ON THE MAXIMA OF SUPREMA OF DEPENDENT GAUSSIAN MODELS

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Abstract: In this paper, we study the asymptotic distribution of the maxima of suprema of dependent Gaussian processes with trend. For different scales of the time horizon we obtain different normalizing functions for the convergence of the maxima. The obtained results not only have potential applications in estimating the delay of certain Gaussian fork-join queueing systems but also provide interesting insights to the extreme value theory for triangular arrays of random variables with row-wise dependence.

Key Words: Extreme value; self-similarity; Gaussian processes; fractional Brownian motion; triangular arrays; Pickands constant; Piterbarg constant.

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1. Introduction

Let $\{X_i(t), t \geq 0\}, i = 1, 2, \ldots,$ be independent copies of a centered self-similar Gaussian process with almost surely (a.s.) continuous sample paths, self-similarity index $H \in (0, 1)$ and variance function t^{2H}, and let $\{X(t), t \geq 0\}$ be another independent centered self-similar Gaussian processes with a.s. continuous sample paths, self-similarity index $H_0 \in (0, 1)$ and variance function t^{2H_0}. We define, for positive constants $\sigma, \sigma_0, c_i, i \geq 1$, $\beta > \max(H, H_0)$, and a deterministic function $T_n > 0$,

$$M_n := \max_{i \leq n} \sup_{t \in [0, T_n]} (\sigma X_i(t) + \sigma_0 X(t) - c_i t^\beta), \qquad n \geq 1. \tag{1}$$

This paper is concerned with asymptotic distributional properties of M_n as $n \to \infty$. More precisely, we aim to establish limit theorems for $\nu_n^{-1}(M_n - \mu_n)$, as $n \to \infty$, for some suitably chosen normalizing functions $\nu_n, \mu_n, n \geq 1$. This work is a continuation of the recent work done in [19], where the case $T_n = \infty$ was discussed. Note that the general kth order statistics were discussed in [19], but to ease the complication we shall only focus on the maxima in this paper; one can check that some results for the kth order statistics can also be established. As in [19], without loss of generality, we assume $\sigma = 1$. Throughout the rest of the paper, we also assume that $\lim_{n\to\infty} T_n \in [0, \infty]$ exists and investigate how different scales of T_n influence the normalizing functions $\nu_n, \mu_n, n \geq 1$ in the limit theorems for the maxima defined in (1).

The motivation for the study of distributional properties of M_n stems from a recent contribution [22] on a Brownian fork-join queueing system, which is a special model of (1) with all the Gaussian processes involved being Brownian motions, $\beta = 1$, $c = c_i, i \geq 1$ and $T_n = \infty$ (hereafter called *Browian model with linear drift*). In their context, M_n (with $T_n = \infty$) models the maximum of stady-state queue lengths (or delay) in a fork-join network of n statistically identical queues which are driven by a common Brownian motion perturbed arrival process and independent Brownian motion perturbed service processes, respectively. The theoretical limit result obtained therein is the key to developing structural insights into the dimensioning of assembly systems; interested readers are referred to [22] for more details on this application. More recently, the tail asymptotics for the delay in such a Brownian fork-join queueing system were studied in [25]. As discussed in [6, 7, 21] and references therein, for a fluid queueing model it is of great interest to consider general Gaussian processes with

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a non-linear trend and study the distributional properties of the transient queue length (i.e., the supremum is taken over a finite time interval instead of $\mathbb{R}_+$). Analogously, the maximum M_n in (1) can be seen as the maximum of n transient queue lengths (or delay over a finite time horizon) in a general Gaussian fork-join queueing system, with the time horizon T_n possibly dependent on n. In this paper, we consider different scales of T_n. This discussion may be intersting from an application point of view, for instance, the system users may be interested in estimating the delay over any short-time or long-time horizon. Note that as in [19] the study in this paper also provides complementary results to the extreme value theory for multivairate Gaussian models in random environment.

Clearly, the study of the maxima $M_n, n \ge 1$ is relevent to the extreme value theory for triangular arrays of random variables with row-wise dependence. Define $\{Y_{kn}, k \le n, n \ge 1\}$ to be a triangular array of random variables and $N_n = \max_{k \le n} Y_{kn}$ to be the row-wise maximum. The extreme value theory for the triangular array $\{Y_{kn}, k \le n, n \ge 1\}$ is concerned with the convergence of the row-wise maxima $N_n, n \ge 1$ under a linear normalization. If $Y_{kn}, k \le n$ is stationary for any fixed n, then we call $N_n, n \ge 1$ *homogeneous* maxima, otherwise, we call it *inhomogeneous* maxima. Current literature on extreme value theory for triangular arrays has been focused on homogeneous maxima, and particularly, the maxima for row-wise independent and indentically distributed triangular arrays (i.e., $Y_{kn}, k \le n$ being independent and identically distributed (IID)); see, e.g., [1, 15, 23]. Some conditions guaranteeing the convergence of normalized maxima to some limit are given by [15] under some differentiation conditions. Particularly, the maxima of row-wise independent Poisson-distributed and related triangular arrays are discussed in [1], and the maxima for some row-wise independent Weibull-(truncated) regular variation mixture distributed triangular arrays are discussed in [23]. Somehow surprisingly, except for the triangular arrays of normal random variables (e.g., [16]), there are very few papers dealing with the homogeneous maxima with row-wise stationary triangular arrays. The only result on this topic that we could find is the recent one obtained in [13] where a Gumbel limit theorem is obtained for normalized maxima under some general conditions (see Theorem 2.1 therein). It turns out that there exists no theory for general (in)homogeneous maxima which covers the convergence of $M_n, n \ge 1$ under some normalization that is interested in this paper. In what follows, if $c = c_i, i \ge 1$, the maxima $M_n, n \ge 1$ is called *homogeneous*, and otherwise, called *inhomogeneous*. We obtain convergence results for suitably normalized $M_n, n \ge 1$ for both homogeneous case and some inhomogeneous case. As we will see, the only possible non-degenerate limit distributions are from the family of Gumbel, Gaussian or a mixture of them. This study provides some interesting examples, which enriches the extreme value theory for triangular arrays of random variables with row-wise dependence.

The rest of the paper is organized as follows: In Section 2 we present some preliminary results concerning the tail asymptotics of the supremum of a class of self-similar Gaussian processes with trend over a threshold-dependent time horizon. The main results on the homogeneous maxima $M_n, n \ge 1$ are given in Section 3. Section 4 discusses some inhomogeneous maxima. All the proofs are presented in Section 5.

2. Prelimanaries

In this section, we mainly discuss the tail asymptotics of the supremum of a self-similar Gaussian process with trend over a threshold-dependent time interval. This study is useful for the construction of normalizing functions for the maxima, and is also of independent interest. The results presented in Proposition 2.1 below generalize some of the existing results obtained in [3], see also [20].

Let $\{X_H(t), t \geq 0\}$ be a centered self-similar Gaussian process with a.s. continuous sample paths, self-similarity index $H \in (0,1)$, variance function t^{2H}. We assume a *local stationarity* of the standardized Gaussian process $\overline{X}_H(t) := X_H(t)/t^H, t > 0$ in a neighbourhood of the point $t = 1$, i.e.,

$$\lim_{s,t\to 1} \frac{\mathbb{E}\left\{(\overline{X}_H(s) - \overline{X}_H(t))^2\right\}}{K^2(|s-t|)} = 1 \tag{2}$$

holds for some positive function $K(\cdot)$ which is regularly varying at 0 with index $\alpha/2 \in (0,1)$. Condition (2) is a common assumption in the literature; see, e.g., [11] and [17]. It is worth noting that the assumption (2) is slightly general than the S2 in [8] where a decent discussion on properties and examples of self-similar Gaussian processes is given. Note that the local stationarity at $t = 1$ and the self-similarity of the random process imply the local stationarity at any point $t = r > 0$, i.e.,

$$\lim_{s,t\to r} \frac{\mathbb{E}\left\{(\overline{X}_H(s) - \overline{X}_H(t))^2\right\}}{K^2(|s-t|)} = r^{-\alpha}. \tag{3}$$

For a threshold-dependent time horizon T_u (to be specified below) and constants $c > 0, \beta > H$, we shall derive asymptotics for

$$\psi_{T_u}(u) := \mathbb{P}\left\{\sup_{t\in[0,T_u]} X_H(t) - ct^\beta > u\right\}, \qquad u \to \infty.$$

Throughout this paper, for two positive functions f, h and some $u_0 \in [-\infty, \infty]$, write $h(u) \sim f(u)$ or $h(u) = f(u)(1 + o(1))$ if $\lim_{u\to u_0} f(u)/h(u) = 1$, write $f(u) = o(h(u))$ if $\lim_{u\to u_0} f(u)/h(u) = 0$, and write $f(u) = O(h(u))$ if $\lim_{u\to u_0} f(u)/h(u) \in (0,\infty)$. Further, we denote by $\overleftarrow{K}(\cdot)$ the asymptotic inverse of $K(\cdot)$, and thus

$$\overleftarrow{K}(K(t)) = K(\overleftarrow{K}(t))(1 + o(1)) = t(1 + o(1)), \quad t \downarrow 0.$$

It follows that $\overleftarrow{K}(\cdot)$ is regularly varying at 0 with index $2/\alpha$; see, e.g., [14].

We shall consider the following scenarios for the threshold-dependent time horizon T_u:

D1: $\lim_{u\to\infty} T_u/u^{1/\beta} = 0$;

D2: $\lim_{u\to\infty} T_u/u^{1/\beta} = s_0 \in (0, t_0)$;

D3: $\lim_{u\to\infty} \frac{T_u - t_0 u^{1/\beta}}{A^{1/2}B^{-1/2}u^{H/\beta+1/\beta-1}} = x \in (-\infty, \infty]$.

Here in the above

$$t_0 = \left(\frac{H}{c(\beta - H)}\right)^{1/\beta},$$

and

$$A = \frac{t_0^H}{1 + ct_0^\beta} = \frac{\beta - H}{\beta}\left(\frac{H}{c(\beta-H)}\right)^{H/\beta}, \quad B = \left(\frac{H}{c(\beta - H)}\right)^{-\frac{H+2}{\beta}} H\beta. \tag{4}$$

Below, by $\{B_{\alpha/2}(t), t \geq 0\}$ we denote a standard fractional Brownian motion (sfBm) with Hurst index $\alpha/2 \in (0,1)$, and

$$\mathrm{Cov}(B_{\alpha/2}(t), B_{\alpha/2}(s)) = \frac{1}{2}(t^\alpha + s^\alpha - \mid t - s \mid^\alpha), \quad t, s \geq 0.$$

The well known Pickands constant $\mathcal{H}_\alpha$ and Piterbarg constant $\mathcal{P}_\alpha^d$ in the Gaussian theory is defined, respectively, by

$$\mathcal{H}_\alpha = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left\{\exp\left(\sup_{t\in[0,T]} (\sqrt{2}B_{\alpha/2}(t) - t^\alpha)\right)\right\} \in (0,\infty).$$

$$\mathcal{P}_\alpha^d = \lim_{T\to\infty} \mathbb{E}\left\{\exp\left(\sup_{t\in[0,T]} (\sqrt{2}B_{\alpha/2}(t) - (1+d)t^\alpha)\right)\right\} \in (0,\infty), \qquad d > 0.$$

We refer to [2, 4, 10, 12, 24] for basic properties of the Pickands, Piterbarg and related constants. In particular, it has been shown that $\mathcal{H}_1 = 1$ and $\mathcal{P}_1^d = 1 + 1/d$, $d > 0$.

Proposition 2.1. *Let $\{X_H(t), t \ge 0\}$ be the self-similar Gaussian process defined as above with (2) satisfied, and assume $c > 0$, $\beta > H$.*

(i). *Further assume that the following limit exists and*

$$\lim_{t \downarrow 0} \frac{t}{K^2(t)} =: Q \in [0, \infty].$$

Then, under scenarios D1 and D2 (i.e., $\lim_{u\to\infty} T_u/u^{1/\beta} = s_0 \in [0, t_0)$), we have

$$\psi_{T_u}(u) = \mathcal{D}_{c_0}\left(\frac{u + cT_u^\beta}{T_u^H}\right)\left(\frac{u + cT_u^\beta}{T_u^H}\right)^{-1} \exp\left(-\frac{(u + cT_u^\beta)^2}{2T_u^{2H}}\right)(1 + o(1)), \quad u \to \infty,$$

where, for $y > 0$,

$$\mathcal{D}_{c_0}(y) = \begin{cases} \frac{\mathcal{H}_\alpha}{2^{1/\alpha}\sqrt{2\pi}(H - c_0\beta)} y^{-2}\left(\overleftarrow{K}(1/y)\right)^{-1}, & \text{if } Q = 0, \\ \frac{1}{\sqrt{2\pi}}\, \mathcal{P}_\alpha^{2(H - c_0\beta)Q}, & \text{if } Q \in (0, \infty), \\ \frac{1}{\sqrt{2\pi}} & \text{if } Q = \infty, \end{cases} \qquad \text{with } c_0 = \frac{cs_0^\beta}{1 + cs_0^\beta}. \tag{5}$$

(ii). *Under scenario D3 (i.e., $\lim_{u\to\infty} \frac{T_u - t_0 u^{1/\beta}}{A^{1/2}B^{-1/2}u^{H/\beta + 1/\beta - 1}} = x \in (-\infty, \infty]$), we have*

$$\psi_{T_u}(u) = \psi_\infty(u)\Phi(x)(1 + o(1)), \quad u \to \infty,$$

where $\Phi(x)$ is the standard normal distribution function and

$$\psi_\infty(u) := \mathbb{P}\left\{\sup_{t\ge 0} X_H(t) - ct^\beta > u\right\} = R(u)\exp\left(-\frac{u^{2\left(1 - \frac{H}{\beta}\right)}}{2A^2}\right)(1 + o(1)), \quad u \to \infty,$$

where (with A, B given in (4))

$$R(u) = \frac{A^{\frac{3}{2} - \frac{2}{\alpha}}\mathcal{H}_\alpha}{2^{\frac{1}{\alpha}}B^{\frac{1}{2}}t_0} \frac{u^{\frac{2H}{\beta} - 2}}{\overleftarrow{K}(u^{\frac{H}{\beta} - 1})}, \quad u > 0. \tag{6}$$

Remark 2.2. *If $\{X_H(t), t \ge 0\}$ is the sfBm with Hurst index $H \in (0, 1)$ and $\beta = 1$, then we can check that $K(t) = t^H$, and the results in Proposition 2.1 reduce to that given in (2.2)-(2.5) in [20] (here the explicit formula for $\mathcal{P}_1^d$ given above should be used).*

3. Homogeneous maxima

We shall consider asymptotic distributional properties of the homogeneous maxima

$$M_n = \max_{i\le n} \sup_{t\in[0,T_n]} (X_i(t) + \sigma_0 X(t) - ct^\beta), \qquad n \ge 1,$$

where we assume $\sigma = 1$ and $c = c_i, i \ge 1$. Precisely, we aim to establish limit theorems for $\nu_n^{-1}(M_n - \mu_n)$, as $n \to \infty$, for some suitably chosen normalizing functions $\nu_n, \mu_n, n \ge 1$. Motivited by the scenarios D1-D3 treated in Section 2, we consider the following scenarios for T_n:

- **S1.** (*Super-short time horizon*) $\lim_{n\to\infty} T_n^H \sqrt{2\log n} = \kappa_0 \in [0, \infty)$;
- **S2.** (*Short time horizon*) $\lim_{n\to\infty} T_n^H\sqrt{2\log n} = \infty$ and $\lim_{n\to\infty} \frac{T_n^{1-H/\beta}}{(\sqrt{2\log n})^{1/\beta}} = 0$;
- **S3.** (*Intermediate time horizon*) $\lim_{n\to\infty} \frac{T_n^{1-H/\beta}}{(\sqrt{2\log n})^{1/\beta}} = \widetilde{s}_0 \in \left(0, \widetilde{t}_0\right)$, with $\widetilde{t}_0 = \left(\frac{t_0^\beta}{1 + ct_0^\beta}\right)^{1/\beta} = \left(\frac{H}{c\beta}\right)^{1/\beta}$;
- **S4.** (*Long time horizon*) $\lim_{n\to\infty} \frac{T_n^{1-H/\beta}}{(\sqrt{2\log n})^{1/\beta}} = \widetilde{t}_0$ and $\lim_{n\to\infty} \frac{T_n - (\widetilde{t}_0^\beta\sqrt{2\log n})^{1/(\beta - H)}}{A^{1/2}B^{-1/2}(2A^2\log n)^{(H+1-\beta)/(2(\beta - H))}} = x_0 \in \mathbb{R}$;

S5. (*Super-long time horizon*) $\lim_{n\to\infty} \frac{T_n^{1-H/\beta}}{(\sqrt{2\log n})^{1/\beta}} = \widetilde{s}_0 > \widetilde{t}_0$.

To determine the correct normalizing functions, we will discuss the following maxima of the corresponding IID sequence

$$\widetilde{M}_n := \max_{i\le n} \sup_{t\in[0,T_n]} (X_i(t) - ct^\beta), \qquad n \ge 1.$$

It is known that normalizing functions for the maxima of a sequence of IID random variables can be retrieved from the tail asymptptics of the random variable; see, e.g., [19]. Motivited by the tail asymptotics discussed in Proposition 2.1, we introduce the following normailizing functions.

Under scenarios S2 and S3:

$$\begin{aligned} b_n &:= T_n^H \left(\sqrt{2\log n} + \frac{1}{\sqrt{2\log n}} \log\left(f_n\left(T_n^H\sqrt{2\log n} - cT_n^\beta\right)\right)\right) - cT_n^\beta,\ n\in\mathbb{N}, \\ a_n &:= \frac{T_n^H}{\sqrt{2\log n}},\ n\in\mathbb{N}. \end{aligned} \tag{7}$$

where, recalling the function $\mathcal{D}_{c_0}(\cdot)$ defined in (5),

$$f_n(w) := \mathcal{D}_{c\widetilde{s}_0^\beta}\left(\frac{w+cT_n^\beta}{T_n^H}\right)\left(\frac{w+cT_n^\beta}{T_n^H}\right)^{-1}, \quad w>0, \tag{8}$$

Under scenarios S4 and S5 (for S5, we set $x_0 = \infty$, i.e., use $d_n(\infty)$):

$$\begin{aligned} d_n(x_0) &:= (2A^2\log n)^{1/\tau}\left(1 + \frac{1}{\tau\log n}\log(R((2A^2\log n)^{1/\tau})\Phi(x_0))\right),\ n\in\mathbb{N}, \\ e_n &:= \frac{(2A^2\log n)^{1/\tau}}{\tau\log n},\ n\in\mathbb{N}, \end{aligned} \tag{9}$$

where $\tau = 2(1-H/\beta)$, and $R(u)$ is defined in (6).

The following proposition shows that these normalizing functions are the required ones for $\widetilde{M}_n, n\ge 1$. Hereafter, $\overset{d}{\to}$ denotes the convergence in distribution, and Λ denotes a standard Gumbel random varaiable, i.e., $\mathbb{P}\{\Lambda\le x\} = \exp(-e^{-x}), x\in\mathbb{R}$.

Proposition 3.1. *We have*

(i). *Under scenarios S2 and S3,*

$$a_n^{-1}(\widetilde{M}_n - b_n) \overset{d}{\to} \Lambda, \qquad n\to\infty.$$

(ii). *Under scenarios S4 and S5,*

$$e_n^{-1}(\widetilde{M}_n - d_n(x_0)) \overset{d}{\to} \Lambda, \qquad n\to\infty.$$

Below is our pricinpal result on the aymptotical distribution for the homogenous maxima $M_n, n\ge 1$ under suitable normalization. In what follows, we denote $\mathcal{N}$ to be a standard normal random variable which is independent of Λ.

Theorem 3.2. *For the homogeneous maxima $M_n, n\ge 1$ defined in* (1) *with $\sigma = 1$ and $c = c_i, i\ge 1$, we have*

(a). *Under scenario S1,*

$$M_n \overset{d}{\to} \kappa_0, \qquad n\to\infty.$$

(b). *Under scenarios S2 and S3,*

(b.i) *If further $\lim_{n\to\infty} \frac{T_n^{H-H_0}}{\sqrt{2\log n}} = 0$, then*

$$\sigma_0^{-1}T_n^{-H_0}(M_n - b_n) \overset{d}{\to} \mathcal{N}, \qquad n\to\infty.$$

(b.ii) *If further* $\lim_{n\to\infty} \frac{T_n^{H-H_0}}{\sqrt{2\log n}} = \infty$, *then*

$$a_n^{-1}(M_n - b_n) \xrightarrow{d} \Lambda, \qquad n \to \infty.$$

(b.iii) *If further* $\lim_{n\to\infty} \frac{T_n^{H-H_0}}{\sqrt{2\log n}} = q_0 \in (0,\infty)$, *then*

$$a_n^{-1}(M_n - b_n) \xrightarrow{d} \Lambda + \frac{\sigma_0}{q_0}\mathcal{N}, \qquad n \to \infty.$$

(c). *Under scenarios S4 and S5,*

(c.i) *If further* $2H - H_0 < \beta$, *then*

$$\sigma_0^{-1} T_n^{-H_0}(M_n - d_n(x_0)) \xrightarrow{d} \mathcal{N}, \quad n \to \infty.$$

(c.ii) *If further* $2H - H_0 > \beta$, *then*

$$e_n^{-1}(M_n - d_n(x_0)) \xrightarrow{d} \Lambda, \quad n \to \infty.$$

(c.iii) *If further* $2H - H_0 = \beta$, *then*

$$e_n^{-1}(M_n - d_n(x_0)) \xrightarrow{d} \Lambda + \frac{\sigma_0 c\beta}{H}\mathcal{N}, \quad n \to \infty.$$

Remarks 3.3. *(a). We remark that the conditions of (b.i)-(b.iii) can be more specific under scenario S3. In fact, under S3 we know that*

$$T_n \sim \left(\widetilde{s}_0^{\beta}\sqrt{2\log n}\right)^{\frac{1}{\beta-H}}, \qquad n \to \infty,$$

and thus the case-specific conditions of (b.i)-(b.iii) can be simplified to $2H - H_0 < \beta$, $2H - H_0 > \beta$ *and* $2H - H_0 = \beta$, *respectively, and further* $q_0 = \widetilde{s}_0^{\beta}$.

(b). It turns out that there is an interesting smooth transition from a_n, b_n, *respectively, to* $e_n, d_n(x_0)$ *in the sense that, if* T_n *satisfies S4 then*

$$b_n \sim T_n^H\sqrt{2\log n} - cT_n^{\beta} \sim (2A^2\log n)^{1/\tau} \sim d_n(x_0), \quad \textit{and} \quad a_n \sim e_n, \qquad n \to \infty.$$

In fact, we can rewrite T_n *as*

$$\begin{aligned} T_n &= \left(\widetilde{t}_0^{\beta}\sqrt{2\log n}\right)^{\frac{1}{\beta-H}} + A^{1/2}B^{-1/2}x_0(2A^2\log n)^{(H+1-\beta)/(2(\beta-H))} + \varepsilon(n) \\ &= t_0(2A^2\log n)^{\frac{1}{2(\beta-H)}}\left(1 + \frac{A^{1/2}B^{-1/2}x_0}{t_0}(2A^2\log n)^{-1/2} + t_0^{-1}\varepsilon(n)(2A^2\log n)^{\frac{-1}{2(\beta-H)}}\right), \end{aligned} \tag{10}$$

where $\varepsilon(n) = o\left((\log n)^{(H+1-\beta)/(2(\beta-H))}\right)$. *Thus,*

$$\begin{aligned} &T_n^H\sqrt{2\log n} - cT_n^{\beta} - (2A^2\log n)^{1/\tau} \\ &= A^{-1}t_0^H(2A^2\log n)^{\frac{\beta}{2(\beta-H)}}\left(1 + Ht_0^{-1}A^{1/2}B^{-1/2}x_0(2A^2\log n)^{-1/2} + Ht_0^{-1}\varepsilon(n)(2A^2\log n)^{\frac{-1}{2(1-H)}} + O((2A^2\log n)^{-1})\right) \\ &\quad -ct_0(2A^2\log n)^{\frac{\beta}{2(\beta-H)}}\left(1 + \beta t_0^{-1}A^{1/2}B^{-1/2}x_0(2A^2\log n)^{-1/2} + \beta t_0^{-1}\varepsilon(n)(2A^2\log n)^{\frac{-1}{2(1-H)}} + O((2A^2\log n)^{-1})\right) \\ &\quad -(2A^2\log n)^{1/\tau} \\ &= o\left((2A^2\log n)^{1/\tau}\right), \qquad n \to \infty, \end{aligned}$$

where in the second equality we have used $A^{-1}t_0^H = 1 + ct_0$. *Similarly, some elementary calculations show that*

$$\lim_{n\to\infty} a_n/e_n = 1.$$

This is an interesting observation which reveals a smooth change of the normalising functions. However, it looks that under scenario S4 the b_n *is not the correct normalising function but the* $d_n(x_0)$ *is.*

Example 3.4. *This example illustrates Theorem 3.2 for $T_n = (\lambda\sqrt{2\log n})^\gamma$, with $\gamma \in \mathbb{R}$ and $\lambda > 0$. We check how the asymptotic distribution of a normalized M_n will change according to different values of γ. There can be a lot of cases to be considered, but we choose to work with one representative case where $\beta > H_0 > 2H$. In this case, we obtain different results according to where the value of γ falls in the following intervals:*

$$-\infty < -\frac{1}{H} < -\frac{1}{H_0 - H} < \frac{1}{\beta - H} < \infty.$$

Precisely, a direct appliation of Theorem 3.2 yeilds the following convergence results, as $n \to \infty$:

(1). *If $\gamma \in \left(-\infty, -\frac{1}{H}\right)$, then $M_n \xrightarrow{d} 0$.*
(2). *If $\gamma = -\frac{1}{H}$, then $M_n \xrightarrow{d} \lambda^{-1}$.*
(3). *If $\gamma \in \left(-\frac{1}{H}, -\frac{1}{H_0 - H}\right)$, then $a_n^{-1}(M_n - b_n) \xrightarrow{d} \Lambda$.*
(4). *If $\gamma = -\frac{1}{H_0 - H}$, then $a_n^{-1}(M_n - b_n) \xrightarrow{d} \Lambda + \sigma_0\lambda^{-1}\mathcal{N}$.*
(5). *If $\gamma \in \left(-\frac{1}{H_0 - H}, \frac{1}{\beta - H}\right)$, then $\sigma_0^{-1}T_n^{-H_0}(M_n - b_n) \xrightarrow{d} \mathcal{N}$.*
(6). *If $\gamma = \frac{1}{\beta - H}$ and $\lambda \in (0, \widetilde{t}_0^\beta)$, then $\sigma_0^{-1}T_n^{-H_0}(M_n - b_n) \xrightarrow{d} \mathcal{N}$.*
(7). *If $\gamma = \frac{1}{\beta - H}$ and $\lambda = \widetilde{t}_0^\beta$, then $\sigma_0^{-1}T_n^{-H_0}(M_n - d_n(0)) \xrightarrow{d} \mathcal{N}$.*
(8). *If $\gamma = \frac{1}{\beta - H}$ and $\lambda \in (\widetilde{t}_0^\beta, \infty)$, or $\gamma > \frac{1}{\beta - H}$, then $\sigma_0^{-1}T_n^{-H_0}(M_n - d_n(\infty)) \xrightarrow{d} \mathcal{N}$.*

4. Inhomogenous maxima

We shall consider convergence results for the inhomogenous maxima

$$M_n = \max_{i \le n} \sup_{t \in [0, T_n]} (X_i(t) + \sigma_0 X(t) - c_i t^\beta), \quad n \ge 1,$$

where we assume $\sigma = 1$. For simplicity, we shall assume that all the $c_i, i \ge 1$ take value from a finite set of distinct values, denoted as

$$\mathcal{S} = \{\hat{c}_1, \hat{c}_2, \cdots, \hat{c}_k\}, \quad \text{where } \ 0 < \hat{c}_1 (= c) < \hat{c}_2 < \cdots < \hat{c}_k < \infty, \tag{11}$$

with some $k \ge 2$. Here, for notational convenience, we use $\hat{c}_1 = c$ which is helpful when we adopt the normalizing functions defined in the previous sections. Further, let $m_j = \#\{i \le n : c_i = \hat{c}_j\}$ and assume that

$$\lim_{n\to\infty} \frac{m_1}{n} =: p_1 \in (0, 1], \quad \lim_{n\to\infty} \frac{m_j}{n} =: p_j \in [0, 1), \quad 2 \le j \le k. \tag{12}$$

Obviously, $\sum_{j=1}^k p_j = 1$.
Similarly to the homogenous case we shall first discuss the maxima of the corresponding IID sequence

$$\widehat{M}_n := \max_{i \le n} \sup_{t \in [0, T_n]} (X_i(t) - c_i t^\beta), \quad n \ge 1. \tag{13}$$

We have the following result for the asympototical distribution of suitably normalized maxima $\widehat{M}_n, n \ge 1$.

Proposition 4.1. *Let $\widehat{M}_n, n \ge 1$ be defined as in* (13)*, with $c_i, i \ge 1$ satisfying* (11) *and* (12)*. We have*

(a). *Under S1,*

$$\widehat{M}_n \xrightarrow{d} \kappa_0, \quad n \to \infty.$$

(b). *Under S2,*

$$a_n^{-1}(\widehat{M}_n - b_n) \xrightarrow{d} \widehat{\Lambda}, \quad n \to \infty,$$

where

$$\widehat{\Lambda} = \begin{cases} \Lambda, & \text{if } \lim_{n\to\infty} T_n^{\beta-H}\sqrt{2\log n} = 0, \\ \Lambda + \log\left(p_1 + \sum_{j=2}^k p_j e^{-(\hat{c}_j - c)q_1}\right), & \text{if } \lim_{n\to\infty} T_n^{\beta-H}\sqrt{2\log n} = q_1 \in (0,\infty), \\ \Lambda + \log p_1, & \text{if } \lim_{n\to\infty} T_n^{\beta-H}\sqrt{2\log n} = \infty. \end{cases}$$

(c). *Under S3,*

$$a_n^{-1}(\widehat{M}_n - b_n) \xrightarrow{d} \Lambda + \log p_1, \qquad n \to \infty.$$

(d). *Under S4 and S5,*

$$e_n^{-1}(\widehat{M}_n - d_n(x_0)) \xrightarrow{d} \Lambda + \log p_1, \qquad n \to \infty.$$

Remarks 4.2. *(a). It is interesting to observe that under S2, the three possible limits (i.e., three values of $\widehat{\Lambda}$) are all from the Gumbel family, where the second one depends on all the constants $\hat{c}_j, 1 \le j \le k$, the third one depends only on the proportion of $\hat{c}_1 = c$, and the first one is not really affected by the more specific information of the trend functions.*
(b). After some algebric calculations for the normalizing functions, one can check that the result for S5 is consistent with the result of Theorem 3.5 in [19].

Below is the main result of this section.

Theorem 4.3. *Let $M_n, n \ge 1$ be the inhomogenous maxima defined as in* (1)*, with $\sigma = 1$, $c_i, i \ge 1$ satisfying* (11) *and* (12)*. We have*

(a). *Under S1,*

$$M_n \xrightarrow{d} \kappa_0, \qquad n \to \infty.$$

(b). *Under S2,*

(b.i) *If further $\lim_{n\to\infty} \frac{T_n^{H-H_0}}{\sqrt{2\log n}} = 0$, then*

$$\sigma_0^{-1} T_n^{-H_0}(M_n - b_n) \xrightarrow{d} \mathcal{N}, \qquad n \to \infty.$$

(b.ii) *If further $\lim_{n\to\infty} \frac{T_n^{H-H_0}}{\sqrt{2\log n}} = \infty$, then*

$$a_n^{-1}(M_n - b_n) \xrightarrow{d} \widehat{\Lambda}, \qquad n \to \infty.$$

(b.iii) *If further $\lim_{n\to\infty} \frac{T_n^{H-H_0}}{\sqrt{2\log n}} = q_0 \in (0,\infty)$, then*

$$a_n^{-1}(M_n - b_n) \xrightarrow{d} \widehat{\Lambda} + \frac{\sigma_0}{q_0}\mathcal{N}, \qquad n \to \infty,$$

with $\widehat{\Lambda}$ defined as in Proposition 4.1(b).

(c). *Under S3,*

(c.i) *If further $\beta > 2H - H_0$, then*

$$\sigma_0^{-1} T_n^{-H_0}(M_n - b_n) \xrightarrow{d} \mathcal{N}, \qquad n \to \infty.$$

(c.ii) *If further $\beta < 2H - H_0$, then*

$$a_n^{-1}(M_n - b_n) \xrightarrow{d} \Lambda + \log p_1, \qquad n \to \infty.$$

(c.iii) *If further $\beta = 2H - H_0$, then*

$$a_n^{-1}(M_n - b_n) \xrightarrow{d} \Lambda + \log p_1 + \frac{\sigma_0}{\tilde{s}_0^{\beta}}\mathcal{N}, \qquad n \to \infty.$$

(d). *Under S4 and S5,*

(d.i) *If further* $\beta > 2H - H_0$*, then*

$$\sigma_0^{-1} T_n^{-H_0}(M_n - d_n(x_0)) \xrightarrow{d} \mathcal{N}, \qquad n \to \infty.$$

(d.ii) *If further* $\beta < 2H - H_0$*, then*

$$e_n^{-1}(M_n - d_n(x_0)) \xrightarrow{d} \Lambda + \log p_1, \qquad n \to \infty.$$

(d.iii) *If further* $\beta = 2H - H_0$*, then*

$$e_n^{-1}(M_n - d_n(x_0)) \xrightarrow{d} \Lambda + \log p_1 + \frac{\sigma_0 c\beta}{H}\mathcal{N}, \qquad n \to \infty.$$

Remark 4.4. *In Theorem 4.3(b), we introduce mixture conditions according to the possible limit values of* $\lim_{n\to\infty} \frac{T_n^{H-H_0}}{\sqrt{2\log n}}$ *and* $\lim_{n\to\infty} T_n^{\beta-H}\sqrt{2\log n}$ *(see also Proposition 4.1(b)). However, not every combination of them is valid. In fact, it can be easily shown that if* $\lim_{n\to\infty} T_n^{\beta-H}\sqrt{2\log n} = q_1 \in (0,\infty)$ *holds then* $\lim_{n\to\infty} \frac{T_n^{H-H_0}}{\sqrt{2\log n}} = 0$*. Thus, we should not expect a result like in (b.ii) and (b.iii) involving* q_1 *and* $\hat{c}_j, 2 \le j \le k$ *in this situation. All other combinations may be possible, as illustrated in the next example.*

Example 4.5. *As in Example 3.4 this example illustrates Theorem 4.3 for* $T_n = (\lambda\sqrt{2\log n})^\gamma$*, with* $\gamma \in \mathbb{R}$ *and* $\lambda > 0$*. We choose to work with two representative cases. The first one is the same as in Example 3.4 where* $\beta > H_0 > 2H$*. In this case,*

$$-\infty < -\frac{1}{H} < -\frac{1}{H_0 - H} < -\frac{1}{\beta - H} < \frac{1}{\beta - H} < \infty.$$

We can show that the same eight convergence results are valid as those for M_n *in Example 3.4. For the second case, we consider* $H_0 < H < \beta < 2H - H_0$*. In this case,*

$$-\infty < -\frac{1}{\beta - H} < -\frac{1}{H} < \frac{1}{H - H_0} < \frac{1}{\beta - H} < \infty.$$

A direct appliation of Theorem 4.3 yeilds the following convergence results, as $n \to \infty$*:*

(1). *If* $\gamma \in \left(-\infty, -\frac{1}{H}\right)$*, then* $M_n \xrightarrow{d} 0$.
(2). *If* $\gamma = -\frac{1}{H}$*, then* $M_n \xrightarrow{d} \lambda^{-1}$.
(3). *If* $\gamma \in \left(-\frac{1}{H}, \frac{1}{H-H_0}\right)$*, then* $\sigma_0^{-1} T_n^{-H_0}(M_n - b_n) \xrightarrow{d} \mathcal{N}$.
(4). *If* $\gamma = \frac{1}{H-H_0}$*, then* $a_n^{-1}(M_n - b_n) \xrightarrow{d} \Lambda + \log p_1 + \sigma_0\lambda^{-1}\mathcal{N}$.
(5). *If* $\gamma \in \left(\frac{1}{H-H_0}, \frac{1}{\beta-H}\right)$*, then* $a_n^{-1}(M_n - b_n) \xrightarrow{d} \Lambda + \log p_1$.
(6). *If* $\gamma = \frac{1}{\beta-H}$ *and* $\lambda \in (0, \widetilde{t}_0^\beta)$*, then* $a_n^{-1}(M_n - b_n) \xrightarrow{d} \Lambda + \log p_1$.
(7). *If* $\gamma = \frac{1}{\beta-H}$ *and* $\lambda = \widetilde{t}_0^\beta$*, then* $e_n^{-1}(M_n - d_n(0)) \xrightarrow{d} \Lambda + \log p_1$.
(8). *If* $\gamma = \frac{1}{\beta-H}$ *and* $\lambda \in (\widetilde{t}_0^\beta, \infty)$*, or* $\gamma > \frac{1}{\beta-H}$*, then* $e_n^{-1}(M_n - d_n(\infty)) \xrightarrow{d} \Lambda + \log p_1$.

5. Further results and proofs

5.1. **Proof of Proposition 2.1:** We first discuss (i). It follows, by the self-similarity, that

$$\begin{aligned}
\psi_{T_u}(u) &= \mathbb{P}\left\{\sup_{t\in[0,1]} X_H(t) - cT_u^{\beta-H}t^\beta > uT_u^{-H}\right\} \\
&= \mathbb{P}\left\{\sup_{t\in[0,1]} \frac{X_H(t)}{1+ct^\beta T_u^\beta/u} > uT_u^{-H}\right\} \\
&= \mathbb{P}\left\{\sup_{t\in[0,1]} \frac{X_H(t)}{1+cT_u^\beta/u - c(1-t^\beta)T_u^\beta/u} > uT_u^{-H}\right\} \\
&= \mathbb{P}\left\{\sup_{t\in[0,1]} \frac{X_H(t)}{1-\frac{cT_u^\beta/u}{1+cT_u^\beta/u}(1-t^\beta)} > \frac{u+cT_u^\beta}{T_u^H}\right\}.
\end{aligned}$$

Note that

$$\lim_{u\to\infty} \frac{cT_u^\beta/u}{1+cT_u^\beta/u} = \frac{cs_0^\beta}{1+cs_0^\beta} = c_0 \ge 0.$$

Now we discuss the case $s_0 > 0$, implying $c_0 > 0$. We can easily see that, for any small $\varepsilon > 0$, the concerned quantity $\psi_{T_u}(u)$ lies between probaiblities

$$\mathbb{P}\left\{\sup_{t\in[0,1]} Z_{\pm\varepsilon}(t) > \frac{u+cT_u^\beta}{T_u^H}\right\}$$

for all large enough u, where

$$Z_{\pm\varepsilon}(t) = \frac{X_H(t)}{1-c_0(1\pm\varepsilon)(1-t^\beta)}, \quad t \ge 0.$$

Since $s_0 < t_0$, we have that, for any sufficiently small $\varepsilon > 0$, the variance function

$$\sigma^2_{Z_{\pm\varepsilon}}(t) := \mathrm{Var}(Z_{\pm\varepsilon}(t)) = \frac{t^{2H}}{(1-c_0(1\pm\varepsilon)(1-t^\beta))^2}, \quad t \ge 0,$$

attains its maximum at the unique point which is 1, and $\sigma_{Z_{\pm\varepsilon}}(1) = 1$. Further,

$$\begin{aligned}
1-\sigma_{Z_{\pm\varepsilon}}(t) &= 1 - \frac{t^H}{1-c_0(1\pm\varepsilon)(1-t^\beta)} \\
&= \frac{1-c_0(1\pm\varepsilon)(1-t^\beta)-t^H}{1-c_0(1\pm\varepsilon)(1-t^\beta)} \\
&= (H-c_0\beta(1\pm\varepsilon))(1-t)(1+o(1)), \quad t\uparrow 1.
\end{aligned}$$

Moreover, for the correlation function $r_{Z_{\pm\varepsilon}}(s,t)$ of $Z_{\pm\varepsilon}$, we have from (2) that

$$1-r_{Z_{\pm\varepsilon}}(s,t) = \frac{1}{2}K^2(|t-s|)(1+o(1)), \quad s,t\uparrow 1.$$

Noting that

$$\lim_{u\to\infty} \frac{u+cT_u^\beta}{T_u^H} = \lim_{u\to\infty} \frac{u^{H/\beta}}{T_u^H}\frac{u}{u^{H/\beta}}\left(1+c\frac{T_u^\beta}{u}\right) = \infty,$$

we can apply Theorem 2.1 of [5], where the γ defined therein is given by

$$\gamma = \lim_{t\downarrow 0} 2(H-c_0\beta(1\pm\varepsilon))\frac{t}{K^2(t)} = 2(H-c_0\beta(1\pm\varepsilon))Q.$$

Consequently, the claim in (i) for $s_0 > 0$ follows from an application of Theorem 2.1 in [5] and letting $\varepsilon \to 0$. Next, we discuss the case $s_0 = 0$, for which $c_0 = 0$. In this case, the concerned quantity $\psi_{T_u}(u)$ satisfies

$$\mathbb{P}\left\{\sup_{t\in[0,1]} X_H(t) > \frac{u + cT_u^\beta}{T_u^H}\right\} \le \psi_{T_u}(u) \le \mathbb{P}\left\{\sup_{t\in[0,1]} \frac{X_H(t)}{1-\varepsilon(1-t)} > \frac{u + cT_u^\beta}{T_u^H}\right\}$$

for all large enough u, with any small $\varepsilon > 0$. Therefore, it can be easily checked that the claim in (i) for $s_0 = 0$ follows similarly as the case $s_0 > 0$.

Now we consider (ii). The claim in (ii) follows by applying similar arguments as for Theorem 2 in [18]. We refer to the proof of Theorem 4.1 of [9] where if we set $S_v = 0$ therein we obtain the case discussed here for $x \in \mathbb{R}$. If $x = \infty$, we have, for any large $M > 0$,

$$T_u - t_0 u^{1/\beta} > M A^{1/2} B^{-1/2} u^{H/\beta + 1/\beta - 1}$$

holds for all large enough u. Thus, applying the result with $x = M < \infty$,

$$\Phi(M) \le \liminf_{u\to\infty} \frac{\psi_{T_u}(u)}{\psi_\infty(u)} \le \limsup_{u\to\infty} \frac{\psi_{T_u}(u)}{\psi_\infty(u)} \le 1.$$

Letting $M \to \infty$, we obtain the required asymptotics for the case $x = \infty$. This completes the proof.

5.2. **Proof of Proposition 3.1.** Consider (i). For any $x \in \mathbb{R}$, we have

$$\begin{aligned}
\mathbb{P}\left\{a_n^{-1}(\widetilde{M}_n - b_n) \le x\right\} &= \left(\mathbb{P}\left\{\sup_{t\in[0,T_n]} X_i(t) - ct^\beta \le b_n + a_n x\right\}\right)^n \\
&= \exp\left(n \log\left(1 - \mathbb{P}\left\{\sup_{t\in[0,T_n]} X_i(t) - ct^\beta > b_n + a_n x\right\}\right)\right)
\end{aligned}$$

From the assumptions we in fact know for any $x \in \mathbb{R}$

$$b_n + a_n x \sim b_n \sim (1 - c\widetilde{s}_0^\beta) T_n^H \sqrt{2\log n} \to \infty \qquad n \to \infty, \tag{14}$$

and thus

$$\lim_{n\to\infty} \mathbb{P}\left\{\sup_{t\in[0,T_n]} X_i(t) - ct^\beta > b_n + a_n x\right\} = 0.$$

If we can show that

$$\lim_{n\to\infty} n\mathbb{P}\left\{\sup_{t\in[0,T_n]} X_i(t) - ct > b_n + a_n x\right\} = e^{-x}, \tag{15}$$

then

$$\begin{aligned}
\mathbb{P}\left\{a_n^{-1}(\widetilde{M}_n - b_n) \le x\right\} &\sim \exp\left(-n\mathbb{P}\left\{\sup_{t\in[0,T_n]} X_i(t) - ct > b_n + a_n x\right\}\right) \\
&\sim \exp\left(-e^{-x}\right), \qquad n \to \infty,
\end{aligned}$$

which is the required result for (i). Next, we prove (15). Since

$$\lim_{n\to\infty} \frac{T_n}{(b_n + a_n x)^{1/\beta}} = \left(\frac{\widetilde{s}_0^\beta}{1 - c\widetilde{s}_0^\beta}\right)^{1/\beta} \in [0, t_0),$$

and

$$\frac{c\left(\frac{\widetilde{s}_0^\beta}{1-c\widetilde{s}_0^\beta}\right)}{1 + c\left(\frac{\widetilde{s}_0^\beta}{1-c\widetilde{s}_0^\beta}\right)} = c\widetilde{s}_0^\beta,$$

we obtain, by Proposition 2.1(i), that

$$\mathbb{P}\left\{\sup_{t\in[0,T_n]} X_i(t)-ct>b_n+a_nx\right\}\sim f_n(b_n+a_nx)\exp\left(-\frac{(b_n+a_nx+cT_n^\beta)^2}{2T_n^{2H}}\right),\qquad n\to\infty,$$

where $f_n(\cdot)$ is given in (8). Moreover, we have

$$\begin{aligned}\frac{b_n+a_nx+cT_n^\beta}{T_n^H} &= \sqrt{2\log n}+\frac{1}{\sqrt{2\log n}}\left[x+\log\left(f_n\left(T_n^H\sqrt{2\log n}-cT_n^\beta\right)\right)\right]\\ &= \sqrt{2\log n}+\frac{1}{\sqrt{2\log n}}\left[x+\log\left(\mathcal{D}_{c\widetilde{s}_0^\beta}\left(\sqrt{2\log n}\right)\left(\sqrt{2\log n}\right)^{-1}\right)\right]\\ &\sim \sqrt{2\log n},\qquad n\to\infty,\end{aligned}$$

and, by the regular variation property of $\mathcal{D}_{c\widetilde{s}_0^\beta}(\cdot)$,

$$\begin{aligned} f_n(b_n+a_nx) &= \mathcal{D}_{c\widetilde{s}_0^\beta}\left(\frac{b_n+a_nx+cT_n^\beta}{T_n^H}\right)\left(\frac{b_n+a_nx+cT_n^\beta}{T_n^H}\right)^{-1}\\ &\sim \mathcal{D}_{c\widetilde{s}_0^\beta}\left(\sqrt{2\log n}\right)\left(\sqrt{2\log n}\right)^{-1},\qquad n\to\infty. \end{aligned}\tag{16}$$

Thus, the claim in (15) follows by some elementary calculations.
Consider (ii). Following the same idea as in (i), we only need to show

$$\lim_{n\to\infty} n\mathbb{P}\left\{\sup_{t\in[0,T_n]} X_i(t)-ct>d_n(x_0)+e_nx\right\}=e^{-x}.\tag{17}$$

Next, we only focus on scenario S4, since similar arguments apply for scenario S5 as well. Note that, by the relations between A, t_0 and $\widetilde{t}_0$ given in (4) and in scenario S3,

$$\left(\widetilde{t}_0^\beta\sqrt{2\log n}\right)^{\frac{1}{\beta-H}}=t_0(2A^2\log n)^{\frac{1}{2(\beta-H)}},$$

and thus

$$\begin{aligned}&T_n-t_0(d_n(x_0)+e_nx)^{1/\beta}\\ &=\left(\widetilde{t}_0^\beta\sqrt{2\log n}\right)^{\frac{1}{\beta-H}}+A^{1/2}B^{-1/2}x_0(2A^2\log n)^{\frac{H+1-\beta}{2(\beta-H)}}(1+o(1))-t_0(2A^2\log n)^{\frac{1}{2(\beta-H)}}\\ &\quad-\frac{t_0A^2}{\beta-H}(2A^2\log n)^{\frac{2H-2\beta+1}{2(\beta-H)}}\left(\log\left(R\left(\left(2A^2\log n\right)^{1/\tau}\right)\Phi(x_0)\right)+x\right)(1+o(1))\\ &=A^{1/2}B^{-1/2}x_0(2A^2\log n)^{\frac{H+1-\beta}{2(\beta-H)}}(1+o(1))\\ &\quad-\frac{t_0A^2}{\beta-H}(2A^2\log n)^{\frac{2H-2\beta+1}{2(\beta-H)}}\left(\log\left(R\left(\left(2A^2\log n\right)^{1/\tau}\right)\Phi(x_0)\right)+x\right)(1+o(1)),\end{aligned}$$

as $n\to\infty$. This implies that, under the assumption of scenario S4,

$$\lim_{n\to\infty}\frac{T_n-t_0(d_n(x_0)+e_nx)^{1/\beta}}{A^{1/2}B^{-1/2}(d_n(x_0)+e_nx)^{H/\beta+1/\beta-1}}=x_0,$$

which also holds, with $x_0=\infty$, under the assumption of scenario S5. Hence, by Proposition 2.1(ii),

$$\mathbb{P}\left\{\sup_{t\in[0,T_n]} X_i(t)-ct>d_n(x_0)+e_nx\right\}\sim\Phi(x_0)R(d_n(x_0)+e_nx)\exp\left(-\frac{(d_n(x_0)+e_nx)^\tau}{2A^2}\right),\qquad n\to\infty,$$

with $R(\cdot)$ given in (6). Therefore, the claim in (17) follows by the regular variation property of $R(\cdot)$ and using some elementary calculations as follows

$$\begin{aligned}\frac{(d_n(x_0)+e_n x)^\tau}{2A^2} &= \log n\left(1+\frac{x+\log\left(R\left((2A^2\log n)^{1/\tau}\right)\Phi(x_0)\right)}{\tau\log n}\right)^\tau\\ &\sim \log n + x + \log\left(R\left((2A^2\log n)^{1/\tau}\right)\Phi(x_0)\right), \quad n\to\infty.\end{aligned}$$

The proof is compete.

5.3. **Proof of Theorem 3.2.** Consider (a). Note that

$$\widetilde{M}_n - \sup_{t\in[0,T_n]}(-\sigma_0 X(t)) \le M_n \le \widetilde{M}_n + \sup_{t\in[0,T_n]}\sigma_0 X(t),$$

and, since $\lim_{n\to\infty} T_n = 0$,

$$\lim_{n\to\infty}\sup_{t\in[0,T_n]}(-\sigma_0 X(t)) = \lim_{n\to\infty}\sup_{t\in[0,T_n]}\sigma_0 X(t) = 0, \qquad a.s.$$

Thus, it is sufficient to show that

$$\widetilde{M}_n \overset{d}{\to} \kappa_0. \tag{18}$$

Notice that, by an application of Theorem 2.1 in [5],

$$\mathbb{P}\left\{\sup_{t\in[0,1]} X_i(t) > u\right\} = \mathcal{D}_0(u)e^{-\frac{u^2}{2}}(1+o(1)), \quad u\to\infty, \tag{19}$$

where $\mathcal{D}_0(\cdot)$ is the regularly varying function given in (5). Defining

$$\mu_n := \sqrt{2\log n} + \frac{\log(\mathcal{D}_0(\sqrt{2\log n}))}{\sqrt{2\log n}}, \qquad n\in\mathbb{N},$$

we obtain, from (19) and Proposition 2.2 in [19], that

$$\sqrt{2\log n}\left(\max_{i\le n}\sup_{t\in[0,1]} X_i(t) - \mu_n\right) \overset{d}{\to} \Lambda, \qquad n\to\infty. \tag{20}$$

Next, we have

$$\max_{i\le n}\sup_{t\in[0,T_n]} X_i(t) - cT_n^\beta \le \widetilde{M}_n \le \max_{i\le n}\sup_{t\in[0,T_n]} X_i(t),$$

and by self-similarity

$$\begin{aligned}\max_{i\le n}\sup_{t\in[0,T_n]} X_i(t) &\overset{d}{=} T_n^H \max_{i\le n}\sup_{t\in[0,1]} X_i(t)\\ &= \frac{T_n^H}{\sqrt{2\log n}}\left(\sqrt{2\log n}\left(\max_{i\le n}\sup_{t\in[0,1]} X_i(t) - \mu_n\right)\right) + T_n^H\mu_n\\ &\overset{d}{\to} \kappa_0, \qquad n\to\infty.\end{aligned}$$

Therefore, the claim in (18) is established and thus the proof for (a) is complete.
Before we give the proof for (b) and (c), we shall derive some preliminary results presented in the following lemma.

Lemma 5.1. *We have, for any small enough $\varepsilon_0\in(0,1)$ and any $x\in\mathbb{R}$,* (i). *Under S2 and S3,*

$$\lim_{n\to\infty} \mathbb{P}\left\{a_n^{-1}\left(\max_{i\le n}\sup_{0\le t\le(1-\varepsilon_0)T_n} X_i(t)-ct^\beta-b_n\right)>x\right\}=0, \tag{21}$$

and

$$a_n^{-1}\left(\max_{i\le n}\sup_{(1-\varepsilon_0)T_n\le t\le T_n} X_i(t)-ct^\beta-b_n\right)\overset{d}{\to}\ \Lambda,\quad n\to\infty. \tag{22}$$

(ii). *Under S4,*

$$\lim_{n\to\infty} \mathbb{P}\left\{e_n^{-1}\left(\max_{i\le n}\sup_{0\le t\le(1-\varepsilon_0)T_n} X_i(t)-ct^\beta-d_n(x_0)\right)>x\right\}=0, \tag{23}$$

and

$$e_n^{-1}\left(\max_{i\le n}\sup_{(1-\varepsilon_0)T_n\le t\le T_n} X_i(t)-ct^\beta-d_n(x_0)\right)\overset{d}{\to}\ \Lambda,\quad n\to\infty. \tag{24}$$

Proof of Lemma 5.1: Consider (i). First, note that

$$\mathbb{P}\left\{a_n^{-1}\left(\max_{i\le n}\sup_{0\le t\le(1-\varepsilon_0)T_n} X_i(t)-ct^\beta-b_n\right)>x\right\}\le n\mathbb{P}\left\{\sup_{0\le t\le(1-\varepsilon_0)T_n} X_i(t)-ct^\beta>b_n+xa_n\right\}.$$

Similarly to the proof of Proposition 3.1, we have, by Proposition 2.1(i), that

$$\begin{aligned}\mathbb{P}\left\{\sup_{0\le t\le(1-\varepsilon_0)T_n} X_i(t)-ct^\beta>b_n+xa_n\right\}&=\mathcal{D}_{\widetilde{c}_0}\left(\frac{b_n+xa_n+cT_n^\beta(1-\varepsilon_0)^\beta}{T_n^H(1-\varepsilon_0)^H}\right)\left(\frac{b_n+xa_n+cT_n^\beta(1-\varepsilon_0)^\beta}{T_n^H(1-\varepsilon_0)^H}\right)^{-1}\\&\quad\times\exp\left(-\frac{(b_n+xa_n+cT_n^\beta(1-\varepsilon_0)^\beta)^2}{2T_n^{2H}(1-\varepsilon_0)^{2H}}\right)(1+o(1)),\end{aligned}$$

as $n\to\infty$, where

$$\widetilde{c}_0=\frac{c(1-\varepsilon_0)^\beta\left(\frac{\widetilde{s}_0^\beta}{1-c\widetilde{s}_0^\beta}\right)}{1+c(1-\varepsilon_0)^\beta\left(\frac{\widetilde{s}_0^\beta}{1-c\widetilde{s}_0^\beta}\right)}=\frac{c(1-\varepsilon_0)^\beta\widetilde{s}_0^\beta}{1-c\widetilde{s}_0^\beta+c(1-\varepsilon_0)^\beta\widetilde{s}_0^\beta}\ge 0. \tag{25}$$

Further, we have

$$\frac{(b_n+xa_n+cT_n^\beta(1-\varepsilon_0)^\beta)^2}{2T_n^{2H}(1-\varepsilon_0)^{2H}}\sim\left(\frac{1-c\widetilde{s}_0^\beta+c\widetilde{s}_0^\beta(1-\varepsilon_0)^\beta}{(1-\varepsilon_0)^H}\right)^2\log n,\qquad n\to\infty,$$

and, since $c\widetilde{s}_0^\beta<c\widetilde{t}_0^\beta=H/\beta$, we have for all sufficiently small $\varepsilon_0>0$,

$$\frac{1-c\widetilde{s}_0^\beta+c\widetilde{s}_0^\beta(1-\varepsilon_0)^\beta}{(1-\varepsilon_0)^H}>1.$$

These, together with the regular variation of $\mathcal{D}_{\widetilde{c}_0}(\cdot)$, imply that

$$\lim_{n\to\infty} n\mathbb{P}\left\{\sup_{0\le t\le(1-\varepsilon_0)T_n} X_i(t)-ct^\beta>b_n+xa_n\right\}=0.$$

Thus, (21) is established. Next, we prove (22). It is sufficient to show that, for any $x\in\mathbb{R}$,

$$\lim_{n\to\infty}\mathbb{P}\left\{a_n^{-1}\left(\max_{i\le n}\sup_{(1-\varepsilon_0)T_n\le t\le T_n} X_i(t)-ct^\beta-b_n\right)>x\right\}=\mathbb{P}\left\{\Lambda>x\right\}.$$

We have, for any $x \in \mathbb{R}$, by Proposition 3.1(i),

$$\begin{aligned}\mathbb{P}\left\{\Lambda > x\right\} &= \lim_{n\to\infty} \mathbb{P}\left\{a_n^{-1}\left(\widetilde{M}_n - b_n\right) > x\right\}\\ &\geq \limsup_{n\to\infty} \mathbb{P}\left\{a_n^{-1}\left(\max_{i\leq n} \sup_{(1-\varepsilon_0)T_n\leq t\leq T_n} X_i(t) - ct^\beta - b_n\right) > x\right\}.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\mathbb{P}\left\{a_n^{-1}\left(\widetilde{M}_n - b_n\right) > x\right\} &\leq \mathbb{P}\left\{a_n^{-1}\left(\max_{i\leq n} \sup_{0\leq t\leq (1-\varepsilon_0)T_n} X_i(t) - ct^\beta - b_n\right) > x\right\}\\ &\quad +\mathbb{P}\left\{a_n^{-1}\left(\max_{i\leq n} \sup_{(1-\varepsilon_0)T_n\leq t\leq T_n} X_i(t) - ct^\beta - b_n\right) > x\right\}.\end{aligned}$$

We have from (21) that the first term on the right-hand side tends to 0, as $n \to \infty$. Thus,

$$\mathbb{P}\left\{\Lambda > x\right\} \leq \liminf_{n\to\infty} \mathbb{P}\left\{a_n^{-1}\left(\max_{i\leq n} \sup_{(1-\varepsilon_0)T_n\leq t\leq T_n} X_i(t) - ct^\beta - b_n\right) > x\right\}.$$

This completes the proof for (i).
Consider (ii). First, since

$$T_n \sim \left(\widetilde{t}_0^\beta \sqrt{2\log n}\right)^{1/(\beta-H)} \sim t_0(d_n(x_0))^{1/\beta}, \qquad n \to \infty,$$

the claim in (23) follows from simialr arguments as Lemma 4.5 (see also Remark 4.6) in [19]. Next, similarly to the proof of (22) we can prove (24) by using Proposition 3.1(ii). This completes the proof of the lemma. □

Proof of Theorem 3.2 continued: Now, we are ready to continue the proof for Theorem 3.2 (b)-(c) below. We shall consider (b.i) and (b.ii)-(b.iii), respectively, followed by some arguments for (c).
Consider (b.i): We need to show that, for any $x \in \mathbb{R}$,

$$\mathbb{P}\left\{T_n^{-H_0}(M_n - b_n) > x\right\} \to \mathbb{P}\left\{\sigma_0 X(1) > x\right\}, \qquad n \to \infty.$$

For any small enough $\varepsilon_0 \in (0,1)$, we have

$$I_1(n,\varepsilon_0,x) \leq \mathbb{P}\left\{T_n^{-H_0}(M_n - b_n) > x\right\} \leq I_1(n,\varepsilon_0,x) + I_2(n,\varepsilon_0,x),$$

where

$$\begin{aligned}I_1(n,\varepsilon_0,x) &= \mathbb{P}\left\{T_n^{-H_0}\left(\max_{i\leq n} \sup_{(1-\varepsilon_0)T_n\leq t\leq T_n} X_i(t) + \sigma_0 X(t) - ct^\beta - b_n\right) > x\right\}\\ I_2(n,\varepsilon_0,x) &= \mathbb{P}\left\{T_n^{-H_0}\left(\max_{i\leq n} \sup_{0\leq t\leq (1-\varepsilon_0)T_n} X_i(t) + \sigma_0 X(t) - ct^\beta - b_n\right) > x\right\}.\end{aligned}$$

For $I_1(n,\varepsilon_0,x)$, we have

$$\begin{aligned}&\mathbb{P}\left\{T_n^{-H_0}\left(\max_{i\leq n} \sup_{(1-\varepsilon_0)T_n\leq t\leq T_n} X_i(t) - ct^\beta - b_n - \sup_{(1-\varepsilon_0)T_n\leq t\leq T_n}(-\sigma_0 X(t))\right) > x\right\}\\ &\leq I_1(n,\varepsilon_0,x)\\ &\leq \mathbb{P}\left\{T_n^{-H_0}\left(\max_{i\leq n} \sup_{(1-\varepsilon_0)T_n\leq t\leq T_n} X_i(t) - ct^\beta - b_n + \sup_{(1-\varepsilon_0)T_n\leq t\leq T_n} \sigma_0 X(t)\right) > x\right\}\end{aligned}$$

We derive by (22) and using the scenario assumption that,

$$\begin{aligned}
&T_n^{-H_0}\left(\max_{i\le n}\sup_{(1-\varepsilon_0)T_n\le t\le T_n} X_i(t)-ct^\beta-b_n\right)\\
&=a_n^{-1}\left(\max_{i\le n}\sup_{(1-\varepsilon_0)T_n\le t\le T_n} X_i(t)-ct^\beta-b_n\right)(T_n^{-H_0}a_n)\\
&\overset{d}{\to} 0,\quad n\to\infty.
\end{aligned}$$

Thus, by self-similarity, the independence of the Gaussian processes and the symmetry of normal distribution, we obtain

$$\begin{aligned}
&\mathbb{P}\left\{\sigma_0 X(1)>x\right\}=\lim_{\varepsilon_0\to 0}\mathbb{P}\left\{-\sup_{1-\varepsilon_0\le t\le 1}(-\sigma_0X(t))>x\right\}\\
&\le \lim_{\varepsilon_0\to0}\liminf_{n\to\infty} I_1(n,\varepsilon_0,x)\le \lim_{\varepsilon_0\to0}\limsup_{n\to\infty} I_1(n,\varepsilon_0,x)\\
&\le \lim_{\varepsilon_0\to0}\mathbb{P}\left\{\sup_{1-\varepsilon_0\le t\le 1}\sigma_0X(t)>x\right\}=\mathbb{P}\left\{\sigma_0X(1)>x\right\}.
\end{aligned}$$

To complete the proof, it is remains to show that

$$\lim_{n\to\infty} I_2(n,\varepsilon_0,x)=0. \tag{26}$$

Since

$$I_2(n,\varepsilon_0,x)\le \mathbb{P}\left\{T_n^{-H_0}\left(\max_{i\le n}\sup_{0\le t\le(1-\varepsilon_0)T_n}X_i(t)-ct^\beta-b_n\right)+\sup_{0\le t\le 1-\varepsilon_0}\sigma_0X(t)>x\right\}$$

and $\sup_{0\le t\le 1-\varepsilon_0}X(t)<\infty$ a.s., it is sufficient to show that, for any $x\in\mathbb{R}$,

$$\lim_{n\to\infty}\mathbb{P}\left\{T_n^{-H_0}\left(\max_{i\le n}\sup_{0\le t\le(1-\varepsilon_0)T_n}X_i(t)-ct^\beta-b_n\right)>x\right\}=0. \tag{27}$$

Next, note that $T_n^{H_0}=o(b_n)$, since otherwise, there exits a subsequence $\{n_k\}_{k\ge1}$ such that $T_{n_k}^{H_0}\ge CT_{n_k}^H\sqrt{2\log n_k}$ holds for some positive constant C. Then, T_{n_k} converges to ∞ and thus, $T_{n_k}^{1-H/\beta}/(\sqrt{2\log n_k})^{1/\beta}\ge CT_{n_k}^{1-H_0/\beta}\to\infty$, which is a contradiction with the assumption of scenarios S2 and S3. For the fixed ε_0 and x, we have by Proposition 2.1(i)

$$\begin{aligned}
\mathbb{P}\left\{\sup_{0\le t\le(1-\varepsilon_0)T_n}X_i(t)-ct>b_n+xT_n^{H_0}\right\}&=\mathcal{D}_{\widetilde{c}_0}\left(\frac{b_n+xT_n^{H_0}+cT_n^\beta(1-\varepsilon_0)^\beta}{T_n^H(1-\varepsilon_0)^H}\right)\left(\frac{b_n+xT_n^{H_0}+cT_n^\beta(1-\varepsilon_0)^\beta}{T_n^H(1-\varepsilon_0)^H}\right)^{-1}\\
&\quad\times\exp\left(-\frac{(b_n+xT_n^{H_0}+cT_n(1-\varepsilon_0))^2}{2T_n^{2H}(1-\varepsilon_0)^{2H}}\right)(1+o(1)),
\end{aligned}$$

as $n\to\infty$, with $\widetilde{c}_0$ given in (25). Similarly to the proof of (21), we can establish (27), and thus the proof for case (b.i) is finished.

<u>Consider (b.ii) and (b.iii):</u> We first introduce the following notation using the indicator function:

$$I_{\{\text{case (b.iii)}\}}=\begin{cases}1, & \text{if condition of case (b.iii) is satisfied, i.e., } \lim_{n\to\infty}\frac{T_n^{H-H_0}}{\sqrt{2\log n}}=q_0\in(0,\infty),\\ 0, & \text{otherwise .}\end{cases}$$

We need to show that, for any $x\in\mathbb{R}$,

$$\lim_{n\to\infty}\mathbb{P}\left\{a_n^{-1}(M_n-b_n)>x\right\}=\mathbb{P}\left\{\Lambda+\sigma_0/q_0X(1)I_{\{\text{case (b.iii)}\}}>x\right\}.$$

We will consider asymptotic upper and lower bounds, respectively. First, for any sufficiently small $\varepsilon_0 \in (0,1)$, we have

$$\begin{aligned}\mathbb{P}\left\{a_n^{-1}(M_n-b_n)>x\right\} \quad &\leq \quad \mathbb{P}\left\{a_n^{-1}\left(\max_{i\leq n}\sup_{(1-\varepsilon_0)T_n\leq t\leq T_n} X_i(t)+\sigma_0X(t)-ct^\beta-b_n\right)>x\right\}\\ &\quad + \mathbb{P}\left\{a_n^{-1}\left(\max_{i\leq n}\sup_{0\leq t\leq(1-\varepsilon_0)T_n} X_i(t)+\sigma_0X(t)-ct^\beta-b_n\right)>x\right\}. \qquad (28)\end{aligned}$$

For the second term above, we have, by self-similarity,

$$\begin{aligned}&\mathbb{P}\left\{a_n^{-1}\left(\max_{i\leq n}\sup_{0\leq t\leq(1-\varepsilon_0)T_n} X_i(t)+\sigma_0X(t)-ct^\beta-b_n\right)>x\right\}\\ &\leq \mathbb{P}\left\{a_n^{-1}\left(\max_{i\leq n}\sup_{0\leq t\leq(1-\varepsilon_0)T_n} X_i(t)-ct^\beta-b_n+\sup_{0\leq t\leq(1-\varepsilon_0)T_n}\sigma_0X(t)\right)>x\right\}\\ &= \mathbb{P}\left\{a_n^{-1}\left(\max_{i\leq n}\sup_{0\leq t\leq(1-\varepsilon_0)T_n} X_i(t)-ct^\beta-b_n\right)+a_n^{-1}T_n^{H_0}\sup_{0\leq t\leq 1-\varepsilon_0}\sigma_0X(t)>x\right\}.\end{aligned}$$

Under the assumption of cases (b.ii)-(b.iii), we have $\lim_{n\to\infty} a_n^{-1}T_n^{H_0}\sup_{0\leq t\leq 1-\varepsilon_0}\sigma_0X(t)<\infty$, and thus by an application of Lemma 5.1(i) we derive that the above term tends to 0 as $n\to\infty$. For the first term on the right-hand side of (28), we have

$$\begin{aligned}&\mathbb{P}\left\{a_n^{-1}\left(\max_{i\leq n}\sup_{(1-\varepsilon_0)T_n\leq t\leq T_n} X_i(t)+\sigma_0X(t)-ct^\beta-b_n\right)>x\right\}\\ &\leq \mathbb{P}\left\{a_n^{-1}\left(\max_{i\leq n}\sup_{(1-\varepsilon_0)T_n\leq t\leq T_n} X_i(t)-ct^\beta-b_n\right)+a_n^{-1}T_n^{H_0}\sup_{1-\varepsilon_0\leq t\leq 1}\sigma_0X(t)>x\right\}.\end{aligned}$$

Thus, by Lemma 5.1(i),

$$\begin{aligned}&\limsup_{n\to\infty}\mathbb{P}\left\{a_n^{-1}\left(\max_{i\leq n}\sup_{(1-\varepsilon_0)T_n\leq t\leq T_n} X_i(t)+\sigma_0X(t)-ct^\beta-b_n\right)>x\right\}\\ &\leq \lim_{\varepsilon_0\to 0}\mathbb{P}\left\{\Lambda+\sigma_0/q_0\sup_{1-\varepsilon_0\leq t\leq 1}X(t)I_{\{\text{case (b.iii)}\}}>x\right\}\\ &= \mathbb{P}\left\{\Lambda+\sigma_0/q_0X(1)I_{\{\text{case (b.iii)}\}}>x\right\},\end{aligned}$$

which gives the required upper bound. For the lower bound, we have for any small $\varepsilon_0 \in (0,1)$,

$$\begin{aligned}\mathbb{P}\left\{a_n^{-1}(M_n-b_n)>x\right\} &\geq \mathbb{P}\left\{a_n^{-1}\left(\max_{i\leq n}\sup_{(1-\varepsilon_0)T_n\leq t\leq T_n} X_i(t)+\sigma_0X(t)-ct^\beta-b_n\right)>x\right\}\\ &\geq \mathbb{P}\left\{a_n^{-1}\left(\max_{i\leq n}\sup_{(1-\varepsilon_0)T_n\leq t\leq T_n} X_i(t)-ct^\beta-b_n\right)-a_n^{-1}T_n^{H_0}\sup_{1-\varepsilon_0\leq t\leq 1}(-\sigma_0X(t))>x\right\}.\end{aligned}$$

By the same derivation as above, we obtain

$$\liminf_{n\to\infty}\mathbb{P}\left\{a_n^{-1}(M_n-b_n)>x\right\}\geq\mathbb{P}\left\{\Lambda+\sigma_0/q_0X(1)I_{\{\text{case (b.iii)}\}}>x\right\}.$$

Hence, the proof for (b.ii) and (b.iii) is finished.

Consider (c). The proof for scenario S4 follows from the same lines as those for (b) above, by noting that

$$\lim_{n\to\infty} e_n^{-1}T_n^{H_0}=\begin{cases}\infty, & \text{if } 2H-H_0<\beta,\\ 0, & \text{if } 2H-H_0>\beta,\\ c\beta/H, & \text{if } 2H-H_0=\beta.\end{cases}$$

Next, we prove the claim for scenario S5. We have, for any small $\varepsilon_0 > 0$,

$$T_n \geq (1+2\varepsilon_0)\left(\widetilde{t}_0^{\beta}\sqrt{2\log n}\right)^{1/(\beta-H)} \geq (1+\varepsilon_0)t_0(d_n(\infty))^{1/\beta}$$

holds for all large enough n. Since, for all large n,

$$\max_{i\leq n}\sup_{t\in\left[(1-\varepsilon_0)t_0(d_n(\infty))^{\frac{1}{\beta}},(1+\varepsilon_0)t_0(d_n(\infty))^{\frac{1}{\beta}}\right]}(X_i(t)+\sigma_0X(t)-ct) \leq M_n \leq \max_{i\leq n}\sup_{t\geq 0}(X_i(t)+\sigma_0X(t)-ct),$$

the proof follows from similar arguments as Theorem 3.1 in [19]. This finishes the proof for (c), and thus completes the proof of the theorem. □

5.4. **Proof of Proposition 4.1:** Consider (a). The proof follows similarly as that for Theorem 3.2(a). Consider (b) and (c). For any $x \in \mathbb{R}$, we have

$$\begin{aligned}\mathbb{P}\left\{a_n^{-1}(\widehat{M}_n-b_n)\leq x\right\} &= \prod_{i=1}^{n}\mathbb{P}\left\{\sup_{t\in[0,T_n]}X_i(t)-c_it^{\beta}\leq b_n+a_nx\right\}\\ &= \exp\left(\sum_{j=1}^{k}m_j\log\left(1-\mathbb{P}\left\{\sup_{t\in[0,T_n]}X_i(t)-\hat{c}_jt^{\beta}>b_n+a_nx\right\}\right)\right)\end{aligned}$$

By (14) we know

$$\lim_{n\to\infty}\mathbb{P}\left\{\sup_{t\in[0,T_n]}X_i(t)-\hat{c}_jt^{\beta}>b_n+a_nx\right\}=0$$

holds uniformly in $j=1,\cdots,k$. This implies, for any small $\varepsilon>0$, that $\mathbb{P}\left\{a_n^{-1}(\widehat{M}_n-b_n)\leq x\right\}$ lies between

$$\exp\left(-(1\pm\varepsilon)\sum_{j=1}^{k}m_j\mathbb{P}\left\{\sup_{t\in[0,T_n]}X_i(t)-\hat{c}_jt^{\beta}>b_n+a_nx\right\}\right)$$

for all large enough n. Similarly to the proof of Proposition 3.1, we obtain further asymptotic bounds for $\mathbb{P}\left\{a_n^{-1}(\widehat{M}_n-b_n)\leq x\right\}$ as follows

$$\exp\left(-(1\pm2\varepsilon)\sum_{j=1}^{k}m_jf_n(b_n+a_nx,\hat{c}_j)\exp\left(-\frac{(b_n+a_nx+\hat{c}_jT_n^{\beta})^2}{2T_n^{2H}}\right)\right), \tag{29}$$

where, with $\mathcal{D}_{c_0}(\cdot)$ defined in (5),

$$f_n(w,\hat{c}_j):=\mathcal{D}_{c\widetilde{s}_0^{\beta}}\left(\frac{w+\hat{c}_jT_n^{\beta}}{T_n^H}\right)\left(\frac{w+\hat{c}_jT_n^{\beta}}{T_n^H}\right)^{-1},\quad w>0. \tag{30}$$

Now, we focus on the summation in the exponent of (29). We write (recall $c=\hat{c}_1$)

$$\begin{aligned}I_1(n,x) &:= m_1f_n(b_n+a_nx,c)\exp\left(-\frac{(b_n+a_nx+cT_n^{\beta})^2}{2T_n^{2H}}\right)\\ I_2(n,x) &:= \sum_{j=2}^{k}m_jf_n(b_n+a_nx,\hat{c}_j)\exp\left(-\frac{(b_n+a_nx+\hat{c}_jT_n^{\beta})^2}{2T_n^{2H}}\right)\end{aligned}$$

Similarly to the proof of Proposition 3.1, we can obtain

$$I_1(n,x)\sim p_1e^{-x}>0,\qquad n\to\infty.$$

Next, we discuss $I_2(n,x)$. We have, uniformly in $j=2,\cdots,k$,

$$\begin{aligned}\frac{b_n+a_nx+\hat{c}_jT_n^\beta}{T_n^H} &= \sqrt{2\log n}+(\hat{c}_j-c)T_n^{\beta-H}+\frac{1}{\sqrt{2\log n}}\left[x+\log\left(f_n\left(T_n^H\sqrt{2\log n}-cT_n^\beta,\hat{c}_j\right)\right)\right]\\ &\sim (1+(\hat{c}_j-c)\widetilde{s}_0^\beta)\sqrt{2\log n},\quad n\to\infty, \qquad (31)\end{aligned}$$

and, by the regular variation property of $\mathcal{D}_{c\widetilde{s}_0^\beta}(\cdot)$, as $n\to\infty$,

$$\begin{aligned}f_n(b_n+a_nx,\hat{c}_j) &= \mathcal{D}_{c\widetilde{s}_0^\beta}\left(\frac{b_n+a_nx+\hat{c}_jT_n}{T_n^H}\right)\left(\frac{b_n+a_nx+\hat{c}_jT_n}{T_n^H}\right)^{-1}\\ &\sim \mathcal{D}_{c\widetilde{s}_0^\beta}\left((1+(\hat{c}_j-c)\widetilde{s}_0^\beta)\sqrt{2\log n}\right)\left((1+(\hat{c}_j-c)\widetilde{s}_0^\beta)\sqrt{2\log n}\right)^{-1}. \qquad (32)\end{aligned}$$

Below we consider the scenarios S2 and S3, seperately, to derive asymptotics for $I_2(n,x)$, as $n\to\infty$.
First consider S2, where $\widetilde{s}_0=0$. It follows that

$$\begin{aligned}\frac{(b_n+a_nx+\hat{c}_jT_n^\beta)^2}{2T_n^{2H}} &= \log n+\frac{1}{2}(\hat{c}_j-c)^2T_n^{2(\beta-H)}+(\hat{c}_j-c)T_n^{\beta-H}\sqrt{2\log n}\\ &\quad+\left(1+\frac{(\hat{c}_j-c)T_n^{\beta-H}}{\sqrt{2\log n}}\right)\left[x+\log\left(f_n\left(T_n^H\sqrt{2\log n}-cT_n^\beta,\hat{c}_j\right)\right)\right]\\ &\quad+\frac{1}{4\log n}\left[x+\log\left(f_n\left(T_n^H\sqrt{2\log n}-cT_n^\beta,\hat{c}_j\right)\right)\right]^2.\end{aligned}$$

Thus, some elementary calculations yield

$$I_2(n,x)\to\begin{cases}(1-p_1)e^{-x}, & \text{if } \lim_{n\to\infty}T_n^{\beta-H}\sqrt{2\log n}=0,\\ \sum_{j=2}^k p_je^{-(\hat{c}_j-c)q_1}e^{-x}, & \text{if } \lim_{n\to\infty}T_n^{\beta-H}\sqrt{2\log n}=q_1\in(0,\infty),\\ 0, & \text{if } \lim_{n\to\infty}T_n^{\beta-H}\sqrt{2\log n}=\infty,\end{cases}\qquad n\to\infty.$$

Next consider S3, where $\widetilde{s}_0\in(0,\widetilde{t}_0)$. It can be checked that, by (31) and (32),

$$I_2(n,x)\to 0,\qquad n\to\infty.$$

Now, inserting the above asymptotics for $I_1(n,x)$ and $I_2(n,x)$ into (29) we obtain the following bounds for all large n,

$$\exp\left(-(1+3\varepsilon)I(x)\right)\le\mathbb{P}\left\{a_n^{-1}(\widehat{M}_n-b_n)\le x\right\}\le\exp\left(-(1-3\varepsilon)I(x)\right)$$

where, under S2,

$$I(x)=\begin{cases}e^{-x}, & \text{if } \lim_{n\to\infty}T_n^{\beta-H}\sqrt{2\log n}=0,\\ \left(p_1+\sum_{j=2}^k p_je^{-(\hat{c}_j-c)q_1}\right)e^{-x}, & \text{if } \lim_{n\to\infty}T_n^{\beta-H}\sqrt{2\log n}=q_1\in(0,\infty),\\ p_1e^{-x}, & \text{if } \lim_{n\to\infty}T_n^{\beta-H}\sqrt{2\log n}=\infty.\end{cases}$$

and, under S3,

$$I(x)=p_1e^{-x}.$$

Therefore, the claims in (b) and (c) are established by letting $\varepsilon\to 0$.
Consider (d). Since the proof under S5 is similar to the proof under S4, we shall only present a proof under S4. Similarly as for (c) above, we conclude, for any small $\varepsilon>0$, that $\mathbb{P}\left\{e_n^{-1}(\widehat{M}_n-d_n(x_0))\le x\right\}$ lies between

$$\exp\left(-(1\pm\varepsilon)\sum_{j=1}^k m_j\mathbb{P}\left\{\sup_{t\in[0,T_n]}X_i(t)-\hat{c}_jt^\beta>d_n(x_0)+e_nx\right\}\right)$$

for all large enough n. Similarly to the proof of Proposition 3.1, by an application of Proposition 2.1(ii) we obtain asymptotic bounds for $\mathbb{P}\left\{e_n^{-1}(\widehat{M}_n - d_n(x_0)) \le x\right\}$ as follows

$$\exp\left(-(1\pm 2\varepsilon)\sum_{j=1}^{k} m_j\Phi(x_0)R(d_n(x_0)+e_nx)\exp\left(-\frac{(d_n(x_0)+e_nx)^\tau}{2A^2}\right)\right).$$

Note that $A = A(c)$ defined in (4), as a function of c, is strictly dicreasing. Thus, using similar arguments as before we can estiblish the result under S4. This completes the proof of the proposition.

5.5. **Proof of Theorem 4.3.** The proof follows by the same arguments as those for Theorem 3.2, for which we should use the Proposition 4.1 as a replacement of Proposition 3.1.

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