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CORRECTION

Correction to: Extremal $GI/GI/1$ queues given two moments: exploiting Tchebycheff systems

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1 The errors

Unfortunately, we have discovered several errors in [2]:

(i) Lemma 5 in Sect. 4 is incorrect. A counterexample is given in Sect. 2 below.
(ii) Theorem 5 in Sect. 5 is incorrect. It would be correct if we could replace $t \geq -M_a$ by $t \geq 0$ in the condition (39) in Theorem 4, but we are not free to do so, because the condition $t \geq -M_a$ is required by the increasing convex stochastic order used in Theorem 4.
(iii) The presentation of Lemma 3 is incorrect, but this is fixable, as explained in Sect. 3.
(iv) Proposition 1 is incorrect, but this is fixable. This proposition becomes correct if the condition $g(0) = 0$ is added, as holds in the intended Erlang example (E_k for $k \geq 2$). The correction is needed because (57) in [2] is missing the term $g(0)h(t)$.

These errors have serious implications. The error in Lemma 5 invalidates the proofs of Theorems 1 and 3. The error in Theorem 5 invalidates the proof of Theorem 2. Thus, Theorems 1–3 become conjectures remaining to be proved or disproved.

The error in the proof of Theorem 1 invalidates the proof of Theorem 8, which invalidates the proof of Theorem 7. However, we have obtained new results, which provide a new proof of Theorem 7, as explained in Sect. 4 below.

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2 Counterexample to Lemma 5

We will work with the two-point distributions as defined in Sect. 2.1 of [2]. Assume that the mean is $m = 1$, the upper limit of the support is $M = 5$ and the squared coefficient of variation is $c^2 = 1$. Let X_0 and X_u be random variables with the extremal two-point cdf's F_0 and F_u, respectively. Then, $P(X_0 = 2) = 1/2 = P(X_0 = 0)$, while $P(X_u = 5) = 1/17$ and $P(X_u = 3/4) = 16/17$. It is known that $X_0 \le_{3-cx} X_u$, as stated in (34) of [2]. Since $E[X_0] = E[X_u] = 1$ and $E[X_0^2] = E[X_u^2] = 2$, we also have $X_0 \le_{2,2} X_u$. However, contrary to Lemma 5 in [2], the ordering $Y_0 \equiv (X_0 - 3/4)^+ \le_{2,2} (X_u - 3/4)^+ \equiv Y_u$ fails to hold. This is easy to see, because Y_0 and Y_u are the two-point distribution with $P(Y_0 = 0) = 1/2 = P(Y_0 = 5/4)$, while $P(Y_u = 0) = 16/17$ and $P(Y_u = 17/4) = 1/17$, so that we have a reverse ordering of the means: $E[Y_0] = 5/8 > 1/4 = E[Y_u] = E[X_u] - 3/4$. For the counterexample to the ordering under consideration, note that $Y_0 + t \ge 0$ and $Y_u + t \ge 0$ for all $t \ge 0$,

$$\begin{aligned} E[(Y_0 + t)^2] &= t^2 + 5t/4 + O(1) \quad \text{and} \\ E[(Y_u + t)^2] &= t^2 + t/2 + O(1) \quad \text{as} \quad t \to \infty, \end{aligned}$$

so that $E[(Y_0 + t)^2] > E[Y_u + t)^2]$ for all t sufficiently large. This contradicts the claim of Lemma 5.

3 Correcting Lemma 3

Lemma 3 is important because it provides a way to apply the theory of Tchebycheff (T) systems from [4], as briefly reviewed in [1] and Section 3 of [2]. However, in the statement of Lemma 3 insufficient care was given to the support of the random variable Y with distribution Γ appearing in (22) of [2]. The support of Y should be chosen so that the integrand $\phi(u)$ appearing in (21) of [2] is not identically 0 for any subinterval of $[0, M_a]$. Hence, the support of Y should be changed from $[0, \infty)$ to a more general interval, i.e., (22) should be replaced by

$$\phi(u) \equiv \int_a^b h((y-u)^+)\, d\Gamma(y) = h(0)\Gamma(u) + \int_{u+}^b h(y-u)\, d\Gamma(y), \quad 0 \le u \le M_a, \tag{1}$$

where

$$-\infty \le a \le 0 < M_a \le b \le \infty, \tag{2}$$

Γ is a cdf of a real-valued random variable Y with a continuous positive density function over the interval $[a, b]$. Then, in Lemma 3 of [2] we should replace (25) by (2) above. The proof also needs to be adjusted accordingly. In particular, the revised proof is:

Proof First, observe that the finite mgf condition implies that all integrals are finite. In each case, we can apply Lemmas 1 and 2 of [2] with (1) and (2). To do so, we apply the Leibniz rule for differentiation of an integral with (1). Using (2), we have

$$\phi(u) = \int_a^b h((y-u)^+)\,\mathrm{d}\Gamma(y) = \int_u^b h(y-u)\,\mathrm{d}\Gamma(y) + h(0)\Gamma(u) \quad \text{and}$$

$$\begin{aligned}\phi^{(1)}(u) &= -\int_u^b h^{(1)}(y-u)\,\mathrm{d}\Gamma(y) - h(0)\gamma(u) + h(0)\gamma(u)\\ &= -\int_u^b h^{(1)}(y-u)\,\mathrm{d}\Gamma(y).\end{aligned} \tag{3}$$

For $h(x) \equiv x$ in condition (i), we have $h^{(1)}(x) = 1$ for all x, so that

$$\phi^{(1)}(u) = -\int_u^b h^{(1)}(y-u)\,\mathrm{d}\Gamma(y) = -\int_u^b \mathrm{d}\Gamma(y) = -(1-\Gamma(u)), \tag{4}$$

so that, by the condition on Γ,

$$\phi^{(2)}(u) = \gamma(u) > 0 \quad \text{and} \quad \phi^{(3)}(u) = \gamma^{(1)}(u) < 0 \quad \text{for} \quad 0 \le u \le M_a. \tag{5}$$

From the form of $\phi^{(3)}(u)$ in (5), we see that the condition on γ is necessary as well as sufficient. We also see that the UB and LB are switched if instead $\gamma^{(1)}(u) > 0$.

Turning to $h(x) = x^2$ in condition (ii), we use $h^{(1)}(0) = 0$ and $h^{(2)}(x) = 2$ for all x with the second line of (3) to get

$$\phi^{(2)}(u) = \int_u^b h^{(2)}(y-u)\,\mathrm{d}\Gamma(y) = 2\int_u^b \mathrm{d}\Gamma(y) = 2(1-\Gamma(u)) > 0, \tag{6}$$

so that $\phi^{(3)}(u) = -2\gamma(u) < 0$ for $0 \le u \le M_a$.

Conditions (iii) and (iv) are both special cases of condition (v), which implies that

$$\phi^{(3)}(u) = -\int_u^b h^{(3)}(y-u)\,\mathrm{d}\Gamma(y) < 0. \tag{7}$$

$\square$

4 Application of Lemma 3 to the higher cumulants

In [3], we have applied the corrected Lemma 3 in [2] to develop new extremal results for the higher cumulants of the steady-state waiting time that provide corrected proofs of Theorems 7 and 8 in [2]. These bounds for higher cumulants are interesting and important because they clearly demonstrate the value of Lemma 3 in [2] and highlight its limitation for treating the mean. In particular, the decreasing pdf condition in Lemma 3 (i) prevents positive results for the mean that we now obtain for the higher cumulants from Lemma 3 (ii) and (iii).

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