & \omega^{2} & \omega^{7} & 1 & 2 & 1 \\
1 & \omega^{7} & \omega^{5} & \omega & 2 & \omega^{5} & \omega & \omega^{7} & 1 & \omega^{2} & \omega^{3} & \omega^{5} & \omega^{3} & \omega^{3} & \omega^{7} \\
\omega^{7} & 0 & \omega & 1 & \omega^{3} & \omega^{6} & \omega^{6} & \omega & \omega^{7} & 2 & \omega^{3} & \omega & \omega^{7} & \omega^{5} & \omega^{6} \\
\omega^{6} & 0 & \omega^{7} & \omega^{5} & \omega^{2} & \omega^{5} & \omega & \omega^{6} & \omega & \omega^{3} & 0 & \omega^{2} & 0 & 0 & 2 \\
2 & 2 & \omega^{5} & \omega^{3} & 2 & \omega^{6} & 0 & \omega & \omega^{6} & 0 & \omega & \omega^{3} & 2 & \omega & \omega^{5} \\
\omega^{5} & \omega & \omega^{3} & \omega^{3} & \omega^{6} & \omega^{5} & 0 & 0 & \omega & \omega^{2} & \omega^{7} & \omega^{7} & \omega^{5} & \omega & 2
\end{array}\right)
$$

is Hermitian self-orthogonal. The dual has parameters $[29,15,11]_{9}$. For their respective $(n, k)$ values,
both $\mathcal{C}$ and $\mathcal{C}^{\perp_{\mathrm{H}}}$ have the best-known minimum distances. We get a $\llbracket 29,1,11 ; 0 \rrbracket_{3}$-code by Proposition 1. The existence of $a \llbracket 29,1+i, 11 ; i \rrbracket_{3}$-code for each $1 \leq i \leq 14$ is guaranteed by Theorem 12.

Theorem 12 allows for the transmission of a larger number of qudits when more pairs of maximally entangled qudits are available, while preserving the minimum distance $\delta$, the total number $n$ of qudits to be sent, as well as the net rate. The main idea is to multiply the columns of the generator matrix by an invertible diagonal matrix to decrease the dimension of the Hermitian hull.

We can use the same approach to try to increase the dimension of the Hermitian hull. This yields the following generalization of the Hermitian construction in Proposition 1.

Theorem 14. Let $\mathcal{C}$ be an $[n, k]_{q^{2}}$-code whose Hermitian dual is $\mathcal{C}^{\perp_{\mathrm{H}}}$. Then there exists an $\llbracket n, \kappa, \delta ; c \rrbracket_{q^{-}}$ code $\mathcal{Q}$ with

$$
c=\min \left\{\operatorname{rank}\left(G \operatorname{diag}\left(b_{1}, \cdots, b_{n}\right) G^{\dagger}\right): b_{i} \in \mathbb{F}_{q}^{*}\right\}
$$

$$
\begin{equation*}
\kappa=n-2 k+c, \tag{19}
\end{equation*}
$$

and $\quad \delta \geq \begin{cases}\mathrm{wt}\left(\mathcal{C}^{\perp_{\mathrm{H}}} \backslash\left(\mathcal{C} \cap \mathcal{C}^{\perp_{\mathrm{H}}}\right)\right), & \text { if } c>2 k-n ; \\ \mathrm{wt}\left(\mathcal{C}^{\perp_{\mathrm{H}}}\right), & \text { if } c=2 k-n .\end{cases}$

Proof. Consider the code $\mathcal{C}^{\prime}$ generated by a matrix $G^{\prime}:=G \operatorname{diag}\left(a_{1}, \cdots, a_{n}\right)$, with $a_{i} \in \mathbb{F}_{q^{2}}^{*}$. By Lemma $4, \operatorname{dim}\left(\operatorname{Hull}_{\mathrm{H}}\left(\mathcal{C}^{\prime}\right)\right)=k-\operatorname{rank}\left(G^{\prime} G^{\prime \dagger}\right)$, where

$$
\begin{align*}
& \operatorname{rank}\left(G^{\prime} G^{\prime \dagger}\right)= \\
& \quad \operatorname{rank}\left(G \operatorname{diag}\left(a_{1}^{q+1}, \cdots, a_{n}^{q+1}\right) G^{\dagger}\right) \tag{21}
\end{align*}
$$

As $a_{i}^{q+1} \in \mathbb{F}_{q}$, it suffices to minimize (21) over all invertible diagonal matrices over $\mathbb{F}_{q}$. By the surjectivity of the norm, given $b_{i} \in \mathbb{F}_{q}$, there exists $b_{i} \in \mathbb{F}_{q^{2}}$ with $b_{i}^{q+1}=a_{i}$.

Concerning the minimum distance $\delta$, first we note that multiplying the coordinates of the code $\mathcal{C}$ with non-zero elements $a_{i}$ does not change its distance or that of its Hermitian dual $\mathcal{C}^{\perp_{\mathrm{H}}}$. Moreover, the Hermitian hull $\operatorname{Hull}_{\mathrm{H}}\left(\mathcal{C}^{\prime}\right)$ contains the transformed vectors of $\operatorname{Hull}_{\mathrm{H}}(\mathcal{C})$. Since the Hermitian hull of $\mathcal{C}^{\prime}$ might be a larger set than that of $\mathcal{C}$, we have

$$
\operatorname{wt}\left(\mathcal{C}^{\prime \perp_{\mathrm{H}}} \backslash\left(\mathcal{C}^{\prime} \cap \mathcal{C}^{\prime \perp_{\mathrm{H}}}\right)\right) \geq \operatorname{wt}\left(\mathcal{C}^{\perp_{\mathrm{H}}} \backslash\left(\mathcal{C} \cap \mathcal{C}^{\perp_{\mathrm{H}}}\right)\right)
$$

The second part of (20) applies in the extremal case when $\mathcal{C}^{\prime \perp_{\mathrm{H}}} \subseteq \mathcal{C}^{\prime}, c=2 k-n$, and $\kappa=0$.

We do not have an efficient method to determine an equivalent code $\mathcal{C}^{\prime}$ that minimizes (19).

In the extremal case of $c=0$, we obtain a Hermitian self-orthogonal code by finding particular solutions to a linear system.

Theorem 15. A given $[n, k, d]_{q^{2}}-\operatorname{code} \mathcal{C}$ is equivalent to a Hermitian self-orthogonal code if and only if there is a vector $\mathbf{b} \in \mathbb{F}_{q}^{n}$, with $b_{i} \neq 0$, such that

$$
\begin{equation*}
\sum_{i=1}^{n} b_{i} x_{i} y_{i}^{q}=0 \text { for all } \mathbf{x}, \mathbf{y} \in \mathcal{C} \tag{22}
\end{equation*}
$$

Proof. Let $\mathcal{C}$ be equivalent to a Hermitian selforthogonal code $\mathcal{C}^{\prime}$. Without loss of generality, we can assume that $\mathcal{C}^{\prime}$ is obtained from $\mathcal{C}$ via multiplication of the coordinates by $a_{i} \in \mathbb{F}_{q^{2}}^{*}$. Setting $b_{i}=a_{i}^{q+1} \in \mathbb{F}_{q}^{*}$ implies (22). On the other hand, if (22) has a solution, then, by the surjectivity of the norm, we find $a_{i} \in \mathbb{F}_{q^{2}}^{*}$ with $b_{i}=a_{i}^{q+1}$. Multiplying the coordinates of $\mathcal{C}$ by $b_{i}$ yields an equivalent Hermitian self-orthogonal code, provided that $a_{i}$ and $b_{i}$ are nonzero for all $i$.

The linear space of all solutions $\mathbf{b} \in \mathbb{F}_{q}^{n}$ of (22) (i.e., allowing zero-coordinates as well) is known as a punctured code [35].

Swapping the roles of $\mathcal{C}$ and $\mathcal{C}^{\perp_{\mathrm{H}}}$ in Proposition 1 enables us to better control the quantum distance. We use this approach in the proof of the next result.

Theorem 16 (Same Entanglement). If there exists a pure $\llbracket n, \kappa, \delta ; c \rrbracket_{q}$-code $\mathcal{Q}$ with $\kappa>0, c>0$ obtained by the Hermitian construction in Proposition 1, then there exists an $\llbracket n+1, \kappa-1, \delta^{\prime} ; c \rrbracket_{q}$-code $\mathcal{Q}^{\prime}$ that is pure to distance $\delta^{\prime}$ with $\delta \leq \delta^{\prime} \leq \delta+1$.

Proof. Let $\mathcal{C}^{\perp_{\mathrm{H}}}$ be the linear code with parameters $[n, n-k, \delta]_{q^{2}}$ used in the Hermitian construction of $\mathcal{Q}$. The dimension of the Hermitian hull is

$$
\operatorname{dim}\left(\mathcal{C} \cap \mathcal{C}^{\perp_{\mathrm{H}}}\right)=\ell=n-k-c
$$

Applying Proposition 10 to $\mathcal{C}^{\perp_{\mathrm{H}}}$ yields an $[n+1, n-$ $\left.k, \delta^{\prime}\right]_{q^{2}}$-code $\mathcal{C}^{\prime \perp_{\mathrm{H}}}$. The dimension of its Hermitian hull is $\ell+1$. Applying the Hermitian construction to $\mathcal{C}^{\prime}$ gives us an $\llbracket n+1, \kappa-1, \delta^{\prime} ; c \rrbracket_{q}$-code $\mathcal{Q}^{\prime}$.

Applying the Hermitian construction on the $[n+$ $\left.1, k, d^{\prime}\right]_{q^{2}}$-code $\mathcal{C}^{\prime}$ from Proposition 10 produces an $\llbracket n+1, \kappa-1, \delta^{\prime} ; c \rrbracket_{q}$-code $\mathcal{Q}^{\prime}$. On the original classical code $\mathcal{C}$, the outcome is an $\llbracket n, \kappa, \delta ; c \rrbracket_{q}$-code $\mathcal{Q}$. The minimum distance $\delta^{\prime}$ of $\mathcal{Q}^{\prime}$ depends on how the extended code $\mathcal{C}^{\prime}$ is built, as illustrated in the following example.

Example 17. Let $\omega$ be a root of $x^{2}+2 x+2 \in \mathbb{F}_{3}[x]$ and let $\mathbb{F}_{9}=\mathbb{F}_{3}(\omega)$. Let $\mathcal{C}$ be the $[5,4,2]_{9}$-code generated by $G=\left(\begin{array}{l}\left.I_{4} B\right) \text { with } B=\left(\begin{array}{ll}2 & 2\end{array} 2\right)^{\top} \text {. Extending }\end{array}\right.$ the matrix $G$ by the column $\left(1 w^{3} w^{2} w^{7}\right)^{\top}$, we obtain $a[6,4,3]_{9}$-code $\mathcal{C}^{\prime}$ with $\operatorname{dim} \operatorname{Hull}\left(\mathcal{C}^{\prime}\right)=1$. Since $\mathcal{C}^{\prime}$ is an MDS code, its Hermitian dual $\mathcal{C}^{\prime \perp_{\mathrm{H}}}$ has parameters $[6,2,5]_{9}$. Using the code $\mathcal{C}^{\prime}$ in Proposition 1 results in a pure $\llbracket 6,1,5 ; 3 \rrbracket_{3}$-code which is optimal by the bounds in the next section. Its net rate is $-1 / 3$ and it improves on the distance of the distance-optimal quantum code $\llbracket 6,1,3 \rrbracket_{3}$ by 2 .

Based on the derived $\left[n+1, k+1, d^{\prime}\right]_{q^{2}}$-code in Proposition 11, we have the following result which allows for the transmission of the same amount of quantum information using a smaller number of pairs of maximally entangled qudits.

Theorem 18 (Less Entanglement). The existence of a pure $\llbracket n, \kappa, \delta ; c \rrbracket_{q}$-code $\mathcal{Q}$, constructed from an $\mathbb{F}_{q^{2}}$ linear code $\mathcal{C}$ based on Proposition 1, implies the existence of an $\llbracket n+1, \kappa, \delta^{\prime} ; c-1 \rrbracket_{q}$-code $\mathcal{Q}^{\prime}$ with $\delta^{\prime} \leq \delta$.

Proof. The desired result follows from Proposition 11 by a method analogous to the one in the proof of Theorem 16.

In Theorem 18, the pure minimum distance of the resulting EAQECC is determined by the choice of the codeword c, which was defined earlier in Proposition 11. To construct a new EAQECC with good minimum distance, one can try all such codewords and then select an EAQECC with the largest minimum distance from all resulting EAQECCs. The next example illustrates such an implementation.

Example 19. Let $\omega$ be a root of $x^{2}+2 x+2 \in \mathbb{F}_{3}[x]$ and let $\mathbb{F}_{9}=\mathbb{F}_{3}(\omega)$. Let $\mathcal{C}$ be the $[16,5,8]_{9}$-code with generator matrix $G$ given by

$$
\left(\begin{array}{cccccccccccccccc}
1 & 0 & 0 & 0 & \omega^{6} & 1 & \omega^{6} & \omega^{5} & 0 & \omega^{6} & \omega^{6} & 1 & \omega^{7} & 1 & \omega & 1 \\
0 & 1 & 0 & 0 & \omega^{7} & 2 & \omega & \omega^{2} & 0 & \omega & 0 & 0 & 1 & \omega^{5} & \omega^{2} & \omega^{3} \\
0 & 0 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & \omega^{5} & \omega^{3} & \omega & 2 & \omega^{2} & \omega^{2} & \omega^{3} \\
0 & 0 & 0 & 1 & \omega^{2} & 1 & \omega^{7} & \omega^{2} & 0 & 2 & \omega & \omega^{2} & \omega & \omega^{6} & \omega & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2
\end{array}\right)
$$

The Hermitian dual code $\mathcal{C}^{\perp_{\mathrm{H}}}$ has parameters $[16,11,5]_{9}$ and the Hermitian hull $\operatorname{Hull}_{\mathrm{H}}(\mathcal{C})$ is a $[16,3,12]_{9}$-code. Following the proof of Proposition 11, we select a codeword
$\mathbf{c}:=\left(1 \omega^{7} \omega \omega^{5} \omega^{6} \omega^{2} 0 \omega^{5} \omega^{3} 2 \omega^{2} 1 \omega^{2} \omega^{5} \omega^{3} \omega^{2}\right)$
of weight 15 in $\mathcal{C}^{\perp_{\mathrm{H}}} \backslash \operatorname{Hull}_{\mathrm{H}}(\mathcal{C})$ such that $\mathbf{c c}^{\dagger}=1$ to obtain the $[17,6,8]$-code $\mathcal{C}^{\prime}$ whose generator matrix is

$$
G^{\prime}=\left(\begin{array}{cc}
G & \mathbf{0}_{5 \times 1} \\
\mathbf{c} & \omega
\end{array}\right)
$$

The Hermitian dual $\mathcal{C}^{\prime \perp_{\mathrm{H}}}$ has parameters $[17,11,5]_{9}$ and $\operatorname{Hull}_{\mathrm{H}}\left(\mathcal{C}^{\prime}\right)$ is a $[17,4,10]_{9}$-code.

We now switch perspective and use the $[16,5,8]_{9-}$ code as the $\mathcal{C}^{\perp_{\mathrm{H}}}$, instead of as the $\mathcal{C}$, in Proposition 1 to construct a pure $\llbracket 16,2,8 ; 8 \rrbracket_{3}$-code $\mathcal{Q}$. Using the derived $[17,6,8]$-code $\mathcal{C}^{\prime}$ as the $\mathcal{C}^{\perp_{\mathrm{H}}}$ in Proposition 1 leads to a pure $\llbracket 17,2,8 ; 7 \rrbracket_{3}$-code $\mathcal{Q}^{\prime}$.

If we have chosen as our codeword $\mathbf{c}$ the vector

$$
\mathbf{c}:=\left(\omega^{7} \omega^{5} \omega 0 \omega^{5} \omega^{5} 02 \omega^{7} \omega^{6} \omega^{2} 2 \omega 2 \omega^{2} \omega^{3}\right)
$$

of weight 14, instead of the one in (23), then the resulting $\mathcal{C}^{\prime}$ would have parameters $[17,6,7]_{9}$. The constructed pure quantum codes $\mathcal{Q}$ and $\mathcal{Q}^{\prime}$ would have parameters $\llbracket 16,2,7 ; 8 \rrbracket_{3}$ and $\llbracket 17,2,7 ; 7 \rrbracket_{3}$, respectively. This highlights the importance of choosing $\mathbf{c}$ such that $d^{\prime}=d$, that is, $d_{0} \geq d-1$, in Proposition 11.

The idea of Lisoněk and Singh in [32] is to start with a length $n$ classical code with a large hull. One then carefully selects a codeword so that it can be used to extend the length by 1 and prevent the quantum distance from deteriorating. Our idea here is similar. The advantage is that we have more freedom in choosing the codeword that may lead to a better quantum distance. The two approaches coincide when the classical ingredient $\mathcal{C}$ is $k$-dimensional over $\mathbb{F}_{q^{2}}$ and its hull has dimension $k-1$.

## 3 Upper Bounds

There is a vast literature on EAQECCs constructed via classical maximum distance separable (MDS) code and (Hermitian) LCD codes. Their parameters and excellent properties allow for a straightforward derivation of the parameters of the corresponding quantum codes. The length of MDS codes, however, are constrained by the cardinality of the underlying finite fields. Using classical MDS codes over $\mathbb{F}_{4}$ for the qubit case and $\mathbb{F}_{9}$ for the qutrit case offer limited insights beyond very small lengths.

The Singleton bound for an $\llbracket n, \kappa, \delta ; c \rrbracket_{q}$-code $\mathcal{Q}$ in [22, Corollary 9] reads

$$
\begin{align*}
& \kappa \leq c+\max \{0, n-2 \delta+2\}  \tag{24}\\
& \kappa \leq n-\delta+1  \tag{25}\\
& \kappa \leq \frac{(n-\delta+1)(c+2 \delta-2-n)}{3 \delta-3-n}, \text { if } \delta-1 \geq \frac{n}{2} \tag{26}
\end{align*}
$$

Codes attaining the bound (24) for $\delta-1 \leq \frac{n}{2}$ or the bound (26) for $\delta-1 \geq \frac{n}{2}$ with equality are called MDS EAQECCs. We note that without the bound (25), the upper bound on the dimension $\kappa$ would be linear in the a priori unbounded number $c$ of maximally entangled pairs of qudits.

To our knowledge, most known families of MDS EAQECCs, e.g., those presented in [11, 12, 16, 24, 33], were built by applying Propositions 1 and 2 on suitably chosen classical MDS codes. In [24], an $[n, k, d]_{q^{2}}$-code $\mathcal{C}$, whose Hermitian dual is an $\left[n, n-k, d^{\prime}\right]_{q^{2}}$-code, is used in Proposition 1 to yield two EAQECCs with parameters

$$
\begin{align*}
& \llbracket n, k-\operatorname{dim}\left(\operatorname{Hull}_{\mathrm{H}}(\mathcal{C})\right), d ; n-k-\operatorname{dim}\left(\operatorname{Hull}_{\mathrm{H}}(\mathcal{C})\right) \rrbracket_{q},  \tag{27}\\
& \llbracket n, n-k-\operatorname{dim}\left(\operatorname{Hull}_{\mathrm{H}}(\mathcal{C})\right), d^{\prime} ; k-\operatorname{dim}\left(\operatorname{Hull}_{\mathrm{H}}(\mathcal{C})\right) \rrbracket_{q} . \tag{28}
\end{align*}
$$

From an $[n, k, n-k+1]_{q^{2}}-\mathrm{MDS}$ code $\mathcal{C}$ and its $[n, n-$ $k, k+1]_{q^{2}}$-Hermitian dual $\mathcal{C}^{\perp_{\mathrm{H}}}$ with $\operatorname{dim}\left(\operatorname{Hull}_{\mathrm{H}}(\mathcal{C})\right)=$ $\operatorname{dim}\left(\operatorname{Hull}_{\mathrm{H}}\left(\mathcal{C}^{\perp_{\mathrm{H}}}\right)\right)=\ell$, one obtains EAQECCs with parameters

$$
\begin{align*}
& \llbracket n, k-\ell, n-k+1 ; n-k-\ell \rrbracket_{q} \text { and }  \tag{29}\\
& \llbracket n, n-k-\ell, k+1 ; k-\ell \rrbracket_{q} . \tag{30}
\end{align*}
$$

In general, only the code with distance $d \leq \frac{n}{2}$ is an MDS EAQECC, whereas, for $d>\frac{n}{2}$ and $\kappa \leq c<n-k$, the bound (26) cannot be achieved with equality.

As shown by Grassl, Huber, and Winter in [22, Theorem 7], any pure $\llbracket n, \kappa, \delta ; c \rrbracket_{q}$-code obeys the bounds

$$
\begin{equation*}
2 \delta \leq n+c-\kappa+2 \tag{31}
\end{equation*}
$$

We show that this bound also applies to EAQECCs that can be obtained by Propositions 1 and 2.

Theorem 20. For any $\llbracket n, \kappa, \delta ; c \rrbracket_{q}$-code $\mathcal{Q}$ obtained by the Hermitian construction in Proposition 1, we have

$$
\begin{equation*}
2 \delta \leq n+c-\kappa+2 \tag{32}
\end{equation*}
$$

Proof. Corresponding to the $\llbracket n, \kappa, \delta ; c \rrbracket_{q}$-code $\mathcal{Q}$, there exists an $[n, n-\kappa-\ell]_{q^{2}}$-code $\mathcal{C}$ such that $\operatorname{dim}_{\mathbb{F}_{q^{2}}}\left(\operatorname{Hull}_{\mathrm{H}}(\mathcal{C})\right)=\ell$. If $\operatorname{Hull}_{\mathrm{H}}(\mathcal{C})$ has generator matrix $\left(\begin{array}{ll}I_{\ell} & R\end{array}\right)$, then $\mathcal{C}^{\perp_{\mathrm{H}}}$ has generator matrix

$$
\left(\begin{array}{cc}
I_{\ell} & R \\
\mathbf{O}_{\kappa \times \ell} & A
\end{array}\right) .
$$

The code generated by

$$
\left(\begin{array}{ll}
\mathbf{O}_{\kappa \times \ell} & A
\end{array}\right)
$$

is a subset of $\mathcal{C}^{\perp_{\mathrm{H}}} \backslash \operatorname{Hull}_{\mathrm{H}}(\mathcal{C})$ and has parameters $[n, \kappa, \geq \delta]_{q^{2}}$. Hence, the linear code generated by the matrix $A$ has parameters

$$
[n-\ell, \kappa, \geq \delta]_{q^{2}}
$$

By the classical Singleton bound, we arrive at

$$
\delta \leq n-\ell-\kappa+1
$$

Since $c=n-\kappa-2 \ell$, we have
$\delta \leq n-\frac{n-\kappa-c}{2}-\kappa+1 \Longleftrightarrow 2 \delta \leq n+c-\kappa+2$.

The codes in the CSS-like subfamily obeys the bound (31) as well.

Theorem 21. For $i=1,2$, let $\mathcal{C}_{i}$ be an $\left[n, k_{i}\right]_{q}$-code. Let $\kappa=n-\left(k_{1}+k_{2}\right)+c$. For any $\llbracket n, \kappa, \delta ; c \rrbracket_{q}$-code $\mathcal{Q}$ obtained by the CSS-like construction in Proposition 2, we have

$$
\begin{equation*}
2 \delta \leq n+c-\kappa+2 \tag{33}
\end{equation*}
$$

Proof. By Proposition 2, there exist two linear codes $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$ with respective parameters $\left[n, k_{1}\right]_{q}$ and $\left[n, k_{2}\right]_{q}$, where $k_{2}=n-\kappa+c-k_{1}$. We denote by $\Delta$ the code $\mathcal{C}_{1} \cap \mathcal{C}_{2}^{\perp}$ and let $\ell=\operatorname{dim}_{\mathbb{F}_{q}}(\Delta)$. Let $\Delta$ be generated by $\left(\begin{array}{ll}I_{\ell} & R\end{array}\right)$. Let

$$
\left(\begin{array}{cc}
I_{\ell} & R \\
\mathbf{O}_{\left(n-\kappa_{2}-\ell\right) \times \ell} & A
\end{array}\right)
$$

generate $\mathcal{C}_{2}^{\perp}$. The $\left[n, n-k_{2}-\ell, \geq \delta\right]_{q}$-code generated by $\left(\mathbf{O}_{\left(n-k_{2}-\ell\right) \times \ell} A\right)$ is a subset of $\mathcal{C}_{2}^{\perp} \backslash \Delta$. Hence, there exists an $\left[n-\ell, n-k_{2}-\ell, \geq \delta\right]_{q}$-code generated by the matrix $A$. By the Singleton bound, we infer that $\delta \leq k_{2}+1$. In a similar manner, starting from the $\beta$-dimensional code $\Gamma=\mathcal{C}_{2} \cap \mathcal{C}_{1}^{\perp}$, one derives an $[n-$ $\left.\beta, n-k_{1}-\beta, \geq \delta\right]_{q}$-code, with $\delta \leq k_{1}+1$. Combining the two inequalities gives us $2 \delta \leq k_{1}+k_{2}+2=n+$ $c-\kappa+2$, as promised.

All these Singleton-type bounds are independent of the alphabet size $q$. The classical bound of Griesmer from [23] leads to a sharper upper bound for lengths $n>q^{2}+1$.

Theorem 22. For any $\llbracket n, \kappa, \delta ; c \rrbracket_{q}$-code $\mathcal{Q}$ obtained by the CSS-like construction in Proposition 2, we have

$$
\frac{n+\kappa+c}{2} \geq \sum_{i=0}^{\kappa-1}\left\lceil\frac{\delta}{q^{i}}\right\rceil
$$

Proof. Let $k_{2}=n-\kappa+c-k_{1}$. By the proof of Theorem 21, there exist two linear codes $\mathcal{A}$ and $\mathcal{B}$ with respective parameters $\left[n-\ell, n-k_{2}-\ell, \geq \delta\right]_{q}$ and $\left[n-\beta, n-k_{1}-\beta, \geq \delta\right]_{q}$, where $\ell$ and $\beta$ are the $\mathbb{F}_{q}$-dimensions of $\mathcal{C}_{1} \cap \mathcal{C}_{2}^{\perp}$ and $\mathcal{C}_{2} \cap \mathcal{C}_{1}^{\perp}$. Let $G_{1}$ and $G_{2}$ be generator matrices of $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$, respectively. Then the dimension of the solution space of $G_{1} G_{2}^{\top} \mathbf{x}^{\top}=\mathbf{0}$ is $k_{2}-\operatorname{rank}\left(G_{1} G_{2}^{\top}\right)$. Since

$$
G_{1} G_{2}^{\top} \mathbf{x}^{\top}=G_{1}\left(\mathbf{x} G_{2}\right)^{\top}=\mathbf{0}
$$

we have $k_{2}-\operatorname{rank}\left(G_{1} G_{2}^{\top}\right)=\beta$. Employing the method analogous to the one we have just used, we arrive at

$$
k_{1}-\operatorname{rank}\left(G_{2} G_{1}^{\top}\right)=\ell \Longrightarrow k_{2}-\beta=k_{1}-\ell
$$

We note that

$$
\begin{equation*}
c=\operatorname{rank}\left(G_{1} G_{2}^{\top}\right)=k_{1}-\ell=k_{2}-\beta \tag{34}
\end{equation*}
$$

Applying the Griesmer bound to $\mathcal{A}$ and $\mathcal{B}$ gives us

$$
\begin{align*}
& n-\ell \geq \sum_{i=0}^{n-k_{2}-\ell-1}\left\lceil\frac{\delta}{q^{i}}\right\rceil=\sum_{i=0}^{n-\left(k_{1}+k_{2}\right)+c-1}\left\lceil\frac{\delta}{q^{i}}\right\rceil \text { and }  \tag{35}\\
& n-\beta \geq \sum_{i=0}^{n-k_{1}-\beta-1}\left\lceil\frac{\delta}{q^{i}}\right\rceil=\sum_{i=0}^{n-\left(k_{1}+k_{2}\right)+c-1}\left\lceil\frac{\delta}{q^{i}}\right\rceil . \tag{36}
\end{align*}
$$

Since $\ell=k_{1}-c$ and $\beta=k_{2}-c$, it follows from (35) and (36) that

$$
\begin{aligned}
2 n-\ell-\beta=2 n-\left(k_{1}+\right. & \left.k_{2}\right)+2 c= \\
& \geq 2 \sum_{i=0}^{n-\left(k_{1}+k_{2}\right)+c-1}\left\lceil\frac{\delta}{q^{i}}\right\rceil .
\end{aligned}
$$

The conclusion follows from $k_{2}=n-\kappa+c-k_{1}$.

Theorem 23. [31] For any $\llbracket n, \kappa, \delta ; c \rrbracket_{q}-E A Q E C C$ obtained by the Hermitian construction of Proposition 1, we have

$$
\begin{equation*}
\frac{n+\kappa+c}{2} \geq \sum_{i=0}^{\kappa-1}\left\lceil\frac{\delta}{q^{2 i}}\right\rceil \tag{37}
\end{equation*}
$$

Proof. We have an $[n-\ell, \kappa, \geq \delta]_{q^{2}}$-code with $\ell=$ $\frac{n-\kappa-c}{2}$ from the proof of Theorem 20. By the Griesmer bound,

$$
n-\ell=\frac{n+\kappa+c}{2} \geq \sum_{i=0}^{k-1}\left\lceil\frac{\delta}{q^{2 i}}\right\rceil
$$

## 4 Computational Results

The results we have derived as well as previously available tools can now be used to search for good entanglement-assisted (EA) qubits and qutrit.

The simplest approach would have been to apply Proposition 1 on $\mathbb{F}_{4}$ and $\mathbb{F}_{9}$-linear codes in the current MAGMA BKLC database of codes with best-known minimum distances [3, 20]. Most codes in the database are LCD codes or codes with small Hermitian hulls. For qutrit codes, in light of Theorems 7 and 12, we prefer codes with large Hermitian hulls, e.g., specially crafted quasi-cyclic codes with large Hermitian hulls based on the construction method in [15, Section III]. Such classical codes yield EAQECCs with a wider range of parameters. The parameters also depend on the minimum distances of their respective dual codes. The choice of classical codes to record in the said database does not take the above into consideration. One can switch the role of the code and its dual in Proposition 1 so that the code from the database provide information on the minimum distance.

Theorem 16 yields good codes on numerous occasions. Determining the matrix $D$, however, is time consuming and the resulting parameters are often already covered by the other construction approaches. Computational evidences indicate that the benefit from applying Theorem 16 occurs when $d^{\prime}=d+1$. Replacing the diagonal matrix on the right hand side of (12) by a matrix of rank $s$ sometimes allows for a more efficient randomized procedure to find a suitable matrix $D$ that eventually leads to a good qutrit code.

For qutrit codes, we use Theorem 14 to determine the minimum number of maximally entangled pairs $c_{\text {min }}, e . g .$, by exhaustive search, or use a randomized search to find a smaller value.

We provide the parameters of the best-performing qubit, for lengths $3 \leq n \leq 64$, and qutrit, for lengths $3 \leq n \leq 36$, of EAQECCs that we can explicitly construct in Tables 1 and 2. Among the parameters $\llbracket n, \kappa, \delta ; c \rrbracket_{q}$, all other parameters being equal, we record the smallest $n$, the largest $\kappa$, the largest $\delta$, and
the smallest $c$ for $q \in\{2,3\}$. The tables are compressed using the propagation rules in this paper and those given in [17, 18, 22]. For ease of reference we list the following eight propagation rules. The first four rules are trivial. Rule (5) is obtained by erasing one position of the original code. Rules (6) to (8) come from Theorem 12, [34, Theorem 7], and [34, Theorem 8], respectively.
(1) length extension: $\llbracket n, \kappa, \delta ; c \rrbracket_{q} \longrightarrow \llbracket n+1, \kappa, \delta ; c \rrbracket_{q}$.
(2) subcode: $\llbracket n, \kappa, \delta ; c \rrbracket_{q} \longrightarrow \llbracket n, \kappa-1, \delta ; c \rrbracket_{q}$.
(3) smaller distance: $\llbracket n, \kappa, \delta ; c \rrbracket_{q} \longrightarrow \llbracket n, \kappa, \delta-1 ; c \rrbracket_{q}$.
(4) requiring more entanglement:
$\llbracket n, \kappa, \delta ; c \rrbracket_{q} \longrightarrow \llbracket n, \kappa, \delta ; c+1 \rrbracket_{q}$.
(5) puncturing, assuming $\delta>1$ and $c<n-\kappa$ : $\llbracket n, \kappa, \delta ; c \rrbracket_{q} \longrightarrow \llbracket n-1, \kappa, \delta-1 ; c \rrbracket_{q}$.
(6) increasing the dimension of a pure $q$-ary quantum code with $q>2$ by using extra entanglement, provided that $c \leq n-\kappa-2$ :
$\llbracket n, \kappa, \delta ; c \rrbracket_{q} \longrightarrow \llbracket n, \kappa+1, \delta ; c+1 \rrbracket_{q}$.
(7) reducing the length by using extra entanglement, provided that $c \leq n-\kappa-2$ :
$\llbracket n, \kappa, \delta ; c \rrbracket_{q} \longrightarrow \llbracket n-1, \kappa, \delta ; c+1 \rrbracket_{q}$.
(8) shortening pure quantum code:
$\llbracket n, \kappa, \delta ; c \rrbracket_{q} \longrightarrow \llbracket n-1, \kappa+1, \delta-1 ; c \rrbracket_{q}$.
The parameters that we have determined in this work can be found in the online record of the bounds on the minimum distance of entanglement-assisted quantum codes [21].

## 5 Concluding Remarks

The use of pre-shared entanglement in quantum error control raises questions. How do entanglementassisted QECCs compare to other QECCs that draw on different resources? In what setups can they be more useful than the others? The enhanced rate or better error-handling capability offered by EAQECCs must be paid for by the additional cost of pre-shared entanglement. On the more practical front, one asks how to best share ebits and how many of them to share.

It is possible for the net rate $\bar{\rho}(Q)$ to be zero or negative. Can such a code be useful in practice? Since shared entanglement $|\phi\rangle_{A B}$ is independent of the message $|\varphi\rangle$, it can be prepared ahead of time and stored to be used as and when needed. In a quantum network, where usage varies over time, Alice and Bob can use periods of low usage to accumulate ebits. These can then be utilized to increase the transmission rate without trading off on the error-correcting power when the network usage grows higher.

Codes with positive net rate can be used as building blocks in the construction of catalytic quantum codes, leading to the quantum analogue of highly-efficient classical codes such as Turbo and LDPC codes [36].

The quantum setup provides a rich ground for coding theorists of the more classical mould to venture into topics hitherto less explored. Instead of focusing on quantum codes that meet the analogue of the Singleton bound, for example, constructing qubit and qutrit codes that have better chances of being implemented in actual quantum devices and networks could take a more focal position.

We identify the following open directions for further investigation.

1. Establish sharper lower and upper bounds on the parameters of best EAQECCs, especially for qubit and qutrit codes.
2. Find the quantum code with the largest rate for a specified quantum distance and hull dimension. The duality can be chosen among suitable choices of inner products, depending on the construction routes.
3. In the classical setting, given a length $n$ and dimension $k$, construct a code with the largest hull and optimal dual distance.

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Table 1：A Concise Version of the Parameters of Good Qubit EAQECCs with $3 \leq n \leq 64$ ．To obtain the full table，one applies the propagation rules in Section 4．We exclude the entries with $c=0$ since a large database for such codes is already available in［20］．

| ［n，$\left.\kappa, \delta ;{ }^{\text {c }}\right]_{2}$ | ［n，$\kappa, \delta ; c \rrbracket_{2}$ | ［n，$\kappa, \delta ; c \rrbracket_{2}$ | 【n，$\kappa, \delta ; c \rrbracket_{2}$ | ［n，$\kappa, \delta ; c \rrbracket_{2}$ | ［n，$\kappa, \delta ; c \rrbracket_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| ［ $3,1,3 ; 2]_{2}$ | ［ $5,4,2 ; 1]_{2}$ | ［ $5,0,4 ; 1]_{2}$ | ［6，0，6；4］${ }_{2}$ | ［7，6，2；1］${ }_{2}$ | ［ $7,4,3 ; 3]_{2}$ |
| 【8， 5,$3 ; 3 \rrbracket_{2}$ | ［8， 4,$4 ; 4 \rrbracket_{2}$ | ［8， 2,$5 ; 4 \rrbracket_{2}$ | ［8， 0,$8 ; 6 \rrbracket_{2}$ | ［9， 8,$2 ; 1 \rrbracket_{2}$ | ［9， 6,$3 ; 3]_{2}$ |
| 【9，4，5；5】 ${ }_{2}$ | ［9， 2,$6 ; 5 \rrbracket_{2}$ | ［10， 6,$4 ; 4 \rrbracket_{2}$ | 【10， 4,$5 ; 4 \rrbracket_{2}$ | 【10，4，6；$\rrbracket_{2}$ | 【10， 0,$10 ; 8 \rrbracket_{2}$ |
| 【11，10， $2 ; 1 \rrbracket_{2}$ | 【11， 7,$3 ; 2 \rrbracket_{2}$ | ［11， 6,$4 ; 3 \rrbracket_{2}$ | 【11， 5,$6 ; 6 \rrbracket_{2}$ | 【12， 9,$3 ; 3 \rrbracket_{2}$ | 【12， 8,$4 ; 4 \rrbracket_{2}$ |
| 【12， 5,$6 ; 5 \rrbracket_{2}$ | 【12， 0,$12 ; 10 \rrbracket_{2}$ | 【13，12， $2 ; 1 \rrbracket_{2}$ | 【13，10，3； $3 \rrbracket_{2}$ | 【13， 9,$4 ; 4 \rrbracket_{2}$ | ［14，0，14；12】 ${ }_{2}$ |
| ［15，14， $2 ; 1 \rrbracket_{2}$ | 【15， 9,$5 ; 6 \rrbracket_{2}$ | 【16，13，3； $3 \rrbracket_{2}$ | ［16，9，4；1】 ${ }_{2}$ | 【16，9， $6 ; 7 \rrbracket_{2}$ | ［16， 0,$16 ; 14 \rrbracket_{2}$ |
| 【17，16， $2 ; 1 \rrbracket_{2}$ | 【17，13，3；2］${ }_{2}$ | 【17，9，6；6】 ${ }_{2}$ | 【17， 0,$8 ; 1 \rrbracket_{2}$ | 【17， 0,$12 ; 9 \rrbracket_{2}$ | 【18，15，3；3】 ${ }_{2}$ |
| 【18，10，5；4 $]_{2}$ | ［18，7，9；11］${ }_{2}$ | 【18， 0,$10 ; 6 \rrbracket_{2}$ | 【18， 0,$18 ; 16 \rrbracket_{2}$ | 【19，18， $2 ; 1 \rrbracket_{2}$ | 【19，13，4；4］${ }_{2}$ |
| ［20，15，3；1】 ${ }_{2}$ |  | ［20，14，5；6】 ${ }_{2}$ | 【20，10，6；4 ${ }_{2}$ | 【20，9，8；9］ 2 | ［20， 0,$20 ; 18 \rrbracket_{2}$ |
| ［21，20，2；1】 ${ }_{2}$ | ［21，16， $4 ; 5 \rrbracket_{2}$ | 【21，15，5；6】 2 | 【21，9，7；$\rrbracket_{2}$ | ［22，16，4；4］${ }_{2}$ | 【22，12，5；4］${ }_{2}$ |
| 【22，0，22； $20 \rrbracket_{2}$ | 【23， 22,$2 ; 1 \rrbracket_{2}$ | 【23，18， $4 ; 5 \rrbracket_{2}$ | 【23，14，5；5】 ${ }_{2}$ | 【23，14，6；7 $]_{2}$ | ［24，18， $4 ; 4 \rrbracket_{2}$ |
| 【24，16，5；6】 ${ }_{2}$ | ［24，16， $6 ; 8 \rrbracket_{2}$ | ［24，0，24；22】 ${ }_{2}$ | ［25，24，2；1］${ }_{2}$ | 【25，20，3；3 $]_{2}$ | 【25，19，4；4］${ }_{2}$ |
| ［25，11， $7 ; 6 \rrbracket_{2}$ | ［25， 0,$14 ; 11 \rrbracket_{2}$ | 【26， 22,$3 ; 4 \rrbracket_{2}$ | 【26，21， $4 ; 5 \rrbracket_{2}$ | 【26，13， $7 ; 7 \rrbracket_{2}$ | ［26，0，26； 24$]_{2}$ |
| 【27， 26,$2 ; 1 \rrbracket_{2}$ | 【27， 23,$3 ; 4 \rrbracket_{2}$ | 【27， 22,$4 ; 5 \rrbracket_{2}$ | 【28， 23,$3 ; 3 \rrbracket_{2}$ | 【28， 23,$4 ; 5 \rrbracket_{2}$ | ［28， 0,$28 ; 26 \rrbracket_{2}$ |
| 【29， 28,$2 ; 1 \rrbracket_{2}$ | 【29， 25,$3 ; 4 \rrbracket_{2}$ | 【29， 23,$4 ; 4 \rrbracket_{2}$ | ［29，0，12；1］${ }_{2}$ | 【30，26，3；4］${ }_{2}$ | 【30，25，4；5】 ${ }_{2}$ |
| ［30， 0,$16 ; 12 \rrbracket_{2}$ | ［30， 0,$30 ; 28 \rrbracket_{2}$ | ［31，30，2；1】 ${ }_{2}$ | 【31， 26,$4 ; 5 \rrbracket_{2}$ | ［31，22，5；5】 2 | ［31，16，6；1】 2 |
| ［31，16，9；13】 ${ }_{2}$ | 【31，9，13；16】 ${ }_{2}$ | ［31， 2,$14 ; 11 \rrbracket_{2}$ | 【32，27，3； $3 \rrbracket_{2}$ | ［32，26，4；4］ 2 | ［32，24，5；6】 2 |
| 【32， 0,$32 ; 30 \rrbracket_{2}$ | ［33，32，2；1］${ }_{2}$ | ［33， 28,$4 ; 5 \rrbracket_{2}$ | 【33， 26,$5 ; 7 \rrbracket_{2}$ | ［34，29，3； $3 \rrbracket_{2}$ | ［34， 28,$4 ; 4 \rrbracket_{2}$ |
| ［34， 26,$5 ; 6 \rrbracket_{2}$ | ［34， 24,$6 ; 8]_{2}$ | ［34， 0,$14 ; 8 \rrbracket_{2}$ | ［34， 0,$34 ; 32 \rrbracket_{2}$ | 【35， 34,$2 ; 1 \rrbracket_{2}$ | 【35，30，4；5】 ${ }_{2}$ |
| ［35， 28,$5 ; 7 \rrbracket_{2}$ | 【35，26， $6 ; 9 \rrbracket_{2}$ | ［35，12，8；1］${ }_{2}$ | ［35， 4,$14 ; 11 \rrbracket_{2}$ | 【36，32，3；4］${ }_{2}$ | 【36，30，4；4］${ }_{2}$ |
| ［36，28，5；6】 ${ }_{2}$ | ［36，26， $6 ; 8 \rrbracket_{2}$ | ［36，17，10；13】 ${ }_{2}$ | ［36，9，17；23］${ }_{2}$ | ［36， 0,$36 ; 34 \rrbracket_{2}$ | ［37，36，2；1］${ }_{2}$ |
| ［37，30，5；7】 ${ }_{2}$ | 【37， 28,$6 ; 9 \rrbracket_{2}$ | ［37，14，13；19】 ${ }_{2}$ | ［38， 28,$6 ; 8 \rrbracket_{2}$ | 【38，16，13； $20 \rrbracket_{2}$ | $\llbracket 38,16,14 ; 22 \rrbracket_{2}$ |
| 【38，14， $8 ; 2 \rrbracket_{2}$ | ［38， 0,$38 ; 36 \rrbracket_{2}$ | ［39， 38,$2 ; 1 \rrbracket_{2}$ | 【39，30， $6 ; 9 \rrbracket_{2}$ | 【39，22，10；17】 ${ }_{2}$ | 【39，16，8； $3 \rrbracket_{2}$ |
| 【39，15，11；12】 ${ }_{2}$ | ［39，13，17；26】 ${ }_{2}$ | 【39，12，9；3］${ }_{2}$ | ［39， 0,$18 ; 15 \rrbracket_{2}$ | 【40，36，3；4 ${ }_{2}$ | 【40，31， 4 ；1］${ }_{2}$ |
| 【40，30，5；4 ${ }_{2}$ | 【40，30， $6 ; 8 \rrbracket_{2}$ | 【40，22，10；16】 ${ }_{2}$ | 【40，20，12；20】 ${ }_{2}$ | 【40，18，13；20】 ${ }_{2}$ | 【40，16，14；20】 ${ }_{2}$ |
| ［40， 0,$40 ; 38 \rrbracket_{2}$ | 【41，40， $2 ; 1 \rrbracket_{2}$ | 【41， 36,$3 ; 3 \rrbracket_{2}$ | 【41，32，5；5】 ${ }_{2}$ | 【41，32， $6 ; 9 \rrbracket_{2}$ | 【41，20，7；1】 ${ }_{2}$ |
| 【41，20，13；21】 ${ }_{2}$ | 【41，18，14；21】 ${ }_{2}$ | ［42， 38,$3 ; 4 \rrbracket_{2}$ | 【42，34， $5 ; 6 \rrbracket_{2}$ | 【42，33， $6 ; 9 \rrbracket_{2}$ | ［42，28， $8 ; 14 \rrbracket_{2}$ |
| 【42，22，12；20】 ${ }_{2}$ | 【42，20，11；16】 ${ }_{2}$ | 【42，20，13；20】 ${ }_{2}$ | 【42，20，14；22】 ${ }_{2}$ | 【42，16，9；6】 ${ }_{2}$ | 【42， 0,$16 ; 8 \rrbracket_{2}$ |
| 【42，0，42；40】 ${ }_{2}$ | 【43， 42,$2 ; 1 \rrbracket_{2}$ | 【43，39， $3 ; 4 \rrbracket_{2}$ | 【43，36，5；7】 ${ }_{2}$ | 【43，34，4；3】 2 | 【43，34， $6 ; 9 \rrbracket_{2}$ |
| 【43， 31,$6 ; 8 \rrbracket_{2}$ | 【43， 29,$8 ; 14 \rrbracket_{2}$ | 【43，21，14； $22 \rrbracket_{2}$ | 【43， 0,$18 ; 13 \rrbracket_{2}$ | 【44，39，3；3】 ${ }_{2}$ | 【44，36，4；4］${ }_{2}$ |
| 【44，29， $8 ; 13 \rrbracket_{2}$ | 【44，21，14；21】 ${ }_{2}$ | 【44， 0,$44 ; 42 \rrbracket_{2}$ | 【45，44， $2 ; 1 \rrbracket_{2}$ | 【45，41，3；4］${ }_{2}$ | 【45，31， $8 ; 14 \rrbracket_{2}$ |
| 【45，24，12；21】 ${ }_{2}$ | 【45，22，7；1］${ }_{2}$ | 【46，41，3； $3 \rrbracket_{2}$ | 【46， 38,$4 ; 4 \rrbracket_{2}$ | 【46， 38,$5 ; 8 \rrbracket_{2}$ | 【46，34，6； $6 \rrbracket_{2}$ |
| 【46，28， $8 ; 12 \rrbracket_{2}$ | 【46，32， $8 ; 14 \rrbracket_{2}$ | 【46，19， $9 ; 5 \rrbracket_{2}$ | 【46，17，10；7】 2 | 【46，25，11；19】 ${ }_{2}$ | 【46，24，12；20】 ${ }_{2}$ |
| 【46，1，15；7】 ${ }_{2}$ | 【46，1，19；17】 ${ }_{2}$ | 【46， 0,$46 ; 44 \rrbracket_{2}$ | 【47，46， $2 ; 1 \rrbracket_{2}$ | 【47，41，3；2】 ${ }_{2}$ | 【47，40， $4 ; 5 \rrbracket_{2}$ |
| 【47，33， $8 ; 14 \rrbracket_{2}$ | 【48，43， $3 ; 3 \rrbracket_{2}$ | 【48， 42,$4 ; 6 \rrbracket_{2}$ | 【48，35， $7 ; 13 \rrbracket_{2}$ | 【48， 33,$8 ; 13 \rrbracket_{2}$ | 【48，23，13；21】 ${ }_{2}$ |
| 【48，16，17；26】 ${ }_{2}$ | ［48， 0,$20 ; 16 \rrbracket_{2}$ | ［48， 0,$48 ; 46 \rrbracket_{2}$ | 【49，48， $2 ; 1 \rrbracket_{2}$ | 【49，45， $3 ; 4 \rrbracket_{2}$ | 【49，43，4；6】 ${ }_{2}$ |
| 【49， 35,$8 ; 14 \rrbracket_{2}$ | ［50，46，3；4］${ }_{2}$ | ［50，43， $4 ; 5 \rrbracket_{2}$ | 【50，36，7；12 $\rrbracket_{2}$ | 【50， 35,$8 ; 13 \rrbracket_{2}$ | ［50，19，9；1］${ }_{2}$ |
| ［50， 0,$50 ; 48 \rrbracket_{2}$ | 【51， 50,$2 ; 1 \rrbracket_{2}$ | 【51，46，3；3 $]_{2}$ | 【51，45，4；6】 ${ }_{2}$ | 【51，43， $5 ; 8 \rrbracket_{2}$ | ［51，34，6；1］${ }_{2}$ |
| ［51，38，7；13】 ${ }_{2}$ | 【51，37， $8 ; 14 \rrbracket_{2}$ | ［51，9，12；2】 ${ }_{2}$ | ［52，48，3；4］${ }_{2}$ | ［52，46，4；6】 ${ }_{2}$ | ［52，44，5；8】 ${ }_{2}$ |
| ［52，17，10；1］${ }_{2}$ | ［52， 0,$22 ; 18 \rrbracket_{2}$ | ［52， 0,$52 ; 50 \rrbracket_{2}$ | ［53，52，2；1］${ }_{2}$ | ［53，48，3；3 $]_{2}$ | ［53，47，4；6】 ${ }_{2}$ |
| ［53，44，5；7】 ${ }_{2}$ | ［53，19，10； $2 \rrbracket_{2}$ | ［53，0，16；1］${ }_{2}$ | 【54， 50,$3 ; 4 \rrbracket_{2}$ | ［54，46，5；8】 2 | ［54，44，6；10】 ${ }_{2}$ |
| 【54，0，54； $52 \rrbracket_{2}$ | 【55，54，2；1】 ${ }_{2}$ | ［55，51，3；4］${ }_{2}$ | 【55， 47,$4 ; 4 \rrbracket_{2}$ | 【55， 47,$5 ; 8 \rrbracket_{2}$ | ［56，52，3；4］${ }_{2}$ |
| ［56，48， $5 ; 8 \rrbracket_{2}$ | 【56，0，24；20】 ${ }_{2}$ | 【56，0，56； $54 \rrbracket_{2}$ | 【57， 56,$2 ; 1 \rrbracket_{2}$ | 【57， 52,$3 ; 3 \rrbracket_{2}$ | 【57，49，4；4］${ }_{2}$ |
| ［57，49， $5 ; 8 \rrbracket_{2}$ | ［58，54，3；4］${ }_{2}$ | ［58， 0,$58 ; 56 \rrbracket_{2}$ | ［59，58，2；1］${ }_{2}$ | 【59，55，3；4］${ }_{2}$ | ［60，55，3；3 ${ }_{2}$ |
| ［60， 0,$60 ; 58 \rrbracket_{2}$ | ［61，60，2；1］${ }_{2}$ | 【61， 57,$3 ; 4 \rrbracket_{2}$ | ［61， 0,$18 ; 1 \rrbracket_{2}$ | 【62，58，3；4］${ }_{2}$ | ［62，51，4；1］${ }_{2}$ |
| 【62， 0,$62 ; 60 \rrbracket_{2}$ | ［63， 62,$2 ; 1 \rrbracket_{2}$ | ［63，58，3；3】 2 | ［63，44， $6 ; 1 \rrbracket_{2}$ | 【63，17，13； $4 \rrbracket_{2}$ | ［63，16，14； $5 \rrbracket_{2}$ |
| 【64，60，3；4］${ }_{2}$ | ［64，53，4；1］${ }_{2}$ | ［64，49，5；1］${ }_{2}$ | ［64，40，10；14】 ${ }_{2}$ | 【64，37， $8 ; 1 \rrbracket_{2}$ | 【64，31，9；1】 ${ }_{2}$ |
| ［64，25，11； $1 \rrbracket_{2}$ | ［64，24，12； $2 \rrbracket_{2}$ | $\llbracket 64,21,16 ; 17 \rrbracket_{2}$ | ［64，17，15；9］${ }^{2}$ | $\llbracket 64,16,16 ; 14 \rrbracket_{2}$ | 【64，12，14； $2 \rrbracket_{2}$ |
| 【64，2，22； $12 \rrbracket_{2}$ | 【64，1，20； $9 \rrbracket_{2}$ | $\llbracket 64,1,23 ; 13 \rrbracket_{2}$ | 【64，1，27； $25 \rrbracket_{2}$ | 【64， 0,$24 ; 14 \rrbracket_{2}$ | $\llbracket 64,0,64 ; 62 \rrbracket_{2}$ |

Table 2：A Concise Version of the Parameters of Good Qutrit EAQECCs with $3 \leq n \leq 36$ ．To obtain the full table，one applies the propagation rules in Section 4．We include entries with $c=0$ since a large database for such codes is not yet currently available online．

| ［n，$\kappa, \delta ; c \rrbracket_{3}$ | $\left\lfloor n, \kappa, \delta ; c \rrbracket_{3}\right.$ | ［n，$\kappa, \delta ; c \rrbracket_{3}$ | ［n，$\kappa, \delta ; c \rrbracket_{3}$ | $\left\lfloor n, \kappa, \delta ; c \rrbracket_{3}\right.$ | ［n，$\kappa, \delta ; c \rrbracket_{3}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| ［3， 0,$3 ; 1]_{3}$ | ［5，4，2；1］${ }_{3}$ | ［ $5,2,3 ; 1]_{3}$ | ［ $5,0,4 ; 1]_{3}$ | ［ $5,0,5 ; 3]_{3}$ | ［6，2，4；2】 ${ }_{3}$ |
| ［7， 3,$3 ; 0 \rrbracket_{3}$ | ［7， 0,$6 ; 3]_{3}$ | ［8， 4,$3 ; 0 \rrbracket_{3}$ | ［8， 0,$7 ; 4 \rrbracket_{3}$ | 【10， 6,$3 ; 0 \rrbracket_{3}$ | ［10， 4,$4 ; 0 \rrbracket_{3}$ |
| 【10， 4,$5 ; 2 \rrbracket_{3}$ | 【10，1，6；1】 ${ }_{3}$ | ［10，2，7；4］${ }_{3}$ | 【10， 0,$8 ; 4 \rrbracket_{3}$ | 【10， 0,$9 ; 6 \rrbracket_{3}$ | ［10， 0,$10 ; 8 \rrbracket_{3}$ |
| ［11， 0,$11 ; 9 \rrbracket_{3}$ | 【12， 0,$12 ; 10 \rrbracket_{3}$ | ［13，4，7；5发 | ［13，1，10；8】 ${ }_{3}$ | ［13， 0,$13 ; 11 \rrbracket_{3}$ | 【14， 8,$3 ; 0 \rrbracket_{3}$ |
| 【14， 5,$6 ; 3 \rrbracket_{3}$ | 【14， 2,$9 ; 6 \rrbracket_{3}$ | ［14， 0,$14 ; 12 \rrbracket_{3}$ | 【15， 9,$3 ; 0 \rrbracket_{3}$ | $\llbracket 15,5,5 ; 0 \rrbracket_{3}$ | 【15， 5,$7 ; 4]_{3}$ |
| ［15，4， $8 ; 5 \rrbracket_{3}$ | 【15，0，12；9】 ${ }_{3}$ | ［15， 0,$15 ; 13 \rrbracket_{3}$ | 【16，10，3；0】 ${ }_{3}$ | 【16，9，4；1］${ }_{3}$ | 【16， 7,$5 ; 1 \rrbracket_{3}$ |
| 【16， 6,$6 ; 2 \rrbracket_{3}$ | 【16， 6,$7 ; 4 \rrbracket_{3}$ | 【16， 5,$8 ; 5 \rrbracket_{3}$ | 【16， 4,$9 ; 6 \rrbracket_{3}$ | 【16， 2,$10 ; 6 \rrbracket_{3}$ | 【16，1，11；7】 ${ }_{3}$ |
| 【16，1，12；9】 ${ }_{3}$ | ［16， 0,$13 ; 10 \rrbracket_{3}$ | ［16， 0,$16 ; 14 \rrbracket_{3}$ | 【17，11，3；0】 ${ }_{3}$ | ［17，10， $4 ; 1 \rrbracket_{3}$ | ［17， 9,$5 ; 2 \rrbracket_{3}$ |
| 【17， 8,$6 ; 3 \rrbracket_{3}$ | ［17， 0,$14 ; 11 \rrbracket_{3}$ | ［17， 0,$17 ; 15 \rrbracket_{3}$ | 【18，12，3；0】 ${ }_{3}$ | 【18，11， $4 ; 1 \rrbracket_{3}$ | 【18，10， $5 ; 2 \rrbracket_{3}$ |
| ［18， 6,$8 ; 4 \rrbracket_{3}$ | 【18， 6,$9 ; 6 \rrbracket_{3}$ | ［18， 4,$10 ; 6 \rrbracket_{3}$ | 【18， 0,$18 ; 16 \rrbracket_{3}$ | 【19，13，3；0】 ${ }_{3}$ | ［19，12，4；1］${ }_{3}$ |
| 【19，11， $5 ; 2 \rrbracket_{3}$ | ［19， 2,$13 ; 11 \rrbracket_{3}$ | ［19，0，19；17】 ${ }_{3}$ | ［19， $0,20,19 \rrbracket_{3}$ | 【20，12，5；2】 ${ }_{3}$ | ［20，10，6；4］${ }_{3}$ |
| ［20，9，7；5】 3 | 【20， 6,$10 ; 6 \rrbracket_{3}$ | 【20，5，11；9】 ${ }_{3}$ | 【20，3，12；9】 ${ }_{3}$ | 【20， 0,$15 ; 12 \rrbracket_{3}$ | ［20， 0,$20 ; 18 \rrbracket_{3}$ |
| 【21，15，3；0】 ${ }_{3}$ | 【21，13， $4 ; 0 \rrbracket_{3}$ | 【21，11，6；4】 3 | 【21，10，7；5我 | ［21，2，14；11】 ${ }_{3}$ | 【21， 2,$15 ; 13 \rrbracket_{3}$ |
| ［21，1，16；14］${ }_{3}$ | ［21， 0,$21 ; 19 \rrbracket_{3}$ | 【22，16，3；0】 ${ }_{3}$ | ［22，15， $4 ; 1 \rrbracket_{3}$ | 【22，12，6；4】 ${ }_{3}$ | ［22，11， $7 ; 5 \rrbracket_{3}$ |
| ［22，1，17；15】 ${ }_{3}$ | 【22，0，22；20】 ${ }_{3}$ | 【23，17，3；0】 ${ }_{3}$ | ［23，16， 4 ；1］$]_{3}$ | 【23，9， $8 ; 6 \rrbracket_{3}$ | ［23，4，13；11］${ }_{3}$ |
| ［23，0，23；21］${ }_{3}$ | 【24，13，6；3】 ${ }_{3}$ | ［24，11，8；7】 ${ }_{3}$ | ［24，6，11； $8 \rrbracket_{3}$ | 【24，6，12；10】 ${ }_{3}$ | ［24，0，19；16】 ${ }_{3}$ |
| ［24， 0,$24 ; 22 \rrbracket_{3}$ | 【25，17，4；0】 ${ }_{3}$ | ［25，15，5；2】 ${ }_{3}$ | 【25，9，10；8】 ${ }_{3}$ | 【25，5，14；12】 ${ }_{3}$ | ［25，3，16；14】 ${ }_{3}$ |
| ［25，2，17；15】3 | ［25， 0,$25 ; 23 \rrbracket_{3}$ | 【26，18， $4 ; 0 \rrbracket_{3}$ | 【26，13， $7 ; 5 \rrbracket_{3}$ | 【26，12， $8 ; 6 \rrbracket_{3}$ | 【26，11，9；7】 ${ }_{3}$ |
| ［26，10，10；8】 3 | 【26，7，12；9】 ${ }_{3}$ | ［26，6，14；12】 ${ }_{3}$ | ［26，3，17；15】 ${ }^{\text {a }}$ | 【26，2，18；16】 ${ }_{3}$ | ［26，0，26；24］${ }_{3}$ |
| 【27，19，4；0】 ${ }_{3}$ | 【27，17，5；2】 ${ }_{3}$ | ［27，15，6；2］${ }_{3}$ | 【27，14，7；5】 ${ }_{3}$ | 【27，13， $8 ; 6 \rrbracket_{3}$ | 【27，12，9；7】 ${ }_{3}$ |
| ［27，11，10；8］ 3 | 【27， 8,$12 ; 9 \rrbracket_{3}$ | ［27， 8,$13 ; 11 \rrbracket_{3}$ | ［27，7，14；12】 ${ }_{3}$ | 【27， 4,$16 ; 13 \rrbracket_{3}$ | ［27，3，18；16】 ${ }_{3}$ |
| ［27，2，19；17】 $]^{3}$ | ［27，1，20；18】 ${ }_{3}$ | 【27，$, 27,25 \rrbracket_{3}$ | 【28，20， $4 ; 0 \rrbracket_{3}$ | 【28，14， $8 ; 6 \rrbracket_{3}$ | 【28，13，9；7】 ${ }_{3}$ |
| ［28，12，10；8］ 3 | ［28，11，11；9］${ }^{\text {d }}$ | ［28，10，13；12】 ${ }_{3}$ | 【28，9，12；9】 ${ }_{3}$ | ［28， 8,$14 ; 12 \rrbracket_{3}$ | 【28， 5,$16 ; 13 \rrbracket_{3}$ |
| ［28，4，17；14］${ }_{3}$ | 【28，4，18；16】 ${ }_{3}$ | 【28， 4,$19 ; 18 \rrbracket_{3}$ | ［28，2，20；18】 3 | 【28，1，21；19】 ${ }_{3}$ | 【28， 0,$28 ; 26 \rrbracket_{3}$ |
| 【29，23，3；0】 ${ }_{3}$ | 【29，21， $4 ; 0 \rrbracket_{3}$ | 【29，19，5；2］${ }_{3}$ | 【29，17，6；2】 ${ }_{3}$ | 【29，16，7；5】 ${ }_{3}$ | 【29， 0,$29 ; 27 \rrbracket_{3}$ |
| 【30，23，4；1】 ${ }_{3}$ | 【30，21，6；5】 ${ }^{\text {a }}$ | 【30，18，7；6】 ${ }_{3}$ | 【30，10，15；16】 ${ }_{3}$ | 【30， 0,$12 ; 0 \rrbracket_{3}$ | ［30， 0,$30 ; 28 \rrbracket_{3}$ |
| 【31，25，3；0】 ${ }_{3}$ | 【31，25，4；2】 3 | 【31，22，5；3】 ${ }_{3}$ | 【31，22，6；5】 ${ }_{3}$ | 【31，0，31；29】 3 | 【32，26，4；2】 ${ }_{3}$ |
| 【32，23，5；3】 ${ }_{3}$ | 【32，23， $6 ; 5 \rrbracket_{3}$ | ［32，0，32；30】 ${ }_{3}$ | 【33，27，3；0】 ${ }_{3}$ | 【33，27，4；2】 3 | 【33，25，5；4］${ }_{3}$ |
| ［33，24，6；5】 ${ }_{3}$ | ［33，14，10；9】 3 | 【33，13，13；14】 ${ }_{3}$ | 【33，13，14；16】 ${ }_{3}$ | 【33，12，15；17】 ${ }_{3}$ | 【33，11，16；18】 ${ }_{3}$ |
| ［33，10，17；19】 ${ }_{3}$ | ［33， 8,$18 ; 19 \rrbracket_{3}$ | 【33， 8,$19 ; 21 \rrbracket_{3}$ | 【33，7，20；22】 ${ }_{3}$ | $\llbracket 33,0,33 ; 31 \rrbracket_{3}$ | 【34，28，3；0】 ${ }_{3}$ |
| 【34，26，5；4］${ }_{3}$ | 【34，25， $6 ; 5 \rrbracket_{3}$ | 【34，12，12；10】 ${ }_{3}$ | ［34， 0,$22 ; 18 \rrbracket_{3}$ | ［34，4，23；24］${ }^{\text {d }}$ | ［34，0，34； $32 \rrbracket_{3}$ |
| 【35，30，3；1】 ${ }_{3}$ | 【35，28，4；1】 ${ }_{3}$ | 【35，26，6；5】 ${ }_{3}$ | ［35，15，10；8】 ${ }^{\text {a }}$ | 【35， 0,$35 ; 33 \rrbracket_{3}$ | 【36，31，3；1】 ${ }_{3}$ |
| 【36，30，4；2】 ${ }_{3}$ | 【36，27，5；3】 ${ }_{3}$ | 【36，27，6；5】 ${ }_{3}$ | 【36，20，7；2】 ${ }_{3}$ | 【36，19，10；11】 ${ }_{3}$ | 【36，18，8；4］${ }_{3}$ |
|  | ［36，15，11； $9 \rrbracket_{3}$ | $\llbracket 36,15,14 ; 17 \rrbracket_{3}$ | ［36，14，13；14］${ }_{3}$ | ［36，14，15；18】 ${ }_{3}$ | 【36，13，12；9】 ${ }^{\text {d }}$ |
| $\begin{aligned} & \llbracket 36,11,16 ; 17 \rrbracket_{3} \\ & \llbracket 36,0,36 ; 34 \rrbracket_{3} \\ & \hline \end{aligned}$ | $\llbracket 36,11,18 ; 21 \rrbracket_{3}$ | $\llbracket 36,10,17 ; 18 \rrbracket_{3}$ | 【36，10，19；22】 ${ }_{3}$ | 【36， 8,$20 ; 22 \rrbracket_{3}$ | 【36， 8,$21 ; 24 \rrbracket_{3}$ |

