# Efficient detection for quantum states containing fewer than $k$ unentangled particles in multipartite quantum systems

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In this paper, we mainly investigate the detection of quantum states containing fewer than $k$ unentangled particles in multipartite quantum systems. Based on calculations about operators, we derive two practical criteria for judging $N$-partite quantum states owning fewer than $k$ unentangled particles. In addition, we demonstrate the effectiveness of our frameworks through some concrete examples, and specifically point out the quantum states having fewer than $k$ unentangled particles that our methods can detect, while other criteria cannot recognize.

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## I. INTRODUCTION

Quantum entanglement [1, 2] is a very special phenomenon in quantum systems and has been regarded as an important resource. Based on the properties of quantum entanglement, it can complete some tasks that cannot be completed by traditional methods, such as quantum communication and quantum computing [3], quantum cryptography [4, 5], quantum dense coding [6, 7], quantum teleportation [8].

In the theory of quantum entanglement, one of the most basic problems is to determine whether a quantum state is entangled or separable, and great efforts have been made to try to solve this problem for a long time [9–17]. In multipartite quantum systems, the entanglement of quantum states is often characterized from different perspectives [18], for example, $k$-nonseparability is according to "How many partitions are separable?"; $k$-partite entanglement is based on "How many particles are entangled?". In the past few years, many entanglement criteria have been proposed in terms of tools [19–31], which can only recognize certain multipartite entanglement, not all multipartite entanglement. There is another way to depict entanglement from the number of unentangled particles [18, 32], that is, the quantum states containing fewer than $k$ unentangled particles in $N$-partite quantum systems. When the quantum states violate the criterion based on the variance [33], the quantum states cannot be constructed by one or more unentangled particles. Using quantum Fisher information, any state that violates the framework has fewer than $k$ unentangled particles [34]. Another criterion gives an inequality that a quantum state owning at least $k$ unentangled particles must satisfy [35]. The number of unentangled particles is closely related to metrological usefulness of quantum states [34], so this prompts us to further explore these quantum states fewer than $k$ unentangled particles.

In this paper, we further research the quantum states fewer than $k$ unentangled particles and develop the detection frame of such states. In Section II, we introduce some important concepts and symbols to be used in the following text. In Section III and Section IV, we present our main results, namely the detection methods of quantum states fewer than $k$ unentangled particles, and demonstrate the practicality and operability of our criteria through concrete examples, respectively.

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## II. PRELIMINARY

In an $N$-partite quantum system with Hilber space $\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_N$, the pure state $|\psi\rangle$ contains $k$ unentangled particles if it can be written as $|\psi\rangle=\bigotimes_{i=1}^{k+1}|\psi_{A_i}\rangle$, where $A_1,A_2,\cdots,A_{k+1}$ constitutes a partition of $\{1,2,\cdots,N\}$, $|\psi_{A_i}\rangle$ is a single-partite state for $1\leq i\leq k$, and $|\psi_{A_{k+1}}\rangle$ owns $N-k$ particles [32, 34]. If a mixed state $\rho$ can be represented as a mixture of the pure states containing $k$ or more unentangled particles (may belong to different partitions), then $\rho$ contains at least $k$ unentangled particles [32, 34], otherwise the state $\rho$ has fewer than $k$ unentangled particles.

For Hilbert space $\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_N$ with dim $\mathcal{H}_i=d_i, i=1,2,\cdots,N$, we first introduce the local permutation operator $P_\alpha$ and global permutation operator $P$,

$$P_\alpha(\bigotimes_{i=1}^N x_i)\otimes(\bigotimes_{i=1}^N y_i)=(\bigotimes_{i\in\alpha}y_i\bigotimes_{i\notin\alpha}x_i)\otimes(\bigotimes_{i\in\alpha}x_i\bigotimes_{i\notin\alpha}y_i),$$

$$P(\bigotimes_{i=1}^N x_i)\otimes(\bigotimes_{i=1}^N y_i)=(\bigotimes_{i=1}^N y_i)\otimes(\bigotimes_{i=1}^N x_i),$$

where $\alpha$ is any subset of $\{1,2,\cdots,N\}$, $\bigotimes_{i=1}^N x_i$ and $\bigotimes_{i=1}^N y_i$ are any operators of Hilbert space $\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_N$ with $x_i$ and $y_i$ acting on $\mathcal{H}_i$. In particular, when $\alpha$ is taken as $\{1,2,\cdots,N\}$ or $\{i\}$ with $1\leq i\leq N$, $P_\alpha$ is abbreviated as $P$, $P_i$, respectively.

## III. DETECTION OF QUANTUM STATES CONTAINING FEWER THAN $k$ UNENTANGLED PARTICLES

Now, we state our main results.

*Theorem 1.* In Hilbert space $\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_N$ with dim $\mathcal{H}_i=d_i, i=1,2,\cdots,N$, if a quantum state $\rho$ contains at least $k$ unentangled particles for $1\leqslant k\leqslant N-1$, then we have

$$(2^{k+1}-2)\sqrt{\mathrm{Tr}\big[(X^\dagger\otimes Y^\dagger)\rho^{\otimes 2}P(X\otimes Y)\big]}\leq\sum_{\{\alpha\}}\sqrt{\mathrm{Tr}\big[(X^\dagger\otimes Y^\dagger)P_\alpha^\dagger\rho^{\otimes 2}P_\alpha(X\otimes Y)\big]},\tag{1}$$

where $\{\alpha\}$ consists of nonempty proper subsets of $\{1,2,\cdots,N\}$, $X=\bigotimes_{i=1}^N x_i$ and $Y=\bigotimes_{i=1}^N y_i$ are any operators of Hilbert space $\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_N$. If a quantum state $\rho$ violates inequality (1), then it contains fewer than $k$ unentangled particles.

*Proof.* First we prove that the inequality (1) is true for any pure state. Suppose that pure state $\rho=|\psi\rangle\langle\psi|$ contains at least $k$ unentangled particles, then there exits a partition $A_1|A_2|\cdots|A_{k+1}$ such that $|\psi\rangle=\bigotimes_{m=1}^{k+1}|\varphi_{A_m}\rangle$. By Cauchy-Schwarz inequality, we have

$$\mathrm{Tr}\big[(X^\dagger\otimes Y^\dagger)\rho^{\otimes 2}P(X\otimes Y)\big]=|\langle\psi|YX^\dagger|\psi\rangle|^2\leq\langle\psi|YY^\dagger|\psi\rangle\langle\psi|XX^\dagger|\psi\rangle.\tag{2}$$

For the partition $A_1|A_2|\cdots|A_{k+1}$, let $\alpha_{j_1,\cdots,j_n}$ be a set of any $n$ subsets $A_{j_t}$ (that is, $\alpha_{j_1,\cdots,j_n}=A_{j_1}\bigcup\cdots\bigcup A_{j_n}$), $\overline{\alpha}_{j_1,\cdots,j_n}$ be complement ($\overline{\alpha}_{j_1,\cdots,j_n}=\{1,2,\cdots,N\}-\alpha_{j_1,\cdots,j_n}=A_{j_{n+1}}\bigcup\cdots\bigcup A_{j_{k+1}}$). Thus we get

$$\begin{aligned}
&\mathrm{Tr}\big[(X^\dagger\otimes Y^\dagger)P^\dagger_{\alpha_{j_1,j_2,\cdots,j_n}}\rho^{\otimes 2}P_{\alpha_{j_1,j_2,\cdots,j_n}}(X\otimes Y)\big]\\
=\;&\mathrm{Tr}\big\{[(\bigotimes_{i\in\alpha_{j_1,j_2,\cdots,j_n}}y_i^\dagger)\otimes(\bigotimes_{i\notin\alpha_{j_1,j_2,\cdots,j_n}}x_i^\dagger)](\bigotimes_{m=1}^{k+1}|\varphi_{A_m}\rangle\langle\varphi_{A_m}|)[(\bigotimes_{i\in\alpha_{j_1,j_2,\cdots,j_n}}y_i)\otimes(\bigotimes_{i\notin\alpha_{j_1,j_2,\cdots,j_n}}x_i)]\big\}\\
&\times\mathrm{Tr}\big\{[(\bigotimes_{i\in\alpha_{j_1,j_2,\cdots,j_n}}x_i^\dagger)\otimes(\bigotimes_{i\notin\alpha_{j_1,j_2,\cdots,j_n}}y_i^\dagger)](\bigotimes_{m=1}^{k+1}|\varphi_{A_m}\rangle\langle\varphi_{A_m}|)[(\bigotimes_{i\in\alpha_{j_1,j_2,\cdots,j_n}}x_i)\otimes(\bigotimes_{i\notin\alpha_{j_1,j_2,\cdots,j_n}}y_i)]\big\}\\
=\;&\Big(\prod_{t=1}^n\langle\varphi_{A_{j_t}}|\bigotimes_{i\in A_{j_t}}y_iy_i^\dagger|\varphi_{A_{j_t}}\rangle\prod_{t=n+1}^{k+1}\langle\varphi_{A_{j_t}}|\bigotimes_{i\in A_{j_t}}x_ix_i^\dagger|\varphi_{A_{j_t}}\rangle\Big)\times\Big(\prod_{t=1}^n\langle\varphi_{A_{j_t}}|\bigotimes_{i\in A_{j_t}}x_ix_i^\dagger|\varphi_{A_{j_t}}\rangle\prod_{t=n+1}^{k+1}\langle\varphi_{A_{j_t}}|\bigotimes_{i\in A_{j_t}}y_iy_i^\dagger|\varphi_{A_{j_t}}\rangle\Big)\\
=\;&\langle\psi|YY^\dagger|\psi\rangle\langle\psi|XX^\dagger|\psi\rangle.
\end{aligned}\tag{3}$$

Eq. (2) and Eq. (3) ensure that the inequality (1) holds for any pure state containing at least $k$ unentangled particles.

Let $\rho$ be a mixed state containing at least $k$ unentangled particles, then it can be rewritten as $\rho = \sum_i p_i|\varphi_i\rangle\langle\varphi_i|$ with the pure state $\rho_i = |\varphi_i\rangle\langle\varphi_i|$ containing at least $k$ unentangled particles. Thus we have

$$\begin{aligned}
&(2^{k+1}-2)\sqrt{\mathrm{Tr}\big[(X^\dagger\otimes Y^\dagger)\rho^{\otimes 2}P(X\otimes Y)\big]}\\
\leq&(2^{k+1}-2)\sum_i p_i\sqrt{\mathrm{Tr}(X^\dagger\otimes Y^\dagger)\rho_i^{\otimes 2}P(X\otimes Y)} && (4)\\
\leq&\sum_i p_i\sum_{\{\alpha\}}\sqrt{\mathrm{Tr}(X^\dagger\otimes Y^\dagger)P_\alpha^\dagger\rho_i^{\otimes 2}P_\alpha(X\otimes Y)} && (5)\\
\leq&\sum_{\{\alpha\}}\sqrt{\Big\{\sum_i p_i\mathrm{Tr}[(\bigotimes_{i\in\alpha}y_i^\dagger)\otimes(\bigotimes_{i\notin\alpha}x_i^\dagger)\rho_i(\bigotimes_{i\in\alpha}y_i)\otimes(\bigotimes_{i\notin\alpha}x_i)]\Big\}\Big\{\sum_i p_i\mathrm{Tr}[(\bigotimes_{i\in\alpha}x_i^\dagger)\otimes(\bigotimes_{i\notin\alpha}y_i^\dagger)\rho_i(\bigotimes_{i\in\alpha}x_i)\otimes(\bigotimes_{i\notin\alpha}y_i)]\Big\}} && (6)\\
=&\sum_{\{\alpha\}}\sqrt{\mathrm{Tr}\big[(X^\dagger\otimes Y^\dagger)P_\alpha^\dagger\rho^{\otimes 2}P_\alpha(X\otimes Y)\big]}.
\end{aligned}$$

Here we have used triangle inequality, validity of inequality (1) for any pure state containing at least $k$ unentangled particles, and Cauchy-Schwarz inequality at the inequalities (4), (5) and (6), respectively. The above proof is exactly what we want.

*Theorem 2.* In a Hilbert space $\mathcal{H}^{\otimes N} = \mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_N$ with $\dim\mathcal{H} = d$, any $N$-partite quantum state $\rho$ containing at least $k$ unentangled particles must satisfy the following inequality,

$$\begin{aligned}
&\sum_{\substack{s,t\in\omega\\1\leq i\neq j\leq N}}\sqrt{\mathrm{Tr}\big[(X_i^{s\dagger}\otimes X_j^{t\dagger})\rho^{\otimes 2}P(X_i^s\otimes X_j^t)\big]}\\
\leq&\sum_{\substack{s,t\in\omega\\1\leq i\neq j\leq N}}\sqrt{\mathrm{Tr}\big[(X_i^{s\dagger}\otimes X_j^{t\dagger})P_i^\dagger\rho^{\otimes 2}P_i(X_i^s\otimes X_j^t)\big]}+T(N-k-1)\sum_{\substack{s\in\omega\\1\leq i\leq N}}\sqrt{\mathrm{Tr}\big[(X_i^{s\dagger}\otimes X_i^{s\dagger})\rho^{\otimes 2}(X_i^s\otimes X_i^s)\big]}
\end{aligned} \tag{7}$$

for $2\leq k\leq N-1$, and

$$\mathrm{Tr}\big[(X_i^{s\dagger}\otimes X_j^{t\dagger})\rho^{\otimes 2}P(X_i^s\otimes X_j^t)\big]\leq\mathrm{Tr}\big[(X_i^{s\dagger}\otimes X_j^{t\dagger})P_i^\dagger\rho^{\otimes 2}P_i(X_i^s\otimes X_j^t)\big] \tag{8}$$

for $k=1$. Here $\omega=\{\omega_1,\cdots,\omega_T\}$ is a set of arbitrary $T$ operators in $\mathcal{H}$, $X=\bigotimes_{i=1}^N x_i$ are any operator with $x_i$ acting on subsystem $\mathcal{H}_i$, $X_i^s = (\bigotimes_{m=1}^{i-1}x_m)\otimes\omega_s\otimes(\bigotimes_{m=i+1}^{N}x_m)$ is the operator with $\omega_s$ acting on subsystem $\mathcal{H}_i$ and $x_m$ acting on subsystem $\mathcal{H}_m$ for $m\neq i$. If a quantum state $\rho$ does not satisfy the above inequality, it contains fewer than $k$ unentangled particles.

*Proof.* Suppose that the quantum state $\rho = |\psi\rangle\langle\psi|$ is pure state where $|\psi\rangle = \bigotimes_{m=1}^{k+1}|\varphi_{A_m}\rangle$ contains at least $k$ unentangled particles under the partition $A_1|\cdots|A_{k+1}$ with the subset $A_{k+1}$ owning $N-k$ particles and each of rest subsets $A_m$ owning 1 particle. After calculations, we can easily obtain

$$\begin{aligned}
&\sqrt{\mathrm{Tr}\big[(X_i^{s\dagger}\otimes X_j^{t\dagger})\rho^{\otimes 2}P(X_i^s\otimes X_j^t)\big]}\\
=&\ |\langle\psi|X_j^tX_i^{s\dagger}|\psi\rangle|\\
\leq&\ \sqrt{\langle\psi|X_j^tX_j^{t\dagger}|\psi\rangle\langle\psi|X_i^sX_i^{s\dagger}|\psi\rangle}\\
\leq&\ \frac{\sqrt{\mathrm{Tr}\big[(X_j^{t\dagger}\otimes X_j^{t\dagger})\rho^{\otimes 2}(X_j^t\otimes X_j^t)\big]}+\sqrt{\mathrm{Tr}\big[(X_i^{s\dagger}\otimes X_i^{s\dagger})\rho^{\otimes 2}(X_i^s\otimes X_i^s)\big]}}{2},
\end{aligned}$$

when $i$ and $j$ are both in the subset $A_{k+1}$, where these two inequalities are true by using Cauchy-Schwarz inequality and the mean inequality; and

$$\begin{aligned}
&\sqrt{\mathrm{Tr}\big[(X_i^{s\dagger}\otimes X_j^{t\dagger})\rho^{\otimes 2}P(X_i^s\otimes X_j^t)\big]}\\
=&\ \Big|\langle\psi|X_j^tX_i^{s\dagger}|\psi\rangle\Big|\\
\leq&\ \sqrt{\mathrm{Tr}\big[(X_i^{s\dagger}\otimes X_j^{t\dagger})P_i^\dagger\rho^{\otimes 2}P_i(X_i^s\otimes X_j^t)\big]}
\end{aligned}$$

when $i$ and $j$ belong to different subsets $A_l$, $A_{l'}$. Based on the above two cases, so we can get

$$
\begin{aligned}
&\sum_{\substack{s,t\in\omega\\1\le i\neq j\le N}} \sqrt{\mathrm{Tr}\big[(X_i^{s\dagger}\otimes X_j^{t\dagger})\rho^{\otimes 2}P(X_i^s\otimes X_j^t)\big]}\\
=&\sum_{\substack{s,t\in\omega\\i,j\in A_{k+1},i\neq j}} \sqrt{\mathrm{Tr}\big[(X_i^{s\dagger}\otimes X_j^{t\dagger})\rho^{\otimes 2}P(X_i^s\otimes X_j^t)\big]}+\sum_{\substack{s,t\in\omega\\i\in A_l,j\in A_{l'},l\neq l'}} \sqrt{\mathrm{Tr}\big[(X_i^{s\dagger}\otimes X_j^{t\dagger})\rho^{\otimes 2}P(X_i^s\otimes X_j^t)\big]}\\
\le&\sum_{\substack{s,t\in\omega\\i,j\in A_{k+1},i\neq j}} \frac{\sqrt{\mathrm{Tr}\big[(X_j^{t\dagger}\otimes X_j^{t\dagger})P_j^\dagger\rho^{\otimes 2}P_j(X_j^t\otimes X_j^t)\big]}+\sqrt{\mathrm{Tr}\big[(X_i^{s\dagger}\otimes X_i^{s\dagger})P_i^\dagger\rho^{\otimes 2}P_i(X_i^s\otimes X_i^s)\big]}}{2}\\
&+\sum_{\substack{s,t\in\omega\\i\in A_l,j\in A_{l'},l\neq l'}} \sqrt{\mathrm{Tr}\big[(X_i^{s\dagger}\otimes X_j^{t\dagger})P_i^\dagger\rho^{\otimes 2}P_i(X_i^s\otimes X_j^t)\big]}\\
\le&\sum_{\substack{s,t\in\omega\\1\le i\neq j\le N}} \sqrt{\mathrm{Tr}\big[(X_i^{s\dagger}\otimes X_j^{t\dagger})P_i^\dagger\rho^{\otimes 2}P_i(X_i^s\otimes X_j^t)\big]}+T(N-k-1)\sum_{\substack{s\in\omega\\1\le i\le N}} \sqrt{\mathrm{Tr}\big[(X_i^{s\dagger}\otimes X_i^{s\dagger})\rho^{\otimes 2}(X_i^s\otimes X_i^s)\big]}.
\end{aligned}
$$

This shows that the inequality (7) holds for any pure state containing at least $k$ unentangled particles.

Next, let $\rho=\sum\limits_m p_m\rho_m=\sum\limits_m p_m|\varphi_{A_m}\rangle\langle\varphi_{A_m}|$ be a mixed state with the pure state $\rho_m=|\varphi_{A_m}\rangle\langle\varphi_{A_m}|$ containing at least $k$ unentangled particles, then we have

$$
\begin{aligned}
&\sum_{\substack{s,t\in\omega\\1\le i\neq j\le N}} \sqrt{\mathrm{Tr}\big[(X_i^{s\dagger}\otimes X_j^{t\dagger})\rho^{\otimes 2}P(X_i^s\otimes X_j^t)\big]}\\
\le&\sum_m p_m\sum_{\substack{s,t\in\omega\\1\le i\neq j\le N}} \sqrt{\mathrm{Tr}\big[(X_i^{s\dagger}\otimes X_j^{t\dagger})\rho_m^{\otimes 2}P(X_i^s\otimes X_j^t)\big]} &(9)\\
\le&\sum_m p_m\sum_{\substack{s,t\in\omega\\1\le i\neq j\le N}} \sqrt{\mathrm{Tr}\big[(X_i^{s\dagger}\otimes X_j^{t\dagger})P_i^\dagger\rho_m^{\otimes 2}P_i(X_i^s\otimes X_j^t)\big]}\\
&+T(N-k-1)\sum_m p_m\sum_{\substack{s\in\omega\\1\le i\le N}} \sqrt{\mathrm{Tr}\big[(X_i^{s\dagger}\otimes X_i^{s\dagger})\rho_m^{\otimes 2}(X_i^s\otimes X_i^s)\big]} &(10)\\
\le&\sum_{\substack{s,t\in\omega\\1\le i\neq j\le N}} \sqrt{\big[\sum_m p_m\mathrm{Tr}(X^\dagger\rho_m X)\big]\big[\sum_m p_m\mathrm{Tr}((X_{ij}^{st})^\dagger\rho_m X_{ij}^{st})\big]}+T(N-k-1)\sum_{\substack{s\in\omega\\1\le i\le N}}\mathrm{Tr}\big[(X_i^s)^\dagger\sum_m p_m\rho_m X_i^s\big] &(11)\\
=&\sum_{\substack{s,t\in\omega\\1\le i\neq j\le N}} \sqrt{\mathrm{Tr}\big[(X_i^{s\dagger}\otimes X_j^{t\dagger})P_i^\dagger\rho^{\otimes 2}P_i(X_i^s\otimes X_j^t)\big]}+T(N-k-1)\sum_{\substack{s\in\omega\\1\le i\le N}} \sqrt{\mathrm{Tr}\big[(X_i^{s\dagger}\otimes X_i^{s\dagger})\rho^{\otimes 2}(X_i^s\otimes X_i^s)\big]},
\end{aligned}
$$

where $X_{ij}^{st}=(\bigotimes\limits_{m\neq i\text{ and }m\neq j}x_m)\otimes w_s\otimes w_t$ with $x_m$, $w_s$ and $w_t$ acting on subsystem $\mathcal{H}_m$, $\mathcal{H}_i$ and $\mathcal{H}_j$, respectively. The inequality (9), (10) and (11) are established by triangle inequality, validity of inequality (7) for any pure state containing at least $k$ unentangled particles, and Cauchy-Schwarz inequality, respectively. This shows that the inequality (7) holds for any mixed state containing at least $k$ unentangled particles. We can similarly prove inequality (8).

## IV. ILLUSTRATION

In this section, we will demonstrate the operability and efficiency of our framework by applying it on typical quantum states. It is worth noting that our criteria has better detection performance in the following two explicit examples.

*Example 1.* Consider the family of 8-qubit quantum states $\rho(p)=p|G_8\rangle\langle G_8|+\frac{1-p}{2^8}\mathbf{1}$, with $|G_8\rangle=\frac{|0\rangle^{\otimes 8}+|1\rangle^{\otimes 8}}{\sqrt{2}}$ being 8-qubit GHZ state.

TABLE I: The thresholds of the quantum states containing fewer than $k$ unentangled particles for $\rho(p)=p|G_8\rangle\langle G_8|+\frac{1-p}{2^8}\mathbf{1}$. When $p_k<p\le 1$ and ${p_k}'<p\le 1$, $\rho(p)$ contains fewer than $k$ unentangled particles captured by our Theorem 1 and observation 5 in Ref.[34], respectively.

| $p$ \ $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $p_k$ | 0.4980 | 0.2485 | 0.1241 | 0.0620 | 0.0310 | 0.0155 | 0.0078 |
| ${p_k}'$ | 0.8015 | 0.6279 | 0.4790 | 0.3550 | 0.2557 | 0.1811 | 0.1315 |

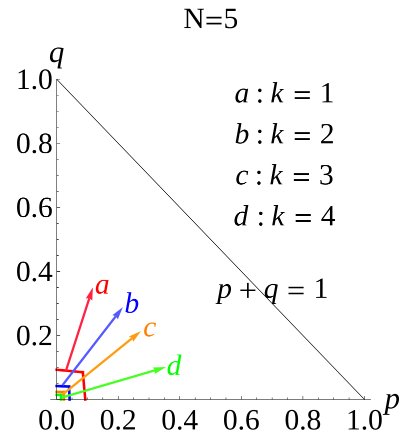


FIG. 1: (Color online) Detection power of Theorem 2 for $\rho(p,q)=p|W\rangle\langle W|+q|\widetilde{W}\rangle\langle\widetilde{W}|+\frac{1-p-q}{d^N}\mathbf{1}$ for $k=1,2,3,4$ when $N=5,d=4$. The area enclosed by red line $a$ ( blue line $b$, orange line $c$, green line $d$ ), the $p$ axis, the $q$ axis and line $p+q=1$ represents quantum states containing fewer than 1 (2, 3, 4) unentangled particles, respectively.

These quantum states $\rho(p)$ contain fewer than $k$ unentangled particles when $p_k<p\le p_k'$, which only can be detected by our Theorem 1 with choosing $x_i=|1\rangle\langle 0|$, $y_i=|0\rangle\langle 0|$, but not by observation 5 in Ref.[34]. This indicates that our Theorem 1 has more efficient detection than observation 5 in Ref.[34] for the family of quantum states $\rho(p)$. The specific values of $p_k$ and $p_k'$ are shown in Table I.

*Example 2.* Considering the family of $N$-partite quantum states,

$$\rho(p,q)=p|W\rangle\langle W|+q|\widetilde{W}\rangle\langle\widetilde{W}|+\frac{1-p-q}{d^N}\mathbf{1},$$

where $d\ge 3$, $|W\rangle=\frac{1}{\sqrt{N(d-1)}}\sum_{i=1}^{d-1}(|0\cdots 00i\rangle+|0\cdots 0i0\rangle+\cdots+|i0\cdots 00\rangle)$ and $|\widetilde{W}\rangle=\sigma^{\otimes N}|W\rangle$ with $\sigma|0\rangle=|1\rangle,\cdots,\sigma|d-2\rangle=|d-1\rangle,\sigma|d-1\rangle=|0\rangle$.

Choose $x_i=|0\rangle\langle 0|$ and $\{\omega_1,\cdots,\omega_T\}=\{|1\rangle\langle 0|,|2\rangle\langle 0|,\cdots,|d-1\rangle\langle 0|\}$ (or $x_i=|1\rangle\langle 1|$ and $\{\omega_1,\cdots,\omega_T\}=\{|0\rangle\langle 1|,|0\rangle\langle 2|,\cdots,|0\rangle\langle d-1|\}$), our Theorem 2 can always detect some quantum states containing fewer than $k$ unentangled particles. Since the observation 5 of Ref.[34] only works for $d=2$, it can't recognize any quantum states containing fewer than $k$ unentangled particles for $d\ge 3$. This indicates that our Theorem 2 is more powerful than observation 5 of Ref.[34] for $N$-partite quantum states $\rho(p,q)$. When $N=5,d=4$, we describe the specific ranges of quantum states containing fewer than $k$ unentangled particles for $k=1,2,3,4$ in Fig. 1.

In particular, for $\rho(p,0)=p|W\rangle\langle W|+\frac{1-p}{d^N}\mathbf{1}$, the quantum states $\rho(p,0)$ contain fewer than $k$ unentangled particles when

$$p>\frac{N(d-1)(2N-k-2)}{kd^N+N(d-1)(2N-k-2)}.$$

Let $p_{(N,k,d)}:=\frac{N(d-1)(2N-k-2)}{kd^N+N(d-1)(2N-k-2)}$. For any $d$, when $1\le k\le N$, the larger $k$ is, the smaller $p_{(N,k,d)}$ is. For any $d$, it's easy to see that $\lim_{N\to+\infty} p_{(N,k,d)} = 0$. This means that as $N$ increases, our Theorem 2 can identify more and more quantum states containing fewer than $k$ ($1 \le k \le N$) unentangled particles for any $d$. For any $N$ and $1 \le k \le N$, it is obvious that $\lim_{d\to+\infty} p_{(N,k,d)} = 0$. This implies that as $d$ increases, our Theorem 2 can detect more and more quantum states containing fewer than $k$ unentangled particles.

## V. CONCLUSIONS

In this work, we propose two practical methods for the detection of quantum states containing fewer than $k$ unentangled particles based on some permutations and operators. These methods are practical in two senses: first, our methods don't involve optimization problems, only require some algebraic operations, this demonstrates their operability; second, our methods can identify some quantum states containing fewer than $k$ unentangled particles that are not recognized by other criteria, this shows our methods can be efficiently applied in practice. As a consequence, the two criteria have good application potential for the detection of quantum states containing fewer than $k$ unentangled particles in multipartite quantum systems.

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