LÉVY-DRIVEN GPS QUEUES WITH HEAVY-TAILED INPUT

K. DĘBICKI, P. LIU, M. MANDJES, AND I. SIERPIŃSKA-TUŁACZ

ABSTRACT. In this paper we derive exact large-buffer asymptotics for a two-class Generalized Processor Sharing (GPS) model, under the assumption that the input traffic streams generated by both classes correspond to heavy-tailed Lévy processes. Four scenarios need to be distinguished, which differ in terms of (i) the level of heavy-tailedness of the driving Lévy processes as well as (ii) the values of the corresponding mean rates relative to the GPS weights. The derived results are illustrated by two important special cases, in which the queues' inputs are modeled by heavy-tailed compound Poisson processes and by α-stable Lévy motions.

KEYWORDS. Lévy process, fluid model, queue, general processor sharing, exact asymptotics.

1. INTRODUCTION

In queueing resources that are shared by multiple traffic streams, smooth streams potentially experience poor performance when they are mixed with less regular streams. Indeed, under a first-come-first-serve (FCFS) discipline, users that correspond to a highly variable input process may negatively affect the quality-of-service of other users. This motivates the attention paid to more sophisticated queueing disciplines, in which firm (per-user) performance guarantees can be given. One such a policy is the *generalized processor sharing* (GPS) discipline. In GPS all users classes are guaranteed a certain service rate, whereas the residual capacity is distributed according to a given allocation rule. The earliest (packet-based) implementations of GPS, usually referred to as *weighted fair queueing* (WFQ), date back to the late 1980s [8].

In many real-life systems, input streams may exhibit rather extreme types of irregularities. For instance in the domain of communication networks, measurement studies show that traffic patterns are typically heavy-tailed, in that there is a relatively high likelihood of an extremely large amount of traffic being generated over a short time interval. Under FCFS all streams would perceive roughly the same performance, which is essentially determined by the input class with the heaviest tail. GPS can be considered as a viable way to remedy this complication, by offering each class a guaranteed service rate.

In this paper we consider a two-class GPS system, in which the inputs are Lévy processes with heavy-tailed marginals; in our context, 'heavy-tailed' refers to the complementary

distribution function having a regularly varying tail. This class of Lévy processes covers many practically relevant processes. In the first place, it contains the class of compound Poisson processes, in which independent and identically distributed (i.i.d.) regularly-varying jobs arrive according to a Poisson process. In the second place, it covers the class of α-stable Lévy motion; this class is particularly relevant, as it appears as the limiting process for random walk models with increments that have infinite variance [14, 15].

Our main findings are the exact asymptotics of the tail distributions of both queues. More specially, with Q_i denoting the stationary workload of the i-th queue, we find explicit functions $f_i(\cdot)$ such that $\mathbb{P}(Q_i > u)/f_i(u) \to 1$ as $u \to \infty$; we write $\mathbb{P}(Q_i > u) \sim f_i(u)$. As it turns out, depending on the interplay between the heaviness of both inputs' tail distributions and the stability of the queues while working in isolation, one can distinguish four scenarios, each of them leading to qualitatively different asymptotics. The resulting asymptotics lend themselves to an intuitive explanation, in that they reveal the most likely way that the workload under consideration exceeds u, for u large. The proofs rely on combining bounds that were derived earlier for related queueing systems, as well as a set of newly derived inequalities. Related results for settings that are special cases of ours can be found in e.g. [2, 9], whereas in [11] the focus is on GPS systems with Gaussian inputs.

The paper is organized as follows. Notation, assumptions and preliminaries are presented in Section 2. Then Section 3 states the main results, in terms of the exact asymptotics for all four scenarios. These results are used in Section 4 to give the corresponding expressions for the compound Poisson and α-stable cases. All proofs are given in Section 5.

2. NOTATION AND MODEL DESCRIPTION

In this paper we consider a queueing system that consists of two queues and one server. Each queue, which has infinite storage capacity, is fed by an own traffic class; the corresponding input processes are assumed to be mutually independent. The total service rate of the server is $c > 0$. Class i is assigned a guaranteed service rate $\phi_i c > 0$ (or 'weight'), where $\phi_1 c + \phi_2 c = c$. This effectively means that if both classes are backlogged, then class i is served at rate $\phi_i c$, for $i = 1, 2$. If class i has no backlog, then the other class obtains the excess service rate.

Throughout this paper, we intensively use the concept of *cumulative input processes*. We define by $Z_i(s, t)$ the cumulative input to queue i in interval $(s, t]$, for $i = 1, 2$ and $s < t$. We assume that

$$Z_i(s, t) = Z_i(t) - Z_i(s), \quad i = 1, 2,$$

where $\{Z_1(t) : t \in \mathbb{R}\}$ and $\{Z_2(t) : t \in \mathbb{R}\}$ are mutually independent Lévy processes.

As pointed out in the introduction, we specially consider the situation in which the Lévy input processes are heavy-tailed. In more concrete terms, this means that in the sequel we impose the following assumptions:

A1 $\mathbb{P}(Z_i(1) > u) \sim u^{-\alpha_i} L_i(u)$, with $\alpha_i > 1$ and $L_i(\cdot)$ slowly varying at ∞, for $i = 1, 2$;
A2 $\mathbb{E}[Z_i(1)] = \mu_i$, with $\mu_1 + \mu_2 = \mu < c$.

We let $\{Q_i(t) : t \geq 0\}$ denote the stationary buffer content processes for class i, for $i = 1, 2$. Observe that condition **A2** guarantees stability of the system, implying existence of the stationary buffer content processes. To shorten the notation we throughout write

$$Q_i \stackrel{\mathrm{d}}{:=} Q_i(0), \quad i = 1, 2.$$

Notice that the system's stability does not rule out that one of the queues 'is in overload' (if it would operate in isolation, that is): one could have that $\mu_i > \phi_i c$ for one of the queues.

We denote by $B_i(s, t)$, for $i = 1, 2$, the amount of service obtained by the i-th class in time interval $(s, t]$. Then there is the obvious identity

(1) $$Q_i(t) = Q_i(s) + Z_i(s, t) - B_i(s, t), \quad \forall s < t.$$

According to Reich's formula [13] (see also [7] in the context of GPS queues) we have the following distributional representation for the stationary workloads:

$$Q_i \stackrel{\mathrm{d}}{=} \sup_{t \geq 0} \{Z_i(-t, 0) - C_i(-t, 0)\},$$

where $C_i(s, t)$ is the amount of the service available to class i in the interval $(s, t]$. The relation $C_i(s, t) \geq \phi_i c\,(t - s)$ holds for all $s < t$.

Additionally, it is convenient to introduce, for $\lambda_i > \mu_i$ and $\lambda > \mu$,

$$Q_i^{\lambda_i}(t) := \sup_{s \geq t} \{Z_i(-s, t) - \lambda_i(t + s)\}, \quad Q^{\lambda}(t) := \sup_{s \geq t} \{Z_1(-s, t) + Z_2(-s, t) - \lambda(t + s)\}.$$

Observe that $Q_i^{\lambda_i} := Q_i^{\lambda_i}(0)$ is distributed as the stationary buffer content of queue i working in isolation, if it were served at rate λ_i all the time. Likewise, $Q^{\lambda} := Q^{\lambda}(0)$ corresponds to the total stationary buffer content of the system, if it were served at rate λ. Since the queues interact symmetrically, we focus on just $\mathbb{P}(Q_1 > u)$, for $u \to \infty$.

3. MAIN RESULTS

In this section we present the main results of the paper. We distinguish four scenarios, that differ in terms of (i) the heaviness of the individual input processes, and (ii) the individual queues being underloaded or overloaded. The proofs of all the results presented in this section are relegated to Section 5.

3.1. **Second queue in overload.** We first consider the scenario that the second queue is unstable when working in isolation: $\mu_2 > \phi_2 c$. In this case, if the input process of the second queue generates traffic at its mean rate (which does not correspond to a rare event), then it will be using its full guaranteed service rate. This pattern would leave the first queue as if working in isolation. Based on this observation, we expect that

$$\mathbb{P}(Q_1 > u) \sim \mathbb{P}(Q_1^{\phi_1 c} > u).$$

The following theorem formalizes this heuristic. Notice that in this scenario we necessarily have $\mu_1 < \phi_1 c$.

Theorem 3.1. *Suppose that Z_1, Z_2 satisfy* **A1-A2**. *If $\mu_2 > \phi_2 c$, then, as $u \to \infty$,*

$$\mathbb{P}(Q_1 > u) \sim \frac{1}{(\phi_1 c - \mu_1)(\alpha_1 - 1)} u^{1-\alpha_1} L_1(u).$$

3.2. **Second queue in underload, first class is heavier.** In the other three scenarios the second queue is stable while working in isolation, i.e., we consider the situation that $\mu_2 < \phi_2 c$. As it turns out, under this condition the interplay between both input processes plays a key role. We first concentrate on the case that the first class is heavier than the second one, i.e., $\alpha_1 < \alpha_2$. Since the second queue is stable while working in isolation and 'is lighter' than the first one, the most likely way to generate a large workload in the first queue does not involve a large buffer content in the second queue. The most probable way the first buffer reaches a large level corresponds to (i) the second class generating traffic at its mean level μ_2, and (ii) the remaining service capacity $c - \mu_2$ being allocated to the first queue. Hence the so-called *reduced-load* equivalence holds in this case, cf. e.g. [3]:

$$\mathbb{P}(Q_1 > u) \sim \mathbb{P}(Q_1^{c-\mu_2} > u).$$

This leads to the following theorem.

Theorem 3.2. *Suppose that Z_1, Z_2 satisfy* **A1-A2**. *If $\mu_2 < \phi_2 c$ and $\alpha_1 < \alpha_2$, then, as $u \to \infty$,*

$$\mathbb{P}(Q_1 > u) \sim \frac{1}{(c-\mu)(\alpha_1 - 1)} u^{1-\alpha_1} L_1(u).$$

3.3. **Second queue in underload, second class is heavier.** In the remaining two scenarios the second queue is stable while working in isolation, and the second class is heavier than the first one, i.e., $\alpha_2 < \alpha_1$. Two cases still need to be distinguished: the first queue being in underload or not.
In this third scenario we suppose that both the first and the second queue are stable while working in isolation, i.e., $\mu_i < \phi_i c$ for $i = 1, 2$ (and, as mentioned above, the second class is the heavier). For this scenario it turns out that again the reduced load equivalence holds:

$$\mathbb{P}(Q_1 > u) \sim \mathbb{P}(Q_1^{c-\mu_2} > u).$$

Intuitively, this means that the most probable way in which queue 1 grows large is as follows: the second class generates traffic roughly at its mean rate, and the first queue builds up as acting in isolation with service rate $c - \mu_2$ (which can be interpreted as the service rate left by the second queue). Although the asymptotics coincide with those obtained in Theorem 3.2, the proof of the upper bound for this case needs an entirely different approach (which motivates why we treat them as separate cases).

Theorem 3.3. *Suppose that Z_1, Z_2 satisfy* **A1-A2**. *If $\mu_1 < \phi_1 c$, $\mu_2 < \phi_2 c$ and $\alpha_2 < \alpha_1$, then, as $u \to \infty$,*

$$\mathbb{P}(Q_1 > u) \sim \frac{1}{(c - \mu)(\alpha_1 - 1)} u^{1-\alpha_1} L_1(u).$$

3.4. **First queue in overload, second class is heavier.** Finally, we consider the scenario that the first queue is in overload (i.e., unstable when working in isolation: $\mu_1 > \phi_1 c$), and the second class is the heavier (i.e., $\alpha_2 < \alpha_1$). We in addition assume that Z_2 be spectrally positive.

In this case the most probable way in which the first queue reaches a high level is such that the first class generates traffic roughly at its average rate μ_1 (which does not correspond to a rare event). Now the crucial issue concerns the fraction of its service rate that is left by the second class to the first class. As it turns out, the most likely behavior of the second queue can be linked with the downstream queue of a fictitious two-node tandem queue fed by Z_2 with service rate $\phi_2 c$ at the upstream queue and service rate $c - \mu_1$ at the downstream queue, in the sense that

$$\mathbb{P}(Q_1 > u) \sim \mathbb{P}(V > u),$$

where

$$V := \sup_{t \geq 0}\{Z_2(-t, 0) - (c - \mu_1)t\} - \sup_{s \geq 0}\{Z_2(-s, 0) - \phi_2 cs\}.$$

This relation was also observed in GPS models with fractional Brownian input in [4], whereas [2] finds a similar relation for the case of heavy tailed on-off input. Combining this with results from [6] on tandem queues with spectrally positive input, we thus arrive at the following asymptotics.

Theorem 3.4. *Suppose that Z_1, Z_2 satisfy* **A1-A2**, *$\mu_1 > \phi_1 c$, $\alpha_2 < \alpha_1$ and Z_2 is spectrally positive with $\alpha_2 \notin \mathbb{N}$. Then, as $u \to \infty$,*

$$\mathbb{P}(Q_1 > u) \sim \left(\frac{\mu_1 - \phi_1 c}{\phi_2 c - \mu_2}\right)^{\alpha_2 - 1} \frac{1}{(c - \mu)(\alpha_2 - 1)} u^{1-\alpha_2} L_2(u).$$

Remark 3.1. In the proof of Theorem 3.4 it plays a crucial role that Z_2 is assumed to be spectrally positive. We strongly believe that this assumption is of a technical nature, in that the statement of Theorem 3.4 is valid for general Z_2. We anticipate, however, that a proof for general Z_2 would be considerably more complicated, and would go along entirely different lines; see also Remark 4.1.

Observe that in the first three scenarios the workload of the first queue inherits the tail behavior of its input process: the complementary distribution function $\mathbb{P}(Q_1 > u)$ essentially behaves as $u^{1-\alpha_1}$. We conclude that in these cases the GPS mechanism succeeds in protecting the first stream against the second stream. Only in the last scenario, $\mathbb{P}(Q_1 > u)$ becomes heavier, which issue to the relatively large weight allocated to the second stream.

4. SPECIAL CASES

In this section we use the general results, as presented in the previous section, to find the asymptotics for GPS systems fed by compound Poisson processes with heavy-tailed jumps (Section 4.1) and by α-stable Lévy input (Section 4.2).

4.1. **Compound Poisson input.** This subsection concentrates on the case of compound Poisson inputs. More concretely, we assume that $Z_i(t)$ is of the form

$$Z_i(t) = \sum_{k=1}^{N_i(t)} B_{k,i}, \quad i = 1, 2.$$

In this definition of $Z_i(t)$, we assume that the processes $N_i(\cdot)$ are independent Poisson processes with rates $\lambda_i > 0$. In addition, $(B_{k,1})_k$ and $(B_{k,2})_k$ are both sequences of i.i.d. non-negative random variables, which are independent of the processes $N_1(\cdot)$ and $N_2(\cdot)$. We denote by B_1, B_2 the generic random variables corresponding to the sequences $B_{k,1}$ and $B_{k,2}$, where $F_1(\cdot)$ and $F_2(\cdot)$ denote their respective distribution functions.

The following proposition translate the findings of the previous section into the setting of the compound Poisson input model.

Proposition 4.1. *Assume that both $Z_1(\cdot)$ and $Z_2(\cdot)$ are independent compound Poisson processes with $\mu_i = \mathbb{E}[Z_i(1)] = \lambda_i \mathbb{E}[B_i]$ and $1 - F_i(x) \sim x^{-\alpha_i} L_i(x)$, as $x \to \infty$, for $i = 1, 2$ and $L_i(\cdot)$ being slowly varying at ∞, with $\alpha_i > 1$.*
1) If $\mu_2 > \phi_2 c$, then, as $u \to \infty$,

$$\mathbb{P}(Q_1 > u) \sim \frac{\lambda_1}{\phi_1 c - \mu_1} \frac{1}{\alpha_1 - 1} u^{1-\alpha_1} L_1(u).$$

2) If $\mu_2 < \phi_2 c$ and $\alpha_1 < \alpha_2$, then, as $u \to \infty$,

$$\mathbb{P}(Q_1 > u) \sim \frac{\lambda_1}{c - \mu} \frac{1}{\alpha_1 - 1} u^{1-\alpha_1} L_1(u).$$

3) If $\mu_2 < \phi_2 c$, $\mu_1 < \phi_1 c$ and $\alpha_2 < \alpha_1$, then, as $u \to \infty$,

$$\mathbb{P}(Q_1 > u) \sim \frac{\lambda_1}{c - \mu} \frac{1}{\alpha_1 - 1} u^{1-\alpha_1} L_1(u).$$

4) If $\mu_1 > \phi_1 c$, $\alpha_2 < \alpha_1$ and Z_2 is spectrally positive, then, as $u \to \infty$,

$$\mathbb{P}(Q_1 > u) \sim \frac{\lambda_2}{c - \mu} \left(\frac{\mu_1 - \phi_1 c}{\phi_2 c - \mu_2} \right)^{\alpha_2 - 1} \frac{1}{\alpha_2 - 1} u^{1-\alpha_2} L_2(u).$$

Proof. The proof follows straightforwardly from Theorems 3.1, 3.2, 3.3 and 3.4, respectively, in combination with Theorem 2.1 in [1]. $\qquad\square$

4.2. α-**stable Lévy input.** In this second subsection we focus on the special case of $Z_1(\cdot)$ and $Z_2(\cdot)$ being independent α_j-stable Lévy motions. This formally means that its law is given in terms of its characteristic function:

$$\log \mathbb{E}e^{i\theta Z_j(1)} = -|\theta|^{\alpha_j}(1 - i\beta_j \mathrm{sign}(\theta) \tan(\pi\alpha_j/2)) + i\mu_j\theta,$$

where $\alpha_j \in (1, 2]$, $\beta_j \in (-1, 1]$, $\mu_j \in \mathbb{R}$, and $\mathrm{sign}(x) := 1_{(0,\infty)}(x) - 1_{(-\infty,0)}(x)$. We write $Z_j \in \mathbb{S}(\alpha_j, \beta_j, \mu_j)$, see e.g., [14] or [5].
Using that

$$\mathbb{P}(Z_j(1) > x) \sim c_{\alpha_j}(1 + \beta_j)x^{-\alpha_j}, \qquad c_\alpha := \frac{1 - \alpha}{2\Gamma(2 - \alpha)\cos(\pi\alpha/2)},$$

see [14], in combination with the results presented in Section 3, we arrive at the following proposition.

Proposition 4.2. *Suppose that $Z_i \in \mathbb{S}(\alpha_i, \beta_i, \mu_i)$, with $\alpha_i \in (1, 2)$ for $i = 1, 2$.*
1) If $\mu_2 > \phi_2 c$, then, as $u \to \infty$,

$$\mathbb{P}(Q_1 > u) \sim \frac{c_{\alpha_1}(1 + \beta_1)}{(\phi_1 c - \mu_1)(\alpha_1 - 1)}u^{1-\alpha_1}.$$

2) If $\mu_2 < \phi_2 c$ and $\alpha_1 < \alpha_2$, then, as $u \to \infty$,

$$\mathbb{P}(Q_1 > u) \sim \frac{c_{\alpha_1}(1 + \beta_1)}{(c - \mu)(\alpha_1 - 1)}u^{1-\alpha_1}.$$

3) If $\mu_2 < \phi_2 c$, $\mu_1 < \phi_1 c$ and $\alpha_2 < \alpha_1$, then, as $u \to \infty$,

$$\mathbb{P}(Q_1 > u) \sim \frac{c_{\alpha_1}(1 + \beta_1)}{(c - \mu)(\alpha_1 - 1)}u^{1-\alpha_1}.$$

4) If $\mu_1 > \phi_1 c$, $\alpha_2 < \alpha_1$ and $\beta_2 = 1$, then, as $u \to \infty$,

$$\mathbb{P}(Q_1 > u) \sim \frac{2c_{\alpha_2}}{(c - \mu)(\alpha_2 - 1)}\left(\frac{\mu_1 - \phi_1 c}{\phi_2 c - \mu_2}\right)^{\alpha_2-1}u^{1-\alpha_2}.$$

Remark 4.1. Complementary to case 4) of Proposition 4.2, for $\beta_2 \in (-1, 1]$ (and $\mu_1 > \phi_1 c$, $\alpha_2 < \alpha_1$), we can find asymptotic upper and lower bounds on $\mathbb{P}(Q_1 > u)$ that are tight up to a constant. In particular, combining the proof of Theorem 3.4 with Theorem 5.3 and Lemma 5.4 in [6] we obtain, as $u \to \infty$,

$$\limsup_{u\to\infty} \mathbb{P}(Q_1 > u)u^{-(1-\alpha_2)} \leq \left(\frac{c_{\alpha_2}(1 + \beta_2)}{(c - \mu)(\alpha_2 - 1)} + \frac{c_{\alpha_2}(1 + \beta_2)}{\phi_2 c}\right)\left(\frac{\mu_1 - \phi_1 c}{\phi_2 c}\right)^{\alpha_2-1},$$

and

$$\liminf_{u\to\infty} \mathbb{P}(Q_1 > u)u^{-(1-\alpha_2)} \geq \frac{c_{\alpha_2}(1 + \beta_2)}{(c - \mu)(\alpha_2 - 1)}\left(\frac{\mu_1 - \phi_1 c}{\phi_2 c}\right)^{\alpha_2-1}.$$

5. Proofs

Before we provide detailed proofs of the results of Section 3, we present some useful lemmas. We begin with the classical result by Port [12], describing the asymptotics of the tail distribution of a single queue that is emptied at rate c.

Lemma 5.1. *Suppose that Z_1 satisfies* **A1**, **A2** *with $c > \mu_1$. Then, as $u \to \infty$,*

$$\mathbb{P}(Q_1^c > u) \sim \frac{1}{c - \mu_1} \frac{1}{\alpha_1 - 1} u^{1-\alpha_1} L_1(u).$$

The following result is due to Willekens [16], describing the asymptotic distribution of the supremum of $Z_1(-t, 0) - ct$ over a finite interval.

Lemma 5.2. *Suppose that Z_1 satisfies* **A1***. Then, for each $T > 0$, as $u \to \infty$,*

$$\mathbb{P}\left(\sup_{t \in [0,T]} \{Z_1(-t, 0) - ct\} > u \right) \sim \mathbb{P}(Z_1(1) > u).$$

Whereas the previous lemma considers the supremum over a finite interval, in the next lemma the interval grows with the exceedance level u. This result may have appeared in some form in the literature, but we decided to include it here, as it has a natural and insightful proof.

Lemma 5.3. *Suppose that Z_1 satisfies* **A1** *with $c > \mu_1$ and $\lim_{u \to \infty} T(u)/u = \infty$. Then, as $u \to \infty$,*

$$\mathbb{P}(Q_1^c > u) \sim \mathbb{P}\left(\sup_{t \in [0,T(u)]} \{Z_1(t) - ct\} > u \right).$$

Proof. Observe that the following trivial inequality holds:

$$\mathbb{P}(\mathscr{E}(u)) \leq \mathbb{P}(Q_1^c > u) \leq \mathbb{P}(\mathscr{E}(u)) + \mathbb{P}(\mathscr{F}(u)),$$

with

$$\mathscr{E}(u) := \left\{ \sup_{t \in [0,T(u)]} \{Z_1(t) - ct\} > u \right\}, \quad \mathscr{F}(u) := \left\{ \sup_{t \geq T(u)} \{Z_1(t) - ct\} > u \right\}.$$

Let $\widetilde{Q}_1^c \stackrel{\mathrm{d}}{=} Q_1^c$, with $\widetilde{Q}_1^c$ being independent of $\{Z_1(t), t \in \mathbb{R}\}$. Then using the independence and stationarity of the increments of Z_1, we have, with $\varepsilon \in (0, c - \mu_1)$,

$$
\begin{aligned}
\mathbb{P}(\mathscr{F}(u)) &= \mathbb{P}\left(Z_1(T(u)) - cT(u) + \sup_{t \geq T(u)} \{Z_1(t) - Z_1(T(u)) - c(t - T(u))\} > u \right) \\
&= \mathbb{P}\left(Z_1(T(u)) + \widetilde{Q}_1^c > u + cT(u) \right) \\
&= \mathbb{P}\left(Z_1(T(u)) - (\mu_1 + \varepsilon)T(u) + \widetilde{Q}_1^c > u + (c - \mu_1 - \varepsilon)T(u) \right).
\end{aligned}
$$

By applying $\mathbb{P}(X + Y \geq z) \leq \mathbb{P}(X \geq fz) + \mathbb{P}(Y \geq (1-f)z)$ for $f \in (0, 1)$, this quantity is in turn bounded from above by, with $\Delta := c - \mu_1 - \varepsilon > 0$,

$$\mathbb{P}\left(Z_1(T(u)) - (\mu_1 + \varepsilon)T(u) \geq \frac{1}{2}\Delta T(u) \right) + \mathbb{P}\left(Q_1^c > u + \frac{1}{2}\Delta T(u) \right).$$

We prove for each of these probabilities that they are $o\left(\mathbb{P}(Q_1^c > u)\right)$ as $u \to \infty$. The first probability is majorized by

$$\mathbb{P}\left(\sup_{t \geq 0}\{Z_1(t) - (\mu_1 + \varepsilon)t\} \geq \frac{1}{2}\Delta T(u) \right) = \mathbb{P}\left(Q_1^{\mu_1 + \varepsilon} \geq \frac{1}{2}\Delta T(u) \right),$$

Now it follows from Lemma 5.1 that, recalling that $u = o(T(u))$,

$$\mathbb{P}\left(Q_1^{\mu_1 + \varepsilon} \geq \frac{1}{2}\Delta T(u) \right) = o\left(\mathbb{P}(Q_1^c > u)\right), \quad \mathbb{P}\left(Q_1^c > u + \frac{1}{2}\Delta T(u) \right) = o\left(\mathbb{P}(Q_1^c > u)\right).$$

Therefore, we conclude that $\mathbb{P}(\mathscr{F}(u)) = o\left(\mathbb{P}(Q_1^c > u)\right)$, which completes the proof. $\square$

Define, for $\lambda < \mu_2$, $T_\lambda(u) := u/\bar{T}_\lambda(u)$, where

$$\bar{T}_\lambda(u) := \sqrt{\mathbb{P}\left(\check{Q}_2^\lambda(0) > u/2 \right) \vee (1/\log u)}, \quad \check{Q}_2^\lambda(s) := \sup_{t \geq s}\{Z_2(s) - Z_2(t) + \lambda(t-s)\}.$$

Lemma 5.4. *Suppose that Z_2 satisfies **A1** with $\lambda < \mu_2$. Then, as $u \to \infty$,*

$$\xi(u) := \mathbb{P}\left(\sup_{s \in [0, T_\lambda(u)]} \check{Q}_2^\lambda(s) > u \right) \to 0.$$

Proof. It is immediate that, with $W(t) := Z_2(t) - \lambda t$,

$$\begin{aligned}
\sup_{s \in [0, T_\lambda(u)]} \check{Q}_2^\lambda(s) &= \sup_{s \in [0, T_\lambda(u)]} \sup_{t \geq s}\{Z_2(s) - Z_2(t) + \lambda(t-s)\} \\
&\leq \sup_{s \in [0, T_\lambda(u)]} \sup_{t \geq 0}\{Z_2(s) - Z_2(t) + \lambda(t-s)\} = \check{Q}_2^\lambda(0) + \sup_{s \in [0, T_\lambda(u)]} W(s).
\end{aligned}$$

Setting

$$g(u) := u\left(\bar{T}_\lambda(u)\right)^{1/2}, \quad h(u) := u\left(\bar{T}_\lambda(u)\right)^{1/3},$$

we have

$$
\begin{aligned}
\zeta(u) \;\; := \;\; & \mathbb{P}\left(\sup_{t\in[0,g(u)]} \check{Q}_2^\lambda(t) > u\right) \leq \mathbb{P}\left(\check{Q}_2^\lambda(0) + \sup_{s\in[0,g(u)]} W(s) > u\right) \\
\leq \;\; & \mathbb{P}\left(\check{Q}_2^\lambda(0) + \sup_{s\in[0,g(u)]} W(s) > u, \; \sup_{s\in[0,g(u)]} W(s) \leq h(u)\right) \\
& + \mathbb{P}\left(\check{Q}_2^\lambda(0) + \sup_{s\in[0,g(u)]} W(s) > u, \; \sup_{s\in[0,g(u)]} W(s) > h(u)\right) \\
\leq \;\; & \mathbb{P}\left(\check{Q}_2^\lambda(0) > u - h(u)\right) + \mathbb{P}\left(\sup_{s\in[0,g(u)]} W(s) > h(u)\right) \\
\leq \;\; & \mathbb{P}\left(\check{Q}_2^\lambda(0) > u/2\right) + \mathbb{P}\left(\sup_{s\in[0,g(u)]} \{Z_2(s) - (\mu_2+1)s\} > h(u) - (\mu_2 - \lambda + 1)g(u)\right) \\
\leq \;\; & \mathbb{P}\left(\check{Q}_2^\lambda(0) > u/2\right) + \mathbb{P}\left(Q_2^{\mu_2+1} > h(u)/2\right).
\end{aligned}
$$

Using the stationarity of $\check{Q}_2^\lambda(t)$, and noting that $m(u) := T_\lambda(u)/g(u) \to \infty$ as $u \to \infty$,

$$
\xi(u) \leq \sum_{i=1}^{[m(u)]+1} \mathbb{P}\left(\sup_{t\in[(i-1)g(u),\, i\, g(u)]} \check{Q}_2^\lambda(t) > u\right) = ([m(u)] + 1)\, \zeta(u).
$$

Now, by applying Lemma 5.1, we have (noting that $h(u) \to \infty$ as $u \to \infty$) that there is a positive constant κ such that

$$
\begin{aligned}
\xi(u) \;\; \leq \;\; & \frac{2T_\lambda(u)}{g(u)}\zeta(u) \leq \frac{2T_\lambda(u)}{g(u)}\left(\mathbb{P}\left(\check{Q}_2^\lambda(0) > u/2\right) + \mathbb{P}\left(Q_2^{\mu_2+1} > h(u)/2\right)\right) \\
\leq \;\; & 2\left(\mathbb{P}\left(\check{Q}_2^\lambda(0) > u/2\right)\right)^{1/4} + \kappa\, (\log u)^{3/4}\, L_2(h(u))(h(u))^{1-\alpha_2} \to 0.
\end{aligned}
$$

This completes the proof.   □

The following lemma plays an important role in the proof of Theorem 3.3.

Lemma 5.5. *Suppose that Z_2 satisfies* **A1** *with $\lambda < \mu_2$. Then*

$$
\frac{Q_2^\lambda(t)}{t} \to 0, \quad \text{a.s.,} \quad \text{as } t \to \infty.
$$

Proof. It suffices to prove that for some $\beta > 0$,

$$
(2) \qquad\qquad \sup_{t\in[n^\beta,(n+1)^\beta]} \frac{Q_2^\lambda(t)}{n^\beta} \to 0, \quad \text{a.s.,} \quad \text{as } n \to \infty.
$$

Let $\beta > (\alpha_2 - 1)^{-1} > 0$ hereafter. Then, with $n_+ := n + 1$, $I_n := [n^\beta, n_+^\beta]$,

$$
\begin{aligned}
\sup_{t \in I_n} Q_2^\lambda(t) &= \sup_{t \in I_n} \sup_{s \leq t} \{Z_2(t) - Z_2(s) - \lambda(t - s)\} \\
&\leq \sup_{t \in I_n} \sup_{s \leq n_+^\beta} \{Z_2(n_+^\beta) - Z_2(s) - \lambda(n_+^\beta - s) + Z_2(t) - Z_2(n_+^\beta) - \lambda(t - n_+^\beta)\} \\
&= Q_2^\lambda(n_+^\beta) + \sup_{t \in I_n} \{W(t) - W(n_+^\beta)\},
\end{aligned}
$$

where, as before, $W(t) := Z_2(t) - \lambda t$. In light of Lemma 5.1 we have that for any $\varepsilon > 0$ and $\gamma \in (1, \beta(\alpha_2 - 1))$ there exists $N_0 \in \mathbb{N}$ such that (where we recall that $\beta > (\alpha_2 - 1)^{-1}$)

$$
\sum_{n=N_0}^\infty \mathbb{P}\left(\frac{Q_2^\lambda(n_+^\beta)}{n^\beta} > \varepsilon\right) = \sum_{n=N_0}^\infty \mathbb{P}\left(Q_2^\lambda(0) > \varepsilon n^\beta\right) \leq \sum_{n=N_0}^\infty n^{-\gamma} < \infty.
$$

Using the Borel-Cantelli lemma, we obtain that $Q_2^\lambda(n_+^\beta)/n^\beta \to 0$ a.s. as $n \to \infty$. In order to establish (2) we are now left to prove that $n^{-\beta} \sup_{t \in I_n} \{W(t) - W(n_+^\beta)\} \to 0$ a.s., as $n \to \infty$. This convergence is established as follows.

By the fact that $Z_2(t)/t \to \mu_2$, a.s., as $t \to \infty$, we have $W(t)/t \to \mu_2 - \lambda$, a.s., as $t \to \infty$, and

$$
\sup_{s,t \geq n} \left\{\frac{W(s)}{s} - \frac{W(t)}{t}\right\} \to 0, \text{ a.s., as } n \to \infty.
$$

Therefore,

$$
\begin{aligned}
0 &\leq \frac{1}{n^\beta} \sup_{t \in I_n} \{W(t) - W(n_+^\beta)\} \leq \frac{n_+^\beta}{n^\beta} \sup_{t \in I_n} \left\{\frac{W(t)}{t} - \frac{W(n_+^\beta)}{t}\right\} \\
&\leq \frac{n_+^\beta}{n^\beta} \left(\sup_{t \in I_n} \left\{\frac{W(t)}{t} - \frac{W(n_+^\beta)}{n_+^\beta}\right\} + |W(n_+^\beta)| \sup_{t \in I_n} \left\{\left|\frac{1}{n_+^\beta} - \frac{1}{t}\right|\right\}\right) \to 0, \text{ a.s.,}
\end{aligned}
$$

as $n \to \infty$. This confirms (2) and thus the proof has been completed. $\qquad \square$

5.1. **Proof of Theorem 3.1.** *Upper bound*: Observe that

$$
(3) \qquad \mathbb{P}(Q_1 > u) \leq \mathbb{P}(Q_1^{\phi_1 c} > u) \sim \frac{1}{(\phi_1 c - \mu_1)(\alpha_1 - 1)} u^{1 - \alpha_1} L_1(u),
$$

by Lemma 5.1.

Lower bound: Since

$$
\begin{aligned}
Q_1 &= \sup_{t \geq 0} \left\{Z_1(-t, 0) + Z_2(-t, 0) - ct - \sup_{s \in [0,t)} \{Z_2(-s, 0) - C_2(-s, 0)\}\right\} \\
&\geq \sup_{t \geq 0} \left\{Z_1(-t, 0) + Z_2(-t, 0) - ct - \sup_{s \in [0,t)} \{Z_2(-s, 0) - \phi_2 cs\}\right\},
\end{aligned}
$$

for any $\varepsilon \in (0,1)$, application of Lemmas 5.3-5.4 yields, with $\lambda = \phi_2 c$,

$$
\mathbb{P}(Q_1 > u) \geq \mathbb{P}\left(\sup_{t \geq 0}\left\{Z_1(-t,0) + Z_2(-t,0) - ct - \sup_{s \in [0,t)}\{Z_2(-s,0) - \phi_2 cs\}\right\} > u\right)
$$

$$
= \mathbb{P}\left(\sup_{t \geq 0}\left\{Z_1(t) - \phi_1 ct - \sup_{s \in [0,t)}\{Z_2(s) - Z_2(t) - \phi_2 c(s-t)\}\right\} > u\right)
$$

$$
\geq \mathbb{P}\left(\sup_{0 \leq t \leq T_\lambda(\varepsilon u)}\left\{Z_1(t) - \phi_1 ct - \sup_{0 \leq s \leq T_\lambda(\varepsilon u), v > s}\{Z_2(s) - Z_2(v) - \phi_2 c(s-v)\}\right\} > u\right)
$$

$$
= \mathbb{P}\left(\sup_{0 \leq t \leq T_\lambda(\varepsilon u)}\{Z_1(t) - \phi_1 ct\} - \sup_{0 \leq s \leq T_\lambda(\varepsilon u)}\check{Q}_2^{\phi_2 c}(s) > u\right)
$$

$$
\geq \mathbb{P}\left(\sup_{0 \leq t \leq T_\lambda(\varepsilon u)}\{Z_1(t) - \phi_1 ct\} - \sup_{0 \leq s \leq T_\lambda(\varepsilon u)}\check{Q}_2^{\phi_2 c}(s) > u, \sup_{0 \leq s \leq T_\lambda(\varepsilon u)}\check{Q}_2^{\phi_2 c}(s) \leq \varepsilon u\right)
$$

$$
\geq \mathbb{P}\left(\sup_{0 \leq t \leq T_\lambda(\varepsilon u)}\{Z_1(t) - \phi_1 ct\} > (1+\varepsilon)u\right)\mathbb{P}\left(\sup_{0 \leq s \leq T_\lambda(\varepsilon u)}\check{Q}_2^{\phi_2 c}(s) \leq \varepsilon u\right)
$$

$$
\sim \mathbb{P}\left(Q_1^{\phi_1 c} > u\right), \quad u \to \infty, \varepsilon \downarrow 0,
$$

which combined with Lemma 5.1 leads to the asymptotic upper bound that matches the lower bound. This completes the proof. $\square$

5.2. **Proof of Theorem 3.2.** *Upper bound*: Combining

$$
(4) \qquad \mathbb{P}(Q_1 > u) \leq \mathbb{P}(Q_1 + Q_2 > u) = \mathbb{P}\left(\sup_{t \geq 0}\{Z_1(-t,0) + Z_2(-t,0) - ct\} > u\right)
$$

with $\mathbb{P}(Z_1(1) + Z_2(1) > u) \sim \mathbb{P}(Z_1(1) > u)$ as $u \to \infty$, together with Lemma 5.1, straight-forwardly gives that

$$
(5) \qquad \mathbb{P}(Q_1 > u) \sim \frac{1}{c-\mu}\int_u^\infty \mathbb{P}(Z_1(1) > x)\mathrm{d}x \sim \frac{1}{(c-\mu)(\alpha_1 - 1)}u^{1-\alpha_1}L_1(u),
$$

as $u \to \infty$.

Lower bound: Let $\varepsilon > 0$ be given. Following the same argument as in the lower bound of the proof of Theorem 3.1, we have, with $\lambda = \mu_2^\varepsilon := \mu_2 - \varepsilon$,

$$\mathbb{P}(Q_1 > u)$$

$$\geq \mathbb{P}\left(\sup_{t \geq 0} \left\{ Z_1(-t, 0) + Z_2(-t, 0) - ct - \sup_{s \in [0, t)} \{Z_2(-s, 0) - \phi_2 cs\} \right\} > u \right)$$

$$= \mathbb{P}\left(\sup_{t \geq 0} \left\{ Z_1(t) - (c - \mu_2^\varepsilon)t - \sup_{s \in [0, t)} \{Z_2(s) - Z_2(t) - \phi_2 cs + \mu_2^\varepsilon t\} \right\} > u \right)$$

$$\geq \mathbb{P}\left(\sup_{t \in [0, T_\lambda(\varepsilon u)]} \left\{ Z_1(t) - (c - \mu_2^\varepsilon)t - \sup_{s \in [0, T_\lambda(\varepsilon u)), s < t} \{Z_2(s) - Z_2(t) + \mu_2^\varepsilon(t - s)\} \right\} > u \right)$$

$$\geq \mathbb{P}\left(\sup_{t \in [0, T_\lambda(\varepsilon u)]} \left\{ Z_1(t) - (c - \mu_2^\varepsilon)t - \sup_{s \in [0, T_\lambda(\varepsilon u)]} \check{Q}_2^{\mu_2^\varepsilon}(s) \right\} > u \right)$$

$$\geq \mathbb{P}\left(\sup_{t \in [0, T_\lambda(\varepsilon u)]} \{Z_1(t) - (c - \mu_2^\varepsilon)t\} > (1 + \varepsilon)u \right) \mathbb{P}\left(\sup_{s \in [0, T_\lambda(\varepsilon u)]} \check{Q}_2^{\mu_2^\varepsilon}(s) < \varepsilon u \right).$$

By Lemmas 5.3–5.4 in combination with Lemma 5.1, we obtain

$$\mathbb{P}(Q_1 > u) \geq \mathbb{P}(Q_1^{c - \mu_2^\varepsilon} > (1 + \varepsilon)u)(1 + o(1)) \sim \frac{(1 + \varepsilon)^{1 - \alpha_1}}{(c - \mu + \varepsilon)(\alpha_1 - 1)} u^{1 - \alpha_1} L_1(u), \quad u \to \infty.$$

Letting $\varepsilon \downarrow 0$, and recalling (5), completes the proof. □

5.3. **Proof of Theorem 3.3.** *Upper bound*: The starting point is the following evident equality:

$$Q_1 = \sup_{t \in [0, u^{1-\varepsilon})} U_1(t) \vee \sup_{t \geq u^{1-\varepsilon}} U_1(t), \quad U_1(t) := Z_1(-t, 0) - C_1(-t, 0).$$

with ε strictly between $\alpha_1 / (1 + \alpha_1)$ and 1. Then, for

$$\mathscr{U}_\varepsilon := \{\forall t \geq u^{1-\varepsilon} : Z_2(-t, 0) + Q_2^{\phi_2 c}(-t) \leq (\mu_2 + \varepsilon)t\}$$

we have

(6) $$\mathbb{P}\left(\sup_{t \geq u^{1-\varepsilon}} U_1(t) > u \right) = \mathbb{P}\left(\sup_{t \geq u^{1-\varepsilon}} U_1(t) > u; \mathscr{U}_\varepsilon \right) + \mathbb{P}\left(\sup_{t \geq u^{1-\varepsilon}} U_1(t) > u; \mathscr{U}_\varepsilon^c \right).$$

It follows from (1) that on the event $\mathscr{U}_\varepsilon$, for $t \geq u^{1-\varepsilon}$,

$$B_2(-t, 0) = Z_2(-t, 0) + Q_2(-t) - Q_2(0) \leq Z_2(-t, 0) + Q_2^{\phi_2 c}(-t) \leq (\mu_2 + \varepsilon)t,$$

which together with the fact that $C_1(s, t) + B_2(s, t) = c(t - s)$ for all $s \leq t$ yields, for $t \geq u^{1-\varepsilon}$,

$$C_1(-t, 0) \geq (c - \mu_2 - \varepsilon)t.$$

Moreover,

$$\mathbb{P}\left(\sup_{t \geq u^{1-\varepsilon}} U_1(t) > u; \mathscr{U}_\varepsilon\right) \leq \mathbb{P}\left(\sup_{t \geq u^{1-\varepsilon}} \{Z_1(-t, 0) - (c - \mu_2 - \varepsilon)t\} > u; \mathscr{U}_\varepsilon\right)$$

$$\text{(7)} \hspace{5cm} \leq \mathbb{P}\left(\sup_{t \geq 0}\{Z_1(-t, 0) - (c - \mu_2 - \varepsilon)t\} > u\right)\mathbb{P}(\mathscr{U}_\varepsilon);$$

the first term in (7) is roughly of the order $u^{1-\alpha_1}$, whereas $\mathbb{P}(\mathscr{U}_\varepsilon) \to 1$, as $u \to \infty$, as a consequence of the law of large numbers and Lemma 5.5.

In addition,

$$\mathbb{P}\left(\sup_{t \geq u^{1-\varepsilon}} U_1(t) > u; \mathscr{U}_\varepsilon^c\right) \leq \mathbb{P}\left(\sup_{t \geq u^{1-\varepsilon}} \{Z_1(-t, 0) - \phi_1 ct\} > u; Y\mathscr{U}_\varepsilon^c\right)$$

$$\text{(8)} \hspace{5cm} \leq \mathbb{P}\left(\sup_{t \geq 0}\{Z_1(-t, 0) - \phi_1 ct\} > u\right)\mathbb{P}(\mathscr{U}_\varepsilon^c),$$

where the first term in (8) essentially vanishes as $u^{1-\alpha_1}$, but $\mathbb{P}(\mathscr{U}_\varepsilon^c) \to 0$ due to the law of large numbers and Lemma 5.5. We conclude it is negligible relative to (7).

Combining (7) and (8) gives that

$$\mathbb{P}\left(\sup_{t \geq u^{1-\varepsilon}} \{Z_1(-t, 0) - C_1(-t, 0)\} > u\right) \leq \mathbb{P}(Q_1^{c-\mu_2-\varepsilon} > u), \quad \text{as } u \to \infty.$$

Now we are left with showing that

$$\mathbb{P}(Q_1 > u) \sim \mathbb{P}\left(\sup_{t \geq u^{1-\varepsilon}} U_1(t) > u\right), \quad \text{as } u \to \infty.$$

Since we have that

$$\text{(9)} \quad \mathbb{P}(Q_1 > u) = \mathbb{P}\left(\sup_{t \geq 0} U_1(t) > u\right) \leq \mathbb{P}\left(\sup_{t \geq u^{1-\varepsilon}} U_1(t) > u\right) + \mathbb{P}\left(\sup_{t \in [0, u^{1-\varepsilon}]} U_1(t) > u\right)$$

and we have already showed that both $\mathbb{P}\left(\sup_{t \geq u^{1-\varepsilon}} U_1(t) > u\right)$ and $\mathbb{P}(Q_1 > u)$ are of order $u^{1-\alpha_1}$, we see that it suffices to prove that the last term in (9) is negligible.

Let

$$S_n := \sup_{s \in [n, n+1]} Z_1(s) - Z_1(n) - \phi_1 c(s - n), \quad n \in \mathbb{N}.$$

Using that $S_n, n \in \mathbb{N}$ are i.i.d., we get

$$
\begin{aligned}
\mathbb{P}\left(\sup_{t\in[0,u^{1-\varepsilon}]} U_1(t) > u\right) \ &\leq \ \mathbb{P}\left(\sup_{t\in[0,u^{1-\varepsilon}]} \{Z_1(-t,0) - \phi_1 ct)\} > u\right) \\
&= \ \mathbb{P}\left(\sup_{t\in[0,u^{1-\varepsilon}]} \{Z_1(t) - \phi_1 ct)\} > u\right) \\
&\leq \ \mathbb{P}\left(\sum_{i=0}^{[u^{1-\varepsilon}]} S_i > u\right) \leq ([u^{1-\varepsilon}]+1)\mathbb{P}\left(S_0 > \frac{u}{[u^{1-\varepsilon}]+1}\right).
\end{aligned}
$$

Hence, Lemma 5.2, in combination with the fact that ε lies strictly between $\alpha_1/(1+\alpha_1)$ and 1, leads to

$$
\mathbb{P}\left(\sup_{s\in[0,u^{1-\varepsilon}]} U_1(s) > u\right) \leq \bar{\kappa}\, L_1(u^\varepsilon)u^{1-\varepsilon-\varepsilon\alpha_1} = o\left(L_1(u)u^{1-\alpha_1}\right),
$$

where $\bar{\kappa}$ is a positive constant. This completes the proof of the upper bound.

Lower bound: The proof of the lower bound is the same as in the proof of Theorem 3.2. Relying on Lemma 5.1, we then obtain the equivalence of the asymptotic upper and lower bound. $\qquad\square$

5.4. **Proof of Theorem 3.4.** Since the proof of this scenario needs a case-specific approach that involves the notion of tandem systems, we begin with some notation and auxiliary results.

For ε such that $\phi_1 c - \mu_1 < \varepsilon < c - \mu$, let

$$
V^\varepsilon := \sup_{t\geq 0}\{Z_2(-t,0) - (c-\mu_1-\varepsilon)t\} - \sup_{s\geq 0}\{Z_2(-s,0) - \phi_2 cs\}.
$$

Recall that

$$
Q_1^d := \sup_{t\geq 0}\{Z_1(-t,0) - dt\}, \quad d > \mu_1
$$

and introduce

$$
\check{Q}_1^d := \sup_{t\geq 0}\{dt - Z_1(-t,0)\}, \quad d < \mu_1.
$$

The following lemma states a straightforward counterpart of Lemma 2.1 in [4].

Lemma 5.6. *For $\varepsilon > 0$ small enough, any u and x, and for $\delta \in (0,1)$, we have*

$$
\mathbb{P}(V^{-\varepsilon} > u+x)\mathbb{P}(\check{Q}_1^{\mu_1-\varepsilon} \leq x) \leq \mathbb{P}(Q_1 > u) \leq \mathbb{P}(V^\varepsilon > (1-\delta)u) + \mathbb{P}(Q_1^{\mu_1+\varepsilon} > \delta u)
$$

A combination of Theorem 4.7 in [10] (see also Theorem 12.9 in [5]) with Lemma 5.1 leads to the following lemma.

Lemma 5.7. *Let* $|\varepsilon| < \min(c - \mu, \mu_1 - \phi_1 c)$ *and* Z_2 *be spectrally positive with* $\alpha_2 \notin \mathbb{N}$. *Then, as* $u \to \infty$,

$$\mathbb{P}(V^\varepsilon > u) \sim \left(\frac{\mu_1 - \phi_1 c + \varepsilon}{\phi_2 c - \mu_2}\right)^{\alpha_2 - 1} \frac{1}{(c - \mu - \varepsilon)(\alpha_2 - 1)} u^{1 - \alpha_2} L_2(u).$$

Proof of Theorem 3.4: Let $\delta \in (0, 1)$ and $\varepsilon > 0$ be such that $\varepsilon < \min(c - \mu, \mu_1 - \phi_1 c)$. Then, following Lemma 5.7, as $u \to \infty$,

$$\mathbb{P}(\check{Q}_1^{\mu_1 - \varepsilon} < \sqrt{u}) \to 1,$$

$$\mathbb{P}(V^{-\varepsilon} > u + \sqrt{u}) \sim \left(\frac{\mu_1 - \phi_1 c - \varepsilon}{\phi_2 c - \mu_2}\right)^{\alpha_2 - 1} \frac{1}{(c - \mu + \varepsilon)(\alpha_2 - 1)} u^{1 - \alpha_2} L_2(u)$$

$$\mathbb{P}(V^\varepsilon > (1 - \delta)u) \sim \left(\frac{\mu_1 - \phi_1 c + \varepsilon}{\phi_2 c - \mu_2}\right)^{\alpha_2 - 1} \frac{1}{(c - \mu - \varepsilon)(\alpha_2 - 1)} (1 - \delta)^{1 - \alpha_2} u^{1 - \alpha_2} L_2(u).$$

Since $\alpha_1 > \alpha_2$, we find by Lemma 5.1 that for each $\delta \in (0, 1)$,

$$\mathbb{P}(Q_1^{\mu_1 + \varepsilon} > \delta u) = o(\mathbb{P}(V^\varepsilon > (1 - \delta)u)),$$

as $u \to \infty$. Thus, by Lemma 5.6, passing with $\varepsilon, \delta \downarrow 0$, we obtain that

$$\mathbb{P}(Q_1 > u) \sim \left(\frac{\mu_1 - \phi_1 c}{\phi_2 c - \mu_2}\right)^{\alpha_2 - 1} \frac{1}{(c - \mu)(\alpha_2 - 1)} u^{1 - \alpha_2} L_2(u).$$

This completes the proof. $\qquad\qquad\square$

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KRZYSZTOF DĘBICKI, MATHEMATICAL INSTITUTE, UNIVERSITY OF WROCŁAW, PL. GRUNWALDZKI 2/4, 50-384 WROCŁAW, POLAND
E-mail address: `Krzysztof.Debicki@math.uni.wroc.pl`

PENG LIU, MATHEMATICAL INSTITUTE, UNIVERSITY OF WROCŁAW, PL. GRUNWALDZKI 2/4, 50-384 WROCŁAW, POLAND AND DEPARTMENT OF ACTUARIAL SCIENCE, UNIVERSITY OF LAUSANNE, UNIL-DORIGNY 1015 LAUSANNE, SWITZERLAND
E-mail address: `liupnankaimath@163.com`

MICHEL MANDJES, KORTEWEG-DE VRIES INSTITUTE FOR MATHEMATICS, UNIVERSITY OF AMSTERDAM, AMSTERDAM, THE NETHERLANDS
E-mail address: `M.R.H.Mandjes@uva.nl`

IWONA SIERPIŃSKA-TUŁACZ, MATHEMATICAL INSTITUTE, UNIVERSITY OF WROCŁAW, PL. GRUNWALDZKI 2/4, 50-384 WROCŁAW, POLAND
E-mail address: `i.sierpinska@gmail.com`