

# Large number of queues in tandem: Scaling properties under back-pressure algorithm

Alexander L. Stolyar  
Bell Labs, Alcatel-Lucent  
600 Mountain Ave., 2C-322  
Murray Hill, NJ 07974  
stolyar@research.bell-labs.com

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## Abstract

We consider a system with  $N$  unit-service-rate queues in tandem, with exogenous arrivals of rate  $\lambda$  at queue 1, under a back-pressure (MaxWeight) algorithm: service at queue  $n$  is blocked unless its queue length is greater than that of next queue  $n + 1$ . The question addressed is how steady-state queues scale as  $N \rightarrow \infty$ . We show that the answer depends on whether  $\lambda$  is below or above the critical value  $1/4$ : in the former case queues remain uniformly stochastically bounded, while otherwise they grow to infinity.

The problem is essentially reduced to the behavior of the system with infinite number of queues in tandem, which is studied using tools from interacting particle systems theory. In particular, the criticality of load  $1/4$  is closely related to the fact that this is the maximum possible flux (flow rate) of a stationary totally asymmetric simple exclusion process.

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## 1 Introduction

In this paper we consider a system with  $N$  single-server queues in tandem, with exogenous Poisson arrival process at queue 1 and customers leaving after service in queue  $N$ . All service times have exponential distribution with unit mean. The waiting room in each queue is unlimited, but the system operates under the following back-pressure policy: service of queue  $n$  is blocked (stopped) unless its queue length is greater than that of “next-hop” queue  $n + 1$ . The system is stable as long as input rate  $\lambda < 1$ . The main question we address is: Given load  $\lambda < 1$  is fixed, how steady-state queue lengths increase (scale) as  $N \rightarrow \infty$ .

Our main motivation comes from the fact that general *back-pressure* (BP) policies (sometimes also called *MaxWeight*), originally introduced by Tassiulas and Ephremides [13] and received much attention in the literature (cf. [12, 4] for recent reviews), are very attractive for application in communication and service networks. This is due to the adaptive nature of BP policies – they can ensure maximum possible network throughput, without a priori knowledge of flow input rates. Key mechanism in BP policies, giving it the adaptivity (and the name), is that the “priority” of a traffic flow  $f$  at a given network node  $n$  is “proportional” to the difference between flow  $f$  queue lengths at node  $n$  and the next node on the route; in particular,

unless this queue-differential is positive, flow  $f$  service at  $n$  can be blocked. This mechanism, however, has a drawback, namely it can lead to a large queue build-up along a flow route, since, roughly speaking, the queue “needs” to increase as we move back from a flow destination node to its source. Such “bad scaling” behavior of BP algorithm is emphasized in [3], and is not very surprising (see also our Proposition 2 and bound (3)). One approach to mitigate this scaling problem in practical systems is proposed in [2, 3], where it is suggested to “run” BP algorithm on virtual queues as opposed to physical ones.

It is of interest to understand fundamental scaling properties of BP algorithm. In this paper, we address this problem for a simple system – a single flow served by  $N$  queues in tandem. We show that, perhaps somewhat surprisingly, the scaling behavior of BP is *not always* bad. Namely, if load is below some critical level,  $\lambda < 1/4$  for our model, all queues remain uniformly stochastically bounded for all  $N$ . (In fact, we show that the stochastic bound has exponentially decaying tail; see Theorem 1(i).) When  $\lambda > 1/4$ , the queues increase to infinity with  $N$  (see Theorem 1(ii)).

The problem of asymptotic behavior of queues as  $N \rightarrow \infty$  is essentially reduced (see Proposition 4) to the behavior of the system with *infinite* number of queues in tandem. Such infinite-tandem model is within the framework of interacting particle systems [7, 8] – methods and results of the corresponding theory will be our main tools. As we will see, the criticality of load  $1/4$  is closely related to the fact that  $1/4$  is the maximum possible flux (average flow rate) of a stationary *totally asymmetric simple exclusion process* (TASEP). In the subcritical case,  $\lambda < 1/4$ , a stationary regime exists such that only a finite (random) number of “left-most” queues can be greater than 1, the rest of the queues have at most one customer and the process “there” behaves like TASEP (see Theorems 8 and 9). In the supercritical case,  $\lambda > 1/4$ , each queue grows without bound with time (see Theorem 10).

Infinite series of queues in tandem is a much studied model in the literature (cf. [11] and references therein), under a variety of assumptions. In particular, [11] studies an infinite system under a class of blocking policies (which are different from the BP policy in our model), where blocking is caused by limited waiting space (finite buffer) in the queues; there, the phenomenon of critical load, below which the system is stable, also exists.

The rest of the paper is organized as follows. Section 2 presents the formal model and main result. The “reduction” of our problem to the behavior of infinite system, and basic properties of the latter, are given in Section 3. The subcritical and supercritical load cases are treated in Sections 4 and 5, respectively. In Section 6 we remark on more general input processes.

## 2 Formal Model and Main Result

Consider a series of  $N$  servers (sites), numbered  $1, \dots, N$ , each with unlimited queueing room. A Poisson flow of customers, of rate  $\lambda > 0$ , arrives at server 1, and each customer has to be served consecutively by the series of servers, from 1 to  $N$ ; after service by  $N$ -th server, a customer leaves the system. The service time of any customer at each server is exponentially distributed, with mean value 1; all service times are independent of each other and of the input process. To be specific, assume that each site serves customers in first-come-first-serve order – given Markov assumptions, this will not limit generality of results.

Let us denote by  $Q_n^{(N)}(t)$ ,  $n = 1, \dots, N$ ,  $t \geq 0$ , the queue length at  $n$ -th server at time  $t$ . The superscript  $N$  indicates the number of servers, which will be the parameter we vary.

Consider the following *back-pressure* (BP) algorithm: site  $n$  is actually serving a customer (which is the head-of-the-line customer from its queue) at time  $t$  if and only if  $Q_n^{(N)}(t) > Q_{n+1}^{(N)}(t)$ . (We use convention that  $Q_{N+1}^{(N)}(t) \equiv 0$ .) In other words, the service at site  $n$  is blocked unless the queue at site  $n + 1$  is smaller.

The random process  $(Q^{(N)}(t) \equiv (Q_n^{(N)}(t), n = 1, 2, \dots, N), t \geq 0)$ , describing evolution of the queues, is a countable irreducible continuous-time Markov chain. Stability of this process - ergodicity of the Markov

chain - is guaranteed under condition  $\lambda < 1$  - this follows from well-known properties of BP algorithms (cf. [4]). Therefore the unique stationary distribution exists; we denote by  $Q^{(N)}(\infty)$  a random system state in the stationary regime.

The question we address is how steady-state queues  $Q_n^{(N)}(\infty)$  grow (scale) as  $N \rightarrow \infty$ , more specifically whether or not they remain stochastically bounded. It is easy to observe that if condition

$$Q_n^{(N)}(t) + 1 \geq \max_{k>n} Q_k^{(N)}(t), \quad \forall n \geq 1, \quad (1)$$

holds for  $t = t_0$ , then it holds for all  $t \geq t_0$  as well; in particular, it does hold in the stationary regime. Thus,  $Q_1^{(N)}(\infty) + 1$  is an upper bound on all queues. Therefore, we can concentrate on the question of whether or not  $Q_1^{(N)}(\infty)$  remains stochastically bounded as  $N \rightarrow \infty$ . (If it does not, it is easy to see that, for any  $n$ ,  $Q_n^{(N)}(\infty)$  goes to infinity in probability.) We show that the answer depends on whether or not the input rate  $\lambda$  is below or above the critical value  $1/4$ . Namely, our main result is the following

**Theorem 1.** (i) *If  $\lambda < 1/4$ , then there exist  $C_1 > 0$  and  $C_2 > 0$  such that, uniformly on  $N$ ,*

$$\mathbb{P}\{Q_1^{(N)}(\infty) > r\} \leq C_1 e^{-C_2 r}. \quad (2)$$

(ii) *If  $\lambda > 1/4$ , then  $Q_1^{(N)}(\infty) \rightarrow \infty$  in probability as  $N \rightarrow \infty$ .*

Statements (i) and (ii) will follow from Theorems 8 and 10, respectively, which concern with the corresponding infinite-tandem system.

Note that the fact that tight uniform bound (2) *cannot* possibly hold for all  $\lambda < 1$  is fairly obvious. Observe that in the stationary regime the average rate at which customers move from site  $n$  to  $n + 1$ , for any  $n \leq N$ , is  $\lambda$ . Therefore, we have

**Proposition 2.** *If  $\lambda < 1$ , then for any  $n = 1, \dots, N$ ,*

$$\mathbb{P}\{Q_n^{(N)}(\infty) > Q_{n+1}^{(N)}(\infty)\} = \lambda.$$

*Then, using (1),*

$$\mathbb{E}Q_n^{(N)}(\infty) - \mathbb{E}Q_{n+1}^{(N)}(\infty) \geq 2\lambda - 1, \quad n = 1, \dots, N.$$

Thus, in the case  $\lambda > 1/2$ , we have at least linear growth of the first queue expected value:

$$\mathbb{E}Q_1^{(N)}(\infty) \geq (2\lambda - 1)N. \quad (3)$$

### 3 Basic Properties of Infinite-Tandem Queues under Back-pressure algorithm

Consider a system just like the one in Section 2, except there is an infinite number of servers (sites) in tandem, indexed  $n = 1, 2, \dots$ . Arriving customers never leave - they just keep moving from site to site, to the "right". We denote by  $Q_n(t)$ ,  $n = 1, 2, \dots$ ,  $t \geq 0$ , the queue length at site  $n$  at time  $t$ , by  $Q(t) \equiv (Q_n(t), n = 1, 2, \dots)$  the state of the entire system at time  $t$ . It will be convenient to assume that the phase space for each queue (site)  $n$  state  $Q_n$  is the compact set  $\bar{\mathbb{Z}}_+ = \mathbb{Z}_+ \cup \{\infty\}$  with  $\mathbb{Z}_+$  being the set of non-negative integers and with metric, e.g.,  $|e^{-i} - e^{-j}|$ . (We use convention  $\infty - 1 = \infty$ , so that the back-pressure algorithm is well-defined.) The state space of Markov process  $(Q(t), t \geq 0)$  is  $\bar{\mathbb{Z}}_+^S$ ,  $S = \{1, 2, \dots\}$ , with product topology; the process is formally defined within the framework of interacting particle systems (cf. Section I.3 of [7], specifically Theorem I.3.9). We will use some slightly abusive notations:  $Q(\cdot)$  for the process  $(Q(t), t \geq 0)$ , and  $Q = (Q_n, n = 1, 2, \dots) \in \bar{\mathbb{Z}}_+^S$  for elements of the phase space; and will denote  $\|Q\| = \sum_n Q_n$ .

Throughout the paper we will also use the following representation of process  $Q(\cdot)$ , which is standard for Markov interacting-particle systems and is convenient for coupling (cf. [7, 8]) of different versions of the processes. First, assume that there is a site associated with each integer  $n \in \mathbb{Z}$ , not just positive  $n$ . (We will use this convention in Section 4.) The underlying probability space is such that there is a unit rate Poisson processes  $\Pi_n$  for each site  $n$ ; these processes are independent from each other and of the input flow Poisson process. Then, if time  $\tau \geq 0$  is a point of the Poisson process  $\Pi_n$ , a customer moves from queue  $n$  to queue  $n + 1$  at  $\tau$  if  $Q_n(\tau-) > Q_{n+1}(\tau-)$ , otherwise the move is suppressed.

The finite system of Section 2, with  $N$  sites, will be viewed as an infinite one, but with the modification that any customer reaching site  $N + 1$  is immediately removed from the system, and with  $Q_n(t) \equiv 0$  for  $n > N$  (and  $n \leq 0$ ).

It is easy to check that an analog of (1) holds for the infinite system as well. Namely, if

$$Q_n(t) + 1 \geq \sup_{k > n} Q_k(t), \quad \forall n \geq 1, \quad (4)$$

holds for  $t = t_0 \geq 0$ , then it holds for all  $t \geq t_0$  as well.

Now we state basic monotonicity properties of the infinite system. (They are easily proved by contradiction, using coupling on the common probability space defined above.) The inequality  $Q \leq Q'$  is understood component-wise; the order relation  $Q \preceq Q'$  means that both  $\|Q\|$  and  $\|Q'\|$  are finite and

$$\sum_{k \geq n} Q_k \leq \sum_{k \geq n} Q'_k, \quad \forall n.$$

**Lemma 3.** *If  $Q(0) \leq Q'(0)$  [respectively,  $Q(0) \preceq Q'(0)$ ], then the processes  $Q(\cdot)$  and  $Q'(\cdot)$  can be coupled so that  $Q(t) \leq Q'(t)$  [respectively,  $Q(t) \preceq Q'(t)$ ] for all  $t \geq 0$ .*

As a corollary of Lemma 3, we obtain the following

**Proposition 4.** *Consider the infinite system and the finite systems, for each  $N = 1, 2, \dots$ , all with zero initial state (with all queues being 0). Then,*

(i) *All corresponding processes can be coupled so that for all  $t \geq 0$ ,*

$$Q^{(1)}(t) \leq Q^{(2)}(t) \leq \dots \leq Q^{(N)}(t) \leq \dots \leq Q(t),$$

$$Q^{(1)}(t) \preceq Q^{(2)}(t) \preceq \dots \preceq Q^{(N)}(t) \preceq \dots \preceq Q(t).$$

(ii) *Process  $Q(\cdot)$  is stochastically non-decreasing in  $t$  (in the sense of  $\leq$  order), and process  $Q^{(N)}(\cdot)$  is stochastically non-decreasing in both  $t$  and  $N$ .*

(iii) *We have convergences in distribution*

$$Q^{(N)}(t) \Rightarrow Q^{(N)}(\infty), \quad \text{as } t \rightarrow \infty, \quad (5)$$

$$Q^{(N)}(\infty) \Rightarrow Q(\infty), \quad \text{as } N \rightarrow \infty, \quad (6)$$

where the sequence in the left-hand side is stochastically non-decreasing and  $Q(\infty)$  just denotes its limit,

$$Q(t) \Rightarrow Q(\infty), \quad \text{as } t \rightarrow \infty. \quad (7)$$

(iv) *The distribution of the limit  $Q(\infty)$  is a stationary distribution (namely, the lower invariant measure) of Markov process  $Q(\cdot)$ .*

(v) *Condition (4) holds for all  $t \geq 0$  and for the stationary state  $Q(\infty)$ .*

Proposition 4(iii) of course implies that  $Q_1^{(N)}(\infty)$  is stochastically non-decreasing, converging in distribution to  $Q_1(\infty)$  and therefore, to prove Theorem 1, we can study the distribution of  $Q_1(\infty)$ . (If it happens that  $Q_1(\infty) = \infty$ , this implies  $Q_1^{(N)}(\infty) \rightarrow \infty$  in probability as  $N \rightarrow \infty$ .)

We will need one more monotonicity property, which is also a corollary of Lemma 3. Its meaning is very simple: if in addition to process  $Q(\cdot)$  we consider another process  $Q'(\cdot)$  which is constructed the same way as  $Q(\cdot)$ , but with some additional exclusions (“obstructions”) on the movement of the customers, then  $Q'(\cdot)$  will stay “behind”  $Q(\cdot)$  in the sense of  $\preceq$  order.

**Proposition 5.** *Consider a fixed realization of the process  $Q(\cdot)$ , which includes a fixed finite initial state  $Q(0)$  and realizations of the input process (at site 1) and of all processes  $\Pi_n$ ,  $n \geq 1$ . Now, suppose further that in the realizations of processes  $\Pi_n$ , some of the points (jumps) are marked as “valid” (in an arbitrary way) and the remaining points are “invalid”. Consider another realization  $Q'(\cdot)$ , with the same initial state  $Q'(0) = Q(0)$ , and constructed in the same way as  $Q(t)$ , except invalid points of processes  $\Pi_n$  are “ignored” (cause no action). Then,*

$$Q'(t) \preceq Q(t), \quad \forall t \geq 0,$$

and, in particular,  $Q'_1(t) \geq Q_1(t)$  for all  $t$ .

## 4 Subcritical case: $\lambda < 1/4$

Suppose  $\lambda < 1/4$ . We will construct a process  $Q'(\cdot)$ , coupled with  $Q(\cdot)$  so that Proposition 5 holds path-wise, and such that we can obtain a stochastic upper bound on  $Q'_1(t)$ .

Consider process  $Q(\cdot)$ , with zero initial state  $\|Q(0)\| = 0$ , constructed on the probability space described in Section 3. We will extend the probability space to define a stationary *totally asymmetric simple exclusion process* (TASEP, cf. Chapter VIII of [7]), with sites being integers  $n \in \mathbb{Z}$ , and particles moving to the “right”. Specifically, let us choose arbitrary (density)  $\rho \leq 1/2$ , such that  $\mu = \rho(1 - \rho) > \lambda$ . Let  $Y_n(t) \in \{0, 1\}$  denote the number of particles of TASEP at site  $n$  at time  $t \geq 0$ . We augment the probability space so that, independently of all other driving processes, at time 0 each site  $n \in \mathbb{Z}$  contains a particle,  $Y_n(0) = 1$ , with probability  $\rho$  and does not contain one,  $Y_n(0) = 0$ , with probability  $1 - \rho$ . The movement of TASEP particles will be driven by the same Poisson processes  $\Pi_n$ , that drive process  $Q(\cdot)$ . (The exogenous input process at site 1 does *not* affect TASEP.) If time  $\tau$  is a point (jump) of  $\Pi_n$  associated with site  $n$ , then the particle located at  $n$  (if any) attempts to jump to site  $n + 1$  – it actually does jump if site  $n + 1$  is empty, and it stays at  $n$  otherwise. It is well known that if the initial state  $Y(0)$  has Benoulli distribution as defined above, then the TASEP process  $Y(\cdot)$  is stationary (cf. Theorem VIII.2.1 of [7]). The flux of this process, i.e. the average departure rate of particles from a given site, is  $\mu = \rho(1 - \rho)$ ; the average speed of a given (“tagged”) particle is  $v = 1 - \rho$ . (For example,  $\rho = 1/2$  gives the maximum possible flux  $\mu = 1/4 > \lambda$ ; this is where the condition  $\lambda < 1/4$  comes from:  $\lambda$  needs to be less than the flux of a stationary TASEP.)

The process  $Q'(\cdot)$  has the same (zero) initial state as  $Q(\cdot)$ , and is constructed the same way as  $Q(\cdot)$  except for an additional exclusion: a customer (particle) from queue (site) 1 cannot move to queue 2 at time  $\tau$  unless a particle of the TASEP jumps from site 1 to 2 at  $\tau$ . We now record basic properties of process  $Q'(\cdot)$ .

**Proposition 6.** (i)  $Q'(t) \preceq Q(t)$  and  $Q'_1(t) \geq Q_1(t)$  for all  $t \geq 0$  (by Proposition 5).

(ii) For any  $n \geq 2$  and any  $t \geq 0$ ,  $Q'_n(t) \leq Y_n(t)$ . In other words, at all sites to the right of 1, process  $Q'(\cdot)$  stays “within TASEP”; in particular, there can be at most one particle in each site  $n \geq 2$ .

(iii) A particle jump from site 1 to 2 in the process  $Q'(\cdot)$  happens at time  $\tau$  if and only if  $Q'_1(\tau-) \geq 1$  and there is a jump of TASEP particle from 1 to 2 at time  $\tau$ .

(iv) Process  $Q'(\cdot)$  is stochastically non-decreasing with  $t$ .

We know from Proposition 5 that  $Q'_1(t)$  is an upper bound of  $Q_1(t)$ . The behavior of queue length  $Q'_1(t)$  is such that it is initially zero,  $Q'_1(0) = 0$ , the input process is Poisson with rate  $\lambda$ , and the “service process” is the stationary process of TASEP particle jumps from site 1 to 2. We denote by  $A(t_1, t_2)$  and  $S(t_1, t_2)$  the number of points (jumps) of the arrival and service processes, respectively, in the interval  $(t_1, t_2]$ ; WLOG we assume that these processes are defined for all real times, that is  $t_1, t_2 \in \mathbb{R}$ ,  $t_1 \leq t_2$ ; their average rates are  $\lambda$  and  $\mu = ES(t_1, t_2)/(t_2 - t_1)$ , respectively. From the large deviations estimate given below in Lemma 7, it also follows that  $S(-s, 0)/s \rightarrow \mu$ ,  $s \rightarrow \infty$ , with probability 1.

From classical Loynes constructions [10] it is known that the distribution of  $Q'_1(t)$  is stochastically non-decreasing with  $t$ , and as  $t \rightarrow \infty$  it weakly converges to the stationary distribution, which in turn is same as that of random variable

$$Q'_1(\infty) \doteq \sup_{s \geq 0} [A(-s, 0) - S(-s, 0)]. \quad (8)$$

$Q'_1(\infty)$  is a proper random variable due to condition  $\lambda < \mu$ , which guarantees that the RHS of (8) is finite w.p.1.

Thus, we see that, as  $t \rightarrow \infty$ ,  $Q'_1(t)$  (and then  $Q_1(t)$ ) remains stochastically bounded by  $Q'_1(\infty)$ . Moreover, the large deviations estimates (Lemma 7) imply exponential bound on the tail of  $Q'_1(\infty)$  distribution. We proceed with the details.

The following fact is a known property of the stationary TASEP defined above. At time 0, let us consider the site with smallest index  $n_0 \geq 2$  that contains a particle, and tag this particle. (Obviously,  $n_0 - 2$  has geometric distribution.) If we consider the point process of jumps of tagged particle in time interval  $[0, \infty)$ , it is a Poisson process of rate  $v = 1 - \rho$  (cf. Corollary VIII.4.9 of [7]). Therefore, the location  $H(t)$  of tagged particle at time  $t \geq 0$  is  $H(t) = 2 + H_1 + H_2(t)$ , where  $H_1$  is geometric random variable with mean  $(1 - \rho)/\rho$ ,  $H_2(t)$  is Poisson r.v. with mean  $vt$ , and  $H_1$  and  $H_2(t)$  are independent. From the stationarity of TASEP we also know that, at any time  $t$ , the total number  $G(n)$  of particles at sites  $2, 3, \dots, n$  is simply the sum of  $n - 1$  independent Bernoulli variables with mean  $\rho$ . Using “separate” large deviations estimates for  $H(t)$  and  $G(n)$ , even though these two r.v. are not independent, we obtain the following

**Lemma 7.** *For any  $\delta > 0$ , there exist  $C_3 > 0$  and  $C_4 > 0$  such that*

$$\mathbb{P}\{|S(0, t) - \mu t| > \delta t\} \leq C_3 e^{-C_4 t}. \quad (9)$$

*Proof.* To prove bound

$$\mathbb{P}\{S(0, t) < (\mu - \delta)t\} \leq C_3 e^{-C_4 t}$$

we can choose  $\epsilon > 0$ , small enough, so that

$$\mathbb{P}\{S(0, t) < (\mu - \delta)t\} \leq \mathbb{P}\{H(t) \leq (v - \epsilon)t\} + \mathbb{P}\{G((v - \epsilon)t) \leq (\rho - \epsilon)(v - \epsilon)t\}.$$

Bound

$$\mathbb{P}\{S(0, t) > (\mu + \delta)t\} \leq C_3 e^{-C_4 t}.$$

is proved similarly. □

**Theorem 8.** *Assume  $\lambda < 1/4$ . There exist  $C_1 > 0$  and  $C_2 > 0$  such that*

$$\mathbb{P}\{Q'_1(\infty) > r\} \leq C_1 e^{-C_2 r}; \quad (10)$$

and then

$$\mathbb{P}\{Q_1(\infty) > r\} \leq C_1 e^{-C_2 r}.$$

**Remark.** Given the large deviations bound (9), the argument to prove (10) is quite standard (see [6, 5]). However, formally, [5] for example, requires a stronger condition, large deviations principle (LDP) for  $S(0, t)/t$ ; we did not find this LDP result in the literature and it is not needed for our purposes. Hence, for completeness, we give a proof of the theorem.

*Proof.* It follows from the definition (8) that for any fixed  $d > 0$

$$Q'_1(\infty) \leq \sup_{k=1,2,\dots} [A(-kd, 0) - S(-(k-1)d, 0)] \equiv A(-d, 0) + \sup_{k=1,2,\dots} [A(-kd, -d) - S(-(k-1)d, 0)] \quad (11)$$

Let us fix  $b > 0$  such that  $b\lambda < 1$ , and for each  $r > 0$  we will choose  $d = br$ . Then we can write

$$\mathbb{P}\{Q'_1(\infty) > r\} \leq \mathbb{P}\{A(-br, 0) \geq r\} + \sum_{k=2,3,\dots} \mathbb{P}\{A(-kd, -d) - S(-(k-1)d, 0) \geq 0\}.$$

If we fix  $\delta > 0$  small enough so that  $\lambda + \delta < \mu - \delta$  and  $(\lambda + \delta)b < 1$ , we have

$$\begin{aligned} \mathbb{P}\{Q'_1(\infty) > r\} &\leq \mathbb{P}\{A(-br, 0) \geq (\lambda + \delta)br\} + \\ &\sum_{k=2,3,\dots} [\mathbb{P}\{A(-kd, -d) \geq (\lambda + \delta)(k-1)d\} + \mathbb{P}\{S(-(k-1)d, 0) \leq (\mu - \delta)(k-1)d\}]. \end{aligned}$$

It remains to apply large deviations bounds on  $A(\cdot)$  and  $S(\cdot)$  (Lemma 7).  $\square$

Let us define

$$\begin{aligned} B(Q(t)) &\doteq \min\{n = 1, 2, \dots \mid Q_n(t) = 0\}, \\ B'(Q(t)) &\doteq \min\{n = 1, 2, \dots \mid \sum_{k=1}^n Q_k(t) < n\}. \end{aligned}$$

$B(Q(t))$  is the “busy interval” of the process  $Q(t)$ : to the right of, and including, site  $B(Q(t))$  all sites have at most one customer and so the (instantaneous) evolution of the process follows the same “rules” as that of TASEP. The interpretation of  $B'(Q(t))$  is as follows: starting from the state  $Q(t)$ , we take a customer from the left-most site with 2 or more customers, and move it to the left-most empty site, and then repeat until all sites have at most 1 customer; then site  $B'(Q(t))$  is the left-most empty site of the modified state.

Obviously,  $B'(Q(t)) \geq B(Q(t))$ ; and it is easy to check that  $Q'(t) \preceq Q(t)$  implies  $B'(Q'(t)) \geq B'(Q(t))$ . Thus,  $B'(Q'(t)) \geq B(Q(t))$ .

**Theorem 9.** *Assume  $\lambda < 1/4$ . There exist  $C_5 > 0$  and  $C_6 > 0$  such that, for all  $t \geq 0$*

$$\mathbb{P}\{B'(Q'(t)) > n\} \leq C_5 e^{-C_6 n},$$

and then

$$\mathbb{P}\{B(Q(t)) > n\} \leq C_5 e^{-C_6 n}.$$

*Proof.* The result is easily derived from the following facts:  $Q'_1(t)$  is stochastically bounded by  $Q'_1(\infty)$ ; the distribution of  $Q'_1(\infty)$  has exponentially decaying tail (Theorem 8); process  $Q'(t)$  “stays within” stationary TASEP of density  $\rho$  at all sites  $n \geq 2$ ; definition of  $B'$ .  $\square$

Theorem 9 illustrates in particular the fact that (when  $\lambda < 1/4$ ) the infinite-tandem system under BP algorithm in stationary regime is such that there only a “small” number of sites (from site 1 to site  $B(Q(t))$ ) where queue can be greater than 1; while all sites to the right of  $B(Q(t))$  have queue of at most one, and therefore the behavior of the process “to the right of  $B(Q(t))$ ” is same as that of TASEP.

## 5 Supercritical case: $\lambda > 1/4$

Note that if  $\lambda > 1/2$  we immediately see from (3) that  $\mathbb{E}Q_1(\infty) = \lim_N \mathbb{E}Q_1^{(N)}(\infty) = \infty$ . Here we prove that, in fact,  $Q_1(\infty)$  is infinite w.p.1, under a weaker condition  $\lambda > 1/4$ . The intuition behind our argument is as follows. Unless  $Q_1(\infty) = \infty$  w.p.1, busy interval  $B(Q(t))$  must be stochastically bounded w.p.1. Then, in stationary regime, all sites “far enough” to the right have at most one customer (particle) in them, and therefore the process “there” behaves as TASEP. However, a stationary TASEP cannot have flux greater than  $1/4$ , while the flux of our process must be  $\lambda > 1/4$ , a contradiction.

**Theorem 10.** *Assume  $\lambda > 1/4$ . Then  $Q_1(\infty) = \infty$ . (And then  $Q_n(\infty) = \infty$  for all  $n \geq 1$ .)*

*Proof.* The proof is by contradiction – assume  $Q_1(\infty)$  is finite with positive probability. It is easy to see that this is possible only if  $Q_1(\infty) < \infty$  with probability 1. (Otherwise, using the facts that  $Q_1(t)$  is stochastically increasing and weakly converges to  $Q_1(\infty)$ , and using coupling, we could show that  $Q_1(\infty) = \infty$  with prob. 1.) This in turn implies that  $B(Q(\infty))$  (which is the weak limit of stochastically non-decreasing r.v.  $B(Q(t))$ ) is also finite w.p.1. (Otherwise, we again could show that  $Q_1(\infty) = \infty$  with prob. 1.)

The stationary version of the process  $Q(\cdot)$  (i.e. the one with stationary distribution equal to that of  $Q(\infty)$ ) we denote by  $\tilde{Q}(\cdot)$ . This process is such that: w.p.1 condition (4) holds for all  $t \geq 0$ , and therefore all  $\tilde{Q}_n(t)$  for all  $t$  are uniformly stochastically upper bounded by  $Q_1(\infty) + 1$  and then finite w.p.1;  $B(\tilde{Q}(t))$  are finite w.p.1 (equally distributed) random variables for all  $t$ ; the flux is equal to  $\lambda$ , namely,  $\mathbb{E}F_n(t)/t = \lambda$  for any  $t > 0$  and  $n \geq 1$ , where  $F_n(t)$  is the number of customers arrivals at site  $n$  in interval  $(0, t]$ . Note that for  $n \geq 2$ , the arrival process  $F_n(\cdot)$  at site  $n$  is the departure process from site  $n - 1$ . This implies that increments of processes  $F_n(\cdot)$ ,  $n \geq 2$ , are stochastically upper bounded by the increments of independent Poisson processes  $\Pi_{n-1}(\cdot)$ .

Consider space-shifted processes  $\{[T_m\tilde{Q}](\cdot), [T_mF](\cdot), [T_m\Pi](\cdot)\}$ ,  $m = 1, 2, \dots$ , where  $[T_m\tilde{Q}]_i(t) = \tilde{Q}_{m+i}(t)$ ,  $i = 1, 2, \dots$ ,  $t \geq 0$ , and  $[T_mF]_i(t)$  and  $[T_m\Pi]_i(t)$  defined similarly. For each  $m$  this process is such that  $[T_m\tilde{Q}](\cdot)$  and the increments of  $[T_mF](\cdot)$  and  $[T_m\Pi](\cdot)$  are stationary. (Note that process  $[T_m\tilde{Q}](\cdot)$  is *not* Markov.) This process is still well-defined if we assume that each component  $[T_m\tilde{Q}]_i(t)$ ,  $[T_mF]_i(t)$  and  $[T_m\Pi]_i(t)$  takes values in (non-compact) space  $\mathbb{Z}_+$  with the usual topology (because we know that they are finite w.p.1), and with corresponding product topology in the space of values of the entire process. Note that, since  $B(\tilde{Q}(t))$  is finite w.p.1 and condition (4) holds, we have

$$\lim_{m \rightarrow \infty} \mathbb{P}\{\sup_{n \geq 1} [T_m\tilde{Q}]_n(t) \leq 1\} = 1, \quad \forall t \geq 0. \quad (12)$$

Then, using properties of process  $\tilde{Q}(\cdot)$  described earlier, in particular the fact that increments of  $F_n(\cdot)$  are bounded by those of  $\Pi_{n-1}(\cdot)$ , it is easy to see that a process consisting of any finite subset of components  $[T_m\tilde{Q}]_i(\cdot)$ ,  $[T_mF]_i(\cdot)$  and  $[T_m\Pi]_i(\cdot)$  is tight (cf. Theorem 15.6 in [1]). Consequently, there exists a subsequence of  $\{m\}$  along which the shifted process converges in distribution to a process  $\{\overline{Q}(\cdot), \overline{F}(\cdot), \overline{\Pi}(\cdot)\}$ , which has the following (easily verifiable) structure and properties:

- (a)  $\overline{Q}(\cdot)$  is stationary, with flux equal  $\lambda$ ;
- (b)  $\overline{Q}_n(t) \leq 1$  for all  $n$  and all  $t$ ;
- (c) the movement of customers between sites is driven by independent, unit rate Poisson processes  $\overline{\Pi}_i(\cdot)$ , according to BP algorithm rules;
- (d) By (b) and (c),  $\overline{Q}(\cdot)$  is a TASEP (with the “exogenous” arrivals at site 1 forming a stationary process, *not* independent of the “rest of the process”).

Consider the following projection of process  $\overline{Q}(\cdot)$ . All particles arriving at site 1 after time 0 and the particle located at site 1 at time 0 (if any), we will call “new” particles, while all particles initially present at sites  $n \geq 2$  are “old”. Let  $Q^*(\cdot)$  denote the process “keeping track” of new particles in  $\overline{Q}(\cdot)$ , namely  $Q_n^*(t) = 1$  if there is a new particle located at site  $n$  at time  $t$ , and  $Q_n^*(t) = 0$  otherwise. The flux of process  $Q^*(\cdot)$  from site 1 to site 2 is obviously equal to the flux of  $\overline{Q}(\cdot)$ , which is  $\lambda$ . We will compare  $Q^*(\cdot)$  to the following TASEP  $Q^{**}(\cdot)$ , coupled to it – with the same Poisson processes driving movement between sites. The initial state of  $Q^{**}(\cdot)$  is:  $Q_1^{**}(0) = 1$  and  $Q_n^{**}(0) = 0$  for  $n \geq 2$ . By definition,  $Q_1^{**}(t) \equiv 1$ , that is if at any time a particle moves from site 1 to 2, it is immediately replaced at site 1 by another particle. Using the path-wise monotonicity properties given in Section 3, it is easy to see that  $Q^*(t) \preceq Q^{**}(t)$ , which implies

$$F_2^*(t) \leq F_2^{**}(t), \quad t \geq 0,$$

where  $F_2^*(t)$  and  $F_2^{**}(t)$  are the numbers of particle arrivals in  $(0, t]$  at site 2 in the processes  $Q^*(\cdot)$  and  $Q^{**}(\cdot)$ , respectively. Then,

$$\liminf_{t \rightarrow \infty} \frac{\mathbb{E}F_2^{**}(t)}{t} \geq \lim_{t \rightarrow \infty} \frac{\mathbb{E}F_2^*(t)}{t} = \lambda. \quad (13)$$

The TASEP  $Q^{**}(\cdot)$  is a special case of one of the processes studied in [9]. It is known (see Theorem 1.8(a) and Theorem 1.7(b) of [9]) that the distribution of  $Q^{**}(t)$  converges to a stationary distribution, with

the corresponding stationary process having flux  $1/4$ . This means that  $\lim_{t \rightarrow \infty} \mathbb{E}F_2^{**}(t)/t = 1/4$ , which contradicts (13) since  $\lambda > 1/4$ . Proof is complete.  $\square$

## 6 Remark on more general input processes

The Poisson assumption on the input process is adopted to simplify exposition. We believe our main results can be easily generalized for the case of a stationary ergodic input process  $A(\cdot)$ , as long as large deviations bound

$$\mathbb{P}\{|A(0, t) - \lambda t| > \delta t\} \leq C_3 e^{-C_4 t}, \quad (14)$$

analogous to (9), holds for any  $\delta > 0$ . Moreover, if (14) does not hold, and we only have the ergodicity of  $A(\cdot)$ , the uniform stochastic boundedness results of Theorem 8 (and then Theorem 1(i)) and Theorem 9 will still hold, except the bounds are proper (finite w.p.1) random variables, not necessarily with exponential tails.

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