Dispatching to parallel servers

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Solutions of Poisson's Equation for First-Policy Improvement

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Abstract Policy iteration techniques for multiple-server dispatching rely on the computation of value functions. In this context, we consider the continuous-space M/G/1-FCFS queue endowed with an arbitrarily-designed cost function for the waiting times of the incoming jobs. The associated relative value function is a solution of Poisson's equation for Markov chains, which in this work we solve in the Laplace transform domain by considering an ancillary, underlying stochastic process extended to (imaginary) negative backlog states. This construction enables us to issue closed-form relative value functions for polynomial and exponential cost functions and for piecewise compositions of the latter, in turn permitting the derivation of interval bounds for the relative value function in the form of power series or trigonometric sums. We review various cost approximation schemes and assess the convergence of the interval bounds these induce on the relative value function. Namely: Taylor expansions (divergent, except for a narrow class of entire functions with low orders of growth), and uniform approximation schemes (polynomials, trigonometric), which achieve optimal convergence rates over finite intervals. This study addresses all the steps to implementing dispatching policies for systems of parallel servers, from the specification of *general* cost functions towards the computation of interval bounds for the relative value functions and the exact implementation of the first-policy improvement step.

Keywords Dispatching · Policy iteration · First-policy improvement · Poisson equation · M/G/1 queue

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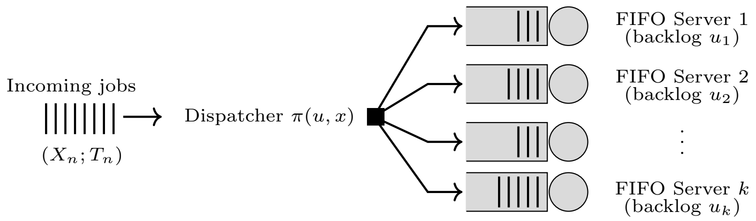

Fig. 1: Size-aware dispatching with i.i.d. service times ($X_n \overset{d}{=} X$ for all n) and i.i.d. exponential inter-arrival times (T_n) to k M/G/1-FCFS servers.

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I Introduction

An essential design aspect for systems of parallel servers resides in the allocation of the processing resources to the impending workload. In the allocation problem, commonly referred to as *dispatching* (also: *task assignment* or *routing*), one server must be assigned to each incoming job in a way so as to minimize a performance metric of interest: typically, the waiting / sojourn times in the server queues or overall power consumption. The dispatching problem is relevant in diverse domains of application including parallel computing (mobile cloud computing, server clusters, supercomputers), industrial logistics (customer service systems), and traffic congestion management (visitor queues, road tolls).

We are interested in systems composed of several first-come, first-served (FCFS) queueing servers operated in parallel, and fed with a sequence of jobs with Markovian arrival times. In our model, illustrated in Figure 1, every new job turning up at the dispatcher is instantly forwarded towards one of the servers, where a penalty is incurred as a function of the backlog (uncompleted work) at the server upon job arrival—server backlog thus coinciding with the waiting time of the job until processing begins. Our objective is to minimize the average cost experienced by the system over an infinite time horizon.

A standard approach for solving this problem is through *policy iteration* (PI), [17,5]. Starting with an inital dispatching policy, PI proceeds in two steps, repeated in turn until a fixed policy is reached: (i) *policy evaluation*, where the mean cost of the considered policy is computed, together with a *relative value function* expressing state sensitivity with respect to the steady-state costs induced by the policy; followed by (ii) *policy improvement*, where the relative value function is exploited to improve the current policy and derive a new, more cost-effective dispatching policy. The policy evaluation step is difficult to implement in continuous state spaces without extensive Monte Carlo simulations. Only the first PI iteration on a tractable, random initial policy is easier to carry out, because the job flow then decomposes into independent Poisson processes for the individual queueing servers, and the relative value function takes a separable form, solution of the so-called Poisson equation. The *first-policy improvement* (FPI) approach (also known as *one-step policy iteration*, and variants) consists of cutting short the policy iteration algorithm after the first iteration. The motivation behind FPI is twofold: it is known that a single iteration of the PI algorithm may produce fine heuristics (see e.g. [29, 42, 36, 39, 6] or [40, §7.5]) and, besides, the Poisson equation for Markov chains admits explicit solutions readily available for effortless PI.

Related work and our contribution. The existence of explicit solutions to the Poisson equation for the waiting times of the M/G/1 queue was pointed out in [14], where a general solution to Poisson's equation was proposed in the form of a fundamental kernel, whose application to the cost function produces solutions of the equation. These solutions proved, in particular, to take closed forms for cost functions given as moments of the waiting time, $f(u) = u^n$. There followed a list of derivations of explicit relative value functions for Markovian queueing systems: both in discrete-space settings where only the number of yet unprocessed jobs at the servers is known to the dispatcher and (typically) the expected sojourn times of the incoming jobs are penalized, [29, 39, 7, 6]; and in 'size-aware' continuous-space settings where the service times of the jobs become available to the dispatcher upon arrival and the actual waiting or sojourn times are penalized, [1, 24, 18, 19, 23]. Recent studies on size-aware dispatching renewed the interest in explicit Poisson solutions, extending their class in [21] to the fixed-deadline cost functions $f(u) = \mathbf{1}_{[\tau,\infty)}(u)$, and to exponential costs in [22], with views on polynomials. In the discrete space setting, the forms $f[u] = u^n a^{-u}$ and $f[u] = \delta[u - a]$ were identified in [12] as candidates for closed-form relative value function, via transform-domain analysis (based on generating functions) of the general solution of Poisson's equation—a methodology in spirit similar to the approach we will use in this study.

In this work we extend the collection of explicit solutions of the continuous-space Poisson equation to $f(u) = u^n e^{-au}$, and we develop a methodology based on complex analysis for solving Poisson's equation that covers a more general class of piecewise continuous cost functions. Our motivation behind piecewise-definite functions is the possibility they offer to derive, under mild conditions for the cost function, tight bounds to the corresponding relative value function, which enable us to perform the FPI step exactly. Our developments depart from previous studies by proposing a comprehensive implementation of FPI in continuous spaces with cost functions of any general kind.

Outline. The paper is structured as follows. Section II introduces the relative value function as the solution of Poisson's equation. This equation is solved in Section III from the viewpoint of complex analysis (III.1); complex analysis which allows us to derive the relative value function of the M/G/1 queue for cost functions of the type $f(u) = u^n e^{-au}$ (III.2), and to provide a method of solution for piecewise-defined costs (III.3), which cover various solutions previously reported in the literature. In Section IV we consider cost functions given as convergent series: successively, Taylor series (4), and uniform approximations by polynomials or trigonometric sums (IV.2); and we propose an algorithm for computing FPI policies based on approximations of the cost functions. We conclude with a full implementation of the FPI dispatcher for the cost function $f(u) = u^2/(a^2 + u^2)$, picked for illustrative purposes, in the case of a two-server system with exponentially distributed service times. We refer to Appendix A for a more detailed presentation of FPI, and to the online supplementary material for ancillary results, implementation details, and many examples of derivations of relative value functions, [9, 8].

Notation. For any real random variable Y, we denote by $X^*(s) = \mathbb{E}[e^{-sY}]$ the Laplace-Stieltjes transform of Y, by μ_Y the probability measure associated with Y, by $F_Y : \mathbb{R} \mapsto [0, 1]$ its cumulative probability distribution, and by $F'_Y : \mathbb{R} \mapsto [0, +\infty]$ its probability density function, with $\text{Prob}(Y \leq y) = \mu_Y((-\infty, y]) = F_Y(y) = \int_{-\infty}^{y} F'_Y(u)\, du$.

II Relative value function of the M/G/1 queue and Poisson equation

The FPI framework lead us to consider an individual server modeled by a continuous-state FCFS-M/G/1 queue. The queue is fed with a sequence of jobs with random arrival times modulated by a Poisson point process with rate $\lambda > 0$, [16, 13]. The dynamics of the queue is modeled by the equation

$$U_{n+1} = \left[U_n + X_n - T_{n+1}\right]^+, \qquad n \geq 0, \tag{Q}$$

where X_n denotes the service time of the nth incoming job, U_n is the coinciding queue backlog upon arrival, and T_{n+1} is the inter-arrival time for X_{n+1}. The random sequence $((U_n, X_n))_{n\in\mathbb{N}}$ produced by (Q) is seen as a Markov decision process (MDP) with transition kernel P.[1]

The service time of every incoming job is assumed as in [41] to be random, conditioned on the activity of the queue at the time of arrival, and independent of the other factors; it is distributed either like the positive random variable X if on arrival the queue is busy processing a previous job, or like a second positive random variable X_0 if the queue is idle (empty), where X_0 may differ from X in distribution, thus accounting for a setup delay that the queue might require to wake up from its idle state. The stability of the queue is guaranteed by a server utilization ratio $\rho = \lambda\mathbb{E}[X]$ less than 1, and by a finite mean service time at $u = 0$, i.e., $\tilde{\rho} = \lambda\mathbb{E}[X_0] < \infty$.

[1] The transition kernel $P(u, x, \mathcal{S}) = \text{Prob}((U_{n+1}, X_{n+1}) \in \mathcal{S} | (U_n, X_n) = (u, x))$ satisfies

$$P(u, x, \mathcal{U} \times \mathcal{X}) = \left\{ \begin{array}{ll} P(u + x, \{0\})\, \mu_{X_0}(\mathcal{X}) & \text{if } \mathcal{U} = \{0\} \\ P(u + x, \mathcal{U})\, \mu_X(\mathcal{X}) & \text{if } \mathcal{U} \not\ni \{0\} \end{array} \right\}, \tag{II.1}$$

where, for all $u \geq 0$, one has $P(u, [0, t]) = e^{-\lambda(u-t)}$ if $t \in [0, u]$, and $P(u, \mathbb{R} \setminus [0, u]) = 0$.

Assumption 1 (Stability) *$\rho < 1$, $\tilde{\rho} < \infty$.*

Ergodicity implies the existence of a unique asymptotic probability distribution $F_{\tilde{W}}$ for the waiting times at the queue, where $\widetilde{W}$ denotes a random variable distributed accordingly. A distinction is made between the actual stationary waiting times, and the waiting times that would ensue with homogeneous service times $X_0 \equiv X$, modeled by the variable W. Now, consider a cost function $f : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}$ quantifying the (expected) penalty $f(u)$ incurred when a job joins the queue at backlog state $u \in \mathbb{R}_{\geq 0}$. We complete our model with assumptions on the costs that guarantee existence of the relative value function.

Assumption 2 (Cost integrability) *$|f|$ is μ_W- and $\mu_{\tilde{W}}$-integrable.*

All in all, the server model considered throughout the paper is:

Server Model *The FCFS-M/G/1 queue* (Q) *with arrival rate λ and service times (X, X_0), endowed with a cost function f, under Assumptions 1 and 2.*

For any $u \in \mathbb{R}_{\geq 0}$ and any time horizon $t \geq 0$, we denote by $V(u,t)$ the (random) total cost incurred over a time interval of the type $[t_0, t_0 + t)$ when the backlog at time t_0 is u. Under Assumption 2, the quantity $V(u,t)$ averaged over the number of arrivals in the time window tends as $t \to \infty$ to the mean cost per job $\bar{f} = \mathbb{E}[f(\tilde{W})]$. The *relative value function* $v : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}$ is then defined by [31,32]

$$v(u) = \lim_{t \to \infty} \left\{ \mathbb{E}[V(u,t)] - \lambda \bar{f} t \right\}, \quad \forall u \geq 0, \tag{VF}$$

as an expression of the state sensitivity of the costs with respect to the steady-state regime. In order to compute (VF), we will regard v as a solution of the following Poisson equation, derived in Appendix B.

Proposition 1 (Poisson equation) *The relative value function* (VF) *rewrites as $v(u) = g(u,0) - f(u) = Pg(u,0) - \bar{f}$ for some $g : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}$ solution of the Poisson equation*

$$g(u,x) = Pg(u,x) + f(u) - \bar{f}, \tag{PE}$$

where $Pg(u,x) := \int g(t,y) P(u,x,d(t,y))$.

All $\mu_{\tilde{W}}$-integrable solutions of (PE) are equal up to an additive constant, [15]. Besides, due to the existence of a strong law of large numbers and a central limit theorem for the costs, [14,15], g and $\bar{f}$ can be estimated empirically, though at the price of extensive numerical simulations. Lastly, and preferably, some solutions of (PE) are known to exist in closed form; deriving explicit solutions of this kind is the direction we will explore in this work.

A general solution to (PE) was given in [14] under the integral form $g(u,x) = \int_0^{+\infty} f(t)\, \Gamma(u,x,dt)\, dt$, where Γ defines the solution kernel of the queue. Although closed-form relative value functions can be inferred from this integral form, it is impractical for a systematic derivation of solutions. In Section III we take a different approach by considering a transform-domain expression of the solutions of PE, obtained by complex analysis of the Poisson equation.

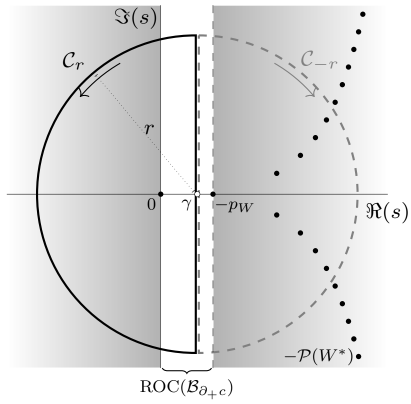


Fig. 2: Convergence of $\mathcal{B}_{\partial_+ c}$ for constant service times $X = x$ and step cost function $f(u) = \mathbf{1}_{[\tau,\infty)}(u)$, with $\tau > 0$: $\mathcal{L}_f(s) = 1/s$ has one pole at $s = 0$ with $\mathrm{ROC}(\mathcal{L}_f) = \{s \in \mathbb{C} \,|\, \Re(s) > 0\}$, while $W^*(-s) = (1-\lambda x)s/[s + \lambda(1 - e^{sx})]$ has an infinity of poles at $s = -\lambda[1 + (1/\rho)\,\mathrm{W}_k\,(-\rho e^{-\rho})]$ for $k \in \mathbb{Z}_0$, where $\rho = \lambda x$ and W_k denotes the kth branch of the product logarithm function, with $-p_W = -\lambda[1 + (1/\rho)\,\mathrm{W}_{-1}\,(-\rho e^{-\rho})] > 0$, [11].

III Closed-form relative value function

In this section we develop the tools that will help us compute relative value functions.

III.1 Characterization of the relative value function

Before proceeding, recall the Pollaczek-Khintchine formula for the Laplace-Stieltjes transform of W, [37,26], which we characterize in Appendix B:

$$W^*(s) = \frac{(1-\rho)s}{s - \lambda(1 - X^*(s))}. \tag{PK}$$

Let p_W denote the dominant singularity of W^* which, in view of Proposition B.1(i), is a real negative pole. In the transform domain, μ_W-integrability of $|f|$ reduces to a condition on the relative positions of the singularities of $W^*(-s)$ and those of $\mathcal{L}_f(s) = \int_0^\infty e^{-su}\, f(u)\, du$, the Laplace transform of f. Concretely, the regions of absolute convergence (ROCs) of $W^*(-s)$ and $\mathcal{L}_f(s)$ (two open half-planes with normal vectors pointing in opposite directions) should have nonempty intersection, i.e., $-p_W \in \mathrm{ROC}(\mathcal{L}_f)$. This condition is illustrated in Figure 2 for the case of constant service times.

For analysis purposes, we now extend the nonnegative process (Q) to negative backlog values by presuming of a (fictitious) stochastic process governed by (PE) over the entire real axis. We set the scene as follows.

First, we let $f(u) = 0$ for $u < 0$, and we complete (II.1) with $P(u, [t, u]) = 1 - e^{-\lambda(u-t)}$ if $u < 0$, thus conjecturing for (Q) the $\mathbb{R}_{<0}$ behaviour

$$U_{n+1} = U_n + X_n - T_{n+1} \quad \text{if} \quad U_n + X_n < 0, \qquad n \geq 0. \tag{Q$^-$}$$

Observe that the so extended Markov process loses the irreducibility of (Q), since the process remains caught in $\mathbb{R}_{\geq 0}$ once it has occupied a nonnegative state. Otherwise, it is expected to drift towards $u = -\infty$, where its chances vanish to ever reach $\mathbb{R}_{\geq 0}$. Next, we consider an ancillary, more tractable transition kernel $\widehat{P}$ of the type (II.1) with *uniform* dynamics for the backlogs:

$$\widehat{P}(u, [t, u]) = 1 - e^{-\lambda(u-t)}, \qquad \forall u \in \mathbb{R}. \tag{III.1}$$

The Poisson equation (PE) then rewrites as the simple form

$$g(u, x) = \widehat{P}g(u, x) + \widehat{f}(u, x), \tag{PE'}$$

where $\widehat{f}(u, x) := \Delta(u + x) + f(u) - \bar{f}\,\mathbf{1}_{[0,+\infty)}(u)$, and $\Delta(u) := (P - \widehat{P})g(u, 0)$. Clearly, (PE') retains the property that its solutions are defined up to a constant. By construction, they also solve (PE) on $\mathbb{R}_{\geq 0}$. The true and virtual parts of these solutions over $\mathbb{R}_{<0}$ are identified by Theorem 1.

Theorem 1 (Extended Poisson equation) *Every solution of* (PE') *has the form* $g(u, x) = \widehat{v}(u + x) + f(u) + r(u + x)\,\mathbf{1}_{(-\infty,-x)}(u)$ *for some* $r : \mathbb{R} \mapsto \mathbb{R}$ *common to all solutions and for* $\widehat{v} : \mathbb{R} \mapsto \mathbb{R}$ *satisfying*

$$\widehat{v}(u) = v(0) + c(u) - \frac{\lambda \bar{f}}{1 - \rho} u\,\mathbf{1}_{[0,+\infty)}(u) + r(u)\,\mathbf{1}_{(-\infty,0)}(u), \quad \forall u \in \mathbb{R}, \tag{$\widehat{\text{S}}$}$$

where the two-sided Laplace transform of the right derivative of c, $\mathcal{B}_{\partial_+ c}(s) = \int_{-\infty}^{\infty} e^{-su}\,\partial_+ c(u)\,du$*, is given on its nonempty region of convergence by*

$$\mathcal{B}_{\partial_+ c}(s) = \frac{\lambda}{(1 - \rho)} W^*(-s)\mathcal{L}_f(s). \tag{C}$$

Theorem 1 can be shown by transform-domain analysis of the solutions of (PE'). The proofs of all the results given in this section are deferred to Appendix B.

The function $\widehat{v}$ in ($\widehat{\text{S}}$) is an extension of the relative value function to the negative backlogs, with $\widehat{v}(u) \equiv v(u)$ if $u \geq 0$. Theorem 1 suggests that the relative value function (VF) characterizes the M/G/1 queue (Q) as much as the imaginary process (Q$^-$) taking place in the negative backlog values. What is more, the hidden negative end of the queue seems to hold the key to solving the associated Poisson equation in the transform domain.

By inverse transformation of ($\widehat{\text{S}}$), we obtain the following results.

Proposition 2 (Relative value function) *Let f be piecewise continuous.*

(i) The relative value function (VF) *is continuous, almost everywhere continuously differentiable, and semi-differentiable with right derivative*

$$\partial_+ v(u) = \lambda \left(f^+(u) - \bar{f} + \mathbb{E}\left[v(u+X) - v(u) \right] \right), \qquad \forall u \in \mathbb{R}_{>0}, \tag{DE}$$

where $f^+(u) := \lim_{t\to u^+} f(t)$*. At* $u = 0$*, one has*

$$v(0) = f(0) - \bar{f} + \mathbb{E}[v(X_0)], \tag{BCa}$$

$$v'(0) = \lambda \left(f^+(0) - f(0) + \mathbb{E}[v(X) - v(X_0)] \right). \tag{BCb}$$

(ii) The relative value function is given by

$$v(u) = v(0) + c(u) - \frac{\lambda \bar{f}}{1-\rho} u, \qquad \forall u \in \mathbb{R}_{\geq 0}, \tag{S}$$

where $c : \mathbb{R} \mapsto \mathbb{R}$ *is continuous, almost everywhere continuously differentiable, and semi-differentiable with right-derivative*

$$\partial_+ c(u) = \frac{\lambda}{1-\rho} \mathbb{E}\left[f(u+W) \right], \qquad \forall u \in \mathbb{R}. \tag{CVF}$$

Equation (DE) in Proposition 2(i) was for instance used in [21] to derive the relative value function of the M/D/1 queue with a step cost function $\mathbf{1}_{[\tau,\infty)}$. However, the expectation of the random jump $v(\cdot + X)$, makes (DE) difficult to solve for v in the general case. The result reported in (ii) is but the expression taken by the kernel solution of [14] in the limit case where the invariant measure of the Poisson equation coincides with the stationary measure of the waiting times. A relation of duality can be observed between (S), where the relative value function follows by cross-correlation of the cost function with the asymptic waiting times, and (DE), where the cost function can be recovered by cross-correlation of the relative value function and the service times. In fact, (DE) and (S) are backlog-domain renditions of the same transform-domain solution (C).

A closer look at (S) tells us that the computation of the relative value function v reduces to the derivation through (CVF) of a related function, denoted c in this work and referred to as the *'core' value function* or, more concisely, *core function*. Intuitively, $c(u)$ corresponds to the expected total cost experienced by the queue from an initial state u until it returns to the empty state 0. By construction, $c(0) = 0$, and the rest of $c(u)$ can be obtained by integration from 0 of its right-derivative $\partial_+ c$, available via (C) or (CVF). Observe that c is fully characterized by λ, X and f^+, independently of the parameters X_0 and $f(0)$, which specify the behavior of the queue at $u = 0$.

The rest of the study is principally concerned with the derivation of the core function, with disregard to the other two terms in (S). Once c is known, the mean cost $\bar{f}$ can be inferred from X_0 and $f(0)$ on condition that $|f|$ is $\mu_{\tilde{W}}$-integrable. Combining (BCb) with (CVF) then yields

$$\bar{f} = \left(\frac{1-\rho}{1-\rho+\tilde{\rho}} \right) \left\{ c'(0)/\lambda + f(0) - f^+(0) + \mathbb{E}[c(X_0)] - \mathbb{E}[c(X)] \right\}. \tag{III.2}$$

Table 1: Explicit core functions for $f = f_{a,n}$, $(a \in \mathcal{P}_W,\ n \in \mathbb{N})$.

$f(u)$	$c'(u)$	$c(u)$
1	$\frac{\lambda}{1-\rho}$	$\frac{\lambda}{1-\rho} u$
e^{-au}	$\frac{\lambda}{1-\rho} W^*(a)\, e^{-au}$	$\frac{\lambda}{1-\rho} W^*(a) \frac{1-e^{-au}}{a}$
u^n	$\frac{\lambda\, n!}{1-\rho} \sum_{k=0}^{n} w_{n-k} \frac{u^k}{k!}$	$\frac{\lambda\, n!}{1-\rho} \sum_{k=0}^{n} w_{n-k} \frac{u^{k+1}}{(k+1)!}$
$u^n e^{-au}$	$\frac{\lambda n!}{1-\rho} \sum_{k=0}^{n} w_{a:n-k} \frac{u^k e^{-au}}{k!}$	$\frac{\lambda n!}{1-\rho} \sum_{t=0}^{n} \left(\sum_{k=t}^{n} \frac{w_{a:n-k}}{a^{k+1}} \right) \left(\delta[t] - \frac{(au)^t e^{-au}}{t!} \right)$

Coefficients:

$$
\begin{aligned}
w_0 &= 1, \\
w_k &= [\lambda/(1-\rho)] \textstyle\sum_{t=0}^{k-1} x_{k-t+1} w_t, \quad (k \geq 1), \\
x_k &= 1/(k!)\, \mathbb{E}[X^k], \quad (k \geq 0),
\end{aligned}
\tag{III.3}
$$

$$
\begin{aligned}
w_{a:0} &= W^*(a), \\
w_{a:1} &= [\lambda/(1-\rho)] \left(W^*(a)/a\right)^2 \left(1 - X^*(a) - a x_{a:1}\right), \\
w_{a:k} &= [\lambda W^*(a)/(1-\rho)a] \left[(1/\lambda - x_{a:1})\, w_{a:k-1} - \textstyle\sum_{t=0}^{k-2} x_{a:k-t} w_{a:t}\right], \quad (k \geq 2), \\
x_{a:k} &= 1/(k!)\, \mathbb{E}[X^k e^{-aX}], \quad (k \geq 0).
\end{aligned}
\tag{III.4}
$$

Note that the CVF and the mean cost are all we need for FPI-dispatching, since the *admission cost* of a job with service time x at state u reduces to

$$
\mathcal{A}(u,x) := f(u) + v(u+x) - v(u) = c(u+x) - c(u) - \left(\frac{\lambda \bar{f}}{1-\rho} \right) x. \tag{AC}
$$

See Appendix A for details on the role of the admission cost in FPI-dispatching.

In Sections III.2-III.3, we exploit these results and derive the causal part of $\partial_+ c$ by inverse transformation of (C).

III.2 Basic solutions: analytic cost functions

The analysis of (C) is straightforward for the cost functions belonging to the class $\Xi := \mathrm{span}(\{f_{a,n} \,|\, a \in \mathbb{C},\ n \in \mathbb{N}\})$, where $\mathrm{span}(S)$ denotes the linear span of a set S, and the function $f_{a,n}$, defined by $f_{a,n}(u) = u^n e^{-au}$, is characterized by the meromorphic Laplace transform $\mathcal{L}_{f_{a,n}}(s) = n!/(s+a)^{n+1}$, which is analytic on the complex plane except for a set of isolated, non-essential singularities, called poles. Observe that the condition of existence of the core function reduces for the cost function $f_{a,n}$ to $a \in \mathcal{P}_W$, where we write $\mathcal{P}_W = \{s \in \mathbb{C} \,|\, \Re(s) < -p_W\}$.

Table 1 provides us with the closed-form core functions for the cost function $f_{a,n}$, obtained after inversion of (C) by integration along a vertical axis in the region of absolute convergence of $\mathcal{B}_{\partial_+ c}$, as we proceed to do now. Let $\gamma \in (a, -p_W)$, and consider the contour $\mathcal{C}_r = \{\gamma + it \,|\, t \in [-r, r]\} \cup \mathcal{A}_r$, where $\mathcal{A}_r = \{\gamma + re^{i\alpha} \,|\, \alpha \in [\frac{\pi}{2}, \frac{3\pi}{2}]\}$ is an arc centered in γ. Since $\lim_{s\to\infty} |W^*(s)| \leq 1$ (cf. Proposition B.1(ii)), we find $\lim_{r\to\infty} W^*(-\gamma - re^{i\alpha}) \mathcal{L}_{f_{a,n}}(\gamma + re^{i\alpha}) = 0$

for $\alpha \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, and the condition of the third Jordan lemma is satisfied [33, §3.1.4, Theorem 1][10, §88]. It follows that integration of $\mathcal{B}_{\partial_+ c}(s)e^{su}$ along the arc $\mathcal{A}_r$ vanishes as $r \to \infty$,

$$\lim_{r\to\infty} \int_{\mathcal{A}_r} W^*(-s)\mathcal{L}_{f_{a,n}}(s)\, e^{su}\, ds = 0, \qquad \forall u \in \mathbb{R}_{\geq 0}, \tag{III.5}$$

and counterclockwise integration of $\mathcal{B}_{\partial_+ c}(s)e^{su}$ on the contour $\mathcal{C}_r$ reduces to computing the residue[2] at the pole of $\mathcal{L}_{f_{a,n}}$. The residue theorem gives

$$\begin{aligned} c'_{a:n}(u) &\overset{(\mathrm{C})}{=} \tfrac{1}{2\pi i} \lim_{t\to\infty} \int_{\gamma-it}^{\gamma+it} \left(\tfrac{\lambda}{1-\rho} W^*(-s) \tfrac{n!}{(s+a)^{n+1}}\right) e^{su}\, ds \\ &\overset{(\mathrm{III.5})}{=} \tfrac{1}{2\pi i} \oint_{\mathcal{C}_r} \left(\tfrac{\lambda}{1-\rho} W^*(-s) \tfrac{n!}{(s+a)^{n+1}}\right) e^{su}\, ds \\ &= \operatorname{Res}_{s=-a}\left(\tfrac{\lambda}{1-\rho} W^*(-s) \tfrac{n!}{(s+a)^{n+1}} e^{su}\right) \\ &\overset{(\mathrm{III.6})}{=} \tfrac{\lambda}{1-\rho} \lim_{s\to-a} \tfrac{1}{n!} \tfrac{d^n}{ds^n}\left[n!\, W^*(-s)e^{su}\right] \\ &= \tfrac{\lambda}{1-\rho} \textstyle\sum_{k=0}^n \binom{n}{k} \left((-1)^{n-k} \tfrac{d^{n-k}}{ds^{n-k}} W^*(a)\right) u^k e^{-au} \\ &= \tfrac{n!\lambda}{1-\rho} \textstyle\sum_{k=0}^n w_{a:n-k} \left(\tfrac{u^k}{k!}\right) e^{-au}, \end{aligned} \tag{III.7}$$

for all $u \in \mathbb{R}_{\geq 0}$, in which

$$w_{a:k} = \frac{(-1)^k}{k!} \frac{d^k}{ds^k} W^*(a)$$

is the kth coefficient of the Taylor expansion of $W^*(-s)$ at a, reducing to $w_{0:k} \equiv w_k$ if $a = 0$. The coefficients $\{w_k\}$ and $\{w_{a:k}\}$ will be referred to as the germ of $W^*(-s)$. In (III.3) and (III.4), they are computed inductively as functions of the coefficients $\{x_k\}$ and $\{x_{a:k}\}$ of the power series of $X^*(s)$. As such, they are finite by analycity of $X^*(s)$ on $\mathcal{P}_W$ (cf. Proposition B.1(i)). See also Proposition B.1(iv)-(v) for a derivation of (III.3) and (III.4), and [9] for expressions of $\{w_k\}$ specific to standard service time distributions. The final expressions[3] for $c'_{a:n}$ and $c_{a:n}$ are reported in Table 1.

Since the operation $f \mapsto c$ is a linear map, observe that all cost functions given as linear combinations of $f_{a,n}$ types are elements of Ξ enjoying explicit relative value functions. Examples include the trigonometric functions cos and sin, which play a part in the developments of Section IV.2, or the set of incomplete gamma functions $\{\Gamma(n+1, a\cdot) \,|\, n \in \mathbb{N},\, a \in \mathbb{C}\}$, which spans Ξ completely.

[2] Recall that the residue of a meromorphic function f at a pole a of order n is given by [10]

$$\operatorname{Res}_{s=a}(f(s)) = \tfrac{1}{(n-1)!} \lim_{s\to a} \tfrac{d^{n-1}}{ds^{n-1}}\left[(s-a)^n f(s)\right]. \tag{III.6}$$

[3] Alternatively, notice that $f_{a,n} = (-1)^n(\delta^n/\delta a^n)\, f_{a,0}$ if $a \in \mathcal{P}_W \setminus \{0\}$. It follows from (CVF) and the Leibniz integral rule that, for $a \in \mathcal{P}_W \setminus \{0\}$ and $n > 0$, $c'_{a:n}(u) = (-1)^n(\delta^n/\delta a^n)c'_{a:0}(u) = (-1)^n[\lambda/(1-\rho)]\,(\delta^n/\delta a^n)[W^*(a)e^{-au}]$, and the expressions for $c'_{a:n}$ can be derived by successive differentiations of $c'_{a:0}$. By continuity arguments, we also find, for $n > 0$, $c'_{0:n}(u) = (-1)^n \lim_{a\to 0}(\delta^n/\delta a^n)[c'_{a:0}(u)]$.

III.3 Piecewise-defined cost functions

Let $f_0, f_1 \in \Xi$, and assume the cost function is given by $f = f_0\,\mathbf{1}_{[0,\tau)} + f_1\,\mathbf{1}_{[\tau,\infty)}$, where $\mathbf{1}.$ denotes the indicator function, or, equivalently,

$$f(u) = f_0(u) + \Delta(u)\,\mathbf{1}_{[\tau,\infty)}(u), \quad \forall u \in \mathbb{R}_{\geq 0}. \tag{III.8}$$

where $\Delta = f_1 - f_0$. Since the Laplace transform of $f(u) = u^n e^{-au}\,\mathbf{1}_{[\tau,\infty)}(u)$ is given on the half-plane $\Re(s) > \Re(-a)$ by

$$\mathcal{L}_f(s) = \int_\tau^\infty u^n e^{-(s+a)u}\,du = n! e^{-a\tau} \sum_{q=0}^n \frac{\tau^q}{q!(s+a)^{n-q+1}}\,e^{-s\tau}, \tag{III.9}$$

we can find ζ such that $\mathcal{L}_f(s) = \mathcal{L}_{f_0}(s) + \zeta(s,\tau)e^{-s\tau}$ with $\lim_{s\to\infty} \zeta(s,\tau) = 0$. If we place γ in the half-plane $\Re\,(s) > 0$ between $-p_W$ and the poles of f_0, f_1, (III.5) becomes in the present setting,

$$\lim_{r\to\infty} \int_{\mathcal{A}_r} W^*(-s)\mathcal{L}_{f_0}(s)\,e^{su}\,ds = 0, \quad \forall u \in \mathbb{R}_{\geq 0},$$

for the first term, and

$$\begin{aligned} &\lim_{r\to\infty} \int_{\mathcal{A}_r} W^*(-s)\zeta(s,\tau)\,e^{s(u-\tau)}\,ds = 0, \qquad \forall u \in (\tau,\infty),\\ &\lim_{r\to\infty} \int_{\mathcal{A}_{-r}} W^*(-s)\zeta(s,\tau)\,e^{s(u-\tau)}\,ds = 0, \quad \forall u \in [0,\tau), \end{aligned}$$

for the second term. Thus, inverse transformation by counterclockwise integration along $\mathcal{C}_r$ still applies for all backlog values $u > \tau$, where

$$\partial_+ c(u) = \frac{\lambda}{1-\rho} \sum_{p\in\mathcal{P}(\mathcal{L}_f)} \operatorname{Res}_{s=p}\big(W^*(-s)\mathcal{L}_f(s)\,e^{su}\big), \quad \forall u \in (\tau,\infty). \tag{III.10}$$

It is clear from (CVF) that the derivative $\partial_+ c(u)$ for $u > \tau$ does not depend on the values of the cost function on the interval $(0,\tau)$. It is therefore equal over (τ,∞) to the derivative of the core function for the analytic cost $f = f_1$, and it can equivalently be derived from (III.7) (or, alternatively, inferred from Table 1) for the cost function $f = f_1$.

For $u < \tau$, however, the terms f_0 and $\Delta\mathbf{1}_{[\tau,\infty)}$ in (III.8) must be treated separately: f_0 by simple inspection of Table 1, and $\Delta\mathbf{1}_{[\tau,\infty)}$ by clockwise integration along the contour $\mathcal{C}_{-r} = \{\gamma + it \,|\, t \in [-r,r]\} \cup \mathcal{A}_{-r}$. The success of this last operation is conditioned by the singularities of $W^*(-s)$, all contained in the interior of $\mathcal{C}_{-r}$ as $r \to \infty$. In our discussion we consider separately the service time distributions for which W^* has a finite set of poles $\mathcal{P}(W^*)$ (e.g. exponential or Erlang service time distributions), and those for which W^* has infinitely many poles (as in discrete service time distributions).

If $\mathcal{P}(W^*)$ is finite, the clockwise integral along $\mathcal{C}_{-r}$ yields $|\mathcal{P}(W^*)|$ residues at the poles of $W^*(-s)$, and we find

$$\begin{aligned} \partial_+ c(u) = {}& \frac{\lambda}{1-\rho} \sum_{p\in\mathcal{P}(\mathcal{L}_{f_0})} \operatorname{Res}_{s=p}\big(W^*(-s)\mathcal{L}_{f_0}(s)\,e^{su}\big)\\ &- \frac{\lambda}{1-\rho} \sum_{p\in\mathcal{P}(W^*)} \operatorname{Res}_{s=-p}\big(W^*(-s)\zeta(s,\tau)\,e^{s(u-\tau)}\big), \quad \forall u \in (0,\tau). \end{aligned} \tag{III.11}$$

If otherwise $\mathcal{P}(W^*)$ is infinite, then the clockwise integral along $\mathcal{C}_{-r}$ cannot be computed directly by the residue theorem, which would issue an infinite sum. This difficulty can nevertheless be overcome whenever W^* rewrites as

$$W^* = W_u^{\Lsh} + W_u^{\Rsh}, \quad \forall u \in (0, \tau), \tag{III.12}$$

where $W_u^{\Lsh}$ and $W_u^{\Rsh}$ are meromorphic, $|\mathcal{P}(W_u^{\Rsh})|$ is finite, and

$$\lim_{r\to\infty} \textstyle\int_{\mathcal{A}_{-r}} W_u^{\Rsh}(-s)\zeta(s,\tau)\, e^{s(u-\tau)}\, ds = \lim_{r\to\infty} \textstyle\int_{\mathcal{A}_r} W_u^{\Lsh}(-s)\zeta(s,\tau)\, e^{s(u-\tau)}\, ds = 0.$$

Then, if we choose $\gamma \in \mathrm{ROC}(\mathcal{B}_{\partial_+ c})$ and consider the pole sets $\mathcal{P}_u^{\Lsh} = \{p \in \mathcal{P}(W_u^{\Lsh}(-\cdot)\, \zeta(\cdot,\tau)), \Re(p) < \gamma\}$ and $\mathcal{P}_u^{\Rsh} = \{p \in \mathcal{P}(W_u^{\Rsh}(-\cdot)\, \zeta(\cdot,\tau)), \Re(p) > \gamma\}$, both finite in cardinality, we find

$$\begin{aligned} \partial_+ c(u) = {} & \tfrac{\lambda}{1-\rho} \textstyle\sum_{p\in\mathcal{P}(\mathcal{L}_{f_0})} \mathrm{Res}_{s=p}\big(W^*(-s)\mathcal{L}_{f_0}(s)\, e^{su}\big) \\ & + \tfrac{\lambda}{1-\rho} \textstyle\sum_{p\in\mathcal{P}_u^{\Lsh}} \mathrm{Res}_{s=p}\big(W_u^{\Lsh}(-s)\, \zeta(s,\tau)\, e^{s(u-\tau)}\big) \\ & - \tfrac{\lambda}{1-\rho} \textstyle\sum_{p\in\mathcal{P}_u^{\Rsh}} \mathrm{Res}_{s=p}\big(W_u^{\Rsh}(-s)\zeta(s,\tau)\, e^{s(u-\tau)}\big), \quad \forall u \in (0,\tau). \end{aligned} \tag{III.13}$$

It is seen in Appendix C that the decomposition proposed in (III.12) is relevant in particular in the case of discrete service time distributions.

IV Relative value function approximations

In the absence of exact expressions for the relative value functions, the FPI step can still be carried out based on relative value function bounds. Suppose that lower and upper bounds, f_- and f_+, are available for f with explicitly computable core functions, denoted by c_- and c_+, respectively. Using the interval arithmetic notation,[4] we write $f \in [f] \equiv [f_-, f_+]$ and, by linearity of the map $f \mapsto c$, we find in $[c] \equiv [c_-, c_+]$ a bounding interval for the core function, while (III.2) provides the bounds $[\bar{f}] \equiv [\bar{f}_-, \bar{f}_+]$ for the mean cost $\bar{f}$.

In the k-server system of Figure 1 with arrival rates $\lambda_1, \ldots, \lambda_k$ and cost functions bounded by $[f_1], \ldots, [f_k]$, the admission cost (AC) inherits the bounds

$$[\mathcal{A}_i](u,x) = [c_i](u_i + x_i) - [c_i](u_i) - \left(\frac{\lambda_i[\bar{f}_i]}{1-\rho_i}\right) x_i, \tag{[AC]}$$

where $[c_1], \ldots, [c_k]$ and $[\bar{f}_1], \ldots, [\bar{f}_k]$ are the corresponding interval bounds for the core function and mean costs. The FPI decision at state (u,x) can be made in favor of a server $i \in \{1, \ldots, k\}$ iff

$$[\mathcal{A}_i](u,x) \le [\mathcal{A}_j](u,x), \quad \forall j \neq i. \tag{D}$$

If otherwise no server satisfies (D), the precision of the interval bounds for the cost functions must be improved until a decision can be made.

In the rest of this section we discuss various cost approximation schemes.

[4] *Interval arithmetic.* We use $[x] \equiv [x_1, x_2]$ to represent an interval on $\mathbb{R}$. We write $[x] \in [\mathbb{R}]$ where $[\mathbb{R}] = \{[x_1, x_2] \,|\, x_1 \le x_2;\, x_1, x_2 \in \mathbb{R}\}$, $a \in [x]$ iff $x_1 \le a \le x_2$, $|[x]| = x_2 - x_1$, and $-[x] = [-x_2, -x_1]$. For $[x], [y] \in [\mathbb{R}]$ we have $[x]+[y] = [x_1+y_1, x_2+y_2]$, $[x] < [y]$ iff $x_2 < y_1$, and $[x] \le [y]$, $[x] > [y]$ and $[x] \ge [y]$ are defined similarly.

IV.1 Analytic cost functions and Taylor series

Due to the availability of explicit relative value functions for the type $f(u) = u^n n$, Taylor/Maclaurin series have been cited as natural candidates for the approximation of analytic cost functions, [20]. Let f be an infinitely smooth real function on $\mathbb{R}_{\geq 0}$ with k-th derivative $f^{(k)}$. For $n \in \mathbb{N}$, consider an interval $[r^{(n)}]$ such that $f \in \hat{f}^{(n)} + [r^{(n)}]$, where $\hat{f}^{(n)}(u) = \sum_{k=0}^{n} f^{(k)}(0)u^k/k!$ is the Taylor polynomial of order n. If $\hat{c}^{(n)}$ denotes the core function associated with $\hat{f}^{(n)}$, and $[\varrho^{(n)}]$ is a bounding interval covering the core functions for all cost functions comprised in $[r^{(n)}]$, then using Table 1 we find

$$\hat{c}^{(n)}(u) = \tfrac{\lambda}{1-\rho} \textstyle\sum_{k=0}^{n} \{\sum_{t=0}^{n-k} w_t\, f^{(k+t)}(0)\} \tfrac{u^{k+1}}{(k+1)!},$$

and, by linearity, $c \in [c^{(n)}]$, where $[c^{(n)}] = \hat{c}^{(n)} + [\varrho^{(n)}]$.

If f is analytic, then $[r^{(n)}]$ vanishes pointwise near $u = 0$ as $n \to \infty$, and our hopes are that the remainder $[\varrho^{(n)}]$ will become small as well, with $[c^{(n)}]$ converging towards c in some sense. The next result, however, claims that a cost function f given as a convergent Taylor series only yields a convergent sequence of core functions if f is entire (i.e., its Taylor series converges everywhere) with order of growth[5] less than the exponential type $|p_W|$, whereas any function f falling outside this restrictive category is expected to produce a divergent sequence for c. A proof of Theorem 2 is given in Appendix E.

Theorem 2 (Taylor series for c**)** *Let f be entire with order of growth ϱ and type σ, so that*

$$f(u) = \sum_{n=0}^{\infty} [f^{(n)}(t)/n!]\,(u-t)^n, \qquad \forall u, t \in \mathbb{R}_{\geq 0}. \tag{IV.1}$$

For $k \in \mathbb{N}$, let $\tilde{c}_k = \lim_{n\to\infty} \tilde{c}_k^{(n)}$, where

$$\tilde{c}_k^{(n)} = [\lambda/(1-\rho)] \sum_{q=0}^{n-k} w_q\, f^{(k+q)}(0), \qquad (n \in \mathbb{N}), \tag{IV.2}$$

and define the functions ψ and χ as

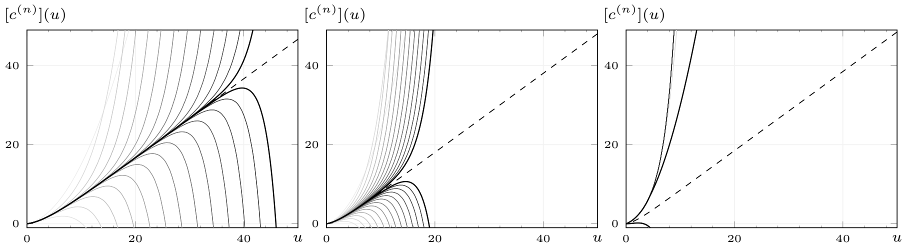
$$\psi(u) = \int_0^u \sum_{k=0}^{\infty} \frac{\tilde{c}_k}{k!}\, \xi^k \, d\xi, \tag{IV.3a}$$

$$\chi(u) = \frac{\lambda}{1-\rho} \int_0^u \sum_{q=0}^{\infty} w_q\, f^{(q)}(\xi)\, d\xi. \tag{IV.3b}$$

(i) If either $\varrho < 1$ or $\varrho = 1$ and $\sigma < |p_W|$, then the coefficients $\tilde{c}_n$ are finite for all n, (IV.3a) *and* (IV.3b) *converge on $\mathbb{R}_{\geq 0}$, and $\psi = \chi = c$.*

(ii) If either $\varrho > 1$ or $\varrho = 1$ and $\sigma > |p_W|$, then (IV.2) *diverges for all k.*

[5] Recall that the *order of growth* of an entire function f [30], defined by $\varrho = \limsup_{r\to\infty} \ln\ln \|f\|_{\infty,r}/\ln r$, where $\|f\|_{\infty,r} = \sup_s\{|f(s)|\,|\,|s| < r\}$, is the infimum of all m such that $f(s) = O(\exp(|s|^m))$, while the *type* of f is defined by $\sigma = \limsup_{r\to\infty} \ln \|f\|_{\infty,r}/r^{\varrho}$. If $\varrho = 1$, then f is said to be of exponential type σ.



(a) $a = 0.5\,(\omega - \lambda)$

(b) $a = 0.8\,(\omega - \lambda)$

(c) $a = 1.0\,(\omega - \lambda)$

Fig. 3: Taylor series for the CVF with exponential service times ($X \sim \text{Exp}(\omega)$; $\omega = 2\lambda$), and cost function $f(u) = 1 - e^{-au}$ of exponential type $|a|$: c (dashed line) and $[c^{(n)}]$ for $n = 1, \dots, 25$. The series converges only if $|a| < |p_W| = \omega - \lambda$.

Equation (IV.3a) is the Taylor series (in convergence conditions) of c at $u = 0$. The coefficients of the series are given by $\{\tilde{c}_k\}$, the sequence of the successive derivatives of $c'(u)$ at 0, obtained by cross-correlation between the sequence $\{f^{(k)}(0)\}$ of the derivatives of f at $u = 0$ and $\{w_k\}$, the germ of W^* at the origin, given in (III.3). Equation (IV.3a) may be understood as an extension of (IV.2) to $u \geq 0$, in the sense that $c'(u)$ is computed directly by cross-correlation of the cost derivatives at u with the the germ of W^* at 0.

The message of Theorem 2 is illustrated by Figure 3, which exposes through an elementary problem the hazards of processing cost functions as Taylor series. Appendix D features an interpretation of this issue in terms of systems and signals, where the coefficients (IV.2) result by linear filtering of coefficients of the Taylor series of f and Theorem 2 is understood as the condition for stability of the system. See [8] and the examples in [9] for all computational details related to Figure 3.

The conclusions of Theorem 2 lead us to consider, in the rest of Section IV, approximations schemes no longer on $\mathbb{R}_{\geq 0}$, where the growth of the functions $\{u^n\}$ as $u \to \infty$ causes divergence, but on finite intervals where the series converge safely.

IV.2 Continuous cost functions and uniform approximation

Assume now that the cost function f is continuous[6], and partition the backlog axis into an interval $(0, \tau)$ where f is approximated precisely (in virtue of the Weierstrass approximation theorem) with respect to the uniform norm $\|f\| = \sup_{u \in [0,\tau]} |f(u)|$ by a finite sum $\hat{f}^{(n)}$ of degree n, and its complement (τ, ∞), where unrefined bounds in Ξ are chosen for f. Bounds for the core function can be inferred from the developments of Section III.3.

[6] Piecewise continuous functions can be treated similarly by partitioning $\mathbb{R}_{>0}$ into as many intervals as required by their discontinuities.

Definition 1 **($\mathcal{W}$)** Given $\tau > 0$ and $\hat{f}, \xi \in \varXi$, we denote by $\mathcal{W}(\hat{f}; \xi)$ the CVF relative to the cost function $f = \hat{f}\,\mathbf{1}_{(0,\tau)} + \xi\,\mathbf{1}_{(\tau,\infty)}$.

Proposition 3 **(Continuous cost)** *Consider the server model of Section II with a cost function f continuous on a nonempty interval $(0, \tau)$, and such that $f \in [f]$, where*

$$[f] = \{\hat{f}^{(n)} + [-\eta^{(n)}, \eta^{(n)}]\}\,\mathbf{1}_{(0,\tau)} + [\xi]\,\mathbf{1}_{(\tau,\infty)}, \tag{IV.4}$$

in which $\hat{f}^{(n)}$ is a finite sum of degree $n \in \mathbb{N}$, $\eta^{(n)} \geq 0$, $[\xi] \equiv [\xi_-, \xi_+]$, and $\hat{f}^{(n)}, \xi_-, \xi_+$ are real elements of $\varXi$. The core function satisfies

$$c \in \left[\mathcal{W}(\hat{f}^{(n)} - \eta^{(n)}; \xi_-), \mathcal{W}(\hat{f}^{(n)} + \eta^{(n)}; \xi_+)\right],$$

where $\mathcal{W}$ (cf. Definition 1) is computed as in Section III.3.

The FPI step can be implemented based on the interval bounds ([AC]) for $[f]$, in place of the actual admission cost (AC), provided that the parameters τ and n chosen for the servers allow for it. Otherwise, the parameter values should be refined (by increasing τ and n) until decision (D) can be made.

A pseudocode for the resulting procedure is given in Algorithm 1, where the cost function f_i of each server i is supplied with a continuum of bounding interval functions $[\xi_i]$ such that, for every $\tau > 0$, $f_i(u_i) \in [\xi_i(\tau)](u_i)$ if $u_i > \tau$. Algorithm 1 infers the FPI decision $\hat{\pi}(u, x)$ at any state (u, x) by gradually decreasing the error tolerance ϵ_t of the admission cost bounds at each server, computed by (IV.4). To guarantee the error margin ϵ_t at a server i, the parameter τ_i is first taken large enough for the approximation error in the $u > \tau_i$ window to be less than $\epsilon_t/2$ (line 1), then the sum $\hat{f}^{(n)}$ is given enough terms for the approximation error in the $0 < u < \tau_i$ window to be less than $\epsilon_t/2$ (line 2), so that the overall precision ϵ_t is secured for the bounds $[\hat{f}_i]$ (line 3). All servers with exceeding admission costs will be ignored (line 4) for the rest of the procedure, which resumes with a smaller margin ϵ_{t+1}.

Uniform approximation. In Table 2 are listed possible approximation schemes for the function f over the interval $(0, \tau)$, together with their respective expression for the finite sum $\hat{f}^{(n)}$ and the associated uniform error bounds $\eta^{(n)}$, given in terms of the *modulus of continuity* of f on $[0, \tau]$, defined by

$$\omega(f; [0, \tau]; \delta) = \sup\{|f(u_1) - f(u_2)| : u_1, u_2 \in [0, \tau], |u_1 - u_2| \leq \delta\}. \tag{IV.5}$$

A simple option for approximating f is the *Bernstein polynomial*, [4]. Its expression in Table 2 rewrites as $b^{(n)}(u) = \mathbb{E}[f(K\tau/n)]$, where $K \sim \mathrm{B}(n, u/\tau)$ denotes the binomial random variable obtained from n trials with success probability u/τ, so that the argument $K\tau/n$ has mean u and variance $O(\tau^2/n)$

[7] In Algorithm 1, the first argument of $\mathcal{A}_i\,(f; \cdot, \cdot)$ (or $[\mathcal{A}_i]\,([f]; \cdot, \cdot)$) indicates the cost function f (resp. the interval function $[f]$) for which the admission cost at server i is computed.

Algorithm 1: FPI with interval relative value functions[7]

Data: $\{(\lambda_1, f_1, [\xi_1]), \ldots, (\lambda_k, f_k, [\xi_k])\}$, $\{\epsilon_t\}$ with $\epsilon_t \downarrow 0$
Input : $(u, x) \in \mathbb{R}^k_{\geq 0} \times \mathbb{R}^k_{\geq 0}$
Output: $\pi \subset \{1, \ldots, k\}$

Initialization: $t \leftarrow 0$, $\pi \leftarrow \{1, \ldots, k\}$, $[\hat{f}_i] \leftarrow (-\infty, \infty)$ for all $i \in \pi$

While $t \leq t_{\max}$ **and** $|\pi| > 1$ **do**
  For $i \in \pi$ **do**
    $\tau_i \leftarrow \arg\inf_{\tau \geq 0} \left\{ \left| [\mathcal{A}_i] \left([\xi_i(\tau)] \, \mathbf{1}_{(\tau,\infty)}; u, x \right) \right| \leq \epsilon_t/2 \right\}$
    $n_i \leftarrow \arg\min_n \left\{ \left| \mathcal{A}_i \left(\mathbf{1}_{(0,\tau_i)}; u, x \right) \right| \leq \epsilon_t/(4\eta^{(n)}) \right\}$
    $[\hat{f}_i] \leftarrow \left\{ \hat{f}_i^{(n_i)} + [-\eta^{(n_i)}, \eta^{(n_i)}] \right\} \mathbf{1}_{(0,\tau_i)} + [\xi_i(\tau_i)] \, \mathbf{1}_{(\tau_i,\infty)}$
  For $i \in \pi$ **do**
    If $\exists j \in \pi \backslash \{i\}$ **such that** $[\mathcal{A}_i] \left([\hat{f}_i]; u, x \right) > [\mathcal{A}_j] \left([\hat{f}_j]; u, x \right)$ **then**
      $\pi \leftarrow \pi \backslash \{i\}$

vanishing uniformly on $[0, \tau]$. It follows by continuity that $b^{(n)}$ converges towards f uniformly on the interval, with error bound $\eta^{(n)} \lesssim \omega\big(f; [0, \tau]; \tau/\sqrt{n}\big)$.

A faster rate, $\eta^{(n)} \lesssim \omega\left(f; [0, \tau]; \tau/\pi n\right)$, can be achieved using *trigonometric sums*, [38]. The scheme $t^{(n)}$ given in Table 2 was derived by developing the Fourier series of the continuous, 2τ-periodic function defined over $[-\tau, \tau]$ by $\check{f}(u) = f(|u|)$, thus obtained by mirroring and replication of the section of f corresponding to $[0, \tau]$. The quantities $\{\check{\alpha}_k\}$ featured in Table 2 are the Fourier coefficients of $\check{f}(u)$, while $\{\varrho_{n,k}\}$ are real parameters that are introduced to guarantee the claimed error bounds.[8] In particular, if for some $\alpha \in (0, 1]$ the cost function satisfies the α-Höldern condition $|f(u_1) - f(u_2)| \leq h|u_1 - u_2|^\alpha$ for all $u_1, u_2 \in [0, \tau]$, then $\omega(f; [0, \tau]; \delta) \leq h\delta^\alpha$, and the trigonometric sum $t^{(n)}$ converges uniformly on $[0, \tau]$ with error bound $\eta^{(n)} = O((\tau/n)^\alpha)$. If f is Lipschitz continuous on $[0, \tau]$ with modulus L, then $\eta^{(n)} < 2L\tau/n$.

The convergence rate of the trigonometric sums is non-improvable without further assumption on f. In the event the function f admits a kth derivative $f^{(k)}$ on $[0, \tau]$, then the uniform error bounds can be further lowered to $O(n^{-k}\omega(f^{(k)}; [0, \tau]; \tau/[2(n-k)]))$ by using the derivatives of f as the targets of approximation, [38]. This is done by considering a different, near-optimal scheme $\hat{t}^{(n)}$, which has the advantage over $t^{(n)}$ to retain the degrees of smoothness of the cost function. The expression for $\hat{t}^{(n)}$ in Table 2 was obtained by computing an approximate trigonometric sum for the 2π-periodic function $\tilde{f}(\theta) = f(\tilde{u}(\theta))$, derived from f via the change of variable $\tilde{u}(\theta) = (\tau/2)\left(1 + \cos(\theta)\right)$. Observe that $\hat{t}^{(n)}$ is fully specified by $\{\tilde{\alpha}_k\}$, the Fourier coefficients of $\tilde{f}$, which for many cost functions can be derived exactly.

[8] In [28, §3], it is suggested to use the parameter values

$$\varrho_{n,0} = 1,\ \ \varrho_{n,1} = \cos\left(\tfrac{\pi}{n+2}\right),\ \ \varrho_{n,k} = \frac{\sum_{q=0}^{n-k} \sin\left(\frac{q+1}{n+2}\pi\right) \sin\left(\frac{q+k+1}{n+2}\pi\right)}{\sum_{q=0}^{n} \sin^2\left(\frac{q+1}{n+2}\pi\right)} \text{ for } k = 2, \ldots, n, \quad \text{(IV.6)}$$

Table 2: Uniform approximation schemes $\hat{f}^{(n)} \in \Xi$ over the interval $[0,\tau]$, and error bounds as functions of the modulus of continuity of f, [4,27,38].

Approx. scheme	$\hat{f}^{(n)}(u)$ for $u \in [0,\tau]$	Error bound $\eta^{(n)}$
Bernstein polynomial	$b^{(n)}(u) = \sum_{k=0}^{n} \Big[\binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} \frac{(-1)^l}{(-\tau)^k} f(\frac{l\tau}{n})\Big] u^k$	$\frac{3}{2}\,\omega\left(f;[0,\tau];\frac{\tau}{\sqrt{n}}\right)$
Trigonometric sum	$t^{(n)}(u) = \breve{\alpha}_0 + 2\sum_{k=1}^{n} \breve{\alpha}_k \cos(\frac{k\pi u}{\tau})$, where $\breve{\alpha}_k = \frac{1}{\tau}\int_0^\tau f(u)\cos(\frac{k\pi u}{\tau})\,du.$	$6\,\omega\left(f;[0,\tau];\frac{\tau}{\pi n}\right)$
Near-optimal polynomial	$\hat{t}^{(n)}(u) = \sum_{k=0}^{n} \gamma(n,k)u^k$, where $\gamma(n,k) = (\frac{2}{\tau})^k \sum_{t=0}^{n-k} \binom{t+k}{k}(-1)^t \bar{\gamma}(n,t+k)$, $\bar{\gamma}(n,t) = \sum_{k\in\bar{\sigma}(n,t)} \varrho_{n,k}\,\beta_k\,\nu(k,\lfloor\frac{t}{2}\rfloor)$, $\bar{\sigma}(n,t) = \begin{Bmatrix}\{t,t{+}2,t{+}4,\dots,n\} & \text{if } n{-}t \text{ even}\\ \{t,t{+}2,t{+}4,\dots,n{-}1\} & \text{if } n{-}t \text{ odd}\end{Bmatrix}$, $\nu(k,q) = (-1)^{\lfloor\frac{k}{2}\rfloor-q}\sum_{t=0}^{q}\binom{k}{2\lfloor\frac{k}{2}\rfloor-2t}\binom{\lfloor\frac{k}{2}\rfloor-t}{\lfloor\frac{k}{2}\rfloor-q}$, $\beta_0 = \tilde{\alpha}_0$ and $\beta_k = 2\tilde{\alpha}_k$ for $k \geq 1$, $\tilde{\alpha}_k = \frac{1}{\pi}\int_0^\tau f(u)\frac{p_k(\frac{2u}{\tau}-1)}{\sqrt{u(\tau-u)}}\,du$, $p_k(x) = \begin{Bmatrix}\sum_{q=0}^{k/2}\nu(k,q)x^{2q} & \text{if } k \text{ even}\\ \sum_{q=0}^{(k-1)/2}\nu(k,q)x^{2q+1} & \text{if } k \text{ odd}\end{Bmatrix}.$	$6\,\omega\left(f;[0,\tau];\frac{\tau}{2n}\right)$
[if f k-times differentiable]	Infer $\hat{f}^{(n)}(u)$ from near-optimal polynomials computed for the first k derivatives of f.	$O\Big(\frac{\omega\left(f^{(k)};[0,\tau];\frac{\tau}{2(n-k)}\right)}{n^k}\Big)$

See Lemma E.1 for closed-form expressions of these coefficients in the case when f is given as a quotient of polynomials.

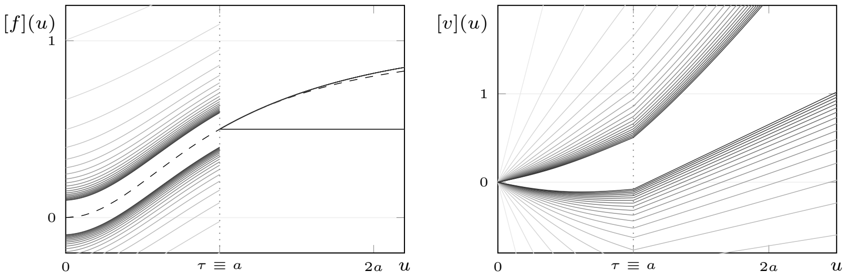
We refer to [9] for an extended discussion on uniform approximation schemes and for detailed derivations of the approximation techniques listed in Table 2.

Case study: quotient cost function. Let $f(u) = u^2/(a^2+u^2)$, where $a > 0$ is a positive parameter, and consider the near-optimal approximation scheme described in Table 2. The Fourier coefficients $\tilde{\alpha}_k$ for f are given by Lemma E.1 with $l(k) \equiv k$. In [9], closed-form expressions for those coefficients are derived from (E.9) after computation of the residues at the complex conjugate poles ia and $-ia$. An exact expression of the modulus of continuity of f is also provided.

In this case study, the cost function f is approximated by (IV.4), with $\hat{f}^{(n)} \equiv \hat{t}^{(n)}$ as stated in Table 2 and $[\xi]$ set to

$$[\xi](u) = \big[f(\tau), -(1-f(\tau))\exp\big\{-\big(\tfrac{f'(\tau)}{1-f(\tau)}\big)(u-\tau)\big\}\big], \quad u \geq \tau,$$

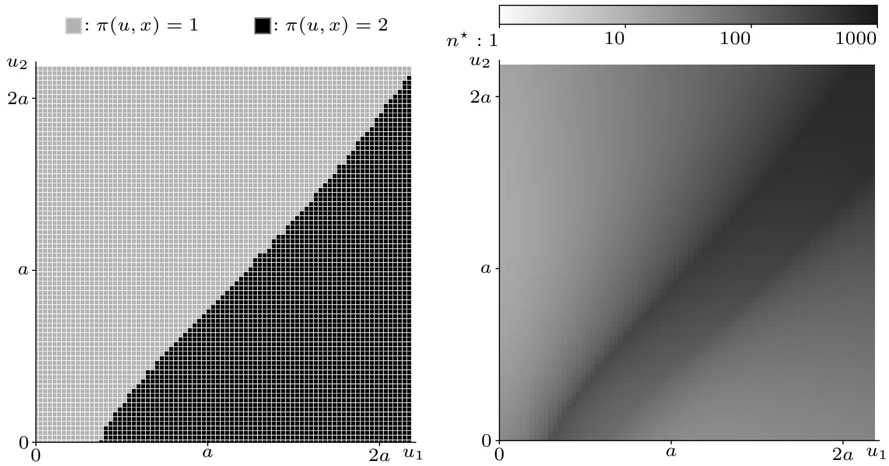
in which $f'(\tau) = 2a^2\tau(a^2+\tau^2)^{-2}$. The intervals produced for f, and for its relative value function in the presence of jobs with exponentially-distributed



(a) Cost function intervals.

(b) Value function intervals (exponentially distributed service times; $\omega = 2$, $\lambda = 1$).

Fig. 4: Intervals for $f(u) = u^2/(a^2 + u^2)$ as per Table 2 ($n = 1, \ldots, 20$).



(a) $\pi(u, x)$ for $x = (1, 2)$.

(b) Minimum order $n^\star(u, x)$ needed for decision, $x = (1, 2)$.

Fig. 5: One-step policy improvement for a two-server system $(1, 2)$ with arrival rates $(\lambda_1, \lambda_2) = (1, 1/2)$, exponentially distributed service times with parameters $(\omega_1, \omega_2) = (2, 1)$, and cost function $f(u) = u^2/(a^2 + u^2)$, [8].

service times are displayed in Figure 4, for fixed τ and $n = 1, \ldots, 20$. The relative value function intervals shown in Figure 4(b) followed from the results in Table 2 and the developments of [9]. The interval gaps can be arbitrarily reduced by increasing both n and τ, as in Algorithm 1.

Consider a system of two parallel servers 1 and 2, with server 1 twice faster than 2. Feed the system a sequence of jobs with arrival rate $\lambda = 3/2$ and service times exponentially distributed with parameters $(\omega_1, \omega_2) = (2, 1)$. Assume that the workload is initially balanced between the two servers, i.e., $(\lambda_1, \lambda_2) = (1, 1/2)$, and let $f(u) = u^2/(a^2 + u^2)$. Figure 5(a) depicts, for a particular job with service times $(x_1, x_2) = (1, 2)$ and for various backlog $u = (u_1, u_2)$, the FPI policy $\pi(u, x)$ issued by Algorithm 1. The quantity $n^\star(u, x)$ displayed in Figure 5(b) is the minimum degree n required by $\hat{t}^{(n)}$ for dispatching at (u, x). This quantity was estimated by reporting the minimum order that allowed for dispatching for a coarse grid of values of the parameter τ. It can be seen that $n^\star(u, x)$ grows with the distance to the origin $u = (0, 0)$, and increases abruptly near the frontiers of the dispatching policy π. The relatively high orders rendered by Figure 5(b) are due to the conservativeness of the uniform error bound $\eta^{(n)} \equiv 6\,\omega(f; [0, \tau]; \tau/(2n))$ for this particular choice of the cost function (cf. Figure 4(a)). In practice, more accurate estimates of the error bound would contribute to reducing the estimation orders. More generally, building the function approximations from the k first derivatives of f, as previously suggested, will significantly accelerate convergence.

V Discussion

Integral transformations of the Poisson equation $g = Pg + f$ have the quality of simplifying the analysis, as they provide a principled framework for the systematic derivation of solutions. Although it is known that the candidate functions for closed-form solutions form a dense set where any f can be approximated with arbitrary precision, one should be cautious that a convergent series for f does not always produce a convergent series for g; Taylor series of f, in particular, are subject to tail effects and most likely to diverge after μ_W-integration with respect to the stationary probability measure of the waiting times.

In the context of first-policy improvement, such tail effects can be avoided by considering approximations of f on finite supports—preferably trigonometric sums, which for Lipschitz-continuous f achieve the convergence rate $O(\tau/n)$ in the number n of approximation terms, improvable to $O(\tau/[(n-k)n^k])$ if f is k-times continuously differentiable—, while using tractable bounds for the larger backlog values. The availability of closed forms for bounding intervals of this type with a diversity of service time distribution models gives the green light to a systematized implementation of the FPI step.

We believe that the techniques developed in this study, combined with well-chosen supervised learning methods, make it possible, in large multiple-server systems, to devise efficient online algorithms for learning FPI policies gradually, as the incoming jobs are dispatched and the (possibly high-dimensional) state space is visited. The design and assessement of FPI dispatching policies in such systems is left to future work.

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A On policy iteration and first-policy improvement

Recall the system depicted in Figure 1, where jobs, arriving according to a Poisson process with rate λ, are dispatched upon arrival towards one of the k servers $(1,\dots,k)$ selected by a (possibly random) dispatching policy $\pi(u,x)$, where $u=(u_1,\dots,u_k)\in\mathbb{R}^k_{\geq 0}$ denotes the server backlog vector and $x=(x_1,\dots,x_k)\in\mathbb{R}^k_{\geq 0}$ are the prospective service times of an incoming job at the servers. By taking snapshots at intitial time $n=0$ and at the job arrival times $(n=1,2,3,\dots)$, the continuous-time system reduces to a MDP, $(\Phi^\pi_n)_{n\in\mathbb{N}}$, with state $\Phi^\pi_n=(U_n,X_n)\in\Omega\equiv\mathbb{R}^k_{\geq 0}\times\mathbb{R}^k_{\geq 0}$, where X_n is the service time vector of the nth job and U_n is the backlog of the system at the time of arrival, and with transition probability kernel $P=(P_1,\dots,P_k)$ such that, for any n and every $(u,x)\in\Omega$, $\mathcal{S}\subset\Omega$,

$$P_i(u,x,\mathcal{S}):=\mathrm{Prob}(\Phi^\pi_{n+1}\in\mathcal{S}|\Phi^\pi_n=(u,x),\,\pi(u,x)=i)=P_i((u_1,\dots,u_i+x,\dots,u_k),0,\mathcal{S}).$$

Assume that the performance of the system is measured by a cost function f, where $f(i,u,x)\equiv f_i(u_i)\,\mathbf{1}_{\mathbb{R}_{>0}}(x)$ models a penalty incurred when a job with service time x joins server i, given backlog state $u = (u_1, \ldots, u_k)$. We would like to minimize the long-run average cost, defined by

$$J_\pi = \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}\left[f\left(\pi(\Phi_n^\pi), \Phi_n^\pi\right)\right],$$

independently of Φ_0^π. The optimality equations of the system are

$$g(u,x) = \min_i \left[f_i(u_i) + P_i g(u,x)\right] - \varsigma, \qquad \text{(OEa)}$$

$$\pi(u,x) \in \arg\min_i \left[f_i(u_i) + P_i g(u,x)\right], \qquad \text{(OEb)}$$

where $P_i g(u,x) = \int_\Omega g(t,y) P_i(u,x,d(t,y)) \equiv P_i g((u_1,\ldots,u_i+x_i,\ldots,u_k),0)$. If one can find $\varsigma^* > 0$, a policy π^*, and an integrable function g such that (g,ς^*) solves (OEa) and π^* satisfies (OEb), then π^* is the optimal policy and

$$\varsigma^* = \lim_{N\to\infty} (1/N) \textstyle\sum_{n=1}^N \mathbb{E}\left[f\left(\pi^*(\Phi_n^{\pi^*}), \Phi_n^{\pi^*}\right)\right]$$

is the optimal cost of the system, [25,2,31,32]. The *policy iteration* algorithm for solving (OE) can be described as follows, [17]. Given an initial policy $\pi^{(0)}$, find, for $k \geq 0$, a function $g^{(k)}$, a mean cost $\varsigma^{(k)}$, and a policy $\pi^{(k+1)}$ satisfying

$$g^{(k)}(u,x) = f(\pi^{(k)}(u,x), u, x) + P_{\pi^{(k)}(u,x)} g^{(k)}(u,x) - \varsigma^{(k)} \qquad \text{(PIa)}$$

$$\pi^{(k+1)}(u,x) \in \arg\min_i \left[f_i(u_i) + v^{(k)}(u_1,\ldots,u_i+x_i,\ldots,u_k)\right] \qquad \text{(PIb)}$$

where (PIa) is the policy evaluation step, (PIb) is the policy improvement step and $v^{(k)}(u) := \int_\Omega g^{(k)}(t,y)\, P(u,0,d(t,y)) - \varsigma^{(k)}$ defines the *relative value function* under policy $\pi^{(k)}$. Under favorable conditions, $(\pi^{(k)}, \varsigma^{(k)})$ eventually converges towards a solution (π^*, ς^*), [31,32]. Solving (PIb), however, is generally difficult.

The first iteration of (PI) may still be implemented easily if the initial policy $\pi^{(0)} \equiv \pi$ is a random Bernoulli-split between the servers. In that case, the Poisson process separates into k independent Poisson processes with rates $\lambda_1, \ldots, \lambda_k$, where $\lambda_1 + \cdots + \lambda_k = \lambda$, and the multiple-server system decomposes into k independent M/G/1 queues with arrival rates $\lambda_1, \ldots, \lambda_k$ and transition probability kernels $P^{(\lambda_1)}, \ldots, P^{(\lambda_k)}$. The relative value function under the initial random policy then takes the additively separable form $v(u) = \sum_{i=1}^k v_i(u_i)$, where $v_i(u_i) := P^{(\lambda_i)} g_i(u_i, 0) - \varsigma_i$ is the relative value function at server i, and g_i satisfies

$$g_i(u_i, x_i) = P^{(\lambda_i)} g_i(u_i, x_i) + f_i(u_i) - \varsigma_i. \qquad (i = 1, \ldots, k) \qquad \text{(FPIa)}$$

Since $f_i(u_i) + v(u_1,\ldots,u_i+x_i,\ldots,u_k) = f_i(u_i) + v(u) + v_i(u_i+x_i) - v_i(u_i)$, (PIb) reduces to

$$\hat{\pi}(u,x) \in \arg\min_i \mathcal{A}_i(u,x), \qquad \text{(FPIb)}$$

where $\mathcal{A}_i(u,x) = f_i(u_i) + v_i(u_i+x_i) - v_i(u_i)$ is the admission cost at server i. The *first-policy improvement* approach then consists in stopping the PI algorithm after a single iteration, by solving (FPIa)-(FPIb). Observe that (FPIa) is an instance of the Poisson equation $g = Pg + f$ under $\int f(u)\, \nu(du) = 0$, where ν denotes the non-trivial measure invariant for the transition kernel P (i.e., $P\nu = \nu$), [34,35,3]. In (FPIa), ν coincides with the asymptotic probability measure of the waiting times at server i.

B Characterization of the value function

Before showing Propositions 1-2, and Theorem 1, we characterize W^* in the complex plane.

Proposition B.1 (Analycity of W^* and pole location) *Under Assumption 1:*

(i) The dominant singularity p_W of W^* (i.e., that with largest real value) is a pole with degree 1 *lying on the negative real axis $\mathbb{R}_{<0}$. The dominant singularity p_X of X^* is real, negative (possibly infinite) and satisfies $p_X < p_W$. X^* is analytic on $\{s \in \mathbb{C} \mid \Re(s) > p_W\}$.*
(ii) W^* is analytic on $\{s \in \mathbb{C}_0 \mid \Re(s) > p_W\}$, where $\lim_{s\to\infty} |W^*(s)| \le 1$.
(iii) One can find $\epsilon > 0$ such that W^* is analytic on $\{s \in \mathbb{C}_0 \setminus \{p_W\} \mid \Re(s) > p_W - \epsilon\}$.
(iv) W^* is analytic in a neighborhood of 0*, where it rewrites as the series*

$$W^*(s) = \textstyle\sum_{k=0}^{\infty} w_k\,(-s)^k,\ \forall s \in \{\sigma \in \mathbb{C}_0 : |\sigma - a| < |p_W|\}, \tag{B.1}$$

in which the coefficients $\{w_k\}$ are given by (III.3) *in Table 1, and satisfy $w_k = \mathbb{E}[W^k]/k!$, for $k \in \mathbb{N}$. The series $\{w_k\}$ is asymptotically geometric with asymptotic rate $|p_W|^{-1}$.*
(v) At any point $a \in \mathbb{C}_0$ where W^* is analytic, W^* rewrites as the series

$$W^*(s) = \textstyle\sum_{k=0}^{\infty} w_{a:k}(a - s)^k,\ \forall s \in \{\sigma \in \mathbb{C} : |\sigma - a| < r_a\}, \tag{B.2}$$

where r_a denotes the distance from a to the closest singularity of W^*. The coefficients $\{w_{a:k}\}$ are given by (III.4) *in Table 1.*

Next, we derive the identities of Section III.1 for the value function.

Proof (Proposition 1) Start the queue at state u. The quantity $V(u,t)$ appearing in (VF) rewrites, for any $T \ge 0$ and for t large enough, as $V(u,t) = V(u,T) + V(U(T), t-T)$, where $U(T)$ denotes the backlog observed after time T. It follows from the Markov property of the system and from the the definition VF of the value function that

$$v(u) = \mathbb{E}[V(u,T) - \lambda \bar{f} T] + \mathbb{E}[v(U(T))]. \tag{B.3}$$

Now, consider the function

$$g(u,x) = \lim_{N\to\infty}\{\textstyle\sum_{n=1}^{N} \mathbb{E}\left[f(U_n) - \bar{f}\right] \mid (U_1, X_1) = (u,x)\} + \bar{f}, \quad \forall u, x \in \mathbb{R}_{\ge 0}, \tag{B.4}$$

which can be verified to satisfy Equation (PE) by application of the Markov property to the MDP. The function g defined by (B.4) can be seen as a discrete-time counterpart of the value function (VF), which follows from (B.4) by using the convention $(U_0, X_0) = (u, 0)$ and setting $v(u) \equiv Pg(u,0) - \bar{f}$ or, equivalently, from (B.3) by defining T as the arrival time of the first job so that $\mathbb{E}[V(u,T) - \lambda \bar{f} T] = -\bar{f}$ and $\mathbb{E}[v(U(T))] \equiv Pg(u,0)$. □

Proof (Theorem 1) A simple calculation reveals that $\Delta(u) = 0$ if $u < 0$, and $\Delta(u) = (P - \widehat{P})g(u,0) = \kappa e^{-\lambda u}$, for $u \ge 0$, with κ satisfying by

$$\kappa = \mathbb{E}\big[g(0, X_0) - \lambda \textstyle\int_{-\infty}^{0} g(u, X)\, e^{\lambda u}\, du\big]. \tag{B.5}$$

We characterize the extended value function $\widehat{v} : u \in \mathbb{R} \mapsto \widehat{v}(u) = g(u,0) - f(u)$ associated with some g solution of (PE'). Note that, by construction, $\widehat{v}$ coincides with the value function on $\mathbb{R}_{\ge 0}$, i.e., $\widehat{v}(u) \equiv v(u)$ if $u \ge 0$. Once $\widehat{v}$ is known, it will be possible to recover g using

$$g(u,x) = g(u+x, 0) - f(u+x) + f(u) = \widehat{v}(u+x) + f(u). \tag{B.6}$$

Consider s in the region of absolute convergence of $\mathcal{B}_{\widehat{v}}$, where the orders of integration in our developments may be permuted. The two-sided Laplace transform of $\widehat{v}$ is given by

$$\begin{aligned}
\mathcal{B}_{\widehat{v}}(s) &= \textstyle\int_{-\infty}^{+\infty} \widehat{v}(u)\, e^{-su}\, du \\
&= \textstyle\int_{-\infty}^{+\infty} [g(u,0) - f(u)]\, e^{-su}\, du \\
&\overset{\text{(PE')}}{=} \textstyle\int_{-\infty}^{+\infty} [\widehat{P}g(u,0) + \Delta(u) - \bar{f}\,\mathbf{1}_{[0,+\infty)}(u)]\, e^{-su}\, du \\
&= \textstyle\int_{-\infty}^{+\infty} e^{-su}\, du \int g(t,x) \widehat{P}(u, 0, d(t,x)) + \int_0^{+\infty} \kappa e^{-(s+\lambda)u}\, du - \frac{\bar{f}}{s} \\
&\overset{\text{(III.1)}}{=} \lambda \mathbb{E}\big[\textstyle\int_{-\infty}^{+\infty} e^{-su}\, du \int_{-\infty}^{u} g(t,X) e^{-\lambda(u-t)}\, dt\big] - \frac{\bar{f}}{s} + \frac{\kappa}{s+\lambda} \\
&= \lambda \mathbb{E}\big[\textstyle\int_{-\infty}^{+\infty} g(t,X)\, e^{\lambda t}\, dt \int_t^{+\infty} e^{-(s+\lambda)u}\, du\big] - \frac{\bar{f}}{s} + \frac{\kappa}{s+\lambda} \\
&\overset{\text{(B.6)}}{=} \tfrac{\lambda}{s+\lambda}\big\{\mathbb{E}\big[\textstyle\int_{-\infty}^{+\infty} \widehat{v}(t+X)\, e^{-st}\, dt\big] + \int_0^{+\infty} f(t)\, e^{-st}\, dt\big\} - \frac{\bar{f}}{s} + \frac{\kappa}{s+\lambda} \\
&= \tfrac{\lambda}{s+\lambda}\big\{X^*(-s)\, \mathcal{B}_{\widehat{v}}(s) + \mathcal{L}_f(s) - \frac{\bar{f}}{s} + \frac{\kappa - \bar{f}}{\lambda}\big\}.
\end{aligned}$$

Solving the above equation for $\mathcal{B}_{\widehat{v}}(s)$ and using $W^*(-s) = 1 + \lambda\mathbb{E}[X^2]/[2(1-\rho)]s + o(s)$ yields, after computations,

$$\begin{aligned}\mathcal{B}_{\widehat{v}}(s) &\overset{\text{(PK)}}{=} \tfrac{\lambda W^*(-s)}{(1-\rho)s}\left[\mathcal{L}_f(s) - \tfrac{\bar{f}}{s} + \tfrac{\kappa-\bar{f}}{\lambda}\right] \\ &= \tfrac{\lambda}{(1-\rho)s}W^*(-s)\mathcal{L}_f(s) - \tfrac{\lambda\bar{f}}{(1-\rho)s^2} + \tfrac{\varepsilon}{s} + \tfrac{h(s)}{s},\end{aligned} \tag{B.7}$$

where $h(s)$ has no singularities on $\{s \in \mathbb{C} \,|\, \Re(s) < -p_W\}$, and ε is given by

$$\varepsilon = \kappa/(1-\rho) - \hat{X}_2^*(\lambda)\bar{f}/(1-\rho)^2, \tag{B.8}$$

with $\hat{X}_2^*(\lambda) := 1 - \lambda\mathbb{E}[X] + \lambda^2\mathbb{E}[X^2]/2$. Since $\widehat{v}$ is expected to be asymptotically flat for $u \to -\infty$, the $-\lambda\bar{f}/[(1-\rho)s^2]$ term in (B.7) is necessarily due to a term $-\lambda\bar{f}/(1-\rho)\,u$ on $\mathbb{R}_{\geq 0}$ in the backlog domain. By inverse transformation of (B.7), we find

$$\widehat{v}(u) = \widehat{v}(0) + c(u) - \tfrac{\lambda\bar{f}}{1-\rho}u\,\mathbf{1}_{[0,+\infty)}(u) + r(u)\,\mathbf{1}_{(-\infty,0)}(u), \qquad \forall u \in \mathbb{R}, \tag{$\widehat{\text{S}}$}$$

where r satisfies $\mathcal{L}_{r(-\cdot)}(s) = -[h(-s)+\varepsilon]/s$. The general form for g follows from (B.6), ($\widehat{\text{S}}$) and $\widehat{v}(u) \equiv v(u)$ on $\mathbb{R}_{\geq 0}$. The non-empty ROC of $\mathcal{B}_{\partial_+\widehat{v}}$ follows from $-p_W \in \mathrm{ROC}(\mathcal{L}_f)$.

It remains to show that the function r is identical for all solutions or, equivalently, that the quantity κ in (B.5) is the same for all g. To see this, consider a solution g_1 of (PE') with associated value function $\widehat{v}_1$ and jump ε_1 at $u = 0$. The value function for every other solution g_2 rewrites as $\widehat{v}_2 = \alpha + (\widehat{v}_1 - \varepsilon_2 + \varepsilon_1)\mathbf{1}_{\mathbb{R}_{<0}} + \widehat{v}_1\mathbf{1}_{\mathbb{R}_{\geq 0}}$, where ε_1, ε_2 and α are constants. We show that $\varepsilon_1 = \varepsilon_2$. If we successively compute the expression (B.8) for ε_1 and ε_2, using (B.5), (B.6) and the extension of (PE), we get, after simplifications, $\varepsilon_2 = \varepsilon_1 - (\varepsilon_1 - \varepsilon_2)X^*(\lambda)/(1-\rho)$. Exploiting twice the strict convexity of e^{-x}, we find $X^*(\lambda) = \mathbb{E}[e^{-\lambda X}] > e^{-\lambda\mathbb{E}[X]} > 1 - \lambda\mathbb{E}[X] = 1 - \rho$. Hence, $X^*(\lambda)/(1-\rho) \neq 1$ and, consequently, $\varepsilon_1 = \varepsilon_2$. □

Proof (Proposition 2) (i) Consider s in the region of absolute convergence of $\mathcal{B}_{\widehat{v}}$. Since $\mathcal{B}_{\partial_+ c}(s) = s\mathcal{B}_c(s)$, (C) rewrites as

$$[s + \lambda(1 - X^*(-s))]\,\mathcal{B}_c(s) \overset{\text{(PK)}}{=} \lambda\mathcal{L}_f(s) + s\mathcal{L}_{r(-\cdot)}(-s)\,[1 + \lambda/s(1 - X^*(-s))]\,,$$

while transformation of ($\widehat{\text{S}}$) gives $\mathcal{B}_c(s) = \mathcal{B}_{\widehat{v}}(s) + \lambda\bar{f}/[(1-\rho)s^2]$. Besides,

$$\begin{aligned}X^*(-s)\mathcal{B}_{\widehat{v}}(s) &= \mathbb{E}[e^{sX}]\int_{-\infty}^{+\infty}\widehat{v}(u)e^{-su}du = \mathbb{E}[\textstyle\int_{-\infty}^{+\infty}\widehat{v}(u)e^{-s(u-X)}du] \\ &= \mathbb{E}[\textstyle\int_{-\infty}^{+\infty}\widehat{v}(t+X)e^{-st}dt] = \int_{-\infty}^{+\infty}\mathbb{E}[\widehat{v}(t+X)]e^{-st}dt = \mathcal{B}_{\mathbb{E}[\widehat{v}(\cdot+X)]}(s).\end{aligned}$$

Combining the above with $\mathcal{B}_{\partial_+\widehat{v}}(s) = s\mathcal{B}_{\widehat{v}}(s)$, we get, after computations,

$$\mathcal{B}_{\partial_+\widehat{v}}(s) = \lambda\left[\mathcal{L}_f(s) - \bar{f}/s + \mathcal{B}_{\mathbb{E}[\widehat{v}(\cdot+X)]}(s) - \mathcal{B}_{\widehat{v}}(s)\right] + \tilde{h}(s), \tag{B.9}$$

where $\tilde{h}(s)$ shows no singularity on $\{s \in \mathbb{C} \,|\, \Re(s) < -p_W\}$, and we have used Proposition B.1(i) and $1 + \lambda/s(1 - X^*(-s)) = 1 - \rho + o(1)$. Inverse Laplace transformation of (B.9) then gives, at every $u \geq 0$ where $\widehat{v}$ is differentiable,

$$\widehat{v}'(u) = \lambda\left(f(u) - \bar{f} + \mathbb{E}\left[\widehat{v}(u+X) - \widehat{v}(u)\right]\right), \tag{B.10}$$

which holds for almost every $u > 0$ by piecewise continuity of f. Since by construction $v(u) \equiv \widehat{v}(u)$ for $u \geq 0$, we find (DE). From Theorem 1, we have

$$\widehat{v}(0) = g(0,0) - f(0) \overset{\text{(PE')}}{=} Pg(0,0) - \bar{f} \overset{\text{(II.1)}}{=} \mathbb{E}[\widehat{v}(X_0)] + f(0) - \bar{f}, \tag{B.11}$$

which yields (BCa). Finally, we find (BCb) by taking the limit of (B.10) as $u \to 0^+$,

$$\widehat{v}'(0) = \lambda\left(f^+(0) - \bar{f} + \mathbb{E}\left[\widehat{v}(X) - \widehat{v}(0)\right]\right) \overset{\text{(B.11)}}{=} \lambda\left(f^+(0) + \mathbb{E}\left[\widehat{v}(X)\right] - \mathbb{E}[\widehat{v}(X_0)] - f(0)\right).$$

(ii) Equation (S) follows directly from $(\widehat{\mathrm{S}})$ and the fact that $v(u) \equiv \widehat{v}(u)$ for $u \geq 0$. It remains to compute $\partial_+ c$. From Theorem 1, we get

$$\begin{aligned} \mathcal{B}_{\partial_+ c}(s) &\overset{(\mathrm{C})}{=} \tfrac{\lambda}{(1-\rho)} W^*(-s) \mathcal{L}_f(s) = \tfrac{\lambda}{(1-\rho)} \mathbb{E}[e^{sW}] \textstyle\int_{-\infty}^{+\infty} f(u) e^{-su} \, du \\ &= \tfrac{\lambda}{(1-\rho)} \mathbb{E}[\textstyle\int_{-\infty}^{+\infty} f(u) e^{-s(u-W)} \, du] = \tfrac{\lambda}{(1-\rho)} \textstyle\int_{-\infty}^{+\infty} \mathbb{E}[f(t+W)] e^{-s(t)} \, dt \\ &= \tfrac{\lambda}{(1-\rho)} \mathcal{B}_{\mathbb{E}[f(\cdot+W)]}(s), \end{aligned} \tag{B.12}$$

where we have used $f(u) = 0$ if $u < 0$. Equation (C) follows by inversion of (B.12). □

C Splitting schemes for the Pollaczek-Khinchin formula

This section includes further details on the transform splitting scheme (III.12), which proves very useful when computing value functions with discrete service time distributions. See also [9] for a step-by-step derivation of the core function in the case of jobs with identical service times.

C.1 Discrete service time distributions

Consider the M/D/1 queue, where all the jobs have equal service time x. In this scenario, the transform W^* is given by (PK) with $X^*(s) = e^{-sx}$, and rewrites as

$$W^*(s) = [\Upsilon(-s)]^m W^*(s) + \tfrac{(1-\lambda x)s}{s-\lambda} \textstyle\sum_{k=0}^{m-1} [\Upsilon(-s)]^k, \quad \forall m \in \mathbb{N}_{>0}, \tag{C.1}$$

where $\Upsilon(s) := [\lambda/(s+\lambda)] \, e^{sx}$. It can be seen that (III.12) holds if

$$W_u^{\circlearrowleft}(s) = [\Upsilon(-s)]^{\tilde{m}(u)} W^*(s), \quad W_u^{\circlearrowright}(s) = \tfrac{s(1-\rho)}{s-\lambda} \textstyle\sum_{k=0}^{\tilde{m}(u)-1} [\Upsilon(-s)]^k,$$

with $\tilde{m}(u) = \lceil (\tau - u)/x \rceil$.

C.2 Degenerate cases

The decomposition scheme (C.1) is not possible for all discrete service time distributions. Consider for instance the geometric service time distribution $F_X(u) = (e^{\varsigma}-1) \sum_{k=1}^{\infty} e^{-k\varsigma} \theta(u - kx)$ for $u \in \mathbb{R}_{\geq 0}$, where $x > 0$ and $\lambda < (1 - e^{-\varsigma})/x$. We have $\mathbb{E}[X] = x/(1 - e^{-\varsigma})$, $X^*(s) = (e^{\varsigma} - 1)/(e^{\varsigma + sx} - 1)$, and W^* degenerates into

$$W^*(s) = \frac{b(e^{\varsigma+sx} - 1)s}{(s-\lambda)e^{\varsigma+sx} - s + \lambda e^{\varsigma}} = b - s(e^{\varsigma+sx} - 1) \, f(s), \tag{C.2}$$

where $b = (1 - e^{-\varsigma} - \lambda x)/(1 - e^{-\varsigma})$ and $f(s) = [(\lambda - s)e^{\varsigma+sx} + s - \lambda e^{\varsigma}]^{-1}$. Although $f(s)$ decreases like $O(r^{-1})$ as $|s| \to \infty$ (i.e., not fast enough for counterclockwise integration along $\mathcal{C}_r$), it decomposes as follows:

$$f(s) = [\tilde{\Upsilon}(-s)]^m f(s) + \textstyle\sum_{k=1}^{m} [\tilde{\Upsilon}(-s)]^k / (\lambda e^{\varsigma} - s), \quad (m = 1, 2, \dots), \tag{C.3}$$

where $\tilde{\Upsilon}(s) = (s + \lambda e^{\varsigma})/(s+\lambda) \, e^{sx-\varsigma}$ is $O(e^{\Re(s)x})$ with just one pole at $-\lambda$. By distributing (C.2) and using (C.3) twice with parameters $m+1$ and m, we find, after computations, that (III.12) holds if we set

$$\begin{aligned} W_u^{\circlearrowleft}(s) &= \lambda b \tfrac{s(e^{\varsigma}-1)(\lambda e^{\varsigma}-s)^{\tilde{m}_l(u)-1}}{(\lambda-s)^{\tilde{m}(u)} \, [s-\lambda-(s-\lambda e^{\varsigma})e^{-(sx+\varsigma)}]} \, e^{-\tilde{m}(u)(sx+\varsigma)}, \\ W_u^{\circlearrowright}(s) &= \tfrac{s-b}{s-\lambda} + \lambda b \tfrac{(e^{\varsigma}-1)-s}{(s-\lambda)(s-\lambda e^{\varsigma})} \textstyle\sum_{k=1}^{\tilde{m}(u)-1} \left(\tfrac{s-\lambda e^{\varsigma}}{s-\lambda}\right)^k e^{-k(sx+\varsigma)}, \end{aligned}$$

with $\tilde{m}(u) = \lceil (\tau - u)/x \rceil$.

D System interpretation of Taylor series CVFs

In view of (C), the core function can be seen as the answer to the input f of a linear filter with transfer function $\lambda/(1-\rho)\,W^*(-s)$. Similarly, the Taylor series (IV.3) for the CVF of an analytic cost function is reminiscent of digital filtering in signal processing. Indeed, for $n \in \mathbb{N}$, consider the vector of derivatives $\tilde{f}^{(n)} = (\tilde{f}_0, \ldots, \tilde{f}_{n-1})$, where $\tilde{f}_k = f^{(k)}(0)$, and the vector $\tilde{c}^{(n)} = (\tilde{c}_0^{(n)}, \ldots, \tilde{c}_{n-1}^{(n)})$, where $\tilde{c}_k^{(n)}$ is defined by (IV.2). Using the matrix inversion lemma, one shows that the Toeplitz, upper triangular matrices

$$X^{(n)} = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ 0 & x_1 & x_2 & \ldots & x_{n-1} \\ 0 & 0 & x_1 & \ldots & x_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & x_1 \end{pmatrix}, \qquad W^{(n)} = \tfrac{\lambda}{1-\rho} \begin{pmatrix} w_0 & w_1 & w_2 & \cdots & w_{n-1} \\ 0 & w_0 & w_1 & \ldots & w_{n-2} \\ 0 & 0 & w_0 & \ldots & w_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & w_0 \end{pmatrix}.$$

satisfy $W^{(n)}(I^{(n)}/\lambda - X^{(n)}) = I^{(n)}$ for all n, where $I^{(n)}$ denotes the identity matrix. Besides, (IV.2) rewrites in matrix form as $\tilde{c}^{(n)} = W^{(n)}\tilde{f}^{(n)}$.

As $n \to \infty$, $\{\tilde{c}_t\}$ becomes the output (at nonnegative times) of the cross-correlation of $\{\tilde{f}_t\}$ with the sequence defined by $\tilde{h}[t] = \lambda/(1-\rho)\,w_t$ for $t \geq 0$, thus giving us an interpretation for analytic functions of Proposition 2(ii), where c' was obtained by cross-correlation of $f(u)$ with $\lambda/(1-\rho)F_W'(u)$. Similarly, (IV.3b) expresses $c'(u)$ as the cross-correlation of the sequence of derivatives of f at u with the sequence $\tilde{h}[t]$.

From our observation follows that the Z-transforms of the sequences satisfy

$$\mathcal{Z}_{\tilde{c}^{(\infty)}}(z) = \mathcal{Z}_{\tilde{h}}(1/z)\,\mathcal{Z}_{\tilde{f}^{(\infty)}}(z), \tag{D.1}$$

where $\mathcal{Z}_h(z) = \sum_{k=0}^{\infty} h[k]z^{-k}$ denotes the Z-transform of a sequence $h[t]$. The vector $\tilde{c}^{(\infty)}$ can be recovered from (D.1) by inverse Z-transform provided that the regions of convergence of $\mathcal{Z}_{\tilde{f}^{(\infty)}}(z)$ and $\mathcal{Z}_{\tilde{h}}(1/z)$ intersect on a non-empty circular band—this condition is to be linked to those of Theorem 2(i).

Conversely, inverting $W^{(n)}$ yields $\tilde{f}^{(n)} = (I^{(n)}/\lambda - X^{(n)})\tilde{c}^{(n)}$ and, as $n \to \infty$,

$$\tilde{f}_k = \tilde{c}_k/\lambda - \textstyle\sum_{t=0}^{\infty}[\tilde{c}_{t+k}/(t+1)!]\,\mathbb{E}[X^{t+1}], \tag{D.2}$$

which provides us with a converse for Theorem 2, where the source cost function of a given core function with germ $\{\tilde{c}_k\}$ can be recovered from c through (D.2), on the condition that c grows slower than the exponential type $|p_X|$—where p_X denotes the dominant pole of X—, in which case $(I^{(n)}/\lambda - X^{(n)})\,\tilde{c}^{(n)}$ converges as $n \to \infty$. Similarly, $\mathcal{Z}_{\tilde{f}^{(\infty)}}(z) = (\mathcal{Z}_{\tilde{h}}(1/z))^{-1}\mathcal{Z}_{\tilde{c}^{(\infty)}}(z)$, where $(\mathcal{Z}_{\tilde{h}}(z))^{-1}$ is the Z-transform of $\delta[t]/\lambda - \mathbb{E}[X^{t+1}]/(t+1)!$.

E Proofs and auxiliary results

Proof (Theorem 2) (i) If ϱ is the order of growth of the entire cost function f, and σ is its type, then for any $\epsilon > 0$, there is $k_\epsilon < \infty$ such that, [30, Lecture 1],

$$\tfrac{1}{k!}|f^{(k)}(0)| < \big(\tfrac{e(\varrho+\epsilon)}{k}\big)^{\frac{k}{\varrho+\epsilon}}, \quad \forall k > k_\epsilon, \tag{E.1a}$$

$$\tfrac{1}{k!}|f^{(k)}(0)| < \big(\tfrac{e(\sigma+\epsilon)\varrho}{k}\big)^{\frac{k}{\varrho}}, \quad \forall k > k_\epsilon. \tag{E.1b}$$

Consider the quantity $\tilde{c}_k = \sum_{q=0}^{\infty} w_q\, f^{(k+q)}(0)$ introduced in (IV.2), as well as

$$\bar{c}_k = \tfrac{\lambda}{1-\rho}\textstyle\sum_{q=0}^{\infty} w_q\,\big|f^{(k+q)}(0)\big|, \quad \forall k \in \mathbb{N}. \tag{E.2}$$

Recall from Proposition B.1-(iv) in Appendix B that $\lim_{k\to\infty} w_{k+1}/w_k = |p_W|^{-1}$. Besides, it can be seen (e.g. using Stirling's approximation for the factorial) that

$$\lim_{k\to\infty} \frac{(k+1)!\left(\frac{esr}{k+l+1}\right)^{\frac{k+l+1}{r}}}{k!\left(\frac{esr}{k+l}\right)^{\frac{k+l}{r}}} = \left\{\begin{array}{l} 0,\ \text{if } r<1 \\ s,\ \text{if } r=1 \\ \infty,\ \text{if } r>1 \end{array}\right\}, \quad \forall l \in \mathbb{N}. \tag{E.3}$$

Equations (E.1a) and (E.1b) tell us that, under the assumptions of (i) and by taking ϵ sufficiently small, one can find a dominant series for $\tilde{c}_k$ and $\bar{c}_k$ that successfully passes the ratio test for convergence due to (E.3), so that both $\tilde{c}_k$ and $\bar{c}_k$ are finite for all k. The finiteness of $\bar{c}_k$ allows us to interchange the integration order in the computation of $\tilde{c}_k$. Noting that $w_q = \mathbb{E}[W^q]/q!$ for all q (cf. Proposition B.1-(iv)), we apply Fubini's theorem and find, for $k \in \mathbb{N}_{\geq 0}$,

$$\tilde{c}_k = \tfrac{\lambda}{1-\rho} \textstyle\sum_{q=0}^{\infty} \mathbb{E}[f^{(k+q)}(0)\tfrac{W^q}{q!}] = \tfrac{\lambda}{1-\rho}\mathbb{E}[\textstyle\sum_{q=0}^{\infty} f^{(k+q)}(0)\tfrac{W^q}{q!}] \overset{\text{(IV.1)}}{=} \tfrac{\lambda}{1-\rho}\mathbb{E}[f^{(k)}(W)]. \tag{E.4}$$

Similarly, we introduce, for $k \in \mathbb{N}$,

$$\begin{aligned}\hat{c}_k = \tfrac{\lambda}{1-\rho}\mathbb{E}\left[|f^{(k)}(W)|\right] &\overset{\text{(IV.1)}}{\leq} \tfrac{\lambda}{1-\rho}\mathbb{E}\left[\textstyle\sum_{q=0}^{\infty}\left|f^{(k+q)}(0)\right|\tfrac{W^q}{q!}\right] \\ &= \tfrac{\lambda}{1-\rho}\textstyle\sum_{q=0}^{\infty} w_q\left|f^{(k+q)}(0)\right| \overset{\text{(E.2)}}{=} \bar{c}_k.\end{aligned} \tag{E.5}$$

and $\hat{c}_k$ is finite as well. Suppose now that $|(d^k/du^k)f(0)| < k!(esr/k)^{k/r}$ for $k > k_\epsilon$—in the case (i), this holds either for some $r < 1$ or for $r = 1$ and some finite s—, and consider the sequence

$$\beta_k = \tfrac{\lambda}{1-\rho}\textstyle\sum_{q=0}^{\infty}(q+k)!\,w_q\left(\tfrac{esr}{q+k}\right)^{\frac{q+k}{r}}, \quad \forall k \in \mathbb{N}. \tag{E.6}$$

It is easy to see that the three sequences $\sum_{k=0}^{\infty}\tilde{c}_k u^{k+1}/(k+1)!$, $\sum_{k=0}^{\infty}\bar{c}_k u^{k+1}/(k+1)!$ and $\sum_{k=0}^{\infty}\hat{c}_k u^{k+1}/(k+1)!$ converge wherever $\sum_{k=0}^{\infty}\beta_k u^{k+1}/(k+1)!$ is convergent. Besides,

$$\beta_{k+1} \overset{\text{(E.6)}}{=} \tfrac{\lambda}{1-\rho}\textstyle\sum_{q=0}^{\infty}\left[\dfrac{(q+k+1)!\left(\frac{esr}{q+k+1}\right)^{\frac{q+k+1}{r}}}{(q+k)!\left(\frac{esr}{q+k}\right)^{\frac{q+k}{r}}}\right](q+k)!\,w_q\left(\tfrac{esr}{q+k}\right)^{\frac{q+k}{r}}. \tag{E.7}$$

In the conditions of (i), we infer from E.3 that the expression between brackets in (E.7) tends to a finite quantity not larger than s, so that, for any $\nu > 0$ one can find a k_ν such that $\beta_{k+1} \leq (\beta_{k_\nu+1} - \beta_{k_\nu}) + (s+\nu)\beta_k$ for $k > k_\nu$. It follows from the ratio test that $\sum_{k=0}^{\infty}\beta_k\xi^k/k!$ converges for $\xi \in \mathbb{R}_{\geq 0}$, and so do $\sum_{k=0}^{\infty}\tilde{c}_k\xi^k/k! = \psi(u)$, $\sum_{k=0}^{\infty}\bar{c}_k\xi^k/k!$ and $\sum_{k=0}^{\infty}\hat{c}_k\xi^k/k!$. This last conclusion, together with (E.4), (E.5), and Fubini's theorem applied to set of natural numbers with the counting measure, yields, for $u \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned}\psi(u) \overset{\text{(E.4)}}{=} \tfrac{\lambda}{1-\rho}\textstyle\int_0^u \sum_{k=0}^{\infty}\mathbb{E}[f^{(k)}(W)\tfrac{\xi^k}{k!}]\,d\xi &= \tfrac{\lambda}{1-\rho}\textstyle\int_0^u \mathbb{E}[\sum_{k=0}^{\infty} f^{(k)}(W)\tfrac{\xi^k}{k!}]\,d\xi \\ &\overset{\text{(IV.1)}}{=} \tfrac{\lambda}{1-\rho}\textstyle\int_0^u \mathbb{E}\left[f(\xi+W)\right]\,d\xi \overset{\text{(CVF)}}{=} c(u),\end{aligned}$$

where the last result follows from Proposition 2(ii). Since

$$\textstyle\sum_{k=0}^{\infty}\bar{c}_k\tfrac{\xi^k}{k!} = \tfrac{\lambda}{1-\rho}\sum_{k=0}^{\infty}\left(\sum_{q=0}^{\infty}|w_q\,f^{(k+q)}(0)\tfrac{\xi^k}{k!}|\right) < \infty, \quad \forall \xi \in \mathbb{R}_{\geq 0},$$

Fubini's theorem applies and one may interchange the order of summation in (IV.3a):

$$\psi(u) = \tfrac{\lambda}{1-\rho}\textstyle\int_0^u \sum_{q=0}^{\infty} w_q\left(\sum_{k=0}^{\infty} f^{(k+q)}(0)\tfrac{\xi^k}{k!}\right)d\xi = \tfrac{\lambda}{1-\rho}\int_0^u \sum_{q=0}^{\infty} w_q\, f^{(q)}(\xi)\,d\xi \overset{\text{(IV.3b)}}{=} \chi(u),$$

which holds for $u \in \mathbb{R}_{\geq 0}$.

(ii) Similarly, for any $\epsilon > 0$, one can find growing sequences of naturals $\{l_k\}$ and $\{m_k\}$ such that, [30, Lecture 1],

$$\tfrac{|f^{(l_k)}(0)|}{l_k!} > \left(\tfrac{e(\varrho-\epsilon)}{l_k}\right)^{\frac{l_k}{\varrho-\epsilon}}, \quad (k \in \mathbb{N}_{\geq 0}), \tag{E.8a}$$

$$\tfrac{|f^{(m_k)}(0)|}{m_k!} > \left(\tfrac{e(\sigma-\epsilon)\varrho}{m_k}\right)^{\frac{m_k}{\varrho}}, \quad (k \in \mathbb{N}_{\geq 0}). \tag{E.8b}$$

Recall the series $\tilde{c}_k$ defined in (IV.2). By taking ϵ sufficiently small in (E.8a) and (E.8b) and using (E.3), we find that the asymptotic ratio between the moduli of two terms of (IV.2) with respective indices $l_q - k, l_{q+1} - k$ (in the case $\varrho > 1$) or $m_q - k, m_{q+1} - k$ (in the case $\varrho = 1$, $\sigma > |p_W|^{-1}$) is greater than one for q taken large enough. Hence, one can find a subsequence of terms of (IV.2) which grows in modulus, and $\tilde{c}_k$ diverges for all k. □

Lemma E.1 (Coefficients $\{\tilde{\alpha}_k\}$ for quotients of polynomials) *Let g_m and h_n be polynomials of degrees m and n, and consider*

$$f(u) = \tfrac{g_m(u)}{h_n(u)}, \quad \forall u \in \mathbb{R}_{\geq 0}.$$

For $\tau > 0$, recall the polynomial p_k given in Table 2 and define $f_k(s) = f(s)\, p_k(2s/\tau - 1)$ under the assumption $\mathcal{P}(f_k) \cap [0, \infty) = \emptyset$. The Fourier coefficients of f satisfy, for $k \geq 0$,

$$\tilde{\alpha}_k = \sqrt{\pi} \textstyle\sum_{q=0}^{l(k)} \frac{\zeta_{-q}\,(-\tau)^q}{q!\Gamma\left(\frac{1}{2}-q\right)} - \sum_{a \in \mathcal{P}(f_k)} \mathrm{Res}_{s=a} \left(f_k(s)\, s^{-\frac{1}{2}}]_{-\pi} (s-\tau)^{-\frac{1}{2}}]_{-\pi} \right), \tag{E.9}$$

where $l(k) = \max(0, m-n+k)$ is the largest nonnegative integer l such that $\lim_{s\to 0} s^l f_k(1/s)$ is finite, and $\{\zeta_q\}$ are the coefficients of the Laurent series at $+\infty$ of the analytic continuation of f_k, i.e.,

$$\zeta_q = \tfrac{1}{(l(k)+q)!} \lim_{s\to 0} \tfrac{d^{l(k)+q}}{ds^{l(k)+q}} \left[s^{l(k)} f_k\left(\tfrac{1}{s}\right) \right], \quad (q = -l(k), \ldots, \infty). \tag{E.10}$$

See [9] for a derivation derivation of Lemma E.1 by the technique of contour integration.