# SOJOURNS OF FRACTIONAL BROWNIAN MOTION QUEUES: TRANSIENT ASYMPTOTICS 

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#### Abstract

We study the asymptotics of sojourn time of the stationary queueing process $Q(t), t \geq 0$ fed by a fractional Brownian motion with Hurst parameter $H \in(0,1)$ above a high threshold $u$. For the Brownian motion case $H=1 / 2$, we derive the exact asymptotics of


$$
\mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}(Q(t)>u+h(u)) d t>x \mid Q(0)>u\right\}
$$

as $u \rightarrow \infty$, where $T_{1}, T_{2}, x \geq 0$ and $T_{2}-T_{1}>x$, whereas for all $H \in(0,1)$, we obtain sharp asymptotic approximations of

$$
\mathbb{P}\left\{\left.\frac{1}{v(u)} \int_{\left[T_{2}(u), T_{3}(u)\right]} \mathbb{I}(Q(t)>u+h(u)) d t>y \right\rvert\, \frac{1}{v(u)} \int_{\left[0, T_{1}(u)\right]} \mathbb{I}(Q(t)>u) d t>x\right\}, \quad x, y>0
$$

as $u \rightarrow \infty$, for appropriately chosen $T_{i}$ 's and $v$. Two regimes of the ratio between $u$ and $h(u)$, that lead to qualitatively different approximations, are considered.
Key Words: sojourn time; fractional Brownian motion; stationary queueing process; exact asymptotics; generalized Berman-type constants.

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## 1. Introduction

Fluid queueing systems with Gaussian-driven structure attained a substantial research interest over the last years; see, e.g., the monograph [1] and references therein. Following the seminal contributions [2-4] the class of fractional Brownian motions (fBm's) is a well legitimated model for the traffic stream in modern communication networks.
Let $B_{H}(t), t \in \mathbb{R}$ be a standard fBm with Hurst index $H \in(0,1)$, that is a Gaussian process with continuous sample paths, zero mean and covariance function satisfying

$$
2 \operatorname{Cov}\left(B_{H}(t), B_{H}(s)\right)=|s|^{2 H}+|t|^{2 H}-|t-s|^{2 H}, \quad s, t \in \mathbb{R} .
$$

Consider the fluid queue fed by $B_{H}$ and emptied with a constant rate $c>0$. Using the interpretation that for $s<t$, the increment $B_{H}(t)-B_{H}(s)$ models the amount of traffic that entered the system in
the time interval $[s, t)$, we define the workload process $Q(t), t \geq 0$ by

$$
\begin{equation*}
Q(t)=B_{H}(t)-c t+\max \left(Q(0),-\inf _{s \in[0, t]}\left(B_{H}(s)-c s\right)\right) \tag{1}
\end{equation*}
$$

The unique stationary solution to the above equation, that is the object of the analysis in this contribution, takes the following form (see. e.g., [1])

$$
\begin{equation*}
\{Q(t), t \geq 0\} \stackrel{d}{=}\left\{\sup _{s \geq t}\left(B_{H}(s)-B_{H}(t)-c(s-t)\right), t \geq 0\right\} \tag{2}
\end{equation*}
$$

The vast majority of the analysis of queueing models with Gaussian inputs deals with the asymptotic results, with particular focus on the asymptotics of the probability

$$
\mathbb{P}\{Q(t)>u\}
$$

as $u \rightarrow \infty$, see e.g., [1, 2, 5-8]. Much less is known about transient characteristics of $Q$, such as

$$
\mathbb{P}\{Q(T)>\omega \mid Q(0)>u\}
$$

with a notable exception for the Brownian motion $(H=1 / 2)$. In particular, in view of [9], see also related works [10-12], it is known that for $H=1 / 2$ and $u, \omega, T>0$

$$
\begin{equation*}
\mathbb{P}\{Q(T)>\omega \mid Q(0)=u\}=\Phi\left(\frac{u-\omega-c T}{\sqrt{T}}\right)+e^{-2 c \omega} \Phi\left(\frac{\omega+u-c T}{\sqrt{T}}\right) \tag{3}
\end{equation*}
$$

and

$$
\begin{align*}
\mathbb{P}\{Q(T)>\omega \mid Q(0)>u\}= & -e^{2 u c} \Phi\left(\frac{-\omega-u-c T}{\sqrt{T}}\right)+e^{-2 c(\omega-u)} \Phi\left(\frac{\omega-u-c T}{\sqrt{T}}\right)  \tag{4}\\
& +\Phi\left(\frac{u-\omega-c T}{\sqrt{T}}\right)+e^{-2 c \omega} \Phi\left(\frac{\omega+u-c T}{\sqrt{T}}\right),
\end{align*}
$$

where $\Phi(\cdot)$ denotes the distribution function of a standard Gaussian random variable. Since $Q(0)$ is exponentially distributed for $H=1 / 2$, (3)-(4) lead to explicit formula for $\mathbb{P}\{Q(0)>u, Q(T)>\omega\}$, which compared with $\mathbb{P}\{Q(0)>u\} \mathbb{P}\{Q(T)>\omega\}$ gives some insight to the dependence structure of the workload process $Q(t), t \geq 0$. Since the general case $H \in(0,1)$ is very complicated, the findings available in the literature concern mainly large deviation-type results; see e.g., [13] where the asymptotics of

$$
\ln (\mathbb{P}\{Q(0)>p u, Q(T u)>q u\}), \quad u \rightarrow \infty
$$

was derived for $H \in(0,1)$. See also [14] for corresponding results the many-source model.
In addition to the conditional probability (4), it is also interesting to know how much time the queue spends above a given threshold during a given time period. This motivates us to consider the following quantity

$$
\begin{equation*}
\mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}(Q(t)>\omega) d t>x \mid Q(0)>u\right\}, \quad x \in\left[0, T_{2}-T_{1}\right) \tag{5}
\end{equation*}
$$

for given non-negative $T_{1}<T_{2}$.

In Section 2, for $H=1 / 2$ we derive exact asymptotics of the above conditional sojourn time by letting $u, \omega=\omega(u) \rightarrow \infty$ in an appropriate way. Specifically, we shall distinguish between two regimes that lead to qualitatively different results:
(i) small fluctuation regime: $\omega=u+w+o(1), w \in \mathbb{R}$, for which the asymptotics of (5) tends to a positive constant as $u \rightarrow \infty$;
(ii) large fluctuation regime: $\omega=(1+a) u+o(u), a \in(-1, \infty)$, for which (5) tends to 0 as $u \rightarrow \infty$ with the speed controlled by $a$
see Propositions 2.1, 2.2 respectively.
Then, in Section 3 for all $H \in(0,1)$ and $x, y$ non-negative we shall investigate approximations, as $u \rightarrow \infty$, of the following conditional sojourn times probabilities
(6) $\mathscr{P}_{T_{1}, T_{2}, T_{3}}^{x, y}(\omega, u):=\mathbb{P}\left\{\frac{1}{v(u)} \int_{\left[T_{2}(u), T_{3}(u)\right]} \mathbb{I}(Q(t)>\omega) d t>\left.y\right|_{\frac{1}{v(u)}} \int_{\left[0, T_{1}(u)\right]} \mathbb{I}(Q(t)>u) d t>x\right\}$,
where $T_{i}(u), i=1,2,3$ and $\omega=u+h(u), v(u)$ are suitably chosen functions, see assumption (T). In Theorem 3.1, complementing the findings of Proposition 2.1, we shall determine

$$
\begin{equation*}
\lim _{u \rightarrow \infty} \mathscr{P}_{T_{1}, T_{2}, T_{3}}^{x, y}\left(u+a u^{2 H-1}, u\right) \tag{7}
\end{equation*}
$$

under some asymptotic restrictions on $T_{i}(u)$ 's and $a \in \mathbb{R}$, which yield a positive and finite limit. The idea of its proof is based on a modification of recently developed extension of the uniform double-sum technique for functionals of Gaussian processes [15]. Then, in Theorem 3.3 we shall obtain approximations of $\mathscr{P}_{T_{1}, T_{2}, T_{3}}^{x, y}((1+a) u, u)$ as $u \rightarrow \infty$, which correspond to the results derived in Proposition 2.2. The main findings of this section go in line with recently derived asymptotics for

$$
\mathbb{P}\left\{\frac{1}{v(u)} \int_{\left[0, T_{1}(u)\right]} \mathbb{I}(Q(t)>u) d t>x\right\}
$$

as $u \rightarrow \infty$, see [15].
Structure of the paper: Section 2 is devoted to the analysis of the exact asymptotics of (5) for the classical model of the Brownian-driven queue, while in Section 3 we shall investigate asymptotic properties of $\mathscr{P}_{T_{1}, T_{2}, T_{3}}^{x, y}(\omega, u)$ for $H \in(0,1)$. Proofs of all the results are deferred to Section 4 and Appendix.

## 2. Preliminary results

In this section we shall focus on the exact asymptotics of (5) for the queueing process (2) driven by the Brownian motion.
Let in the following for $T_{2}-T_{1}>x \geq 0$ and $w \in \mathbb{R}$

$$
\begin{equation*}
\mathcal{C}\left(T_{1}, T_{2}, x ; w\right)=2 c \int_{-\infty}^{w} e^{2 c y} \mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}\left(B_{1 / 2}(t)-c t>y\right) d t>x\right\} d y \in(0, \infty) . \tag{8}
\end{equation*}
$$

We begin with a small fluctuation result concerning the case when the distance between $u$ and $\omega=\omega(u)$ in (5) is asymptotically constant. Below the term $o(1)$ is means for $u \rightarrow \infty$.

Proposition 2.1. If $H=1 / 2$ and $T_{2}-T_{1}>x \geq 0$, then for $\omega(u)=u+w+o(1), w \in \mathbb{R}$

$$
\mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}(Q(t)>\omega(u)) d t>x \mid Q(0)>u\right\} \sim e^{-2 c w} \mathcal{C}\left(T_{1}, T_{2}, x ; w\right)
$$

as $u \rightarrow \infty$.
Next, we consider the large fluctuation scenario, i.e., $\omega=\omega(u)$ in (5) is asymptotically proportional to $u$.

Proposition 2.2. Suppose that $H=1 / 2, T_{2}-T_{1}>x \geq 0$ and $\omega(u)=(1+a) u+o(u)$.
(i) If $a \in(-1,0)$, then as $u \rightarrow \infty$

$$
\mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}(Q(t)>\omega(u)) d t>x \mid Q(0)>u\right\} \sim 1 .
$$

(ii) If $a>0$, then as $u \rightarrow \infty$

$$
\mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}(Q(t)>\omega(u)) d t>x \mid Q(0)>u\right\} \sim e^{-2 c(\omega(u)-u)} \mathcal{C}\left(T_{1}, T_{2}, x ; \infty\right) .
$$

Both (i) and (ii) in Proposition 2.2 also hold if $T_{1}, T_{2}$ depend on $u$ in such a way that as $u \rightarrow \infty$, these converge to positive constants $T_{1}<T_{2}$ with $T_{2}-T_{1}>x \geq 0$.

## 3. Main results

This section is devoted to the asymptotic analysis of (6) for the queueing process $Q$ defined in (2) with fBm input $B_{H}, H \in(0,1)$. Before proceeding to the main results of this contribution, we introduce some notation and assumptions. Let $W_{H}(t)=\sqrt{2} B_{H}(t)-|t|^{2 H}, t \in \mathbb{R}$ and define for

$$
x \geq 0, y \geq 0
$$

and $\lambda \in \mathbb{R}, \mathscr{T}_{1}>0,0<\mathscr{T}_{2}<\mathscr{T}_{3}<\infty$

$$
\overline{\mathcal{B}}_{H}^{x, y}\left(\mathscr{T}_{1} ; \lambda, \mathscr{T}_{2}, \mathscr{T}_{3}\right)=\int_{\mathbb{R}} e^{z} \mathbb{P}\left\{\int_{\left[0, \mathscr{T}_{1}\right]} \mathbb{I}\left(W_{H}(t)>z\right) d t>x, \int_{\left[\mathscr{T}_{2}, \mathscr{T}_{3}\right]} \mathbb{I}\left(W_{H}(t)>z+\lambda\right) d t>y\right\} d z
$$

and set

$$
\overline{\mathcal{B}}_{H}^{x}\left(\mathscr{T}_{1}\right)=\int_{\mathbb{R}} e^{z} \mathbb{P}\left\{\int_{\left[0, \mathscr{\mathscr { T }}_{1}\right]} \mathbb{I}\left(W_{H}(t)>z\right) d t>x\right\} d z
$$

Further, given $H \in(0,1), c>0, u>0$ let

$$
\begin{equation*}
A=\left(\frac{H}{c(1-H)}\right)^{-H} \frac{1}{1-H}, \quad t^{*}=\frac{H}{c(1-H)}, \quad \Delta(u)=2^{\frac{1}{2 H}} t^{*} A^{-\frac{1}{H}} u^{-\frac{1-H}{H}} \tag{9}
\end{equation*}
$$

and set

$$
v(u)=u \Delta(u) .
$$

In the rest of this section, for a given function $h$, we analyse the asymptotics of $\left.\mathscr{P}_{T_{1}, T_{2}, T_{3}}^{x, y}(\omega)(u), u\right)$ defined in (6) with $\omega(u)=u+h(u)$ as $u \rightarrow \infty$, where $T_{i}(u)$ 's depend on $u$ in such a way that
(T) $\lim _{u \rightarrow \infty} \frac{T_{i}(u)}{v(u)}=\mathscr{T}_{i} \in(0, \infty)$, for $i=1,2,3$ with $\mathscr{T}_{1}>x$ and $\mathscr{T}_{3}-\mathscr{T}_{2}>y$
is satisfied.
We note in passing that for $H=1 / 2, v(u)=u \Delta(u)=2^{\frac{1}{2 H}} t^{*} A^{-\frac{1}{H}}$ is a constant. Hence, under (T), we have $T_{i}(u) \rightarrow \mathscr{C}_{i} \in(0, \infty)$. Thus ( $\mathbf{T}$ ) included the model considered in Propositions 2.1 and 2.2. We shall consider two scenarios that depend on the relative size of $h(u)$ with respect to $u$ :
$\diamond$ small fluctuation case: $|h(u)|$ is relatively small with respect to $u$, i.e., $h(u)=\lambda u^{2 H-1}$ with $\lambda \in \mathbb{R}$ and $H \in(0,1)$, which leads to $\lim _{u \rightarrow \infty} \mathscr{P}_{T_{1}, T_{2}, T_{3}}^{x, y}(u+h(u), u)>0 ;$
$\diamond$ large fluctuation case: $h(u)=a u$ is proportional to $u$, which leads to $\mathscr{P}_{T_{1}, T_{2}, T_{3}}^{x, y}(u+h(u), u) \rightarrow 0$ if $h(u)>0$ and $\mathscr{P}_{T_{1}, T_{2}, T_{3}}^{x, y}(u+h(u), u) \rightarrow 1$ if $h(u)<0$ as $u \rightarrow \infty$.
Small fluctuation regime. We begin with the case when $h(u)$ is relatively small with comparison to $u$ and thus the conditional probability $\mathscr{P}_{T_{1}, T_{2}, T_{3}}^{x, y}(u+h(u), u)$ is cut away from 0 , as $u \rightarrow \infty$.

Theorem 3.1. If ( $\mathbf{T}$ ) holds, then with $Q$ defined in (2) and $\lambda \in \mathbb{R}$

$$
\begin{equation*}
\lim _{u \rightarrow \infty} \mathscr{P}_{T_{1}, T_{2}, T_{3}}^{x, y}\left(u+\frac{\lambda}{A^{2}(1-H)} u^{2 H-1}, u\right)=\frac{\overline{\mathcal{B}}_{H}^{x, y}\left(\mathscr{T}_{1} ; \lambda, \mathscr{T}_{2}, \mathscr{T}_{3}\right)}{\overline{\mathcal{B}}_{H}^{x}\left(\mathscr{T}_{1}\right)} \in(0, \infty) . \tag{10}
\end{equation*}
$$

Remark 3.2. (i) In the case of Brownian motion with $H=1 / 2$, function $v(u)=1 /\left(2 c^{2}\right)$ does not depend on $u$ and the above reads

$$
\begin{equation*}
\lim _{u \rightarrow \infty} \mathscr{P}_{T_{1}, T_{2}, T_{3}}^{x, y}\left(u+\frac{\lambda}{2 c}, u\right)=\frac{\overline{\mathcal{B}}_{H}^{x, y}\left(\mathscr{T}_{1} ; \lambda, \mathscr{T}_{2}, \mathscr{T}_{3}\right)}{\overline{\mathcal{B}}_{H}^{x}\left(\mathscr{T}_{1}\right)} \in(0, \infty) . \tag{11}
\end{equation*}
$$

Since $v(u)$ is constant in this case, we can take $T_{i}=2 c^{2} \mathscr{T}_{i}>0, i \leq 3$ in (11). In the particular case that $x=0, H=1 / 2$ we have

$$
\begin{aligned}
& \lim _{u \rightarrow \infty} \mathbb{P}\left\{\int_{\left[T_{2}, T_{3}\right]} \mathbb{I}\left(Q(t)>u+\frac{\lambda}{2 c}\right) d t>\left.2 c^{2} y\right|_{t \in\left[0, T_{1}\right]} Q(t)>u\right\}=\frac{\overline{\mathcal{B}}_{H}^{0, y}\left(\mathscr{T}_{1} ; \lambda, \mathscr{T}_{2}, \mathscr{T}_{3}\right)}{\overline{\mathcal{B}}_{H}^{0}\left(\mathscr{T}_{1}\right)} \in(0, \infty) \\
& \quad \text { and taking } y=0 \text { yields } \\
& \quad \lim _{u \rightarrow \infty} \mathbb{P}\left\{\sup _{t \in\left[T_{2}, T_{3}\right]} Q(t)>u+\left.\frac{\lambda}{2 c}\right|_{t \in\left[0, T_{1}\right]} Q(t)>u\right\}=\frac{\overline{\mathcal{B}}_{H}^{0,0}\left(\mathscr{T}_{1} ; \lambda, \mathscr{T}_{2}, \mathscr{T}_{3}\right)}{\overline{\mathcal{B}}_{H}^{0}\left(\mathscr{T}_{1}\right)} \in(0, \infty) .
\end{aligned}
$$

(ii) It follows from Theorem 3.1 that for $h(u)=o\left(u^{2 H-1}\right)$

$$
\begin{equation*}
\lim _{u \rightarrow \infty} \mathscr{P}_{T_{1}, T_{2}, T_{3}}^{x, y}(u+h(u), u)=\frac{\overline{\mathcal{B}}_{H}^{x, y}\left(\mathscr{T}_{1} ; 0, \mathscr{T}_{2}, \mathscr{T}_{3}\right)}{\overline{\mathcal{B}}_{H}^{x}\left(\mathscr{T}_{1}\right)} \in(0, \infty) . \tag{12}
\end{equation*}
$$

Notably, if $H \in(1 / 2,1)$, then $T_{i}(u) \sim \mathscr{T}_{i} u^{(2 H-1) / H}$ as $u \rightarrow \infty$ for $i=1,2,3$. Hence $\lim _{u \rightarrow \infty}\left(T_{2}(u)-T_{1}(u)\right)=\infty$ and one can take $h(u) \rightarrow \infty$, as $u \rightarrow \infty$. Thus, the insensitivity of limit (12) on $h(u)$ is yet another manifestation of the long range dependence property of $Q$ inherited from the input process $B_{H}$. This observation goes in line with the Piterbarg property

$$
\lim _{u \rightarrow \infty} \frac{\mathbb{P}\left\{\sup _{t \in[0, T(u)]} Q(t)>u\right\}}{\mathbb{P}\{Q(0)>u\}}=1
$$

derived in [16] and the strong Piterbarg property see [17], namely

$$
\lim _{u \rightarrow \infty} \frac{\mathbb{P}\left\{\inf _{t \in[0, T(u)]} Q(t)>u\right\}}{\mathbb{P}\{Q(0)>u\}}=1
$$

where $T(u)=o\left(u^{(2 H-1) / H}\right)$ as $u \rightarrow \infty$.
Large fluctuation regime. Suppose next that $h(u)=a u, a \neq 0$. It appears that in this case the fluctuation $h(u)$ substantially influences the asymptotics of $\mathscr{P}_{T_{1}, T_{2}, T_{3}}^{x, y}(u+h(u), u)$ as $u \rightarrow \infty$. We point out the lack of symmetry with respect to the sign of $a$ in the results given in the following theorem, which is due to the non-reversibility in time of the queueing process $Q$, i.e., the fact that

$$
\mathbb{P}\{Q(s)>u, Q(t)>v\} \neq \mathbb{P}\{Q(t)>u, Q(s)>v\}
$$

for $u \neq v$.
Theorem 3.3. Let $Q$ be defined in (2) and set $\tilde{a}=(1+a)^{(1-2 H) / H}$. Suppose that (T) holds.
(i) If $a \in(-1,0)$, then

$$
\lim _{u \rightarrow \infty} \mathscr{P}_{T_{1}, T_{2}, T_{3}}^{x, y}((1+a) u, u)=1
$$

(ii) If $a>0$, then

$$
\limsup _{u \rightarrow \infty} \frac{\mathscr{P}_{T_{1}, T_{2}, T_{3}}^{x, y}((1+a) u, u)}{\exp \left(-\frac{A^{2}\left((1+a)^{2-2 H}-1\right)}{2} u^{2-2 H}\right)} \leq \tilde{a}^{1-H} \frac{\overline{\mathcal{B}}_{H}^{\tilde{a} y}\left(\tilde{a}\left(\mathscr{T}_{3}-\mathscr{T}_{2}\right)\right)}{\overline{\mathcal{B}}_{H}^{x}\left(\mathscr{T}_{1}\right)}
$$

and

$$
\liminf _{u \rightarrow \infty} \frac{\mathscr{P}_{T_{1}, T_{2}, T_{3}}^{x, y}((1+a) u, u)}{\exp \left(-\frac{A^{2}\left((1+a)^{2-2 H}-1\right)}{2} u^{2-2 H}\right)} \geq \tilde{a}^{1-H} \frac{\overline{\mathcal{B}}_{H}^{\tilde{a} x, \tilde{a} y}\left(\tilde{a} \mathscr{T}_{1} ; 0, \tilde{a} \mathscr{T}_{2}, \tilde{a} \mathscr{T}_{3}\right)}{\overline{\mathcal{B}}_{H}^{x}\left(\mathscr{T}_{1}\right)} .
$$

Remark 3.4. Theorem 3.3 straightforwardly implies that

$$
\lim _{u \rightarrow \infty} \frac{\ln \left(\mathscr{P}_{T_{1}, T_{2}, T_{3}}^{x, y}((1+a) u, u)\right)}{u^{2-2 H}}=-\frac{1}{2} A^{2}\left((1+a)^{2-2 H}-1\right), \quad \forall a>0
$$

## 4. Proofs

In this section we present detailed proofs of Proposition 2.1, 2.2 and Theorem 3.1, 3.3.
4.1. Proof of Proposition 2.1. Recall that by (1)

$$
Q(t)=B_{1 / 2}(t)-c t+\max \left(Q(0),-\inf _{s \in[0, t]}\left(B_{1 / 2}(s)-c s\right)\right)
$$

where $Q(0)$ is independent of $B_{1 / 2}(t)-c t$ and $\inf _{s \in[0, t]}\left(B_{1 / 2}(s)-c s\right)$ for $t>0$. By [18, Eq. (5)] we have

$$
\begin{equation*}
\mathbb{P}\{Q(0)>u\}=\mathbb{P}\left\{\sup _{t \geq 0}\left(B_{1 / 2}(t)-c t\right)>u\right\}=e^{-2 c u}, \quad u \geq 0 \tag{13}
\end{equation*}
$$

Hence it suffices to analyse

$$
\mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}(Q(t)>\omega(u)) d t>x, Q(0)>u\right\} .
$$

We note first that

$$
\begin{align*}
& \mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}(Q(t)>\omega(u)) d t>x, Q(0)>u\right\} \\
& \quad \geq \mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}\left(Q(0)+B_{1 / 2}(t)-c t>\omega(u)\right) d t>x, Q(0)>u\right\} \tag{14}
\end{align*}
$$

Moreover, we have

$$
\begin{align*}
\mathbb{P} & \left\{\int_{T_{1}}^{T_{2}} \mathbb{I}(Q(t)>\omega(u)) d t>x, Q(0)>u\right\} \\
= & \mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}(Q(t)>\omega(u)) d t>x, Q(0)>u, \sup _{s \in\left[0, T_{2}\right]}\left(c s-B_{1 / 2}(s)\right) \leq u\right\} \\
& +\mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}(Q(t)>\omega(u)) d t>x, Q(0)>u, \sup _{s \in\left[0, T_{2}\right]}\left(c s-B_{1 / 2}(s)\right)>u\right\} \\
= & P_{1}(u)+P_{2}(u) . \tag{15}
\end{align*}
$$

For $P_{1}(u)$ we have the following upper bound

$$
P_{1}(u) \leq \mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}\left(Q(0)+B_{1 / 2}(t)-c t>\omega(u)\right) d t>x, Q(0)>u\right\}
$$

and for $P_{2}(u)$ by Borell-TIS inequality (see, e.g., [19])

$$
\begin{equation*}
P_{2}(u) \leq \mathbb{P}\left\{\sup _{s \in\left[0, T_{2}\right]}\left(c s-B_{1 / 2}(s)\right)>u\right\} \leq e^{-C u^{2}} \tag{16}
\end{equation*}
$$

for some $C>0$ and sufficiently large $u$.
Next, we note that

$$
\begin{aligned}
& \mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}\left(Q(0)+B_{1 / 2}(t)-c t>\omega(u)\right) d t>x, Q(0)>u\right\} \\
& \quad=2 c \int_{u}^{\infty} e^{-2 c y} \mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}\left(y+B_{1 / 2}(t)-c t>\omega(u)\right) d t>x\right\} d y \\
& \quad=2 c e^{-2 c \omega(u)} \int_{u-\omega(u)}^{\infty} e^{-2 c y} \mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}\left(B_{1 / 2}(t)-c t>-y\right) d t>x\right\} d y \\
& \quad=2 c e^{-2 c \omega(u)} \int_{-\infty}^{\omega(u)-u} e^{2 c y} \mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}\left(B_{1 / 2}(t)-c t>y\right) d t>x\right\} d y \\
& \quad=e^{-2 c \omega(u)} \mathcal{C}\left(T_{1}, T_{2}, x ; \omega(u)-u\right),
\end{aligned}
$$

where $\mathcal{C}\left(T_{1}, T_{2}, x ; z\right)$ is defined in (8). Hence, by (16) applied to (14) and (15), we arrive at

$$
\begin{equation*}
\mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}(Q(t)>\omega(u)) d t>x, Q(0)>u\right\} \sim e^{-2 c u} e^{-2 c w} \mathcal{C}\left(T_{1}, T_{2}, x ; w\right) \tag{17}
\end{equation*}
$$

as $u \rightarrow \infty$. Finally, by (13) we get

$$
\mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}(Q(t)>\omega(u)) d t>x \mid Q(0)>u\right\} \sim e^{-2 c w} \mathcal{C}\left(T_{1}, T_{2}, x ; w\right)
$$

as $u \rightarrow \infty$. This completes the proof.
4.2. Proof of Proposition 2.2. The idea of the proof is the same as the proof of Proposition 2.1. Since by Borell-TIS inequality

$$
\begin{aligned}
\mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}\left(B_{1 / 2}(t)-c t>y\right) d t>x\right\} & \leq \mathbb{P}\left\{\sup _{t \in\left[T_{1}, T_{2}\right]}\left(B_{1 / 2}(t)-c t\right)>y\right\} \\
& \leq C_{1} \exp \left(-C_{2} y^{2}\right)
\end{aligned}
$$

for some positive constants $C_{1}, C_{2}$, we conclude that

$$
\mathcal{C}\left(T_{1}, T_{2}, x ; \infty\right)=\int_{-\infty}^{\infty} e^{2 c y} \mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}\left(B_{1 / 2}(t)-c t>y\right) d t>x\right\} d y<\infty .
$$

Thus if $a \in(-1,0)$, then as $u \rightarrow \infty$

$$
\begin{aligned}
& \mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}(Q(t)>\omega(u)) d t>x, Q(0)>u\right\} \\
& \sim 2 c e^{-2 c \omega(u)} \int_{-\infty}^{\omega(u)-u} e^{2 c y} \mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}\left(B_{1 / 2}(t)-c t>y\right) d t>x\right\} d y \\
& \sim e^{-2 c u},
\end{aligned}
$$

where we used that uniformly for $y \in(-\infty, \omega(u)-u]$

$$
\lim _{u \rightarrow \infty} \mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}\left(B_{1 / 2}(t)-c t>y\right) d t>x\right\}=1
$$

Similarly for $a>0$, we have that

$$
\mathbb{P}\left\{\int_{T_{1}}^{T_{2}} \mathbb{I}(Q(t)>\omega(u)) d t>x, Q(0)>u\right\} \sim e^{-2 c \omega(u)} \mathcal{C}\left(T_{1}, T_{2}, x ; \infty\right)
$$

as $u \rightarrow \infty$. Thus, combining the above with (13), we complete the proof.
4.3. Proof of Theorem 3.1. We begin with a result which is crucial for the proof and of some interests on its own right. Recall that $Q$ is defined in (2). For $B_{H}, B_{H}^{\prime}$ two independent fBm's with Hurst indexes $H$, we set

$$
\begin{equation*}
W_{H}(t)=\sqrt{2} B_{H}(t)-|t|^{2 H}, \quad W_{H}^{\prime}(t)=\sqrt{2} B_{H}^{\prime}(t)-|t|^{2 H} \tag{18}
\end{equation*}
$$

and

$$
V_{H}(S)=\sup _{s \in[0, S]} W_{H}(s), \quad t \in \mathbb{R}, S \geq 0
$$

Define further for all $x, y$ non-negative and $\lambda \in \mathbb{R}$, the generalized Berman-type constants by

$$
\begin{aligned}
& \mathcal{B}_{H}^{x, y}\left(T_{1} ; \lambda, T_{2}, T_{3}\right)([0, S]) \\
& =\int_{\mathbb{R}} e^{w} \mathbb{P}\left\{\int_{\left[0, T_{1}\right]} \mathbb{I}\left(W_{H}^{\prime}(t)+V_{H}(S)>w\right) d t>x, \int_{\left[T_{2}, T_{3}\right]} \mathbb{I}\left(W_{H}^{\prime}(t)+V_{H}(S)>w+\lambda\right) d t>y\right\} d w .
\end{aligned}
$$

Denote further by $\mathcal{H}_{2 H}$ the Pickands constant corresponding to $B_{H}$, i.e.,

$$
\mathcal{H}_{2 H}=\lim _{S \rightarrow \infty} S^{-1} \mathbb{E}\left\{e^{V_{H}(S)}\right\}=\mathbb{E}\left\{\frac{\sup _{t \in \mathbb{R}} e^{W_{H}(t)}}{\int_{t \in \mathbb{R}} e^{W_{H}(t)} d t}\right\} \in(0, \infty) .
$$

Lemma 4.1. For all $T_{1}, T_{2}, T_{3}$ positive, $\lambda \in \mathbb{R}$, and all $x, y$ non-negative we have

$$
\mathcal{B}_{H}^{x, y}\left(T_{1} ; \lambda, T_{2}, T_{3}\right):=\lim _{S \rightarrow \infty} S^{-1} \mathcal{B}_{H}^{x, y}\left(T_{1} ; \lambda, T_{2}, T_{3}\right)([0, S])=\mathcal{H}_{2 H} \overline{\mathcal{B}}_{H}^{x, y}\left(T_{1} ; \lambda, T_{2}, T_{3}\right) \in(0, \infty)
$$

It is worth mentioning that both sides of equation in the above lemma is equal to zero if $x \geq T_{1}$ or $y \geq T_{3}-T_{2}$. Hence it is valid for all nonnegative $x$ and $y$.
Proof of Lemma 4.1 First note that for any $S>0$ we have using the Fubini-Tonelli theorem and the independence of $V_{H}$ and $W_{H}^{\prime}$

$$
\begin{aligned}
& \mathcal{B}_{H}^{x, y}\left(T_{1} ; \lambda, T_{2}, T_{3}\right)([0, S]) \\
& =\mathbb{E}\left\{\int_{\mathbb{R}} e^{w} \mathbb{I}\left(\int_{\left[0, T_{1}\right]} \mathbb{I}\left(W_{H}^{\prime}(t)+V_{H}(S)>w\right) d t>x, \int_{\left[T_{2}, T_{3}\right]} \mathbb{I}\left(W_{H}^{\prime}(t)+V_{H}(S)>w+\lambda\right) d t>y\right) d w\right\} \\
& =\mathbb{E}\left\{e^{V_{H}(S)} \int_{\mathbb{R}} e^{w} \mathbb{I}\left(\int_{\left[0, T_{1}\right]} \mathbb{I}\left(W_{H}^{\prime}(t)>w\right) d t>x, \int_{\left[T_{2}, T_{3}\right]} \mathbb{I}\left(W_{H}^{\prime}(t)>w+\lambda\right) d t>y\right) d w\right\} \\
& =\mathbb{E}\left\{e^{V_{H}(S)}\right\} \int_{\mathbb{R}} e^{w \mathbb{P}}\left\{\int_{\left[0, T_{1}\right]} \mathbb{I}\left(W_{H}(t)>w\right) d t>x, \int_{\left[T_{2}, T_{3}\right]} \mathbb{I}\left(W_{H}(t)>w+\lambda\right) d t>y\right\} d w \\
& \\
& \leq \mathbb{E}\left\{e^{V_{H}(S)}\right\} \int_{\mathbb{R}} e^{w \mathbb{P}}\left\{\int_{\left[0, T_{1}\right]} \mathbb{I}\left(W_{H}(t)>w\right) d t>0\right\} d w \\
& =\mathbb{E}\left\{e^{V_{H}(S)}\right\} \int_{0}^{\infty} e^{w} \mathbb{P}\left\{V_{H}\left(T_{1}\right)>w\right\} d w \\
& =\mathbb{E}\left\{e^{V_{H}(S)}\right\} \mathbb{E}\left\{e^{V_{H}\left(T_{1}\right)}\right\} .
\end{aligned}
$$

Hence the claim follows by the definition of the Pickands constant and the sample continuity of $V_{H}$.

Let in the following

$$
B=\left(\frac{H}{c(1-H)}\right)^{-H-2} H
$$

and recall that

$$
\begin{equation*}
\Delta(u)=2^{\frac{1}{2 H}} t^{*} A^{-\frac{1}{H}} u^{-\frac{1-H}{H}}, \quad v(u)=u \Delta(u) . \tag{19}
\end{equation*}
$$

Applying [15, Lem 4.1] we obtain the following result.

Proposition 4.2. If ( $\mathbf{T}$ ) holds, then
(20) $\mathbb{P}\left\{\frac{1}{v(u)} \int_{\left[0, T_{1}(u)\right]} \mathbb{I}(Q(t)>u) d t>x\right\} \sim \mathcal{H}_{2 H} \overline{\mathcal{B}}_{H}^{x}\left(\mathscr{T}_{1}\right) \frac{\sqrt{2 \pi}(A B)^{-1 / 2}}{u^{1-H} \Delta(u)} \Psi\left(A u^{1-H}\right), \quad u \rightarrow \infty$.

The next proposition plays a key role in the proof of Theorem 3.1.
Proposition 4.3. If $(\mathbf{T})$ holds, then for all $\lambda \in \mathbb{R}, \tau=\lambda /\left(A^{2}(1-H)\right)$

$$
\begin{align*}
& \mathbb{P}\left\{\frac{1}{v(u)} \int_{\left[T_{2}(u), T_{3}(u)\right]} \mathbb{I}\left(Q(t)>u+\tau u^{2 H-1}\right) d t>y, \frac{1}{v(u)} \int_{\left[0, T_{1}(u)\right]} \mathbb{I}(Q(t)>u) d t>x\right\} \\
& \quad \sim \mathcal{H}_{2 H} \overline{\mathcal{B}}_{H}^{x, y}\left(\mathscr{T}_{1} ; \lambda, \mathscr{T}_{2}, \mathscr{T}_{3}\right) \frac{\sqrt{2 \pi}(A B)^{-1 / 2}}{u^{1-H} \Delta(u)} \Psi\left(A u^{1-H}\right), \quad u \rightarrow \infty . \tag{21}
\end{align*}
$$

Hereafter, for any non-constant random variable $Z$, we denote $\bar{Z}=Z / \sqrt{\operatorname{Var}(Z)}$.
Proof of Proposition 4.3 Using the self-similarity of $B_{H}$, i.e.,

$$
\left\{B_{H}(u t), t \in \mathbb{R}\right\} \stackrel{d}{=}\left\{u^{H} B_{H}(t), t \in \mathbb{R}\right\}, \quad u>0
$$

we have with $\Delta(u)$ given in (9) and $\widetilde{u}=u+\tau u^{2 H-1}$

$$
\begin{aligned}
\mathbb{P}\{ & \left.\frac{1}{v(u)} \int_{\left[0, T_{1}(u)\right]} \mathbb{I}(Q(t)>u) d t>x, \frac{1}{v(u)} \int_{\left[T_{2}(u), T_{3}(u)\right]} \mathbb{I}(Q(t)>\widetilde{u}) d t>y\right\} \\
= & \mathbb{P}\left\{\frac{1}{\Delta(u)} \int_{\left[0, T_{1}(u) / u\right]} \mathbb{I}\left(\sup _{s \geq t}\left(u^{H}\left(B_{H}(s)-B_{H}(t)\right)-c u(s-t)\right)>u\right) d t>x,\right. \\
& \left.\frac{1}{\Delta(u)} \int_{\left[T_{2}(u) / \widetilde{u}, T_{3}(u) / \widetilde{u}\right]} \mathbb{I}\left(\sup _{s \geq t}\left(\widetilde{u}^{H}\left(B_{H}(s)-B_{H}(t)\right)-c \widetilde{u}(s-t)\right)>\widetilde{u}\right) d t>y\right\} \\
= & \mathbb{P}\left\{\frac{1}{\Delta(u)} \int_{\left[0, \bar{T}_{1}(u)\right]} \mathbb{I}\left(\sup _{s \geq t} Z(s, t)>u_{\star}\right) d t>x,\right. \\
& \left.\frac{1}{\Delta(u)} \int_{\left[\bar{T}_{2}(u), \bar{T}_{3}(u)\right]} \mathbb{I}\left(\sup _{s \geq t} Z(s, t)>\widetilde{u}_{\star}\right) d t>y\right\},
\end{aligned}
$$

where

$$
Z(s, t)=A \frac{B_{H}(s)-B_{H}(t)}{1+c(s-t)}
$$

and

$$
u_{\star}=A u^{1-H}, \quad \widetilde{u}_{\star}=A \widetilde{u}^{1-H}, \quad \bar{T}_{1}(u)=T_{1}(u) / u, \bar{T}_{i}(u)=T_{i}(u) / \widetilde{u}, i=2,3 .
$$

Note that as $u \rightarrow \infty$

$$
\begin{equation*}
\widetilde{u}_{\star}=u_{\star}+\frac{\lambda}{u_{\star}}, \quad \widetilde{u}_{\star}^{2} \sim u_{\star}^{2}+2 \lambda+o(1) . \tag{22}
\end{equation*}
$$

Direct calculation shows that

$$
\max _{s \geq t} \sqrt{\operatorname{Var}(Z(s, t))}=\max _{s \geq t} \frac{A(s-t)^{H}}{1+c(s-t)}=1
$$

and the maximum is attained for all $s, t$ such that

$$
s-t=t^{*}=\frac{H}{c(1-H)}
$$

and

$$
\begin{equation*}
1-A \frac{t^{H}}{1+c t} \sim \frac{B}{2 A}\left(t-t^{*}\right)^{2}, \quad t \rightarrow t^{*} . \tag{23}
\end{equation*}
$$

Moreover, we have

$$
\begin{equation*}
\lim _{\delta \rightarrow 0} \sup _{\left|s-t-t^{*}\right|,\left|s^{\prime}-t^{\prime}-t^{*}\right|<\delta,\left|s-s^{\prime}\right|<\delta}\left|\frac{1-\operatorname{Cor}\left(Z(s, t), Z\left(s^{\prime}, t^{\prime}\right)\right)}{\left|s-s^{\prime}\right|^{2 H}+\left|t-t^{\prime}\right|^{2 H}}-2^{-1}\left(t^{*}\right)^{-2 H}\right|=0 . \tag{24}
\end{equation*}
$$

In the following we tacitly assume that

$$
S>\max (x, y)
$$

Observe that

$$
\begin{aligned}
\pi_{1}(u) & \leq \mathbb{P}\left\{\frac{1}{\Delta(u)} \int_{\left[0, \bar{T}_{1}(u)\right]} \mathbb{I}\left(\sup _{s \geq t} Z(s, t)>u_{\star}\right) d t>x, \frac{1}{\Delta(u)} \int_{\left[\bar{T}_{2}(u), \bar{T}_{3}(u)\right]} \mathbb{I}\left(\sup _{s \geq t} Z(s, t)>\widetilde{u}_{\star}\right) d t>y\right\} \\
& \leq \pi_{1}(u)+\pi_{2}(u)
\end{aligned}
$$

where

$$
\begin{aligned}
& \pi_{1}(u)= \mathbb{P}\left\{\frac{1}{\Delta(u)} \int_{\left[0, \bar{T}_{1}(u)\right]} \mathbb{I}\left(\sup _{\left|s-t^{*}\right| \leq(\ln u) / u^{1-H}} Z(s, t)>u_{\star}\right) d t>x,\right. \\
&\left.\frac{1}{\Delta(u)} \int_{\left[\bar{T}_{2}(u), \bar{T}_{3}(u)\right]} \mathbb{I}\left(\sup _{\left|s-t^{*}\right| \leq(\ln u) / u^{1-H}} Z(s, t)>\widetilde{u}_{\star}\right) d t>y\right\}, \\
& \pi_{2}(u)=\mathbb{P}\left\{\begin{array}{l}
\left.\sup _{t \in\left[0, T^{*}(u)\right]\left|s-t^{*}\right| \geq(\ln u) /\left(2 u^{1-H}\right), s \geq t} Z(s, t)>\hat{u}\right\},
\end{array}\right.
\end{aligned}
$$

with $\overline{T^{*}}(u)=\max \left(\bar{T}_{1}(u), \bar{T}_{2}(u), \bar{T}_{3}(u)\right)$ and $\hat{u}=\min \left(u_{\star}, \widetilde{u}_{*}\right)$.
$\diamond$ Upper bound of $\pi_{2}(u)$. Next, for some $T>0$ we have

$$
\pi_{2}(u) \leq \mathbb{P}\left\{\sup _{t \in\left[0, \overline{T^{*}}(u)\right]\left|s-t^{*}\right| \geq(\ln u) /\left(2 u^{1-H}\right), t \leq s \leq T} Z(s, t)>\hat{u}\right\}+\mathbb{P}\left\{\sup _{t \in\left[0, \overline{T^{*}}(u)\right]} \sup _{s \geq T} Z(s, t)>\hat{u}\right\} .
$$

In view of (23) for $u$ sufficiently large

$$
\sup _{t \in\left[0, \overline{T^{*}}(u)\right]\left|s-t^{*}\right| \geq(\ln u) /\left(2 u^{1-H}\right), t \leq s \leq T} \sup \operatorname{Var}(Z(s, t)) \leq 1-\mathbb{Q}\left(\frac{\ln u}{u^{1-H}}\right)^{2}
$$

and by (24)

$$
\mathbb{E}\left\{\left(Z(s, t)-Z\left(s^{\prime}, t^{\prime}\right)\right)^{2}\right\} \leq \mathbb{Q}_{1}\left(\left|s-s^{\prime}\right|^{H}+\left|t-t^{\prime}\right|^{H}\right), \quad t \in\left[0, \overline{T^{*}}(u)\right],\left|s-t^{*}\right| \geq(\ln u) /\left(2 u^{1-H}\right), t \leq s \leq T
$$

Hence, in light of [20, Thm 8.1] for all $u$ large enough

$$
\mathbb{P}\left\{\sup _{t \in\left[0, \overline{T^{*}}(u)\right]\left|s-t^{*}\right| \geq(\ln u) /\left(2 u^{1-H}\right), t \leq s \leq T} \sup Z(s, t)>\hat{u}\right\} \leq \mathbb{Q}_{2} u^{\frac{4(1-H)}{H}} \Psi\left(\frac{\hat{u}}{\sqrt{1-\mathbb{Q}\left(\frac{\ln u}{\left.u^{1-H}\right)^{2}}\right.}}\right)
$$

Moreover, for $T$ sufficiently large

$$
\sqrt{\operatorname{Var}(Z(s, t))}=\frac{A(s-t)^{H}}{1+c(s-t)} \leq \frac{2 A}{c}(T+k)^{-(1-H)}, \quad s \in[T+k, T+k+1], t \in\left[0, \overline{T^{*}}(u)\right] .
$$

Hence for some $\varepsilon \in(0,1)$ (set $c_{\varepsilon}=(1+\varepsilon) c$ )

$$
\begin{aligned}
\mathbb{P}\left\{\sup _{t \in\left[0, T^{*}(u)\right]} \sup _{s \geq T} Z(s, t)>\hat{u}\right\} & \leq \sum_{k=0}^{\infty} \mathbb{P}\left\{\sup _{t \in\left[0, \overline{T^{*}}(u)\right]} \sup _{s \in[T+k, T+k+1]} Z(s, t)>\hat{u}\right\} \\
& \leq \sum_{k=0}^{\infty} \mathbb{P}\left\{\sup _{t \in\left[0, \overline{T^{*}}(u)\right]} \sup _{s \in[T+k, T+k+1]} \bar{Z}(s, t)>\frac{1}{2} c_{\varepsilon}(T+k)^{(1-H)} u^{1-H}\right\}
\end{aligned}
$$

Additionally, for $T$ sufficiently large and $k \geq 0$, we have

$$
\mathbb{E}\left\{\left(\bar{Z}(s, t)-\bar{Z}\left(s^{\prime}, t^{\prime}\right)\right)^{2}\right\} \leq \mathbb{Q}_{3}\left(\left|s-s^{\prime}\right|^{H}+\left|t-t^{\prime}\right|^{H}\right), \quad s, s^{\prime} \in[T+k, T+k+1], t, t^{\prime} \in[0,1] .
$$

Thus by [20, Thm 8.1] for all $T$ and $u$ sufficiently large we have

$$
\begin{aligned}
\mathbb{P}\left\{\sup _{t \in\left[0, \bar{T}^{*}(u)\right]} \sup _{s \geq T} Z(s, t)>\hat{u}\right\} & \leq \sum_{k=0}^{\infty} \mathbb{P}\left\{\sup _{t \in[0,1]} \sup _{s \in[T+k, T+k+1]} \bar{Z}(s, t)>\frac{1}{2} c_{\varepsilon}(T+k)^{(1-H)} u^{1-H}\right\} \\
& \leq \sum_{k=0}^{\infty} \mathbb{Q}_{4} u^{\frac{4(1-H)}{H}} \Psi\left(\frac{1}{2} c_{\varepsilon}(T+k)^{(1-H)} u^{1-H}\right) \\
& \leq \sum_{k=0}^{\infty} \mathbb{Q}_{4} u^{\frac{4(1-H)}{H}} e^{-\frac{1}{2}\left(\frac{1}{2} c_{\varepsilon}(T+k)^{(1-H)} u^{1-H}\right)^{2}} \\
& \leq \sum_{k=0}^{\infty} \mathbb{Q}_{4} u^{\frac{4(1-H)}{H}} \int_{T-1}^{\infty} e^{-\frac{1}{2}\left(\frac{1}{2} c_{\varepsilon} z^{(1-H)} u^{1-H}\right)^{2}} d z \\
& \leq \mathbb{Q}_{4} u^{\frac{4(1-H)}{H}} \Psi\left(\mathbb{Q}_{5}(T u)^{1-H}\right) .
\end{aligned}
$$

Therefore we conclude that for all $u, T$ sufficiently large

$$
\begin{equation*}
\pi_{2}(u) \leq \mathbb{Q}_{2} u^{\frac{4(1-H)}{H}} \Psi\left(\frac{\hat{u}}{\left.\sqrt{1-\mathbb{Q}\left(\frac{\ln u}{\left.u^{1-H}\right)^{2}}\right.}\right)+\mathbb{Q}_{4} u^{\frac{4(1-H)}{H}} \Psi\left(2 A u^{1-H}\right) . . . . ~ . ~ . ~}\right. \tag{25}
\end{equation*}
$$

$\diamond \underline{\text { Upper bound of } \pi_{1}(u)}$. Given a positive integer $k$ and $u>0$ define

$$
I_{k}(u)=[k \Delta(u) S,(k+1) \Delta(u) S], \quad N(u)=\left[\frac{\ln u}{u^{1-H} \Delta(u) S}\right]+1 .
$$

It follows that

$$
\begin{aligned}
\pi_{1}(u)= & \mathbb{P}\left\{\frac{1}{\Delta(u)} \int_{\left[0, \bar{T}_{1}(u)\right]} \mathbb{I}\left(\sup _{|s| \leq(\ln u) / u^{1-H}} Z\left(s+t^{*}, t\right)>u_{\star}\right) d t>x,\right. \\
& \left.\frac{1}{\Delta(u)} \int_{\left[\bar{T}_{2}(u), \bar{T}_{3}(u)\right]} \mathbb{I}\left(\sup _{|s| \leq(\ln u) / u^{1-H}} Z\left(s+t^{*}, t\right)>\widetilde{u}_{\star}\right) d t>y\right\} \\
\leq & \Sigma_{1}^{+}(u)+2 \Sigma \Sigma_{1}(u)+2 \Sigma \Sigma_{2}(u),
\end{aligned}
$$

where

$$
\begin{aligned}
& \Sigma_{1}^{+}(u)=\sum_{k=-N(u)-1}^{N(u)+1} \mathbb{P}\left\{\frac{1}{\Delta(u)} \int_{\left[0, \bar{T}_{1}(u)\right]} \mathbb{I}\left(\sup _{s \in I_{k}(u)} Z\left(s+t^{*}, t\right)>u_{\star}\right) d t>x,\right. \\
& \left.\frac{1}{\Delta(u)} \int_{\left[\bar{T}_{2}(u), \bar{T}_{3}(u)\right]} \mathbb{I}\left(\sup _{s \in I_{k}(u)} Z\left(s+t^{*}, t\right)>\widetilde{u}_{\star}\right) d t>y\right\} \\
& \leq \sum_{k=-N(u)-1}^{N(u)+1} \mathbb{P}\left\{\int_{\left[0, \mathscr{T}_{1}+\epsilon\right]} \mathbb{I}\left(\sup _{s \in[0, S]} Z_{u, k}(s, t)>u_{k}^{-}\right) d t>x,\right. \\
& \left.\int_{\left[\mathscr{T}_{2}-\epsilon, \mathscr{T}_{3}+\epsilon\right]} \mathbb{I}\left(\sup _{s \in[0, S]} Z_{u, k}(s, t)>\widetilde{u}_{k}^{-}\right) d t>y\right\}, \\
& \Sigma \Sigma_{1}(u)=\sum_{|k|,|l| \leq N(u)+1, l=k+1} \mathbb{P}\left\{\sup _{t \in\left[0, T^{*}\right], s \in[k S,(k+1) S]} Z\left(\Delta(u) s+t^{*}, \Delta(u) t\right)>\widetilde{u}_{\star},\right. \\
& \left.\sup _{t \in\left[0, T^{*}\right], s \in[l S,(l+1) S]} Z\left(\Delta(u) s+t^{*}, \Delta(u) t\right)>u_{\star}\right\}, \\
& \Sigma \Sigma_{2}(u)=\sum_{|k|,|l| \leq N(u)+1, l \geq k+2} \mathbb{P}\left\{\sup _{t \in\left[0, T^{*}\right], s \in[k S,(k+1) S]} Z\left(\Delta(u) s+t^{*}, \Delta(u) t\right)>\widetilde{u}_{\star},\right. \\
& \left.\sup _{t \in\left[0, T^{*}\right], s \in[l S,(l+1) S]} Z\left(\Delta(u) s+t^{*}, \Delta(u) t\right)>u_{\star}\right\},
\end{aligned}
$$

with

$$
\begin{gathered}
T^{*}=\max \left(\mathscr{T}_{1}+\epsilon, \mathscr{T}_{2}-\epsilon, \mathscr{T}_{3}+\epsilon\right), \epsilon<\mathscr{T}_{2}, \quad \Delta(u)=C u^{-\frac{1-H}{H}}, \quad C=2^{\frac{1}{2 H}} t^{*} A^{-\frac{1}{H}} \\
Z_{u, k}(s, t)=\bar{Z}\left(t^{*}+\Delta(u)(k S+s), \Delta(u) t\right) \\
u_{k}^{-}=u_{\star}\left(1+\frac{(1-\epsilon) B}{2 A} \Delta^{2}(u) \eta_{k, S}\right), \quad \eta_{k, S}=\inf _{s \in[k S,(k+1) S], t \in\left[0, T_{*}\right]}(s-t)^{2} \\
\widetilde{u}_{k}^{-}=\widetilde{u}_{\star}\left(1+\frac{(1-\epsilon) B}{2 A} \Delta^{2}(u) \eta_{k, S}\right)
\end{gathered}
$$

Since the maximal value of $k$ is $N(u)=\left[\frac{\ln u}{u^{1-H} \Delta(u) S}\right]+1$ and $\eta_{k, S}$ is non-negative and bounded up to some constant by $k^{2} S^{2}$ using further (22) we have

$$
\begin{equation*}
u_{k}^{-}=u_{\star}\left(1+o\left(u^{H-1} \ln u\right)\right), \quad{\widetilde{u_{k}}}^{-}=\left(u_{\star}+\lambda / u_{\star}\right)\left(1+o\left(u^{H-1} \ln u\right)\right)=u_{k}^{-}+\lambda_{u, k} / u_{k}^{-} \tag{26}
\end{equation*}
$$

where $o\left(u^{H-1} \ln u\right)$ does not depend on $k, S$ and further

$$
\lim _{u \rightarrow \infty} \sup _{|k| \leq N(u)}\left|\lambda-\lambda_{u, k}\right|=0
$$

We analyse next the uniform asymptotics of
$p_{k}(u):=\mathbb{P}\left\{\int_{\left[0, \mathscr{T}_{1}+\epsilon\right]} \mathbb{I}\left(\sup _{s \in[0, S]} Z_{u, k}(s, t)>u_{k}^{-}\right) d t>x, \int_{\left[\mathscr{T}_{2}-\epsilon, \mathscr{T}_{3}+\epsilon\right]} \mathbb{I}\left(\sup _{s \in[0, S]} Z_{u, k}(s, t)>u_{k}^{-}+\lambda_{u, k} / u_{k}^{-}\right) d t>y\right\}$
as $u \rightarrow \infty$ with respect to $|k| \leq N(u)+1$. In order to apply Lemma 5.1 in Appendix, we need to check conditions C1-C3 therein. The first condition C1 follows immediately from (26). The second condition C2 is a consequence of (24), while C3 follows from (26). Consequently, using further (26), the application of the aforementioned lemma is justified and we obtain

$$
\begin{equation*}
\lim _{u \rightarrow \infty} \sup _{|k| \leq N(u)+1}\left|\frac{p_{k}(u)}{\Psi\left(u_{k}^{-}\right)}-\mathcal{B}_{H}^{x, y}\left(\mathscr{T}_{1}+\epsilon ; \lambda, \mathscr{T}_{2}-\epsilon, \mathscr{T}_{3}+\epsilon\right)([0, S])\right|=0 . \tag{27}
\end{equation*}
$$

Hence

$$
\begin{align*}
\Sigma_{1}^{+}(u) & \leq \sum_{|k| \leq N(u)+1} \mathcal{B}_{H}^{x, y}\left(\mathscr{T}_{1}+\epsilon ; \lambda, \mathscr{T}_{2}-\epsilon, \mathscr{T}_{3}+\epsilon\right)([0, S]) \Psi\left(u_{k}^{-}\right) \\
& \leq \mathcal{B}_{H}^{x, y}\left(\mathscr{T}_{1}+\epsilon ; \lambda, \mathscr{T}_{2}-\epsilon, \mathscr{T}_{3}+\epsilon\right)([0, S]) \Psi\left(u_{\star}\right) \sum_{|k| \leq N(u)+1} e^{-A^{2} u^{2(1-H)} \times \frac{(1-\epsilon) B}{2 A} \Delta^{2}(u) \times(k S)^{2}} \\
& \sim \frac{\mathcal{B}_{H}^{x, y}\left(\mathscr{T}_{1}+\epsilon ; \lambda, \mathscr{T}_{2}-\epsilon, \mathscr{T}_{3}+\epsilon\right)([0, S])}{S} \frac{\sqrt{2}(A B)^{-1 / 2}(1-\epsilon)^{-1 / 2}}{u^{1-H} \Delta(u)} \Psi\left(u_{\star}\right) \int_{\mathbb{R}} e^{-t^{2}} d t \\
& \sim \frac{\mathcal{B}_{H}^{x, y}\left(\mathscr{T}_{1} ; \lambda, \mathscr{T}_{2}, \mathscr{T}_{3}\right)([0, S])}{S} \frac{\sqrt{2 \pi}(A B)^{-1 / 2}}{u^{1-H} \Delta(u)} \Psi\left(u_{\star}\right), \quad u \rightarrow \infty, \epsilon \rightarrow 0 . \tag{28}
\end{align*}
$$

$\underline{\text { Upper bound of } \Sigma \Sigma_{1}(u)}$. Suppose for notational simplicity that $\lambda=0$. Then $\widetilde{u}_{\star}=u_{\star}$ and

$$
\Sigma \Sigma_{1}(u) \leq \sum_{|k| \leq N(u)+1}\left(q_{k, 1}(u)+q_{k, 2}(u)\right)
$$

where

$$
\begin{aligned}
& q_{k, 1}(u)=\mathbb{P}\left\{\sup _{t \in\left[0, T_{3}^{*}\right], s \in[k S,(k+1) S]} Z_{u}(s, t)>u_{\star} \sup _{t \in\left[0, T_{3}^{*}\right], s \in[(k+1) S,(k+1) S+\sqrt{S}]} Z_{u}(s, t)>u_{\star}\right\} \\
& \leq \mathbb{P}\left\{\sup _{t \in\left[0, T_{3}^{*}\right], s \in[(k+1) S,(k+1) S+\sqrt{S}]} \bar{Z}_{u}(s, t)>u_{k+1}^{-}\right\}, \\
& q_{k, 2}(u)=\mathbb{P}\left\{\sup _{t \in\left[0, T_{3}^{*}\right], s \in[k S,(k+1) S]} Z_{u}(s, t)>u_{\star}, \sup _{t \in\left[0, T_{3}^{*}\right], s \in[(k+1) S+\sqrt{S},(k+2) S]} Z_{u}(s, t)>u_{\star}\right\} \\
& \leq \mathbb{P}\left\{\sup _{t \in\left[0, T_{3}^{*}\right], s \in[k S,(k+1) S]} \bar{Z}_{u}(s, t)>u_{k}^{-}, \sup _{t \in\left[0, T_{3}^{*}\right], s \in[(k+1) S+\sqrt{S},(k+2) S]} \bar{Z}_{u}(s, t)>u_{k+1}^{-}\right\},
\end{aligned}
$$

with

$$
Z_{u}(s, t)=Z\left(t^{*}+\Delta(u) s, \Delta(u) t\right) .
$$

Analogously as in (27), we have that

$$
\lim _{u \rightarrow \infty} \sup _{|k| \leq N(u)+1}\left|\frac{\mathbb{P}\left\{\sup _{t \in\left[0, T_{3}^{*}\right], s \in[(k+1) S,(k+1) S+\sqrt{S}]} \bar{Z}_{u}(s, t)>u_{k+1}^{-}\right\}}{\Psi\left(u_{k+1}^{-}\right)}-\overline{\mathcal{B}}_{H}^{0}\left(T_{3}^{*}\right) \overline{\mathcal{B}}_{H}^{0}(\sqrt{S})\right|=0 .
$$

Thus in view of (28)

$$
\begin{aligned}
\sum_{|k| \leq N(u)+1} q_{k, 1}(u) & \leq \sum_{|k| \leq N(u)+1} \overline{\mathcal{B}}_{H}^{0}\left(T_{3}^{*}\right) \overline{\mathcal{B}}_{H}^{0}(\sqrt{S}) \Psi\left(u_{k+1}^{-}\right) \\
& \leq \frac{\overline{\mathcal{B}}_{H}^{0}\left(T_{3}^{*}\right) \overline{\mathcal{B}}_{H}^{0}(\sqrt{S})}{S} \frac{\sqrt{2 \pi}(A B)^{-1 / 2}}{u^{1-H} \Delta(u)} \Psi\left(u_{\star}\right), \quad u \rightarrow \infty .
\end{aligned}
$$

Additionally, in light of (24) for $u$ sufficiently large
(29) $\left|s-s^{\prime}\right|^{2 H}+\left|t-t^{\prime}\right|^{2 H} \leq 2\left(u_{\star}\right)^{2}\left(1-\operatorname{Cor}\left(\bar{Z}_{u}(s, t), \bar{Z}_{u}\left(s^{\prime}, t^{\prime}\right)\right)\right) \leq 4\left(\left|s-s^{\prime}\right|^{2 H}+\left|t-t^{\prime}\right|^{2 H}\right)$
for all $|s|,\left|s^{\prime}\right| \leq \frac{2 \ln u}{u^{1-H} \Delta(u)}, t, t^{\prime} \in\left[0, T_{*}\right]$. Thus by [21, Cor 3.1] there exist two positive constants $\mathcal{C}, \mathcal{C}_{1}$ such that for $u$ sufficiently large and $S>1$

$$
q_{k, 2}(u) \leq \mathcal{C} S^{4} e^{-\mathcal{C}_{1} S^{\frac{H}{2}}} \Psi\left(u_{k, k+1}^{-}\right), \quad u_{k, l}^{-}=\min \left(u_{k}^{-}, u_{l}^{-}\right) .
$$

Hence

$$
\begin{aligned}
\sum_{|k| \leq N(u)+1} q_{k, 2}(u) & \leq \sum_{|k| \leq N(u)+1} \mathcal{C} S^{4} e^{-\mathcal{C}_{1} S^{\frac{H}{2}}} \Psi\left(u_{k, k+1}^{-}\right) \\
& \leq \mathcal{C} S^{3} e^{-\mathcal{C}_{1} S^{\frac{H}{2}}} \frac{\sqrt{2 \pi}(A B)^{-1 / 2}}{u^{1-H} \Delta(u)} \Psi\left(u_{\star}\right), \quad u \rightarrow \infty
\end{aligned}
$$

Therefore we conclude that

$$
\begin{equation*}
\Sigma \Sigma_{1}(u) \leq\left(\frac{\overline{\mathcal{B}}_{H}^{0}\left(T_{3}^{*}\right) \overline{\mathcal{B}}_{H}^{0}(\sqrt{S})}{S}+\mathcal{C} S^{3} e^{-\mathcal{C}_{1} S^{\frac{H}{2}}}\right) \frac{\sqrt{2 \pi}(A B)^{-1 / 2}}{u^{1-H} \Delta(u)} \Psi\left(u_{\star}\right), \quad u \rightarrow \infty \tag{30}
\end{equation*}
$$

Note that if $\lambda \neq 0$, the bound derived in (30) changes only by a multiplication by some constant, which does not affect the negligibility of $\Sigma \Sigma_{1}(u)$.
Upper bound of $\Sigma \Sigma_{2}(u)$. In light of (29) and applying [21, Cor 3.1], we have that

$$
\begin{align*}
\Sigma \Sigma_{2}(u) & \leq \sum_{|k|,|l| \leq N(u)+1, l \geq k+2} \mathcal{C} S^{4} e^{-\mathcal{C}_{1}|l-k-1|^{H} S^{H}} \Psi\left(u_{k, l}^{-}\right) \\
& \leq \sum_{|k| \leq N(u)+1} \mathcal{C} S^{4} \Psi\left(u_{k}^{-}\right) \sum_{l=1}^{\infty} e^{-\mathcal{C}_{1} l^{H} S^{H}} \\
& \leq \sum_{|k| \leq N(u)+1} \mathcal{C} S^{4} e^{-\mathbb{Q}_{6} S^{H}} \Psi\left(u_{k}^{-}\right) \\
& \leq \mathcal{C} S^{3} e^{-\mathbb{Q}_{6} S^{H}} \frac{\sqrt{2 \pi}(A B)^{-1 / 2}}{u^{1-H} \Delta(u)} \Psi\left(u_{\star}\right), \quad u \rightarrow \infty . \tag{31}
\end{align*}
$$

Consequently, as $u \rightarrow \infty$

$$
\begin{align*}
\pi_{1}(u) \leq & \left(\frac{\mathcal{B}_{H}^{x, y}\left(\mathscr{T}_{1} ; \lambda, \mathscr{T}_{2}, \mathscr{T}_{3}\right)([0, S])}{S}+\frac{\overline{\mathcal{B}}_{H}^{0}\left(T_{3}^{*}\right) \overline{\mathcal{B}}_{H}^{0}(\sqrt{S})}{S}+\mathcal{C} S^{3}\left[e^{-\mathcal{C}_{1} S^{\frac{H}{2}}}+e^{-\mathbb{Q}_{6} S^{H}}\right]\right) \\
& \times \frac{\sqrt{2 \pi}(A B)^{-1 / 2}}{u^{1-H} \Delta(u)} \Psi\left(u_{\star}\right) . \tag{32}
\end{align*}
$$

$\diamond \underline{\text { Lower bound of } \pi_{1}(u)}$. Again for notation simplicity we assume $\lambda=0$. Observe that

$$
\begin{aligned}
& \frac{1}{\Delta(u)} \int_{\left[0, \bar{T}_{1}(u)\right]} \mathbb{I}\left(\sup _{\left|s-t-t^{*}\right| \leq(\ln u) / u^{1-H}} Z(s, t)>u_{\star}\right) d t \\
& \geq \sum_{|k| \leq N(u)} \frac{1}{\Delta(u)} \int_{\left[0, \bar{T}_{1}(u)\right]} \mathbb{I}\left(\sup _{s \in I_{k}(u)} Z\left(s+t^{*}, t\right)>u_{\star}\right) d t \\
& \quad-\sum_{|k|,|l| \leq N(u), k<l} \frac{1}{\Delta(u)} \int_{\left[0, T^{*}(u) / u\right]} \mathbb{I}\left(\sup _{s \in I_{k}(u)} Z\left(s+t^{*}, t\right)>u_{\star}, \sup _{s \in I_{l}(u)} Z\left(s+t^{*}, t\right)>u_{\star}\right) d t \\
& :=F_{1}(u)-F_{2}(u), \\
& \frac{1}{\Delta(u)} \int_{\left[\bar{T}_{2}(u), \bar{T}_{3}(u)\right]} \mathbb{I}\left(\sup _{\left|s-t-t^{*}\right| \leq(\ln u) / u^{1-H}} Z(s, t)>u_{\star}\right) d t \\
& \geq \sum_{|k| \leq N(u)} \frac{1}{\Delta(u)} \int_{\left[\bar{T}_{2}(u), \bar{T}_{3}(u)\right]} \mathbb{I}\left(\sup _{s \in I_{k}(u)} Z\left(s+t^{*}, t\right)>u_{\star}\right) d t \\
& \quad-\sum_{|k|, l \mid \leq N(u), k<l} \frac{1}{\Delta(u)} \int_{\left[0, \overline{T^{*}}(u)\right]} \mathbb{I}\left(\sup _{s \in I_{k}(u)} Z\left(s+t^{*}, t\right)>u_{\star}, \sup _{s \in I_{l}(u)} Z\left(s+t^{*}, t\right)>u_{\star}\right) d t \\
& :=F_{3}(u)-F_{2}(u) .
\end{aligned}
$$

Hence, for $0<\epsilon<1\left(\right.$ write $\left.s_{\epsilon}=(1+\epsilon) s\right)$ )

$$
\begin{aligned}
\pi_{1}(u) & \geq \mathbb{P}\left\{F_{1}(u)-F_{2}(u)>x, F_{3}(u)-F_{2}(u)>y\right\} \\
& \geq \mathbb{P}\left\{F_{1}(u)>x_{\epsilon}, F_{3}(u)>x_{\epsilon}, F_{2}(u)<\epsilon \min (x, y)\right\} \\
& \geq \mathbb{P}\left\{F_{1}(u)>x_{\epsilon}, F_{3}(u)>y_{\epsilon}\right\}-\mathbb{P}\left\{F_{2}(u) \geq \epsilon \min (x, y)\right\}
\end{aligned}
$$

Note that

$$
\begin{aligned}
& \mathbb{P}\left\{F_{1}(u)>x_{\epsilon}, F_{3}(u)>y_{\epsilon}\right\} \\
& \geq \\
& \geq \mathbb{P}\left\{\exists|k| \leq N(u): \frac{1}{\Delta(u)} \int_{\left[0, \bar{T}_{1}(u)\right]} \mathbb{I}\left(\sup _{s \in I_{k}(u)} Z\left(s+t^{*}, t\right)>u_{\star}\right) d t>x_{\epsilon}, F_{3}(u)>y_{\epsilon}\right\} \\
& \geq \\
& \sum_{|k| \leq N(u)} \mathbb{P}\left\{\frac{1}{\Delta(u)} \int_{\left[0, \bar{T}_{1}(u)\right]} \mathbb{I}\left(\sup _{s \in I_{k}(u)} Z\left(s+t^{*}, t\right)>u_{\star}\right) d t>x_{\epsilon}, F_{3}(u)>y_{\epsilon}\right\} \\
& \\
& -\Sigma \Sigma_{1}(u)-\Sigma \Sigma_{2}(u) \\
& \geq \Sigma_{1}^{-}(u)-\Sigma \Sigma_{1}(u)-\Sigma \Sigma_{2}(u),
\end{aligned}
$$

and

$$
\mathbb{P}\left\{F_{2}(u) \geq \epsilon \min (x, y)\right\} \leq \mathbb{P}\left\{F_{2}(u)>0\right\} \leq \Sigma \Sigma_{1}(u)+\Sigma \Sigma_{2}(u),
$$

where
$\Sigma_{1}^{-}(u)=\sum_{k=-N(u)}^{N(u)} \mathbb{P}\left\{\int_{\left[0, \mathscr{F}_{1}\right]} \mathbb{I}\left(\sup _{s \in[0, S]} Z_{u, k}(s, t)>u_{k}^{-}\right) d t>x_{\epsilon}, \int_{\left[\mathscr{F}_{2}, \mathscr{F}_{3}\right]} \mathbb{I}\left(\sup _{s \in[0, S]} Z_{u, k}(s, t)>u_{k}^{-}\right) d t>y_{\epsilon}\right\}$.

Hence

$$
\pi_{1}(u) \geq \Sigma_{1}^{-}(u)-2 \Sigma \Sigma_{1}(u)-2 \Sigma \Sigma_{2}(u)
$$

Analogously as in (28), it follows that

$$
\Sigma_{1}^{-}(u) \sim \frac{\mathcal{B}_{H}^{x, y}\left(\mathscr{T}_{1} ; \lambda, \mathscr{T}_{2}, \mathscr{T}_{3}\right)([0, S])}{S} \frac{\sqrt{2 \pi}(A B)^{-1 / 2}}{u^{1-H} \Delta(u)} \Psi\left(u_{\star}\right), \quad u \rightarrow \infty, \epsilon \rightarrow 0
$$

which together with the upper bound of $\Sigma \Sigma_{i}, i=1,2$ leads to

$$
\begin{align*}
\pi_{1}(u) \geq & \left(\frac{\mathcal{B}_{H}^{x, y}\left(\mathscr{T}_{1} ; \lambda, \mathscr{T}_{2}, \mathscr{T}_{3}\right)([0, S])}{S}-\frac{2 \overline{\mathcal{B}}_{H}^{0}\left(T_{3}^{*}\right) \overline{\mathcal{B}}_{H}^{0}(\sqrt{S})}{S}-2 \mathcal{C} S^{3}\left[e^{-\mathcal{C}_{1} S^{\frac{H}{2}}}+e^{-\mathbb{Q}_{6} S^{H}}\right]\right) \\
& \times \frac{\sqrt{2 \pi}(A B)^{-1 / 2}}{u^{1-H} \Delta(u)} \Psi\left(u_{\star}\right), \quad u \rightarrow \infty \tag{33}
\end{align*}
$$

Next by Lemma 4.1, we have

$$
\lim _{S \rightarrow \infty} \frac{\mathcal{B}_{H}^{x, y}\left(\mathscr{T}_{1} ; \lambda, \mathscr{T}_{2}, \mathscr{T}_{3}\right)([0, S])}{S}=\mathcal{B}_{H}^{x, y}\left(\mathscr{T}_{1} ; \lambda, \mathscr{T}_{2}, \mathscr{T}_{3}\right) \in(0, \infty), \quad \lim _{S \rightarrow \infty} \frac{\overline{\mathcal{B}}_{H}^{0}\left(T_{3}^{*}\right) \overline{\mathcal{B}}_{H}^{0}(\sqrt{S})}{S}=0
$$

Thus letting $S \rightarrow \infty$ in (32) and (33) yields

$$
\pi_{1}(u) \sim \mathcal{B}_{H}^{x, y}\left(\mathscr{T}_{1} ; \lambda, \mathscr{T}_{2}, \mathscr{T}_{3}\right) \frac{\sqrt{2 \pi}(A B)^{-1 / 2}}{u^{1-H} \Delta(u)} \Psi\left(u_{\star}\right), \quad u \rightarrow \infty
$$

which combined with (25) leads to

$$
\begin{aligned}
& \mathbb{P}\left\{\frac{1}{v(u)} \int_{\left[T_{2}(u), T_{3}(u)\right]} \mathbb{I}\left(Q(t)>u+\tau u^{2 H-1}\right) d t>y, \frac{1}{v(u)} \int_{\left[0, T_{1}(u)\right]} \mathbb{I}(Q(t)>u) d t>x\right\} \\
& \\
& \sim \mathcal{B}_{H}^{x, y}\left(\mathscr{T}_{1} ; \lambda, \mathscr{T}_{2}, \mathscr{T}_{3}\right) \frac{\sqrt{2 \pi}(A B)^{-1 / 2}}{u^{1-H} \Delta(u)} \Psi\left(u_{\star}\right), \quad u \rightarrow \infty
\end{aligned}
$$

establishing the proof.

Proof of Theorem 3.1 Clearly, for all $x, y$ non-negative with $\widetilde{u}=u+\tau u^{2 H-1}$

$$
\mathscr{P}_{T_{1}, T_{2}, T_{3}}^{x, y}(\widetilde{u}, u)=\frac{\mathbb{P}\left\{\frac{1}{v(u)} \int_{\left[0, T_{1}(u)\right]} \mathbb{I}(Q(t)>u) d t>x, \frac{1}{v(u)} \int_{\left[T_{2}(u), T_{3}(u)\right]} \mathbb{I}(Q(t)>\widetilde{u}) d t>y\right\}}{\mathbb{P}\left\{\frac{1}{v(u)} \int_{\left[0, T_{1}(u)\right]} \mathbb{I}(Q(t)>u) d t>x\right\}}
$$

The asymptotics of the denominator and the nominator are derived in Proposition 4.2 and Proposition 4.3 , respectively. Hence, using further (22) establishes the claim.
4.4. Proof of Theorem 3.3. Case $a \in(-1,0)$. Observe that

$$
\begin{aligned}
& \mathbb{P}\left\{\frac{1}{v(u)} \int_{\left[T_{2}(u), T_{3}(u)\right]} \mathbb{I}(Q(t)>(1+a) u) d t>y, \frac{1}{v(u)} \int_{\left[0, T_{1}(u)\right]} \mathbb{I}(Q(t)>u) d t>x\right\} \\
& =\mathbb{P}\left\{\frac{1}{v(u)} \int_{\left[0, T_{1}(u)\right]} \mathbb{I}(Q(t)>u) d t>x\right\} \\
& \\
& \quad-\mathbb{P}\left\{\frac{1}{v(u)} \int_{\left[T_{2}(u), T_{3}(u)\right]} \mathbb{I}(Q(t) \leq(1+a) u) d t>T_{3}(u)-T_{2}(u)-y, \frac{1}{v(u)} \int_{\left[0, T_{1}(u)\right]} \mathbb{I}(Q(t)>u) d t>x\right\} \\
& =: \quad P_{1}(u)-P_{2}(u) .
\end{aligned}
$$

Next, recalling that $T^{*}(u)=\max \left(T_{1}(u), T_{2}(u), T_{3}(u)\right)$ and using that $T^{*}(u) \sim C u^{(2 H-1) / H}$ as $u \rightarrow \infty$ for some $C>0$, we obtain

$$
\begin{aligned}
P_{2}(u) & \leq \mathbb{P}\left\{\inf _{t \in\left[T_{2}(u), T_{3}(u)\right]} Q(t) \leq(1+a) u, \sup _{t \in\left[0, T_{1}(u)\right]} Q(t)>u\right\} \\
& \leq \mathbb{P}\left\{\text { there exist } t, s \in\left[0, T^{*}(u)\right], Q(t)-Q(s) \geq-a u\right\} \\
& \leq \mathbb{P}\left\{\sup _{0 \leq t \leq s \leq T^{*}(u)}\left(B_{H}(t)-B_{H}(s)-c(t-s)\right)>-a u\right\} \\
& \leq \mathbb{P}\left\{\sup _{0 \leq t \leq s \leq 1} T^{* H}(u)\left(B_{H}(t)-B_{H}(s)\right)>-a u\right\} \\
& \leq \mathrm{C}_{1} e^{-\mathrm{C}_{2} u^{4-4 H}}
\end{aligned}
$$

for some $C_{1}, C_{2}>0$, where the third inequality is because of (1) and the last inequality above is due to Borell-TIS inequality. Hence, in view of Proposition 4.2, $P_{2}(u)=o\left(P_{1}(u)\right)$ as $u \rightarrow \infty$, which leads to

$$
\mathbb{P}\left\{\left.\frac{1}{v(u)} \int_{\left[T_{2}(u), T_{3}(u)\right]} \mathbb{I}(Q(t)>(1+a) u) d t>y \right\rvert\, \frac{1}{v(u)} \int_{\left[0, T_{1}(u)\right]} \mathbb{I}(Q(t)>u) d t>x\right\} \sim 1
$$

as $u \rightarrow \infty$.
Case $a>0$. First, we consider the asymptotic upper bound. We note that

$$
\mathscr{P}_{T_{1}, T_{2}, T_{3}}^{x, y}((1+a) u, u) \leq \frac{\mathbb{P}\left\{\frac{1}{v(u)} \int_{\left[T_{2}(u), T_{3}(u)\right]} \mathbb{I}(Q(t)>(1+a) u) d t>y\right\}}{\mathbb{P}\left\{\frac{1}{v(u)} \int_{\left[0, T_{1}(u)\right]} \mathbb{I}(Q(t)>u) d t>x\right\}}
$$

and

$$
\begin{aligned}
\mathbb{P} & \left\{\frac{1}{v(u)} \int_{\left[T_{2}(u), T_{3}(u)\right]} \mathbb{I}(Q(t)>(1+a) u) d t>y\right\} \\
& =\mathbb{P}\left\{\frac{1}{v((1+a) u)} \int_{\left[T_{2}(u), T_{3}(u)\right]} \mathbb{I}(Q(t)>(1+a) u) d t>(1+a)^{(1-2 H) / H} y\right\},
\end{aligned}
$$

where, by (T) we have

$$
\lim _{u \rightarrow \infty} \frac{T_{i}(u)}{v((1+a) u)}=\mathscr{T}_{i}(1+a)^{(1-2 H) / H}, \quad i=1,2 .
$$

Consequently, by the stationarity of $Q(t), t \geq 0$ and Proposition 4.2, with $\tilde{a}=(1+a)^{(1-2 H) / H}$ we obtain

$$
\begin{aligned}
\mathbb{P} & \left\{\frac{1}{v(u)} \int_{\left[T_{2}(u), T_{3}(u)\right]} \mathbb{I}(Q(t)>(1+a) u) d t>y\right\} \\
& =\mathbb{P}\left\{\frac{1}{v(u)} \int_{\left[0, T_{3}(u)-T_{2}(u)\right]} \mathbb{I}(Q(t)>(1+a) u) d t>y\right\} \\
& \sim \mathcal{H}_{2 H} \overline{\mathcal{B}}_{H}^{\tilde{a} y}\left(\left(\mathscr{T}_{3}-\mathscr{T}_{2}\right) \tilde{a}\right) \frac{\sqrt{2 \pi}(A B)^{-1 / 2}}{(1+a)^{1-H} u^{1-H} \Delta((1+a) u)} \Psi\left(A((1+a) u)^{1-H}\right)
\end{aligned}
$$

as $u \rightarrow \infty$. Hence

$$
\limsup _{u \rightarrow \infty} \frac{\mathscr{P}_{T_{1}, T_{2}, T_{3}}^{x, y}((1+a) u, u)}{\exp \left(-\frac{A^{2}\left((1+a)^{2+2 H}-1\right)}{2} u^{2-2 H}\right)} \leq \tilde{a}^{1-H} \frac{\overline{\mathcal{B}}_{H}^{\tilde{a} y}\left(\tilde{a}\left(\mathscr{T}_{3}-\mathscr{T}_{2}\right)\right)}{\overline{\mathcal{B}}_{H}^{x}\left(\mathscr{T}_{1}\right)} .
$$

For the proof of the asymptotic lower bound we have

$$
\begin{aligned}
& \mathscr{P}_{T_{1}, T_{2}, T_{3}}^{x, y}((1+a) u, u) \\
& \geq \frac{\mathbb{P}\left\{\frac{1}{v(u)} \int_{\left[T_{2}(u), T_{3}(u)\right]} \mathbb{I}(Q(t)>(1+a) u) d t>y, \frac{1}{v(u)} \int_{\left[0, T_{1}(u)\right]} \mathbb{I}(Q(t)>(1+a) u) d t>x\right\}}{\mathbb{P}\left\{\frac{1}{v(u)} \int_{\left[0, T_{1}(u)\right]} \mathbb{I}(Q(t)>u) d t>x\right\}} .
\end{aligned}
$$

Then, following the same line of arguments as for the asymptotic upper bound, by Proposition 4.3 we obtain

$$
\liminf _{u \rightarrow \infty} \frac{\mathscr{P}_{T_{1}, T_{2}, T_{3}}^{x, y}((1+a) u, u)}{\exp \left(-\frac{A^{2}\left((1+a)^{2-2 H}-1\right)}{2} u^{2-2 H}\right)} \geq \tilde{a}^{1-H} \frac{\overline{\mathcal{B}}_{H}^{\tilde{a} x, \tilde{a} y}\left(\tilde{a} \mathscr{T}_{1} ; 0, \tilde{a} \mathscr{T}_{2}, \tilde{a} \mathscr{T}_{3}\right)}{\overline{\mathcal{B}}_{H}^{x}\left(\mathscr{T}_{1}\right)} .
$$

## 5. Appendix

In this Section we present a lemma that plays a crucial lemma for proof of Proposition 4.3. Consider next

$$
\xi_{u, j}(s, t), \quad(s, t) \in E=[0, S] \times[0, T], \quad j \in S_{u}
$$

a family of centered Gaussian random fields with continuous sample paths and unit variance, where $S_{u}$ is a countable index set. For $S>0,0<b_{1}, b_{2}, b_{3} \leq T, b_{1}>x \geq 0$ and $b_{3}-b_{2}>y \geq 0$, we are interested in the uniform asymptotics of
$p_{u, j}\left(S ; \lambda_{u, j}\right)=\mathbb{P}\left\{\int_{\left[0, b_{1}\right]} \mathbb{I}\left(\sup _{s \in[0, S]} \xi_{u, j}(s, t)>g_{u, j}\right) d t>x, \int_{\left[b_{2}, b_{3}\right]} \mathbb{I}\left(\sup _{s \in[0, S]} \xi_{u, j}(s, t)>g_{u, j}+\lambda_{u, j} / g_{u, j}\right) d t>y\right\}$
with respect to $j \in S_{u}$, as $u \rightarrow \infty$, where $g_{u, j}$ 's and $\lambda_{u, j}$ 's are given constants depending on $u$ and $j$. Suppose next that $S_{u}$ 's are finite index. The following assumptions will be imposed in the lemma below:

C1: $g_{u, j}, j \in S_{u}, u>$ are constants satisfying

$$
\lim _{u \rightarrow \infty} \inf _{j \in S_{u}} g_{u, j}=\infty
$$

C2: There exists $\alpha \in(0,2]$ such that

$$
\lim _{u \rightarrow \infty} \sup _{j \in S_{u}} \sup _{(s, t) \neq\left(s^{\prime}, t^{\prime}\right),(s, t),\left(s^{\prime}, t^{\prime}\right) \in E}\left|g_{u, j}^{2} \frac{1-\operatorname{Corr}\left(\xi_{u, j}(s, t), \xi_{u, j}\left(s^{\prime}, t^{\prime}\right)\right)}{\left|s-s^{\prime}\right|^{\alpha}+\left|t-t^{\prime}\right|^{\alpha}}-1\right|=0 .
$$

C3: The sequence $\lambda_{u, j}$ is such that

$$
\lim _{u \rightarrow \infty} \sup _{j \in S_{u}}\left|\lambda_{u, j}-\lambda\right|=0
$$

for some $\lambda \in \mathbb{R}$.
We state next a modification of [15, Lem 4.1].
Lemma 5.1. Let $\left\{\xi_{u, j}(s, t),(s, t) \in E, j \in S_{u}\right\}$ be a family of centered Gaussian random fields defined as above. If C1-C3 holds, then for all $S>0,0<b_{1}, b_{2}, b_{3} \leq b, b_{1}>x \geq 0$ and $b_{3}-b_{2}>y \geq 0$ we have

$$
\begin{equation*}
\lim _{u \rightarrow \infty} \sup _{j \in S_{u}}\left|\frac{p_{u, j}\left(S ; \lambda_{u, j}\right)}{\Psi\left(g_{u, j}\right)}-\mathcal{B}_{\alpha / 2}^{x, y}\left(b_{1} ; \lambda, b_{2}, b_{3}\right)([0, S])\right|=0 . \tag{34}
\end{equation*}
$$

Proof of Lemma 5.1 The proof of Lemma 5.1 follows by similar argumentation as given in the proof of [15, Lemma 4.1]. For completeness, we present details of the main steps of the argumentation. Let

$$
\chi_{u, j}(s, t):=g_{u, j}\left(\xi_{u, j}(s, t)-\rho_{u, j}(s, t) \xi_{u, j}(0,0)\right), \quad(s, t) \in E,
$$

and

$$
f_{u, j}(s, t, w):=w \rho_{u, j}(s, t)-g_{u, j}^{2}\left(1-\rho_{u, j}(s, t)\right), \quad(s, t) \in E, w \in \mathbb{R}
$$

where $\rho_{u, j}(s, t)=\operatorname{Cov}\left(\xi_{u, j}(s, t), \xi_{u, j}(0,0)\right)$. Conditioning on $\xi_{u, j}(0,0)$ and using the fact that $\xi_{u, j}(0,0)$ and $\xi_{u, j}(s, t)-\rho_{u, j}(s, t) \xi_{u, j}(0,0)$ are mutually independent, we obtain

$$
\begin{aligned}
& p_{u, j}\left(S ; \lambda_{u, j}\right) \\
&= \frac{e^{-g_{u, j}^{2} / 2}}{\sqrt{2 \pi} g_{u, j}} \int_{\mathbb{R}} \exp \left(-w-\frac{w^{2}}{2 g_{u, j}^{2}}\right) \mathbb{P}\left\{\int_{0}^{b_{1}} \mathbb{I}\left(\sup _{s \in[0, S]}\left(g_{u, j}\left(\xi_{u, j}(s, t)-g_{u, j}\right)\right)>0\right) d t>x,\right. \\
&\left.\int_{b_{2}}^{b_{3}} \mathbb{I}\left(\sup _{s \in[0, S]}\left(g_{u, j}\left(\xi_{u, j}(s, t)-g_{u, j}\right)-\lambda_{u, j}\right)>0\right) d t>y \mid \xi_{u, j}(0,0)=g_{u, j}+w g_{u, j}^{-1}\right\} d w \\
&= \frac{e^{-g_{u, j}^{2} / 2}}{\sqrt{2 \pi} g_{u, j}} \int_{\mathbb{R}} \exp \left(-w-\frac{w^{2}}{2 g_{u, j}^{2}}\right) \mathbb{P}\left\{\int_{0}^{b_{1}} \mathbb{I}\left(\sup _{s \in[0, S]}\left(\chi_{u, j}(s, t)+f_{u, j}(s, t, w)\right)>0\right) d t>x,\right. \\
&\left.\int_{b_{2}}^{b_{3}} \mathbb{I}\left(\sup _{s \in[0, S]}\left(\chi_{u, j}(s, t)+f_{u, j}(s, t, w)-\lambda_{u, j}\right)>0\right) d t>y\right\} d w \\
&:= \frac{e^{-g_{u, j}^{2} / 2}}{\sqrt{2 \pi} g_{u, j}} \int_{\mathbb{R}} \exp \left(-w-\frac{w^{2}}{2 g_{u, j}^{2}}\right) \mathcal{I}_{u, j}(w ; x, y) d w,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{I}_{u, j}(w ; x, y)=\mathbb{P}\{ & \int_{0}^{b_{1}} \mathbb{I}\left(\sup _{s \in[0, S]}\left(\chi_{u, j}(s, t)+f_{u, j}(s, t, w)\right)>0\right) d t>x, \\
& \left.\int_{b_{2}}^{b_{3}} \mathbb{I}\left(\sup _{s \in[0, S]}\left(\chi_{u, j}(s, t)+f_{u, j}(s, t, w)-\lambda_{u, j}\right)>0\right) d t>y\right\} .
\end{aligned}
$$

Noting that

$$
\lim _{u \rightarrow \infty} \sup _{j \in S_{u}}\left|\frac{\frac{e^{-g_{u, j}^{2} / 2}}{\sqrt{2 \pi} g_{u, j}}}{\Psi\left(g_{u, j}\right)}-1\right|=0
$$

and for any $M>0$

$$
\lim _{u \rightarrow \infty} \inf _{|w| \leq M} e^{-\frac{w^{2}}{2 g_{u, j}^{2}}}=1
$$

we can establish the claim if we show that

$$
\begin{equation*}
\lim _{u \rightarrow \infty} \sup _{j \in S_{u}}\left|\int_{\mathbb{R}} \exp (-w) \mathcal{I}_{u, j}(w, x, y) d w-\mathcal{B}_{\alpha / 2}^{x, y}\left(b_{1} ; \lambda, b_{2}, b_{3}\right)([0, S])\right|=0 . \tag{35}
\end{equation*}
$$

Weak convergence. We next show the weak convergence of $\left\{\chi_{u, j}(s, t)+f_{u, j}(s, t, w),(s, t) \in E\right\}$ as $u \rightarrow \infty$. By C1 and C2 we have, for $(s, t),\left(s^{\prime}, t^{\prime}\right) \in E$, as $u \rightarrow \infty$, uniformly with respect to $j \in S_{u}$

$$
\begin{aligned}
& \operatorname{Var}\left(\chi_{u, j}(s, t)-\chi_{u, j}\left(s^{\prime}, t^{\prime}\right)\right)=g_{u, j}^{2}\left(\mathbb{E}\left\{\xi_{u, j}(s, t)-\xi_{u, j}\left(s^{\prime}, t^{\prime}\right)\right\}^{2}-\left(\rho_{\xi_{u, j}}(s, t)-\rho_{\xi_{u, j}}\left(s^{\prime}, t^{\prime}\right)\right)^{2}\right) \\
& \rightarrow 2 \operatorname{Var}\left(\zeta(s, t)-\zeta\left(s^{\prime}, t^{\prime}\right)\right),
\end{aligned}
$$

where $\zeta(s, t)=B_{\alpha / 2}(s)+B_{\alpha / 2}^{\prime}(t),(s, t) \in E$ with $B$ and $B^{\prime}$ being independent fBm's. This implies that the finite-dimensional distributions of $\left\{\chi_{u, j}(s, t),(s, t) \in E\right\}$ weakly converge to that of $\{\sqrt{2} \zeta(s, t),(s, t) \in E\}$ as $u \rightarrow \infty$ uniformly with respect to $j \in S_{u}$. Moreover, it follows from $\mathbf{C} 2$ that, for $u$ sufficiently large
$\operatorname{Var}\left(\chi_{u, j}(s, t)-\chi_{u, j}\left(s^{\prime}, t^{\prime}\right)\right) \leq g_{u, j}^{2} \mathbb{E}\left\{\xi_{u, j}(s, t)-\xi_{u, j}\left(s^{\prime}, t^{\prime}\right)\right\}^{2} \leq 4\left(\left|s-s_{1}\right|^{\alpha}+\left|t-t_{1}\right|^{\alpha}\right),(s, t),\left(s_{1}, t_{1}\right) \in E$.
This implies that uniform tightness of $\left\{\chi_{u, j}(s, t),(s, t) \in E\right\}$ for large $u$ with respect to $j \in S_{u}$. Hence $\left\{\chi_{u, j}(s, t),(s, t) \in E\right\}$ weakly converges to $\{\sqrt{2} \zeta(s, t),(s, t) \in E\}$ as $u \rightarrow \infty$ uniformly with respect to $j \in S_{u}$. Additionally, by C1-C2, $\left\{f_{u, j}(s, t, w),(s, t) \in E\right\}$ converges to $\left\{w-|s|^{\alpha}-|t|^{\alpha},(s, t) \in E\right\}$ uniformly with respect to $j \in S_{u}$. Therefore, we conclude that as $u \rightarrow \infty,\left\{\chi_{u, j}(s, t)+f_{u, j}(s, t, w),(s, t) \in\right.$ $E\}$ weakly converges to $\left\{\sqrt{2} \zeta(s, t)+w-|s|^{\alpha}-|t|^{\alpha},(s, t) \in E\right\}$ uniformly with respect to $j \in S_{u}$. Then continuous mapping theorem implies that

$$
\left\{z_{u, j}(t, w)=\sup _{s \in[0, S]}\left(\chi_{u, j}(s, t)+f_{u, j}(s, t, w)\right), t \in[0, b]\right\}
$$

weakly converges to

$$
\left\{z(t)+w=\sup _{s \in[0, S]}\left(\sqrt{2} \zeta(s, t)+w-|s|^{\alpha}-|t|^{\alpha}\right), t \in[0, b]\right\}
$$

uniformly with respect to $j \in S_{u}$ for each $w \in \mathbb{R}$.
Repeating the arguments, in view of $\mathbf{C} 3$ the same convergence holds for $\chi_{u, j}(s, t)+f_{u, j}(s, t, w)+\lambda_{u, j}$. In order to show the weak convergence of

$$
\left(\int_{0}^{b_{1}} \mathbb{I}\left(z_{u, j}(t, w)>0\right) d t, \int_{b_{2}}^{b_{3}} \mathbb{I}\left(z_{u, j}(t, w)-\lambda_{u, j}>0\right) d t\right), \quad u \rightarrow \infty
$$

we have to prove that

$$
\left(\int_{0}^{b_{1}} \mathbb{I}(f(t)>0) d t, \int_{b_{2}}^{b_{3}} \mathbb{I}(f(t)>\lambda) d t\right)
$$

is a continuous functional from $C\left(\left[0, b_{1}\right] \cup\left[b_{2}, b_{3}\right]\right)$ to $\mathbb{R}^{2}$ except a zero probability subset of $C\left(\left[0, b_{1}\right] \cup\right.$ $\left.\left[b_{2}, b_{3}\right]\right)$ under the probability induced by $\left\{z(t)+w, t \in\left[0, b_{1}\right] \cup\left[b_{2}, b_{3}\right]\right\}$. The idea of the proof follows from Lemma 4.2 of [22]. Observe that the discontinuity set is

$$
E^{*}=\left\{f \in C\left(\left[0, b_{1}\right] \cup\left[b_{2}, b_{3}\right]\right): \int_{\left[0, b_{1}\right]} \mathbb{I}(f(t)=0) d t>0 \text { or } \int_{\left[b_{1}, b_{2}\right]} \mathbb{I}(f(t)=\lambda) d t>0\right\} .
$$

Note that for any $c \in \mathbb{R}$

$$
\int_{\mathbb{R}} \mathbb{E}\left(\int_{\left[0, b_{1}\right] \cup\left[b_{2}, b_{3}\right]} \mathbb{I}(z(t)+w=c) d t\right) d w=\int_{\left[0, b_{1}\right] \cup\left[b_{2}, b_{3}\right]} \int_{\mathbb{R}} \mathbb{P}\{z(t)+w=c\} d w d t=0 .
$$

Hence $E^{*}$ has probability zero under the probability induced by $\left\{z(t)+w, t \in\left[0, b_{1}\right] \cup\left[b_{2}, b_{3}\right]\right\}$ for a.e. $w \in \mathbb{R}$. Application of the continuous mapping theorem yields that

$$
\left(\int_{0}^{b_{1}} \mathbb{I}\left(z_{u, j}(t, w)>0\right) d t, \int_{b_{2}}^{b_{3}} \mathbb{I}\left(z_{u, j}(t, w)>\lambda\right) d t\right)
$$

weakly converges to

$$
\left(\int_{0}^{b_{1}} \mathbb{I}(z(t)+w>0) d t, \int_{b_{2}}^{b_{3}} \mathbb{I}(z(t)+w>\lambda) d t\right)
$$

as $u \rightarrow \infty$, uniformly with respect to $j \in S_{u}$ for a.e. $w \in \mathbb{R}$.
Convergence on continuous points. Let

$$
\mathcal{I}(w ; x, y):=\mathbb{P}\left\{\int_{0}^{b_{1}} \mathbb{I}(z(t)+w>0) d t>x, \int_{b_{2}}^{b_{3}} \mathbb{I}(z(t)+w>\lambda) d t>y\right\}
$$

Using similar arguments as in the proof of [23, Thm 1.3.1], we show that (35) holds for continuity points $(x, y)$ with $x, y>0$, i.e.,

$$
\lim _{\varepsilon \rightarrow 0} \int_{\mathbb{R}}(\mathcal{I}(w ; x+\varepsilon, y+\varepsilon)-\mathcal{I}(w ; x-\varepsilon, y-\varepsilon)) e^{-w} d w=0
$$

Note that for all $x, y>0$

$$
\begin{align*}
\mathcal{I}(w ; x, y) & \leq \mathbb{P}\left\{\sup _{(s, t) \in E} \sqrt{2} \zeta(s, t)-|s|^{\alpha}-|t|^{\alpha}>-w\right\} \\
& \leq \mathbb{P}\left\{\sup _{(s, t) \in E} \sqrt{2} \zeta(s, t)>-w+C\right\} \\
& \leq C_{1} e^{-C w^{2}}, w<-M \tag{36}
\end{align*}
$$

for $M$ sufficiently large, where $C, C_{1}$ are positive constants and in the last inequality, we used the Piterbarg inequality [20, Thm 8.1]. Hence the dominated convergence theorem gives

$$
\int_{\mathbb{R}}(\mathcal{I}(w ; x+, y+)-\mathcal{I}(w ; x-, y-)) e^{-w} d w=0
$$

This implies that if $(x, y)$ is a continuity point, then $\mathcal{I}(w ;)$ is continuous at $(x, y)$ for a.e. $w \in \mathbb{R}$. Hence if $(x, y)$ is a continuity point, then

$$
\begin{equation*}
\lim _{u \rightarrow \infty} \sup _{j \in S_{u}}\left|\mathcal{I}_{u, j}(w ; x, y)-\mathcal{I}(w ; x, y)\right|=0 \text {, for a.e. } w \in \mathbb{R} . \tag{37}
\end{equation*}
$$

Applying again the Piterbarg inequality, analogously as in (36), we obtain

$$
\begin{equation*}
\sup _{j \in S_{u}} \mathcal{I}_{u, j}(w ; x, y) \leq C_{1} e^{-C w^{2}}, w<-M \tag{38}
\end{equation*}
$$

for $M$ and $u$ sufficiently large. Consequently, in view of (36), (37) and (38), the dominated convergence theorem establishes (35).
Continuity of $\mathcal{B}_{\alpha / 2}^{x, y}\left(b_{1} ; \lambda, b_{2}, b_{3}\right)([0, S])$. Clearly, $\mathcal{B}_{\alpha / 2}^{x, y}\left(b_{1} ; \lambda, b_{2}, b_{3}\right)([0, S])$ is right-continuous at $(x, y)=$ $\overline{(0,0)}$. We next focus on its continuity over $\left(\left[0, b_{1}\right) \times\left[0, b_{3}-b_{2}\right)\right) \backslash\{(0,0)\}$. To show $\mathcal{B}_{\alpha / 2}^{x, y}\left(b_{1} ; \lambda, b_{2}, b_{3}\right)([0, S])$ is continuous at $(x, y) \in\left(0, b_{1}\right) \times\left(0, b_{3}-b_{2}\right)$, it suffices to prove that

$$
\int_{\mathbb{R}} e^{-w}\left(\mathbb{P}\left\{\int_{0}^{b_{1}} \mathbb{I}(z(t)+w>0) d t=x\right\}+\mathbb{P}\left\{\int_{b_{2}}^{b_{3}} \mathbb{I}(z(t)+w>\lambda) d t=y\right\}\right) d w=0
$$

Denote $A_{w}=\left\{z_{\kappa}(t): \int_{0}^{b_{1}} \mathbb{I}\left(z_{\kappa}(t)+w>0\right) d t=x\right\}$, where $z_{\kappa}(t)=z(t)(\kappa)$ with $\kappa \in \Omega$ the sample space. In light of the continuity of $z_{\kappa}(t)$, if $\int_{0}^{b_{1}} \mathbb{I}\left(z_{\kappa}(t)+w>0\right) d t=x$ for $x \in\left(0, b_{1}\right)$ and $w^{\prime}>w$, then

$$
\int_{0}^{b_{1}} \mathbb{I}\left(z_{\kappa}(t)+w^{\prime}>0\right) d t>x .
$$

Hence $A_{w} \cap A_{w^{\prime}}=\emptyset$ if $w \neq w^{\prime}$. Noting that the continuity of $z(s)$ guarantees the measurability of $A_{w}$, and

$$
\sup _{\Lambda \subset \mathbb{R}, \# \Lambda<\infty} \sum_{w \in \Lambda} \mathbb{P}\left\{A_{w}\right\} \leq 1,
$$

where $\# \Lambda$ stands for the cardinality of the set $\Lambda$.
Note in passing the important fact that $\mathbb{P}$-measurability of $A_{w}$ is a consequence of the Fubini-Tonelli theorem. Next, it follows that

$$
\left\{w: w \in \mathbb{R} \text { such that } \mathbb{P}\left\{A_{w}\right\}>0\right\}
$$

is a countable set, which implies that for $x \in\left(0, b_{1}\right)$

$$
\int_{\mathbb{R}} \mathbb{P}\left\{A_{w}\right\} e^{-w} d w=0
$$

Using similar argument, we can show

$$
\int_{\mathbb{R}} e^{-w} \mathbb{P}\left\{\int_{b_{2}}^{b_{3}} \mathbb{I}(z(t)+w>\lambda) d t=y\right\} d w=0
$$

Therefore, we conclude that $\mathcal{B}_{\alpha / 2}^{x, y}\left(b_{1} ; \lambda, b_{2}, b_{3}\right)([0, S])$ is continuous at $(x, y) \in\left(0, b_{1}\right) \times\left(0, b_{3}-b_{2}\right)$. Analogously, we can show the continuity on $\{0\} \times\left(0, b_{3}-b_{2}\right)$ and $\left(0, b_{1}\right) \times\{0\}$. This completes the proof.

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