

On parallel implementation of Sequential Monte Carlo methods: the island particle model

Christelle Vergé · Cyrille Dubarry · Pierre

Del Moral · Eric Moulines

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Abstract The approximation of the Feynman-Kac semigroups by systems of interacting particles is a very active research field, with applications in many different areas. In this paper, we study the parallelization of such approximations. The total population of particles is divided into sub-populations, referred to as *islands*. The particles within each island follow the usual selection / mutation dynamics. We

Pierre Del Moral

Centre INRIA Bordeaux Sud Ouest - 351 Cours de la Libération, 33405 Talence Cedex, E-mail:
pierre.del-moral@inria.fr

Cyrille Dubarry

SAMOVAR, CNRS UMR 5157 - Institut Télécom/Télécom SudParis, 9 rue Charles Fourier,
91000 Evry

Eric Moulines

LTCI, CNRS UMR 8151 - Institut Télécom/Télécom ParisTech, 46 rue Barrault, 75634 Paris
Cedex 13, France, E-mail: eric.moulines@telecom-paristech.fr

Christelle Vergé

ONERA - The French Aerospace Lab, F-91761 Palaiseau,

CNES - 18 avenue Edouard Belin, 31401 Toulouse Cedex 9, E-mail: christelle.verge@onera.fr

show that the evolution of each island is also driven by a Feynman-Kac semigroup, whose transition and potential can be explicitly related to ones of the original problem. Therefore, the same genetic type approximation of the Feynman-Kac semi-group may be used at the island level; each island might undergo selection / mutation algorithm. We investigate the impact of the population size within each island and the number of islands, and study different type of interactions. We find conditions under which introducing interactions between islands is beneficial. The theoretical results are supported by some Monte Carlo experiments.

Keywords Particle approximation of Feynman-Kac flow, Island models, parallel implementation

1 Introduction

Numerical approximation of Feynman-Kac semigroups by systems of interacting particles is a very active field of researchs. Interacting particle systems are increasingly used to sample complex high dimensional distributions in a wide range of applications including nonlinear filtering, data assimilation problems, rare event sampling, hidden Markov chain parameter estimation, stochastic control problems, financial mathematics; see for example [8], [2], [4], [1], [6] and the references therein.

Let $(\mathbb{E}_n, \mathcal{E}_n)_{n \geq 0}$ be a sequence of measurable spaces. Denote by $\mathcal{B}_b(\mathbb{E}_n)$ the Banach space of all bounded and measurable real valued functions f on \mathbb{E}_n , equipped with the uniform norm. Let $(g_n)_{n \in \mathbb{N}}$ be a sequence of measurable *potential functions*, $g_n : \mathbb{E}_n \rightarrow \mathbb{R}^+$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. In the sequel, all the processes are defined on this probability space. Let $(X_n)_{n \in \mathbb{N}}$ be a non-homogenous

Markov chain on the sequence of state-spaces $(\mathbb{E}_n)_{n \in \mathbb{N}}$ with initial distribution η_0 on $(\mathbb{E}_0, \mathcal{E}_0)$ and Markov kernels $(M_n)_{n \in \mathbb{N}^*}$ ¹. We associate to the sequences of potential functions $(g_n)_{n \in \mathbb{N}}$ and Markov kernels $(M_n)_{n \in \mathbb{N}^*}$ the sequence of *Feynman-Kac measures*, defined for all $n \geq 1$ and for any $f_n \in \mathcal{B}_b(\mathbb{E}_n)$ by

$$\eta_n(f_n) \stackrel{\text{def}}{=} \gamma_n(f_n) / \gamma_n(1), \quad (1)$$

$$\gamma_n(f_n) \stackrel{\text{def}}{=} \mathbb{E} \left[f_n(X_n) \prod_{0 \leq p < n} g_p(X_p) \right] \quad (2)$$

$$= \int \gamma_0(dx_0) \left[\prod_{0 \leq p < n} g_p(x_p) M_{p+1}(x_p, dx_{p+1}) \right] f_n(x_n), \quad (3)$$

where we have set by convention $\eta_0(f_0) = \gamma_0(f_0) \stackrel{\text{def}}{=} \mathbb{E}[f_0(X_0)]$.

The sequences of distributions $(\eta_n)_{n \geq 0}$ and $(\gamma_n)_{n \geq 0}$ are approximated sequentially using interacting particle systems (IPS). Such particle approximations are often referred to as sequential Monte Carlo (SMC) methods. The IPS consists in approximating for each $n \in \mathbb{N}$ the probability η_n by a set of N_1 *particles* $(X_n^i)_{i=1}^{N_1}$ which are generated recursively. Typically, the update of the particles may be decomposed into a mutation and a selection step. For example, the bootstrap algorithm proceeds as follows. In the *selection step* the particles are first sampled with weights proportional to the potential functions. In the *mutation step*, a new generation of particles $(X_{n+1}^i)_{i=1}^{N_1}$ is generated from the selected particles using the kernel M_{n+1} . The asymptotic behavior of such particle approximation is now well understood (see [4] and [6]).

Feynman-Kac measures appear naturally in the filtering problem for Hidden Markov Model (HMM). Recall that a HMM is a pair of discrete time random

¹ a Markov kernel on $\mathbb{E}_n \times \mathcal{E}_{n+1}$ is a function $M_{n+1} : \mathbb{E}_n \times \mathcal{E}_{n+1} \rightarrow [0; 1]$, such that, for all $x_n \in \mathbb{E}_n$, $A_{n+1} \mapsto M_{n+1}(x_n, A_{n+1})$ is a probability measure on $(\mathbb{E}_{n+1}, \mathcal{E}_{n+1})$ and for any $A_{n+1} \in \mathcal{E}_{n+1}$, $x_n \mapsto M_{n+1}(x_n, A_{n+1})$ is a measurable function.

processes $(X, Y) = (X_n, Y_n)_{n \in \mathbb{N}}$, where $(X_n)_{n \geq 0}$ is the hidden state process (often called signal) and $(Y_n)_{n \geq 0}$ are the observations. To fix the ideas, X_n and Y_n take values in $\mathbb{X} \subset \mathbb{R}^k$ and $\mathbb{Y} \subset \mathbb{R}^l$. The state sequence is assumed to be a Markov chain with transition probability density $m(x, x')$ and initial density m_0 (both with respect to some common dominating measure μ). In this case, for all $n \geq 0$, $\mathbb{E}_n = \mathbb{X}$ and for all $A \in \mathcal{B}(\mathbb{X})$, $M_n(x, A) = \int_A m(x, x') \mu(dx')$, where $\mathcal{B}(\mathbb{X})$ is the Borel σ -field. The observations $(Y_n)_{n \geq 0}$ are conditionally independent given X and for all $n \in \mathbb{N}^*$, Y_n has a conditional density $g(X_n, \cdot)$ with respect to a reference measure ν such that $\mathbb{P}(Y_n \in B | X_n) = \int_B g(X_n, y) \nu(dy)$, for all $B \in \mathcal{B}(\mathbb{Y})$. Here the potential functions are the likelihood of the observations $g_n(x) = g(x, Y_n)$. In such settings, γ_n is the joint distribution of X_n and Y_0, \dots, Y_{n-1} , η_n is the predictive distribution of X_n conditionally on Y_0, \dots, Y_{n-1} , and $\gamma_n(1)$ is the likelihood of the sequence of observations Y_0, \dots, Y_{n-1} .

Particle filtering is computationally an intensive method. Parallel computations provides an appealing solution to tackle this issue (see [9] and the references therein for an in-depth description of parallelization of Bayesian computations). The basic idea to implement interacting particle system in parallel goes as follows: instead of considering a single large batch of $N = N_1 N_2$ particles, the population is divided into N_2 batches of N_1 particles. These batches are referred in the sequel to as *islands*. The terminology *island* is borrowed from dynamic populations theory (like the genetic type interacting particle model). The particles within each island are selected and mutates, as described above. We might also introduce interactions among *islands*.

In this paper we introduce the *island particle models*. As we will see below, we may cast the island particle model in the Feynman-Kac framework, with appro-

priately defined potentials and transition kernels. The key observation is that the marginal distribution of the island Feynman-Kac model w.r.t. any individual coincide with (3). This interpretation allows to use the interacting particle model at the island level.

The study of the island particle model gives rise to several challenging theoretical questions. In this paper, we investigate the impact of the number of particles in each island N_1 compared to the number of islands N_2 for a given total number of particles $N \stackrel{\text{def}}{=} N_1 N_2$, for the double bootstrap algorithm, where the bootstrap mechanism is used both within and between the islands. We focus on the asymptotic bias and variance when both N_1 and N_2 goes to infinity. Fluctuation theorem and non-asymptotic results will be present in a forthcoming paper. We also investigate when and why introducing interactions at the island level improves the accuracy of the particle approximation. Intuitively, the trade-off might be understood as follows. When the N_2 islands are run independently, the bias induced in each island only depends on their population size N_1 ; when N_1 is small compared to the total number N , the bias will be large (and is of course not reduced by averaging across the islands). To reduce the bias, introducing an interaction between the islands is beneficial. However, this interaction increases the variance, due to the selection step. If we consider the mean squared error, the interaction is beneficial when the improvement associated to the bias correction is not offset by the variance increase. When the number of particles N_1 within each island is *small* and the number of islands N_2 is *large*, then the interaction is typically beneficial. On the contrary, when $N_2 \ll N_1$, the interaction between islands may increase the mean squared error. We then propose a method, based on a generalization

of the effective sample size, this time computed at the island level, which always achieve a lower mean squared error than the independent island model.

The paper is organized as follows. In section 2 the interacting particle approximation of the Feynman-Kac model is first reviewed. The island Feynman-Kac model is then introduced. We first investigate the double bootstrap algorithm, in which selection and mutation are applied at each iteration within and across the islands. The asymptotic bias and variance of this algorithm is presented in section 3. The Feynman-Kac interpretation of the island model leads to several interacting island algorithms, based on different approximations of Feynman-Kac flows. Some of these are introduced and analyzed in section 4. Some numerical experiments are reported to support our findings and illustrate the impact of the numbers of islands and particles within each island in section 5.

2 Algorithm derivation

In this section, we introduce the island particle model. We first briefly recall the bootstrap approximation of Feynman-Kac measures.

According to the definitions (1) and (3) of the sequences of the Feynman-Kac measures $(\eta_n)_{n \in \mathbb{N}}$ and $(\gamma_n)_{n \in \mathbb{N}}$, for all $f_{n+1} \in \mathcal{B}_b(\mathbb{E}_{n+1})$ we get

$$\gamma_{n+1}(f_{n+1}) = \eta_{n+1}(f_{n+1})\gamma_{n+1}(1),$$

and since,

$$\gamma_{n+1}(1) = \gamma_n(g_n) = \eta_n(g_n)\gamma_n(1),$$

an easy induction shows that

$$\gamma_{n+1}(1) = \prod_{0 \leq p < n+1} \eta_p(g_p)$$

and then,

$$\gamma_{n+1}(f_{n+1}) = \eta_{n+1}(f_{n+1}) \prod_{0 \leq p < n+1} \eta_p(g_p). \quad (4)$$

Moreover, the sequence $(\eta_n)_{n \in \mathbb{N}}$ satisfy a nonlinear recursive relation. Indeed,

$$\eta_{n+1}(f_{n+1}) = \frac{\gamma_n(g_n M_{n+1} f_{n+1})}{\gamma_n(g_n M_{n+1} 1)} = \frac{\eta_n(g_n M_{n+1} f_{n+1})}{\eta_n(g_n)}. \quad (5)$$

Let $\mathcal{P}(\mathbb{E}_n)$ be the set of probability measures on \mathbb{E}_n . Using the Boltzmann-Gibbs transformation $\Psi_n : \mathcal{P}(\mathbb{E}_n) \rightarrow \mathcal{P}(\mathbb{E}_n)$, defined for all $\mu_n \in \mathcal{P}(\mathbb{E}_n)$ by

$$\Psi_n(\mu_n)(dx_n) \stackrel{\text{def}}{=} \frac{g_n(x_n) \mu_n(dx_n)}{\mu_n(g_n)}, \quad (6)$$

the recursion (5) may be rewritten as

$$\eta_{n+1} = \Psi_n(\eta_n) M_{n+1}. \quad (7)$$

The sequence of probability $(\eta_n)_{n \in \mathbb{N}}$ can be approximated using the bootstrap algorithm. Other approximations can also be considered as well, but we only introduce the bootstrap for notational simplicity. Let N_1 be a positive integer. For any nonnegative integer n we denote by

$$(\mathbf{E}_n, \mathcal{E}_n) \stackrel{\text{def}}{=} (\mathbb{E}_n^{N_1}, \mathcal{E}_n^{\otimes N_1}), \quad (8)$$

the product space (the dependence of \mathbf{E}_n and \mathcal{E}_n in N_1 is implicit). Thereafter, we omit to write the σ -field \mathcal{E}_n when there will be no confusion. We define the Markov kernel $M_{n+1}(\mathbf{x}_n, d\mathbf{x}_{n+1})$ from \mathbf{E}_n into \mathbf{E}_{n+1} as follows: for any $\mathbf{x}_n = (x_n^1, \dots, x_n^{N_1}) \in \mathbf{E}_n$, we set

$$M_{n+1}(\mathbf{x}_n, d\mathbf{x}_{n+1}) \stackrel{\text{def}}{=} \prod_{1 \leq i \leq N_1} \sum_{j=1}^{N_1} \frac{g_n(x_n^j)}{\sum_{k=1}^{N_1} g_n(x_n^k)} M_{n+1}(x_n^j, dx_{n+1}^i). \quad (9)$$

In other words, this transition can be interpreted as follows:

- In the *selection step*, the components of the vector \mathbf{x}_n are selected with probabilities proportional to their potential $\{g_n(x_n^i)\}_{i=1}^{N_1}$;
- In the *mutation step*, the selected coordinates move conditionally independently to new positions using the Markov kernel M_{n+1} .

Let us introduce the particles and their evolution. Define by $(\mathbf{X}_n)_{n \geq 0}$ the Markov chain where for each $n \in \mathbb{N}$,

$$\mathbf{X}_n = (X_n^1, \dots, X_n^{N_1}) \in \mathbb{E}_n, \quad (10)$$

with initial distribution $\boldsymbol{\eta}_0 \stackrel{\text{def}}{=} \eta_0^{\otimes N_1}$ and transition kernel M_{n+1} . Denote by m^{N_1} the empirical measure on \mathbb{E}_n , defined as the kernel on $\mathbb{E}_n \times \mathbb{E}_n$ by

$$m^{N_1}(\mathbf{x}_n, dz_n) \stackrel{\text{def}}{=} \frac{1}{N_1} \sum_{i=1}^{N_1} \delta_{x_n^i}(dz_n),$$

where δ_{x_n} is the dirac mass at $x_n \in \mathbb{E}_n$. Equation (4) suggests the following N_1 -particle approximations of the measures η_n and γ_n respectively defined for $f_n \in \mathcal{B}_b(\mathbb{E}_n)$ by

$$\eta_n^{N_1}(f_n) \stackrel{\text{def}}{=} m^{N_1} f_n(\mathbf{X}_n) = \frac{1}{N_1} \sum_{i=1}^{N_1} f_n(\mathbf{X}_n^i) \quad (11)$$

$$\gamma_n^{N_1}(f_n) \stackrel{\text{def}}{=} \eta_n^{N_1}(f_n) \prod_{0 \leq p < n} \eta_p^{N_1}(g_p) = \eta_n^{N_1}(f_n) \gamma_n^{N_1}(1). \quad (12)$$

For $\mathbf{x}_n = (x_n^1, \dots, x_n^{N_1}) \in \mathbb{E}_n$, define the potential function

$$\mathbf{g}_n(\mathbf{x}_n) \stackrel{\text{def}}{=} m^{N_1} g_n(\mathbf{x}_n) = \frac{1}{N_1} \sum_{i=1}^{N_1} g_n(x_n^i). \quad (13)$$

The sequences of transition kernels $(M_n)_{n \in \mathbb{N}}$ and potential functions $(\mathbf{g}_n)_{n \in \mathbb{N}}$ given by (9) and (13), respectively, define the Feynman-Kac process. The associated sequences of Feynman-Kac measures are defined, for each $f_n \in \mathcal{B}_b(\mathbb{E}_n)$, by

the following recursions

$$\boldsymbol{\eta}_0(\mathbf{f}_0) \stackrel{\text{def}}{=} \boldsymbol{\gamma}_0(\mathbf{f}_0) = \mathbb{E}[\mathbf{f}_0(\mathbf{X}_0)] , \quad (14)$$

$$\boldsymbol{\eta}_n(\mathbf{f}_n) \stackrel{\text{def}}{=} \boldsymbol{\gamma}_n(\mathbf{f}_n)/\boldsymbol{\gamma}_n(1), \quad \text{for all } n \geq 1, \quad (15)$$

$$\boldsymbol{\gamma}_n(\mathbf{f}_n) \stackrel{\text{def}}{=} \mathbb{E} \left[\mathbf{f}_n(\mathbf{X}_n) \prod_{0 \leq p < n} \mathbf{g}_p(\mathbf{X}_p) \right], \quad \text{for all } n \geq 1. \quad (16)$$

where $(\mathbf{X}_n)_{n \geq 0}$ is a Markov chain with initial distribution $\boldsymbol{\gamma}_0$ and transition kernel \mathbf{M}_n . The key result, justifying the introduction of the island particle models, is the following theorem which links $(\eta_n, \gamma_n)_{n \geq 0}$ and $(\boldsymbol{\eta}_n, \boldsymbol{\gamma}_n)_{n \geq 0}$.

Theorem 1 For any $\mathbf{f}_n \in \mathcal{B}_b(\mathbf{E}_n)$ of the form $\mathbf{f}_n(\mathbf{x}_n) = N_1^{-1} \sum_{i=1}^{N_1} f_n(x_n^i)$ where $f_n \in \mathcal{B}_b(\mathbb{E}_n)$,

$$\boldsymbol{\gamma}_n(\mathbf{f}_n) = \gamma_n(f_n) \quad \text{and} \quad \boldsymbol{\eta}_n(\mathbf{f}_n) = \eta_n(f_n) . \quad (17)$$

Proof See subsection 6.1.

The Feynman-Kac model $(\boldsymbol{\gamma}_n, \boldsymbol{\eta}_n)_{n \geq 0}$ can be approximated by an interacting particle system *at the island level*. We first describe the *double bootstrap* algorithm where the bootstrap is also applied across the islands (this algorithm shares some similarities with [3]). This is only one of the many possible algorithms that can be derived from this interpretation of the Feynman-Kac model at the island level; see section 4 for other approximations.

Define by $\mathcal{P}(\mathbf{E}_n)$ the set of probabilities measures on \mathbf{E}_n . One can easily check that the sequence of measures $(\boldsymbol{\eta}_n)_{n \geq 0}$ satisfies the following recursion

$$\boldsymbol{\eta}_{n+1} = \boldsymbol{\Psi}_n(\boldsymbol{\eta}_n) \mathbf{M}_{n+1} , \quad (18)$$

where $\Psi_n : \mathcal{P}(\mathbb{E}_n) \rightarrow \mathcal{P}(\mathbb{E}_n)$ is the Boltzmann-Gibbs transformation defined for any $\mu_n \in \mathcal{P}(\mathbb{E}_n)$ by

$$\Psi_n(\mu_n)(d\mathbf{x}_n) \stackrel{\text{def}}{=} \frac{\mathbf{g}_n(\mathbf{x}_n) \mu_n(d\mathbf{x}_n)}{\mu_n(\mathbf{g}_n)}.$$

Let N_2 be a positive integer. We define the Markov kernel \mathcal{M}_{n+1} from $(\mathbb{E}_n^{N_2}, \mathcal{E}_n^{\otimes N_2})$ to $(\mathbb{E}_{n+1}^{N_2}, \mathcal{E}_{n+1}^{\otimes N_2})$ as follows: for any $(\mathbf{x}_n^1, \dots, \mathbf{x}_n^{N_2}) \in \mathbb{E}_n^{N_2}$ and $(\mathbf{x}_{n+1}^1, \dots, \mathbf{x}_{n+1}^{N_2}) \in \mathbb{E}_{n+1}^{N_2}$, we put

$$\begin{aligned} \mathcal{M}_{n+1}((\mathbf{x}_n^1, \dots, \mathbf{x}_n^{N_2}), d(\mathbf{x}_{n+1}^1, \dots, \mathbf{x}_{n+1}^{N_2})) \\ \stackrel{\text{def}}{=} \prod_{1 \leq i \leq N_2} \sum_{j=1}^{N_2} \frac{\mathbf{g}_n(\mathbf{x}_n^j)}{\sum_{k=1}^{N_2} \mathbf{g}_n(\mathbf{x}_n^k)} \mathcal{M}_{n+1}(\mathbf{x}_n^j, d\mathbf{x}_{n+1}^i). \end{aligned} \quad (19)$$

For each $n \in \mathbb{N}$, $(\mathbf{X}_n^1, \dots, \mathbf{X}_n^{N_2}) \in \mathbb{E}_n^{N_2}$ is a population of N_2 interacting islands each with N_1 individuals. The process $\{(\mathbf{X}_n^1, \dots, \mathbf{X}_n^{N_2})\}_{n \geq 0}$ is a Markov chain with the transition kernel $(\mathcal{M}_{n+1})_{n \geq 0}$.

In this interpretation, the N_2 -particle model defined above can be seen as an interacting particle approximation of the island Feynman-Kac measures $\{(\boldsymbol{\eta}_n, \boldsymbol{\gamma}_n)\}_{n \geq 0}$.

The transition \mathcal{M}_{n+1} can be interpreted as follows:

- In the *selection step*, we sample randomly N_2 islands among the current islands $(\mathbf{X}_n^i)_{1 \leq i \leq N_2} \in \mathbb{E}_n^{N_2}$ with probability proportional to the empirical mean of the potentials in each island $\mathbf{g}_n(\mathbf{X}_n^i) = N_1^{-1} \sum_{j=1}^{N_1} g_n(X_n^{i,j})$, $1 \leq i \leq N_2$.
- In the *mutation transition*, the selected islands are independently updated using the Markov transition \mathcal{M}_{n+1} .

Also observe that for $N_1 = 1$, every island has a single particle. In this situation, the island Feynman-Kac model coincides with the N_2 -particle model associated with the Feynman-Kac measures η_n .

Denote by \mathbf{m}^{N_2} the empirical measure defined for any $\mathbf{f}_n \in \mathcal{B}_b(\mathbb{E}_n)$ and $(\mathbf{x}_n^1, \dots, \mathbf{x}_n^{N_2}) \in \mathbb{E}_n^{N_2}$ by

$$\mathbf{m}^{N_2} \mathbf{f}_n(\mathbf{x}_n^1, \dots, \mathbf{x}_n^{N_2}) \stackrel{\text{def}}{=} \frac{1}{N_2} \sum_{i=1}^{N_2} \mathbf{f}_n(\mathbf{x}_n^i).$$

The N_2 -particle approximations of the measures η_n and γ_n are defined for any $\mathbf{f}_n \in \mathcal{B}_b(\mathbb{E}_n)$ by

$$\eta_n^{N_2}(\mathbf{f}_n) \stackrel{\text{def}}{=} \mathbf{m}^{N_2} \mathbf{f}_n(\mathbf{X}_n^1, \dots, \mathbf{X}_n^{N_2}), \quad (20)$$

$$\gamma_n^{N_2}(\mathbf{f}_n) \stackrel{\text{def}}{=} \eta_n^{N_2}(\mathbf{f}_n) \prod_{0 \leq p < n} \eta_p^{N_2}(\mathbf{g}_p) = \eta_n^{N_2}(\mathbf{f}_n) \gamma_n^{N_2}(1). \quad (21)$$

3 Asymptotic analysis of the double bootstrap algorithm

The bootstrap particle approximation of the Feynman-Kac semigroup can be studied using the techniques introduced in [4] and further developed in [6]. For $\ell \in \mathbb{N}$, consider the finite kernel $Q_{\ell+1}$ from $(\mathbb{E}_\ell, \mathcal{E}_\ell)$ into $(\mathbb{E}_{\ell+1}, \mathcal{E}_{\ell+1})$ given for all $x_\ell \in \mathbb{E}_\ell$ by

$$Q_{\ell+1}(x_\ell, dx_{\ell+1}) \stackrel{\text{def}}{=} g_\ell(x_\ell) M_{\ell+1}(x_\ell, dx_{\ell+1}). \quad (22)$$

For $p < n$, define by $Q_{p,n}$ the finite kernel from $(\mathbb{E}_p, \mathcal{E}_p)$ into $(\mathbb{E}_n, \mathcal{E}_n)$ as the following product

$$Q_{p,n} \stackrel{\text{def}}{=} Q_{p+1} Q_{p+2} \dots Q_n, \quad (23)$$

and set by convention $Q_{n,n} \stackrel{\text{def}}{=} I_n$ where I_n is the identity kernel on $(\mathbb{E}_n, \mathcal{E}_n)$. With this definition, the linear semigroup associated with the sequence of unnormalized Feynman-Kac measures $(\gamma_n)_{n \in \mathbb{N}}$ may be equivalently expressed as follows

$$\gamma_n = \gamma_p Q_{p,n}. \quad (24)$$

Algorithm 1 Bootstrap within bootstrap island filter

- 1: Initialization:
 - 2: **for** i from 1 to N_2 **do**
 - 3: Sample N_1 independent random variables $\mathbf{X}_0^i = (X_0^{i,j})_{j=1}^{N_1}$ from η_0 .
 - 4: **end for**
 - 5: **for** p from 0 to $n - 1$ **do**
 - 6: Selection step between islands:
 - 7: Sample $\mathbf{I}_p = (I_p^i)_{i=1}^{N_2}$ multinomially with probability proportional to $\left(\frac{1}{N_1} \sum_{j=1}^{N_1} g_p(X_p^{i,j})\right)_{i=1}^{N_2}$.
 - 8: Island mutation step:
 - 9: **for** i from 1 to N_2 **do**
 - 10: Particle selection within each island:
 - 11: Sample $\mathbf{J}_p^i = (J_p^{i,j})_{j=1}^{N_1}$ multinomially with probability proportional to $\left(g_p(X_p^{i,j})\right)_{j=1}^{N_1}$.
 - 12: Particle mutation:
 - 13: For $1 \leq j \leq N_1$, sample conditionally independently $X_{p+1}^{i,j}$ from the Markov kernel $M_{p+1}(X_p^{i,j}, L_p^{i,j}, \cdot)$, where $L_p^{i,j} = J_p^{i,j}$.
 - 14: **end for**
 - 15: **end for**
 - 16: Approximate $\eta_n(f_n)$ by $\frac{1}{N_1 N_2} \sum_{i=1}^{N_2} \sum_{j=1}^{N_1} f_n(X_n^{i,j})$.
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For any $x_p \in \mathbb{E}_p$, $A_n \in \mathcal{E}_n$, $Q_{p,n}$ may be written as the following conditional expectation,

$$Q_{p,n}(x_p, A_n) = \mathbb{E} \left[\mathbf{1}_{A_n}(X_n) \prod_{p \leq q < n} g_q(X_q) \middle| X_p = x_p \right],$$

where $(X_n)_{n \geq 0}$ is the non-homogenous Markov chain on the sequence of state-spaces $(\mathbb{E}_n, \mathcal{E}_n)_{n \geq 0}$ with initial distribution η_0 and Markov kernels $(M_n)_{n \geq 1}$.

According to (1), $\eta_n = \gamma_n/\gamma_n(1)$ implies that $\eta_n = \gamma_p Q_{p,n}/\gamma_p Q_{p,n}(1)$. Denote by Φ_{n+1} the mapping from $\mathcal{P}(\mathbb{E}_n)$ to $\mathcal{P}(\mathbb{E}_{n+1})$ given, for any $\mu_n \in \mathcal{P}(\mathbb{E}_n)$ by

$$\Phi_{n+1}(\mu_n) \stackrel{\text{def}}{=} \Psi_n(\mu_n) M_{n+1} = \frac{\mu_n Q_{n+1}}{\mu_n Q_{n+1}(1)}. \quad (25)$$

Since $\eta_p = \gamma_p/\gamma_p(1)$, these relations may be equivalently rewritten as

$$\eta_n = \frac{\eta_p Q_{p,n}}{\eta_p Q_{p,n}(1)} = \Phi_{p,n}(\eta_p), \quad (26)$$

where $\Phi_{p,n} = \Phi_n \circ \Phi_{n-1} \circ \dots \circ \Phi_{p+1}$ is the nonlinear semigroup associated to the normalized Feynman-Kac measures $(\eta_n)_{n \geq 0}$. This nonlinear semigroup may be associated to the *potential kernels*

$$P_{p,n} \stackrel{\text{def}}{=} \frac{Q_{p,n}}{\eta_p Q_{p,n}(1)} = \frac{\gamma_p(1)}{\gamma_n(1)} Q_{p,n}, \quad (27)$$

and therefore

$$\eta_n = \eta_p P_{p,n}. \quad (28)$$

For $\ell \in \mathbb{N}$, consider the finite kernel $Q_{\ell+1}$ from $(\mathbb{E}_\ell, \mathcal{E}_\ell)$ into $(\mathbb{E}_{\ell+1}, \mathcal{E}_{\ell+1})$ for any $\mathbf{x}_\ell \in \mathbb{E}_\ell$ by

$$Q_{\ell+1}(\mathbf{x}_\ell, d\mathbf{x}_{\ell+1}) \stackrel{\text{def}}{=} g_\ell(\mathbf{x}_\ell) M_{\ell+1}(\mathbf{x}_\ell, d\mathbf{x}_{\ell+1}),$$

where M_ℓ is defined in (9) and g_ℓ in (13). For $p \leq n$, define by $Q_{p,n}$ the finite kernel from $(\mathbb{E}_p, \mathcal{E}_p)$ into $(\mathbb{E}_n, \mathcal{E}_n)$ by the equation $Q_{p,n} \stackrel{\text{def}}{=} Q_{p+1} Q_{p+2} \dots Q_n$.

Note that, for any $\mathbf{x}_p \in \mathbb{E}_p$, $\mathbf{A}_n \in \mathcal{E}_n$,

$$Q_{p,n}(\mathbf{x}_p, A_n) = \mathbb{E} \left[\mathbb{1}_{\mathbf{A}_n}(\mathbf{X}_n) \prod_{p \leq q < n} g_q(\mathbf{X}_q) \middle| \mathbf{X}_p = \mathbf{x}_p \right],$$

where $(\mathbf{X}_n)_{n \geq 0}$ is the island Markov chain defined in (10). With this notation, we may rewrite (14) as $\boldsymbol{\gamma}_n = \boldsymbol{\gamma}_p Q_{p,n}$. According to (15), $\eta_n = \boldsymbol{\gamma}_n/\boldsymbol{\gamma}_n(1)$ implies that $\eta_n = \boldsymbol{\gamma}_p Q_{p,n}/\boldsymbol{\gamma}_p Q_{p,n}(1)$, and then

$$\eta_n = \frac{\eta_p Q_{p,n}}{\eta_p Q_{p,n}(1)} = \eta_p P_{p,n},$$

where $P_{p,n}$ are given by

$$P_{p,n} \stackrel{\text{def}}{=} \frac{Q_{p,n}}{\eta_p Q_{p,n}(1)} = \frac{\gamma_p(1)}{\gamma_n(1)} Q_{p,n} .$$

According to Theorem 1, $\boldsymbol{\gamma}_p(1) = \gamma_p(1)$ and $\boldsymbol{\gamma}_n(1) = \gamma_n(1)$, which implies that

$$P_{p,n} = \frac{\gamma_p(1)}{\gamma_n(1)} Q_{p,n} .$$

To analyse the fluctuation of the interacting particle approximation $(\eta_n^{N_1})_{n \geq 0}$ around their limiting values $(\eta_n)_{n \geq 0}$, we introduced the *local sampling errors*. We first decompose the difference $\gamma_n^{N_1} - \gamma_n$ as follows

$$\gamma_n^{N_1} - \gamma_n = \sum_{p=1}^n \left[\gamma_p^{N_1} Q_{p,n} - \gamma_{p-1}^{N_1} Q_{p-1,n} \right] + \gamma_0^{N_1} Q_{0,n} - \gamma_n . \quad (29)$$

For any $p \geq 1$, note that

$$\begin{aligned} \gamma_{p-1}^{N_1} Q_p &= \gamma_{p-1}^{N_1}(1) \eta_{p-1}^{N_1} Q_p = \gamma_{p-1}^{N_1}(1) \eta_{p-1}^{N_1}(g_{p-1}) \Phi_p(\eta_{p-1}^{N_1}) \\ &= \gamma_{p-1}^{N_1}(1) \frac{\gamma_{p-1}^{N_1}(g_{p-1})}{\gamma_{p-1}^{N_1}(1)} \Phi_p(\eta_{p-1}^{N_1}) = \gamma_{p-1}^{N_1}(g_{p-1}) \Phi_p(\eta_{p-1}^{N_1}) = \gamma_p^{N_1}(1) \Phi_p(\eta_{p-1}^{N_1}) . \end{aligned}$$

Plugging in this relation in the local error yields to

$$\begin{aligned} \gamma_p^{N_1} Q_{p,n} - \gamma_{p-1}^{N_1} Q_{p-1,n} &= \gamma_p^{N_1} Q_{p,n} - \gamma_{p-1}^{N_1} Q_p Q_{p,n} \\ &= \left(\gamma_p^{N_1} - \gamma_p^{N_1}(1) \Phi_p(\eta_{p-1}^{N_1}) \right) Q_{p,n} = \gamma_p^{N_1}(1) \left(\eta_p^{N_1} - \Phi_p(\eta_{p-1}^{N_1}) \right) Q_{p,n} , \end{aligned}$$

which, together with (29), imply that,

$$W_n^{\gamma, N_1} \stackrel{\text{def}}{=} \sqrt{N_1} \left[\gamma_n^{N_1} - \gamma_n \right] = \sum_{p=0}^n \gamma_p^{N_1}(1) W_p^{N_1} Q_{p,n} , \quad (30)$$

where the local errors $(W_p^{N_1})_{p \geq 0}$ are defined by

$$W_0^{N_1} = \sqrt{N_1}(\eta_0^{N_1} - \eta_0) \quad \text{and} \quad W_p^{N_1} = \sqrt{N_1} \left[\eta_p^{N_1} - \Phi_p(\eta_{p-1}^{N_1}) \right], \quad \text{for all } p \geq 1 . \quad (31)$$

The following results, adapted from [4, Corollary 9.3.1, pp. 295-298], establishes the convergence of $(W_p^{N_1})_{1 \leq p \leq n}$ to centered Gaussian fields.

Theorem 2 For the bootstrap filter, for any fixed time horizon $n \geq 1$, the sequence $(W_p^{N_1})_{1 \leq p \leq n}$ converges in law, as N_1 goes to infinity, to a sequence of n independent centered Gaussian random fields $(W_p)_{0 \leq p \leq n}$ with variance given, for any bounded function $f_p \in \mathcal{B}_b(\mathbb{E}_p)$, and $1 \leq p \leq n$, by

$$\mathbb{E} \left[W_p(f_p)^2 \right] = \eta_p \left[(f_p - \eta_p f_p)^2 \right]. \quad (32)$$

Now, consider the sequence of random fields $(W_n^{\eta, N_1})_{n \geq 0}$ defined for any function $f_n \in \mathcal{B}_b(\mathbb{E}_n)$ by

$$W_n^{\eta, N_1}(f_n) \stackrel{\text{def}}{=} \sqrt{N_1} \left[\eta_n^{N_1} - \eta_n \right] (f_n) = \sqrt{N_1} \eta_n^{N_1} [f_n - \eta_n(f_n)] \quad (33)$$

$$= \sqrt{N_1} \frac{\gamma_n^{N_1}(f_n - \eta_n(f_n))}{\gamma_n^{N_1}(1)}. \quad (34)$$

Using the fact that $\gamma_n(f_n - \eta_n(f_n)) = 0$ and (30), we may write

$$W_n^{\eta, N_1}(f_n) = \sqrt{N_1} \frac{(\gamma_n^{N_1} - \gamma_n)(f_n - \eta_n(f_n))}{\gamma_n^{N_1}(1)} = \frac{W_n^{\gamma, N_1}(f_n - \eta_n(f_n))}{\gamma_n^{N_1}(1)}. \quad (35)$$

The decomposition (30) and (33), combined with the Slutsky's lemma, imply the following asymptotic decomposition (which remains valid for more general algorithms than the bootstrap algorithm)

Theorem 3 Assume that the sequence of local errors $(W_p^{N_1})_{1 \leq p \leq n}$ converges in law, as N_1 goes to infinity, to a sequence of n independent centered Gaussian random fields $(W_p)_{1 \leq p \leq n}$. Then, the sequence of random fields $(W_n^{\gamma, N_1})_{N_1 \geq 0}$ converges in law, as N_1 goes to infinity, to the Gaussian random fields W_n^γ defined for any bounded function f_n in $\mathcal{B}_b(\mathbb{E}_n)$ by

$$W_n^\gamma(f_n) \stackrel{\text{def}}{=} \sum_{p=0}^n \gamma_p(1) W_p(Q_{p,n} f_n) = \gamma_n(1) \sum_{p=0}^n W_p(P_{p,n} f_n), \quad (36)$$

where $P_{p,n}$ is defined in (27). The sequence of random fields $(W_n^{\eta, N_1})_{N_1 \geq 0}$ converges in law, as N_1 goes to infinity, to the Gaussian random fields W_n^η defined for any function

$f_n \in \mathcal{B}_b(\mathbb{E}_n)$ by

$$W_n^\eta(f_n) \stackrel{\text{def}}{=} \sum_{p=0}^n W_p(P_{p,n}(f_n - \eta_n(f_n))). \quad (37)$$

The asymptotic bias and variance for the single island interacting particle approximation of the sequence of Feynman-Kac measure formulated in the forthcoming theorem result almost immediately from Theorem 2.

Theorem 4 *Assume that the sequence of local errors $(W_p^{N_1})_{1 \leq p \leq n}$ converges in law, as N_1 goes to infinity, to a sequence of n independent centered Gaussian random fields $(W_p)_{1 \leq p \leq n}$. Then, for any time horizon $n \geq 0$ and any bounded function $f_n \in \mathcal{B}_b(\mathbb{E}_n)$, we have*

$$\begin{aligned} \lim_{N_1 \rightarrow \infty} N_1 \mathbb{E} \left[\eta_n^{N_1}(f_n) - \eta_n(f_n) \right] &= B_n(f_n), \\ \lim_{N_1 \rightarrow \infty} N_1 \text{Var} \left(\eta_n^{N_1}(f_n) \right) &= V_n(f_n), \end{aligned}$$

with

$$B_n(f_n) \stackrel{\text{def}}{=} - \sum_{p=0}^n \mathbb{E} [W_p(P_{p,n}(1))W_p(P_{p,n}(f_n - \eta_n(f_n)))] , \quad (38)$$

$$V_n(f_n) \stackrel{\text{def}}{=} \sum_{p=0}^n \mathbb{E} \left[\{W_p(P_{p,n}(f_n - \eta_n(f_n)))\}^2 \right]. \quad (39)$$

When the bootstrap algorithm is applied, we get the following expressions for $B_n(f_n)$ and $V_n(f_n)$ using Theorem 2:

$$B_n(f_n) = - \sum_{p=0}^n \eta_p(P_{p,n}(1)P_{p,n}(f_n - \eta_n(f_n))) , \quad (40)$$

$$V_n(f_n) = \sum_{p=0}^n \eta_p \left(P_{p,n}(f_n - \eta_n(f_n))^2 \right). \quad (41)$$

Proof See subsection 6.2.

We now compute the bias and the variance for the double bootstrap algorithm. The asymptotic behavior of the bias and the variance is derived in the following theorem using techniques adapted from [4].

Theorem 5 For the double bootstrap algorithm, for any time horizon $n \geq 0$ and any $f_n \in \mathcal{B}_b(\mathbb{E}_n)$, we have

$$\begin{aligned} \lim_{N_1 \rightarrow \infty} \lim_{N_2 \rightarrow \infty} N_1 N_2 \mathbb{E} \left[\boldsymbol{\eta}_n^{N_2}(m^{N_1} f_n) - \eta_n(f_n) \right] &= B_n(f_n) + \tilde{B}_n(f_n), \\ \lim_{N_1 \rightarrow \infty} \lim_{N_2 \rightarrow \infty} N_1 N_2 \text{Var} \left(\boldsymbol{\eta}_n^{N_2}(m^{N_1} f_n) \right) &= V_n(f_n) + \tilde{V}_n(f_n), \end{aligned}$$

where $B_n(f_n)$ and $V_n(f_n)$ are defined respectively in (38) and in (39), and where $\tilde{B}_n(f_n)$ and $\tilde{V}_n(f_n)$ are given by:

$$\begin{aligned} \tilde{B}_n(f_n) &\stackrel{\text{def}}{=} - \sum_{\ell=0}^n (n-\ell) \mathbb{E} \left[W_\ell(P_{\ell,n}(1)) W_\ell(P_{\ell,n}(f_n - \eta_n(f_n))) \right] \\ &\quad + \sum_{\ell=0}^n \mathbb{E} \left[W_\ell \left(\sum_{p=\ell}^n P_{\ell,p}(1) \right) W_\ell(P_{\ell,n}(f_n - \eta_n(f_n))) \right], \end{aligned} \quad (42)$$

$$\tilde{V}_n(f_n) \stackrel{\text{def}}{=} \sum_{\ell=0}^n (n-\ell) \mathbb{E} \left[W_\ell(P_{\ell,n}(f_n - \eta_n(f_n)))^2 \right]. \quad (43)$$

When the bootstrap algorithm is applied, we get the following expressions for $\tilde{B}_n(f_n)$ and $\tilde{V}_n(f_n)$ using Theorem 2:

$$\begin{aligned} \tilde{B}_n(f_n) &= - \sum_{\ell=0}^n (n-\ell) \eta_\ell \left((P_{\ell,n}(1) - \eta_n(1)) P_{\ell,n}(f_n - \eta_n(f_n)) \right) \\ &\quad + \sum_{\ell=0}^n \eta_\ell \left(\left(\sum_{p=\ell}^n (P_{\ell,p}(1) - \eta_p(1)) \right) P_{\ell,n}(f_n - \eta_n(f_n)) \right), \end{aligned} \quad (44)$$

$$\tilde{V}_n(f_n) = \sum_{\ell=0}^n (n-\ell) \eta_\ell \left(P_{\ell,n}(f_n - \eta_n(f_n)) \right)^2. \quad (45)$$

Proof See subsection 6.3.

We can also consider the case where the N_2 islands are kept independent (a bootstrap filter is still applied within each island, but there is no interaction between islands). To that purpose, denote by $(\tilde{\mathbf{X}}_n^i)_{i=1}^{N_2}$ N_2 independent islands of size N_1 , each evolving using the bootstrap filter and, define the estimator of $\eta_n(f_n)$ for

any $\mathbf{f}_n \in \mathcal{B}_b(\mathbb{E}_n)$, given by the empirical mean across islands

$$\tilde{\eta}_n^{N_2}(\mathbf{f}_n) \stackrel{\text{def}}{=} \frac{1}{N_2} \sum_{i=1}^{N_2} \mathbf{f}_n(\tilde{\mathbf{X}}_n^i).$$

For functions \mathbf{f}_n on \mathbb{E}_n of the form $\mathbf{f}_n(\mathbf{x}_n) = m^{N_1} f_n(\mathbf{x}_n)$, with $f_n \in \mathcal{B}_b(\mathbb{E}_n)$, we have

$$\tilde{\eta}_n^{N_2}(\mathbf{f}_n) = \frac{1}{N_2} \sum_{i=1}^{N_2} m^{N_1} f_n(\tilde{\mathbf{X}}_n^i) = \frac{1}{N_1 N_2} \sum_{i=1}^{N_2} \sum_{j=1}^{N_1} f_n(\tilde{\mathbf{X}}_n^{i,j}).$$

The asymptotic behavior of the bias and variance of $m^{N_1} f_n(\tilde{\mathbf{X}}_n^i)$ may be easily deduced from the one of $\eta_n^{N_1}(f_n)$; Theorem 4 implies that

Theorem 6 *For any time horizon $n \geq 0$ and any $f_n \in \mathcal{B}_b(\mathbb{E}_n)$, we have*

$$\begin{aligned} \lim_{N_1 \rightarrow \infty} N_1 \left\{ \mathbb{E} \left[\tilde{\eta}_n^{N_2}(m^{N_1} f_n) \right] - \eta_n(f_n) \right\} &= B_n(f_n), \\ \lim_{N_1 \rightarrow \infty} N_1 N_2 \text{Var} \left(\tilde{\eta}_n^{N_2}(m^{N_1} f_n) \right) &= V_n(f_n), \end{aligned}$$

where $B_n(f_n)$ and $V_n(f_n)$ are defined respectively in (38) and (39).

The variance of the particle approximation is inversely proportional to $N_1 N_2$, but because the islands do not interact, the bias is independent of N_2 and is inversely proportional to N_1 .

As shown by Theorem 5 and Theorem 6, a trade-off has to be made between the bias and the variance to decide which of the two estimators $\eta_n^{N_2}$ and $\tilde{\eta}_n^{N_2}$ is the best. We can compare the mean squared error (MSE) when the islands interact or when they are kept independent. The MSE for independent islands is given by $\frac{V_n(f_n)}{N_1 N_2} + \frac{B_n(f_n)^2}{N_1^2}$ whereas the MSE for the double bootstrap is given by $\frac{V_n(f_n) + \tilde{V}_n(f_n)}{N_1 N_2}$. Therefore,

$$\frac{V_n(f_n) + \tilde{V}_n(f_n)}{N_1 N_2} < \frac{V_n(f_n)}{N_1 N_2} + \frac{B_n(f_n)^2}{N_1^2} \Leftrightarrow N_1 < \frac{B_n(f_n)^2}{\tilde{V}_n(f_n)} N_2.$$

Consequently, the double bootstrap algorithm outperforms the independent islands when the number of particles N_1 within each island is small compared to the number of islands N_2 ; the interaction improves the bias (which is independent of N_2 when the islands are kept independent). On the contrary, when N_1 is larger than N_2 , the variance increase introduced by the interaction (because of the selection step) may be larger than the bias reduction.

4 Extensions

In section 3 we have described and analyzed an interacting island model where the bootstrap algorithm is used both within and across the islands. Of course, other IPS approximations may be considered within and across islands. We will describe how the results of the previous sections may be adapted. The IPS approximation of each individual island may be cast in the Feynman-Kac framework. This section is devoted to check these conditions for various IPS approximations.

4.1 Epsilon-bootstrap interaction

ϵ -bootstrap interaction is a variant of the bootstrap, in which the selection step is slightly modified: only a fraction of the particles are resampled. Let ϵ_n be a non-negative constant such that $\epsilon_n \|g_n\|_\infty \in [0, 1]$, where $\|g_n\|_\infty = \sup_{x_n \in \mathbb{E}_n} |g_n(x_n)|$. For any measure $\mu_n \in \mathcal{P}(\mathbb{E}_n)$, define S_{n,μ_n} the Markov kernel on $(\mathbb{E}_n, \mathcal{E}_n)$ given for $x_n \in \mathbb{E}_n$ and $A_n \in \mathcal{E}_n$ by

$$S_{n,\mu_n}(x_n, A_n) \stackrel{\text{def}}{=} \epsilon_n g_n(x_n) \delta_{x_n}(A_n) + (1 - \epsilon_n g_n(x_n)) \Psi_n(\mu_n)(A_n), \quad (46)$$

where Ψ_n is defined in (6). ϵ -bootstrap interaction algorithm proceeds as follows.

At iteration n , a particle X_n^i is kept with a probability equal to $\epsilon_n g_n(X_n^i)$ or

resampled with a probability $1 - \epsilon_n g_n(X_n^i)$. Resampling a particle consists in replacing it by a particle selected at random in the current population with weights proportional to their potential $(g_n(X_n^1), \dots, g_n(X_n^{N_1}))$. Then, each selected particle is independently updated according to the Markov kernel M_{n+1} . When $\epsilon_n = 0$, all the particles are resampled, which correspond to the bootstrap filter. Define the Markov kernel $\mathbf{M}_{n+1}(\mathbf{x}_n, d\mathbf{x}_{n+1})$ from \mathbf{E}_n into \mathbf{E}_{n+1} by

$$\mathbf{M}_{n+1}(\mathbf{x}_n, d\mathbf{x}_{n+1}) \stackrel{\text{def}}{=} \prod_{1 \leq i \leq N_1} S_{n, \eta_n^{N_1}} M_{n+1}(x_n^i, dx_{n+1}^i). \quad (47)$$

Consider a Markov chain $(\mathbf{X}_n)_{n \geq 0}$ where for each $n \in \mathbb{N}$, $\mathbf{X}_n = (X_n^1, \dots, X_n^{N_1}) \in \mathbf{E}_n$, with initial distribution η_0 and transition kernel \mathbf{M}_{n+1} . Define the same approximations of the measures η_n and γ_n as in (11) and (12). Then, consider the island Feynman-Kac model associated to the Markov chain (14) and the potential function (13). The associated sequence $\{(\boldsymbol{\eta}_n, \boldsymbol{\gamma}_n)\}_{n \geq 0}$ of Feynman-Kac measures is given for all $\mathbf{f}_n \in \mathcal{B}_b(\mathbf{E}_n)$ by

$$\boldsymbol{\eta}_0(\mathbf{f}_0) \stackrel{\text{def}}{=} \boldsymbol{\gamma}_0(\mathbf{f}_0) = \mathbb{E}[\mathbf{f}_0(\mathbf{X}_0)], \quad (48)$$

$$\boldsymbol{\eta}_n(\mathbf{f}_n) \stackrel{\text{def}}{=} \boldsymbol{\gamma}_n(\mathbf{f}_n) / \boldsymbol{\gamma}_n(1), \quad \text{for all } n \geq 1, \quad (49)$$

$$\boldsymbol{\gamma}_n(\mathbf{f}_n) \stackrel{\text{def}}{=} \mathbb{E} \left[\mathbf{f}_n(\mathbf{X}_n) \prod_{0 \leq p < n} \mathbf{g}_p(\mathbf{X}_p) \right], \quad \text{for all } n \geq 1. \quad (50)$$

We may establish the following extension of Theorem 1. Let $\{(x_p^1, \dots, x_p^{N_1})\}_{0 \leq p \leq n}$ be a population of particles generated by the ϵ -bootstrap interaction algorithm specified by (47); then,

Theorem 7 *For any $\mathbf{f}_n \in \mathcal{B}_b(\mathbf{E}_n)$ of the form $\mathbf{f}_n(\mathbf{x}_n) = N_1^{-1} \sum_{i=1}^{N_1} f_n(x_n^i)$ with $f_n \in \mathcal{B}_b(\mathbb{E}_n)$, we get*

$$\boldsymbol{\gamma}_n(\mathbf{f}_n) = \gamma_n(f_n) \quad \text{and} \quad \boldsymbol{\eta}_n(\mathbf{f}_n) = \eta_n(f_n). \quad (51)$$

Proof See subsection 6.4.

For each $n \in \mathbb{N}$, let $(\mathbf{X}_n^1, \dots, \mathbf{X}_n^{N_2}) \in \mathbb{E}_n^{N_2}$ be a population of N_2 islands each of N_1 individuals. The process $(\mathbf{X}_n^1, \dots, \mathbf{X}_n^{N_2})$ is a Markov chain evolving according to selection and mutation steps, defined as follows

- *Selection step*: each island \mathbf{X}_n^i is kept with a probability equal to $\epsilon_n g_n(\mathbf{X}_n^i)$ or resampled with a probability $1 - \epsilon_n g_n(\mathbf{X}_n^i)$. Resampling an island consists in replacing it by an island selected at random in the current population with weights proportional to their potential $(g_n(\mathbf{X}_n^1), \dots, g_n(\mathbf{X}_n^{N_1}))$.
- *Mutation step*: each selected island is updated independently according to the Markov transition M_{n+1} .

These islands particles allow to build the N_2 -particle approximation of the measures η_n and γ_n , for any $f_n \in \mathcal{B}_b(\mathbb{E}_n)$, as

$$\eta_n^{N_2}(f_n) \stackrel{\text{def}}{=} \frac{1}{N_2} \sum_{i=1}^{N_2} f_n(\mathbf{X}_n^i),$$

$$\gamma_n^{N_2}(f_n) \stackrel{\text{def}}{=} \eta_n^{N_2}(f_n) \prod_{0 \leq p < n} \eta_p^{N_2}(g_p) = \eta_n^{N_2}(f_n) \gamma_n^{N_2}(1).$$

For this selection scheme, the following results, adapted from [4, Corollary 9.3.1, pp. 295-298], establishes the convergence of $(W_p^{N_1})_{1 \leq p \leq n}$ to centered Gaussian fields:

Theorem 8 *For the ϵ_n -bootstrap filter, for any fixed time horizon $n \geq 1$, the sequence $(W_p^{N_1})_{1 \leq p \leq n}$ defined in (31) converges in law, as N_1 goes to infinity, to a sequence of n independent centered Gaussian random fields $(W_p)_{0 \leq p \leq n}$ with variance given, for any bounded function $f_p \in \mathcal{B}_b(\mathbb{E}_p)$, and $1 \leq p \leq n$, by*

$$\mathbb{E} \left[W_p(f_p)^2 \right] = \eta_{p-1} \left[S_{p-1, \eta_{p-1}} M_p f_p^2 - (S_{p-1, \eta_{p-1}} M_p f_p)^2 \right]. \quad (52)$$

This variance is smaller than the variance of the bootstrap algorithm.

Proposition 1 *The asymptotic variance of $\eta_n^{N_1}$ is smaller with respect to a non-zero sequence $(\epsilon_p)_{0 \leq p \leq n-1}$ introduced in (46) than in the bootstrap algorithm.*

Proof The proof is given in subsection 6.5.

For example, for $\epsilon_p = \left(\text{essup}_{\eta_p}(g_p)\right)^{-1}$, $0 \leq p \leq n$ the asymptotic variance of $\eta_n^{N_1}(f_n)$, $0 \leq p \leq n$ is lower than for the bootstrap. We can also adapt it at the island level. For instance, Algorithm 2 describes the $\epsilon_p = \left(\max_{1 \leq j \leq N_1} g_p(\mathbf{X}_p^j)\right)^{-1}$ - bootstrap islands interaction with ESS filter within the islands.

4.2 Effective Sample Size interaction

We describe the particle approximation of the probabilities $(\eta_n)_{n \geq 0}$ using the effective sample size (ESS) method introduced in [10]; see also [11], [5] and [7]. The difference with the bootstrap filter stems from the selection step of the current particles which is not performed at each step, but only when the importance weights do not satisfy some appropriately defined criterion. Contrary to the bootstrap filter, we now keep both the particles and the weights. Denote by x_n^i a particle and w_n^i its associated weight, assumed to be nonnegative. For a weighted sample $\{(w_n^i, x_n^i)\}_{i=1}^{N_1}$, the criterion

$$\left(\sum_{i=1}^{N_1} w_n^i g_n(x_n^i)\right)^2 / \sum_{i=1}^{N_1} (w_n^i g_n(x_n^i))^2$$

is the *effective sample size* (ESS). The algorithm goes as follows. When the ESS is less than αN_1 , for some $\alpha \in (0, 1)$, the particles are multinomially resampled with probabilities proportional to their weights times their potential functions and the weights are all reset to 1. When the ESS is greater than αN_1 , then the weights

are simply multiplied by the potential function. The selected particles are then updated using the transition kernel M_{n+1} . For any nonnegative integer p we set

$(\mathbf{E}_p, \mathcal{E}_p) \stackrel{\text{def}}{=} ((\mathbb{E}_p \times \mathbb{R}^+)^{N_1}, (\mathcal{E}_p \otimes \mathcal{B}(\mathbb{R}^+))^{\otimes N_1})$. Introduce the following set

$$\Theta_{n,\alpha} = \left\{ \mathbf{x}_n = [(x_n^1, w_n^1), \dots, (x_n^{N_1}, w_n^{N_1})] \in \mathbf{E}_n \left| \frac{\left(\sum_{i=1}^{N_1} w_n^i g_n(x_n^i) \right)^2}{\sum_{i=1}^{N_1} (w_n^i g_n(x_n^i))^2} \geq \alpha N_1 \right. \right\}.$$

Define the Markov kernel M_{n+1} from \mathbf{E}_n into \mathbf{E}_{n+1} by

$$\begin{aligned} M_{n+1}(\mathbf{x}_n, d\mathbf{x}_{n+1}) &\stackrel{\text{def}}{=} \mathbf{1}_{\Theta_{n,\alpha}}(\mathbf{x}_n) \left[\prod_{1 \leq i \leq N_1} \delta_{w_n^i g_n(x_n^i)}(dw_{n+1}^i) M_{n+1}(x_n^i, dx_{n+1}^i) \right] \\ &+ \mathbf{1}_{\Theta_{n,\alpha}^c}(\mathbf{x}_n) \left[\prod_{1 \leq i \leq N_1} \delta_1(dw_{n+1}^i) \sum_{j=1}^{N_1} \frac{w_n^j g_n(x_n^j)}{\sum_{k=1}^{N_1} w_n^k g_n(x_n^k)} M_{n+1}(x_n^j, dx_{n+1}^j) \right], \end{aligned} \quad (53)$$

where $\mathbf{x}_n = [(x_n^1, w_n^1), \dots, (x_n^{N_1}, w_n^{N_1})] \in \mathbf{E}_n$ and $\Theta_{n,\alpha}^c$ is the complement of $\Theta_{n,\alpha}$.

We define a Markov chain $(\mathbf{X}_n)_{n \geq 0}$ where for each $n \in \mathbb{N}$,

$$\mathbf{X}_n = [(X_n^1, \omega_n^1), \dots, (X_n^{N_1}, \omega_n^{N_1})] \in \mathbf{E}_n, \quad (54)$$

with initial distribution $\eta_0 \stackrel{\text{def}}{=} (\eta_0 \otimes \delta_1)^{\otimes N_1}$ and transition kernel M_{n+1} . Equation (4) suggests the following N_1 -particle approximations of the measures η_n and γ_n defined for $f_n \in \mathcal{B}_b(\mathbb{E}_n)$ by

$$\eta_n^{N_1}(f_n) \stackrel{\text{def}}{=} \frac{1}{\sum_{i=1}^{N_1} \omega_n^i} \sum_{i=1}^{N_1} \omega_n^i f_n(X_n^i) = m^{N_1} f_n(\mathbf{X}_n), \quad (55)$$

$$\gamma_n^{N_1}(f_n) \stackrel{\text{def}}{=} \eta_n^{N_1}(f_n) \prod_{0 \leq p < n} \eta_p^{N_1}(g_p) = \eta_n^{N_1}(f_n) \gamma_n^{N_1}(1), \quad (56)$$

where m^{N_1} stands for the operator given for any $f_n \in \mathcal{B}_b(\mathbb{E}_n)$ by

$$m^{N_1} f_n(\mathbf{x}_n) \stackrel{\text{def}}{=} \frac{1}{\sum_{i=1}^{N_1} w_n^i} \sum_{i=1}^{N_1} w_n^i f_n(x_n^i).$$

For $\mathbf{x}_n = ((x_n^1, w_n^1), \dots, (x_n^{N_1}, w_n^{N_1})) \in \mathbf{E}_n$, define the potential function

$$\mathbf{g}_n(\mathbf{x}_n) \stackrel{\text{def}}{=} m^{N_1} g_n(\mathbf{x}_n) = \frac{1}{\sum_{i=1}^{N_1} w_n^i} \sum_{i=1}^{N_1} w_n^i g_n(x_n^i). \quad (57)$$

We consider the island Feynman-Kac model associated to the Markov chain (53) and the potential function (57). The associated sequence $\{(\eta_n, \gamma_n)\}_{n \geq 0}$ of Feynman-Kac measures is given for all $\mathbf{f}_n \in \mathcal{B}_b(\mathbb{E}_n)$ by

$$\eta_0(\mathbf{f}_0) \stackrel{\text{def}}{=} \gamma_0(\mathbf{f}_0) = \mathbb{E}[\mathbf{f}_0(\mathbf{X}_0)] , \quad (58)$$

$$\eta_n(\mathbf{f}_n) \stackrel{\text{def}}{=} \gamma_n(\mathbf{f}_n) / \gamma_n(1), \quad \text{for all } n \geq 1, \quad (59)$$

$$\gamma_n(\mathbf{f}_n) \stackrel{\text{def}}{=} \mathbb{E} \left[\mathbf{f}_n(\mathbf{X}_n) \prod_{0 \leq p < n} \mathbf{g}_p(\mathbf{X}_p) \right] , \quad \text{for all } n \geq 1. \quad (60)$$

Theorem 9 For a particle system $\mathbf{x}_n = ((x_n^1, w_n^1), \dots, (x_n^{N_1}, w_n^{N_1})) \in \mathbb{E}_n$ generated by the ESS algorithm and for any $\mathbf{f}_n \in \mathcal{B}_b(\mathbb{E}_n)$ of the form

$$\mathbf{f}_n(\mathbf{x}_n) = \left(\sum_{i=1}^{N_1} w_n^i \right)^{-1} \sum_{i=1}^{N_1} w_n^i f_n(x_n^i)$$

where $f_n \in \mathcal{B}_b(\mathbb{E}_n)$,

$$\gamma_n(\mathbf{f}_n) = \gamma_n(f_n) \quad \text{and} \quad \eta_n(\mathbf{f}_n) = \eta_n(f_n) . \quad (61)$$

Proof See subsection 6.6.

For each $n \in \mathbb{N}$, let $(\mathbf{X}_n^1, \dots, \mathbf{X}_n^{N_2}) \in \mathbb{E}_n^{N_2}$ be a population of N_2 islands each of N_1 individuals. We associate to each island, a weight Ω_n^i , for $i \in \{1, \dots, N_2\}$. We can also make the islands interact using an ESS criterion.

The process $((\mathbf{X}_n^1, \Omega_n^1), \dots, (\mathbf{X}_n^{N_2}, \Omega_n^{N_2}))$ is a Markov chain which evolves according to selection and mutation steps, defined as follows

- *Selection step*: if the ESS criterion $\left(\sum_{i=1}^{N_2} \Omega_n^i \mathbf{g}_n(\mathbf{X}_n^i) \right)^2 / \sum_{i=1}^{N_2} (\Omega_n^i \mathbf{g}_n(\mathbf{X}_n^i))^2$ is larger than βN_2 for one $\beta \in (0, 1)$, we do not resample the islands and we update the weights thanks to the potential function $\Omega_{n+1}^i = \Omega_n^i \mathbf{g}_n(\mathbf{X}_n^i)$; otherwise, we

resample the islands with probability proportional to $\{\Omega_n^i \mathbf{g}_n(\mathbf{X}_n^i)\}_{i=1}^{N_2}$ and the weights are all reset to 1.

- *Mutation step*: each selected island is updated independently according to the Markov transition \mathbf{M}_{n+1} .

These islands particles allow to define the N_2 -particle approximation of the measures $\boldsymbol{\eta}_n$ and $\boldsymbol{\gamma}_n$, for any $\mathbf{f}_n \in \mathcal{B}_b(\mathbb{E}_n)$, as

$$\boldsymbol{\eta}_n^{N_2}(\mathbf{f}_n) \stackrel{\text{def}}{=} \frac{1}{\sum_{i=1}^{N_2} \Omega_n^i} \sum_{i=1}^{N_2} \Omega_n^i \mathbf{f}_n(\mathbf{X}_n^i),$$

$$\boldsymbol{\gamma}_n^{N_2}(\mathbf{f}_n) \stackrel{\text{def}}{=} \boldsymbol{\eta}_n^{N_2}(\mathbf{f}_n) \prod_{0 \leq p < n} \boldsymbol{\eta}_p^{N_2}(\mathbf{g}_p) = \boldsymbol{\eta}_n^{N_2}(\mathbf{f}_n) \boldsymbol{\gamma}_n^{N_2}(1).$$

Algorithm 3 describes the ESS within ESS island filter.

5 Numerical simulations

Example 1 (Linear Gaussian Model) In order to assess numerically the previous results, we now consider the Linear Gaussian Model (LGM) defined by:

$$X_{p+1} = \phi X_p + \sigma_u U_p, \quad Y_p = X_p + \sigma_v V_p,$$

where $X_0 \sim \mathcal{N}(0, \sigma_u^2 / (1 - \phi^2))$, $\{U_p\}_{p \geq 1}$ and $\{V_p\}_{p \geq 1}$ are independent sequences of i.i.d. standard Gaussian random variables, independent of X_0 . In the simulations, we have used $n = 20$ observations, generated using the model with $\phi = 0.9$, $\sigma_u = 0.6$ and $\sigma_v = 1$. We focus on the prediction problem, consisting in computing the predictive distribution of the state X_n given Y_0, \dots, Y_{n-1} . This problem can be cast in the Feynman-Kac framework by setting for all $p \geq 0$

$$M_{p+1}(x_p, dx_{p+1}) = \frac{1}{\sqrt{2\pi}\sigma_u} \exp\left[-\frac{(x_{p+1} - \phi x_p)^2}{2\sigma_u^2}\right] dx_{p+1},$$

$$g_p(x_p) = \frac{1}{\sqrt{2\pi}\sigma_v} \exp\left[-\frac{(y_p - x_p)^2}{2\sigma_v^2}\right].$$

We estimate the predictive mean of the latent state $\mathbb{E}[X_n|Y_0, \dots, Y_{n-1}]$. We compare the results obtained for different interactions across the islands and for different values of N_1 and N_2 ; in all the simulations, the bootstrap filter is used within the islands. We have run the simulations independently 250 times and we have compared these estimators with the value computed using the Kalman filter.

Figure 1 displays the boxplots of the 250 values of these estimators. As expected,

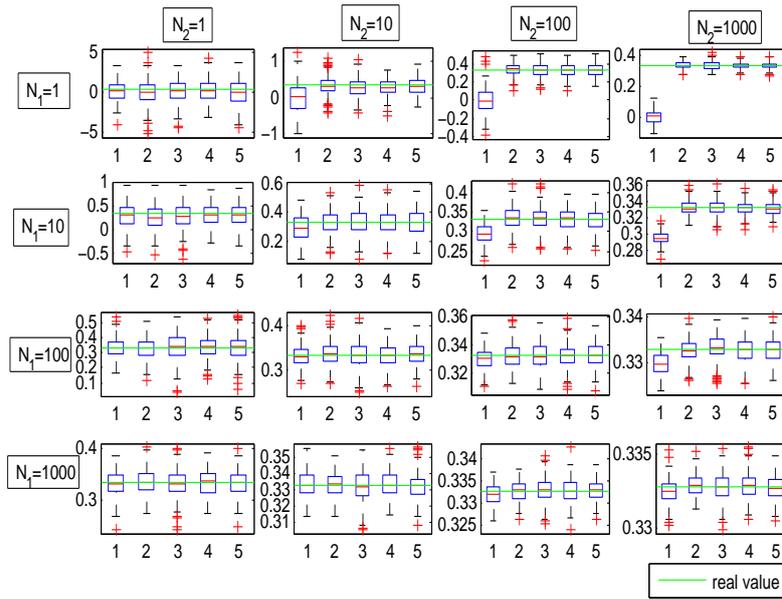


Fig. 1: Comparison of different interactions across the islands with bootstrap within each island for the LGM (1) Bootstrap/independent; (2) Bootstrap/ESS; (3) Bootstrap/Bootstrap; (4) Bootstrap/ $(1/\|g_n\|_\infty)$ -bootstrap; (5) Bootstrap/ $\text{essup}_{\eta_p}^{N_1}(g_n)$ -bootstrap

for small values of N_1 compared to N_2 , the bias of independent islands is large compared to cases where islands interact; on the contrary, the variance is smaller for independent islands than for bootstrap island interaction. In this example,

$N_1 \backslash N_2$	1		10		100		1000	
1	0	20	77	200	825	2000	8264	20000
10	0	20	47	200	636	2000	7122	20000
100	0	20	19	200	297	2000	3609	20000
1000	0	20	7	200	107	2000	1373	20000

Table 1: Island interaction number using bootstrap within ϵ_p -bootstrap and double bootstrap for the LGM.

the type of interaction between islands does not have a significant impact on the dispersion of the estimator (the bias is negligible).

An important aspect for the efficiency of the algorithms is the number of interactions between islands. The smaller this number is, the quicker the algorithm will be. The number of interactions in the bootstrap case is nN_2 . We have compared the island interaction number for the ϵ_p -bootstrap and the ESS interactions w.r.t. the bootstrap one, when we apply the bootstrap filter within the islands. We have computed the empirical number of interactions over the 250 simulations; the results are respectively given in tables 1 and 2. For a given number of islands, the island interaction number for the ESS and the ϵ_p -bootstrap decrease when the island size grows, whereas it is constant for the bootstrap. The island interaction number is always much smaller using the ESS or the ϵ_p -bootstrap than the bootstrap, across the islands. Moreover, as soon as the number of particles in each island is large enough, the ESS is no longer resampling the islands.

Theorem 8 assures that the variance is smaller using the ϵ_p -bootstrap than the bootstrap interaction. The variance gain using ϵ_p -bootstrap or ESS instead of

$N_1 \backslash N_2$	1		10		100		1000	
1	0	20	86	200	945	2000	9056	20000
10	0	20	19	200	230	2000	2408	20000
100	0	20	0	200	0	2000	0	20000
1000	0	20	0	200	0	2000	0	20000

Table 2: Island interaction number using bootstrap within ESS and double bootstrap for the LGM.

$N_1 \backslash N_2$	10		100		1000	
10	9.5	18.7	13.2	20.5	22.8	1.7
100	25.4	26.1	26.1	18.5	13.5	22.4
1000	28.2	34.3	19.5	33.8	25.9	26.5

Table 3: Percentage of the variance gain using bootstrap within ϵ_p -bootstrap on the left side and ESS within bootstrap on the right side, compared to the double bootstrap, in the LGM example.

bootstrap across the islands is given in table 3. The bootstrap interaction is applied within the islands. The variance is significantly reduced using the ϵ_p -bootstrap or the ESS interaction across the islands, instead of the bootstrap, up to 34 percent variance reduction.

Example 2 (Stochastic volatility model) We consider the stochastic volatility model:

$$X_{p+1} = \alpha X_p + \sigma U_{p+1}, \quad Y_p = \beta e^{\frac{X_p}{2}} V_p,$$

where $X_0 \sim \mathcal{N}(0, \sigma^2/(1 - \alpha^2))$, $\{U_p\}_{p \geq 0}$ and $\{V_p\}_{p \geq 0}$ are independent sequences of standard Gaussian random variables independent of X_0 . In the simulations, we have used $n = 100$ observations, generated using the model with $\alpha = 0.98$, $\sigma = 0.5$ and $\beta = 1$. We estimate the predictive mean of the latent state X_n given the observations Y_0, \dots, Y_{n-1} . This problem can be cast in the Feynman-Kac framework by setting for all $p \geq 0$

$$M_{p+1}(x_p, dx_{p+1}) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x_{p+1} - \alpha x_p)^2}{2\sigma^2}\right] dx_{p+1},$$

$$g_p(x_p) = \frac{1}{\sqrt{2\pi}\beta} \exp\left[-\frac{x_p/2 - y_p^2 e^{-x_p}}{2\beta^2}\right].$$

We have computed this quantity using a single run of bootstrap filter with 10^6 particles. In the following results, we always consider bootstrap interaction within each island, and we compare different interactions across the islands, for several values of N_1 and N_2 . We have run the simulations independently 250 times. Figure 2 displays the boxplots of the 250 values of these estimators. The behavior of the different methods is similar to the one observed for the Linear Gaussian Model example.

We have compared the island interaction number for the ϵ_p -bootstrap and the ESS interactions w.r.t. the bootstrap one, when we apply the bootstrap filter within the islands. We have computed the empirical number of interactions over the 250 simulations; the results are respectively given in tables 4 and 5. The number of interactions in the bootstrap case is nN_2 . The same phenomena are observed as for the Linear Gaussian Model example.

The variance gain using the ϵ_p -bootstrap or the ESS instead of the bootstrap across the islands is given in table 6. The bootstrap interaction is applied within the islands. The variance is significantly reduced using the ϵ_p -bootstrap or the ESS

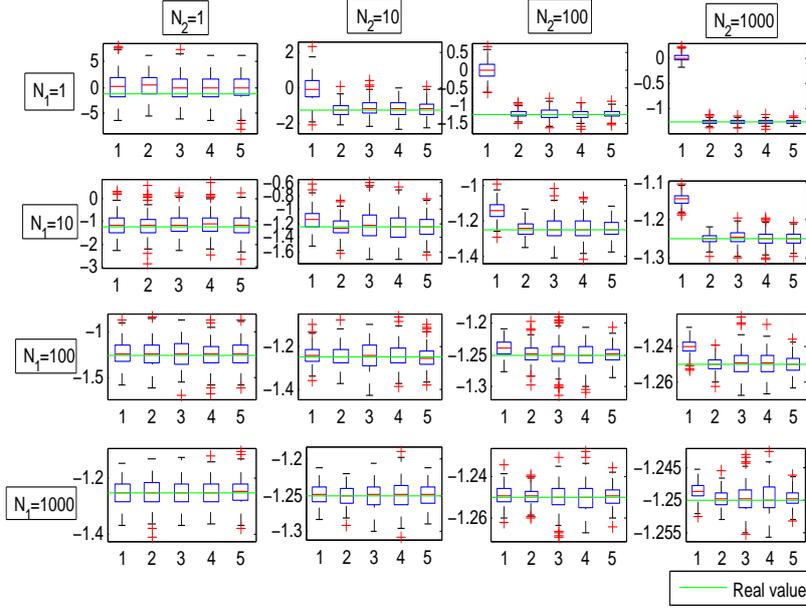


Fig. 2: Comparison of different interactions across the islands with bootstrap within each island for the Stochastic volatility model (1) Bootstrap/independent; (2) Bootstrap/ESS; (3) Bootstrap/Bootstrap; (4) Bootstrap/ $(1/\|g_n\|_\infty)$ -bootstrap; (5) Bootstrap/ $\text{essup}_{\eta_p^{N_1}}(g_n)$ -bootstrap

interaction across the islands, instead of the bootstrap, up to 66 percent variance reduction.

6 Proofs

6.1 Proof of Theorem 1

Using (11) and (13), $\mathbf{g}_n(\mathbf{X}_n)$ may be expressed as $\mathbf{g}_n(\mathbf{X}_n) = \eta_n^{N_1}(g_n)$ where \mathbf{X}_n and $\eta_n^{N_1}$ are defined in (11) and (10), respectively. Similarly, for any $\mathbf{f}_n \in \mathcal{B}_b(\mathbb{E}_n)$ of the form $\mathbf{f}_n(\mathbf{x}_n) = N_1^{-1} \sum_{i=1}^{N_1} f_n(x_n^i)$ where $f_n \in \mathcal{B}_b(\mathbb{E}_n)$, $\mathbf{f}_n(\mathbf{X}_n)$ is given by

$N_1 \backslash N_2$	1	10	100	1000
1	0 100	332 1000	4021 10000	42185 100000
10	0 100	221 1000	3069 10000	34789 100000
100	0 100	100 1000	1523 10000	18647 100000
1000	0 100	36 1000	577 10000	7332 100000

Table 4: Island interaction number using bootstrap within ϵ_p -bootstrap and double bootstrap for the Stochastic volatility model.

$N_1 \backslash N_2$	1	10	100	1000
1	0 100	301 1000	3514 10000	36108 100000
10	0 100	109 1000	1229 10000	12096 100000
100	0 100	15 1000	186 10000	1956 100000
1000	0 100	0 1000	0 10000	0 100000

Table 5: Island interaction number using bootstrap within ESS and double bootstrap for the Stochastic volatility example.

$\mathbf{f}_n(\mathbf{X}_n) = \eta_n^{N_1}(f_n)$. Note that

$$\boldsymbol{\gamma}_n(\mathbf{f}_n) \stackrel{\text{def}}{=} \mathbb{E} \left[\mathbf{f}_n(\mathbf{X}_n) \prod_{0 \leq p < n} \mathbf{g}_p(\mathbf{X}_p) \right] = \mathbb{E} \left[\eta_n^{N_1}(f_n) \prod_{0 \leq p < n} \eta_p^{N_1}(g_p) \right], \quad (62)$$

and since by (12), it suffices to prove that $\gamma_n^{N_1}(f_n)$ is an unbiased estimator of $\gamma_n(f_n)$, i.e.

$$\mathbb{E} \left[\gamma_n^{N_1}(f_n) \right] = \gamma_n(f_n). \quad (63)$$

$N_1 \backslash N_2$	N_2		10		100		1000	
	10	100	10	100	10	100	10	100
10	44.2	57.8	35.3	57.2	30.4	50.7		
100	46.4	49.3	52.2	44.6	46.8	65		
1000	30.4	41.7	49.6	66.9	55.8	61.4		

Table 6: Percentage of the variance gain using bootstrap within ϵ_p -bootstrap on the left side and ESS within bootstrap on the right side, compared to the double bootstrap, in the Stochastic volatility example.

Define the filtration $\mathcal{F}_n^{N_1} \stackrel{\text{def}}{=} \sigma(\mathbf{X}_p, 0 \leq p \leq n)$. Note that

$$\begin{aligned} \mathbb{E} \left[\eta_p^{N_1}(f_p) \middle| \mathcal{F}_{p-1}^{N_1} \right] &= \frac{1}{N_1} \sum_{i=1}^{N_1} \mathbb{E} \left[f_p(X_p^i) \middle| \mathcal{F}_{p-1}^{N_1} \right] = \mathbb{E} \left[f_p(X_p^1) \middle| \mathcal{F}_{p-1}^{N_1} \right] \\ &= \frac{\sum_{i=1}^{N_1} g_{p-1}(X_{p-1}^i) M_p f_p(X_{p-1}^i)}{\sum_{i=1}^{N_1} g_{p-1}(X_{p-1}^i)} = \frac{\eta_{p-1}^{N_1}(Q_p f_p)}{\eta_{p-1}^{N_1}(g_{p-1})}, \end{aligned} \quad (64)$$

where Q_p is defined in (22).

By the definition of $\gamma_n^{N_1}$ given in (12), we have

$$\begin{aligned} \mathbb{E} \left[\gamma_n^{N_1}(f_n) \right] &= \mathbb{E} \left[\mathbb{E} \left[\eta_n^{N_1}(f_n) \middle| \mathcal{F}_{n-1}^{N_1} \right] \prod_{0 \leq p < n} \eta_p^{N_1}(g_p) \right] \\ &= \mathbb{E} \left[\frac{\eta_{n-1}^{N_1}(Q_n f_n)}{\eta_{n-1}^{N_1}(g_{n-1})} \prod_{0 \leq p < n} \eta_p^{N_1}(g_p) \right] \\ &= \mathbb{E} \left[\eta_{n-1}^{N_1}(Q_n f_n) \prod_{0 \leq p < n-1} \eta_p^{N_1}(g_p) \right]. \end{aligned}$$

By iterating this step we get

$$\begin{aligned} \mathbb{E} \left[\gamma_n^{N_1}(f_n) \right] &= \mathbb{E} \left[\eta_0^{N_1}(Q_1 \cdots Q_n f_n) \right] = \mathbb{E} \left[Q_1 \cdots Q_n f_n(X_0^1) \right] \\ &= \gamma_0 Q_1 \cdots Q_n f_n = \gamma_n(f_n). \end{aligned}$$

6.2 Proof of Theorem 4

We preface the proof by the following Lemma.

Lemma 1 *For any $f_n^1, f_n^2 \in \mathcal{B}_b(\mathbb{E}_n)$, the pair $(W_n^{\gamma, N_1}(f_n^1), W_n^{\eta, N_1}(f_n^2))$ converges in law, as N_1 tends to infinity, to $(W_n^\gamma(f_n^1), W_n^\eta(f_n^2))$. In addition, for any polynomial function $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}$, we have:*

$$\lim_{N_1 \rightarrow \infty} \mathbb{E} \left[\Phi \left(W_n^{\gamma, N_1}(f_n^1), W_n^{\eta, N_1}(f_n^2) \right) \right] = \mathbb{E} \left[\Phi \left(W_n^\gamma(f_n^1), W_n^\eta(f_n^2) \right) \right].$$

Proof For any $(\alpha, \beta) \in \mathbb{R}^2$ by the definitions (30) of W_n^{γ, N_1} and (33) of W_n^{η, N_1} we have

$$\begin{aligned} & \alpha W_n^{\gamma, N_1}(f_n^1) + \beta W_n^{\eta, N_1}(f_n^2) \\ &= \sum_{p=0}^n \left[\alpha \gamma_p^{N_1}(1) W_p^{N_1}(Q_{p,n} f_n^1) + \beta \frac{\gamma_p^{N_1}(1)}{\gamma_n^{N_1}(1)} W_p^{N_1}(Q_{p,n}(f_n^2 - \eta_n(f_n^2))) \right]. \end{aligned}$$

As in the proof of Theorem 2, a simple application of Slutsky's Lemma allows to show that $\alpha W_n^{\gamma, N_1}(f_n^1) + \beta W_n^{\eta, N_1}(f_n^2)$ converges in law to $\alpha W_n^\gamma(f_n^1) + \beta W_n^\eta(f_n^2)$.

The proof follows from [4, Theorem 7.4.4], using that for any $p \geq 1$,

$$\sup_{N_1 \geq 1} \mathbb{E} \left[\left| W_n^{\gamma, N_1}(f_n^1) \right|^p \right]^{1/p} \leq c_p(n) \|f_n^1\|, \quad (65)$$

$$\sup_{N_1 \geq 1} \mathbb{E} \left[\left| W_n^{\eta, N_1}(f_n^2) \right|^p \right]^{1/p} \leq c_p(n) \|f_n^2\|, \quad (66)$$

for some finite constant $c_p(n)$ depending only on p and n .

Proof of Theorem 4 Consider first the bias term. We decompose the error as follows using (33):

$$\begin{aligned} N_1 \left[\eta_n^{N_1}(f_n) - \eta_n(f_n) \right] &= \sqrt{N_1} W_n^{\eta, N_1}(f_n) = \sqrt{N_1} \frac{\gamma_n(1)}{\gamma_n^{N_1}(1)} W_n^{\gamma, N_1} \left(\frac{f_n - \eta_n(f_n)}{\gamma_n(1)} \right) \\ &= \sqrt{N_1} \left[\frac{\gamma_n(1)}{\gamma_n^{N_1}(1)} - 1 \right] W_n^{\gamma, N_1} \left(\frac{f_n - \eta_n(f_n)}{\gamma_n(1)} \right) + \sqrt{N_1} W_n^{\gamma, N_1} \left(\frac{f_n - \eta_n(f_n)}{\gamma_n(1)} \right). \end{aligned}$$

Since $W_n^{\gamma, N_1} = \sqrt{N_1} [\gamma_n^{N_1} - \gamma_n]$, Theorem 1 shows that, the expectation of the second term on the RHS of the previous equation, is zero. By noting that

$$\left[\frac{\gamma_n(1)}{\gamma_n^{N_1}(1)} - 1 \right] = -\frac{1}{\gamma_n^{N_1}(1)} [\gamma_n^{N_1} - \gamma_n](1) = -\frac{1}{\sqrt{N_1}} \frac{1}{\gamma_n^{N_1}(1)} W_n^{\gamma, N_1}(1),$$

where W_n^{γ, N_1} is defined in (30), we get

$$\begin{aligned} N_1 \mathbb{E} \left[\eta_n^{N_1}(f_n) - \eta_n(f_n) \right] &= -\mathbb{E} \left[\frac{1}{\gamma_n^{N_1}(1)} W_n^{\gamma, N_1}(1) W_n^{\gamma, N_1} \left(\frac{f_n - \eta_n(f_n)}{\gamma_n(1)} \right) \right] \\ &= -\frac{1}{\gamma_n(1)} \mathbb{E} \left[W_n^{\gamma, N_1}(1) W_n^{\eta, N_1}(f_n) \right], \end{aligned}$$

where W_n^{η, N_1} is given in (33). According to Lemma 1:

$$\lim_{N_1 \rightarrow \infty} N_1 \mathbb{E} \left[\eta_n^{N_1}(f_n) - \eta_n(f_n) \right] = -\frac{1}{\gamma_n(1)} \mathbb{E} [W_n^\gamma(1) W_n^\eta(f_n)] = B_n(f_n), \quad (67)$$

by the definitions of W_n^γ and W_n^η . Consider now the variance. We use the decomposition

$$\text{Var} \left(\eta_n^{N_1}(f_n) \right) = \mathbb{E} \left[\left(\eta_n^{N_1}(f_n) - \eta_n(f_n) \right)^2 \right] - \left\{ \mathbb{E} \left[\eta_n^{N_1}(f_n) - \eta_n(f_n) \right] \right\}^2.$$

Using (67), we get $\left\{ \mathbb{E} \left[\eta_n^{N_1}(f_n) - \eta_n(f_n) \right] \right\}^2 = O(N_1^{-2})$. From the definition (33) of W_n^{η, N_1} , it follows $\mathbb{E} \left[\left(\eta_n^{N_1}(f_n) - \eta_n(f_n) \right)^2 \right] = N_1^{-1} \mathbb{E} \left[W_n^{\eta, N_1}(f_n)^2 \right]$, implying that $\lim_{N_1 \rightarrow \infty} N_1 \text{Var} \left(\eta_n^{N_1}(f_n) \right) = \mathbb{E} \left[W_n^\eta(f_n)^2 \right] = V_n(f_n)$, by the definition of W_n^η and using again Lemma 1.

6.3 Proof of Theorem 5

We preface the proof of Theorem 5 by the following result on the usual Feynman-Kac model.

Lemma 2 For any time horizon $n \geq 0$ and any functions $f_n^1, f_n^2 \in \mathcal{B}_b(\mathbb{E}_n)$ such that $\eta_n(f_n^1) = 0$, we have

$$\begin{aligned} \lim_{N_1 \rightarrow \infty} N_1 \mathbb{E} \left[\eta_n^{N_1}(f_n^1) \eta_n^{N_1}(f_n^2) \prod_{p=0}^{n-1} \eta_p^{N_1}(g_p) \right] \\ = \gamma_n(1) \sum_{p=0}^n \mathbb{E} \left[W_p(P_{p,n}(f_n^1)) W_p(P_{p,n}(f_n^2 - \eta_n(f_n^2))) \right]. \end{aligned}$$

Proof By the definition (12) of $\gamma_n^{N_1}$ we have $\eta_n^{N_1}(f_n^1) \prod_{p=0}^{n-1} \eta_p^{N_1}(g_p) = \gamma_n^{N_1}(f_n^1)$, and, according to (63), $\mathbb{E} \left[\gamma_n^{N_1}(f_n^1) \right] = \gamma_n(f_n^1) = \gamma_n(1) \eta_n(f_n^1) = 0$, so that we get

$$\begin{aligned} \mathbb{E} \left[\eta_n^{N_1}(f_n^1) \eta_n^{N_1}(f_n^2) \prod_{p=0}^{n-1} \eta_p^{N_1}(g_p) \right] &= \mathbb{E} \left[\gamma_n^{N_1}(f_n^1) \eta_n^{N_1}(f_n^2) \right] \\ &= \mathbb{E} \left[\gamma_n^{N_1}(f_n^1) \left(\eta_n^{N_1}(f_n^2) - \eta_n(f_n^2) \right) \right] + \mathbb{E} \left[\gamma_n^{N_1}(f_n^1) \right] \eta_n(f_n^2) \\ &= \mathbb{E} \left[\left(\gamma_n^{N_1}(f_n^1) - \gamma_n(f_n^1) \right) \left(\eta_n^{N_1}(f_n^2) - \eta_n(f_n^2) \right) \right] = \frac{1}{N_1} \mathbb{E} \left[W_n^{\gamma, N_1}(f_n^1) W_n^{\eta, N_1}(f_n^2) \right]. \end{aligned}$$

Then, Lemma 1 gives

$$\lim_{N_1 \rightarrow \infty} N_1 \mathbb{E} \left[\eta_n^{N_1}(f_n^1) \eta_n^{N_1}(f_n^2) \prod_{p=0}^{n-1} \eta_p^{N_1}(g_p) \right] = \mathbb{E} \left[W_n^\eta(f_n^2) W_n^\gamma(f_n^1) \right],$$

where W_n^γ and W_n^η are given by (36) and (37).

Lemma 3 For any time horizon $n \geq 1$, and any linear function $\mathbf{f}_n \in \mathcal{B}_b(\mathbf{E}_n)$ of the form

$$\mathbf{f}_n(\mathbf{x}_n) = m^{N_1} f_n(\mathbf{x}_n), \quad \text{where } f_n \in \mathcal{B}_b(\mathbb{E}_n),$$

we have

$$\mathbf{Q}_n \mathbf{f}_n(\mathbf{x}_{n-1}) = m^{N_1} Q_n f_n(\mathbf{x}_{n-1}), \quad (68)$$

$$\mathbf{Q}_{p,n} \mathbf{f}_n(\mathbf{x}_p) = m^{N_1} Q_{p,n} f_n(\mathbf{x}_p), \quad \text{for any } p \leq n, \quad (69)$$

$$\mathbf{P}_{p,n} \mathbf{f}_n(\mathbf{x}_p) = m^{N_1} P_{p,n} f_n(\mathbf{x}_p) \quad \text{for any } p \leq n. \quad (70)$$

Proof We have from (64)

$$\begin{aligned} M_n \mathbf{f}_n(\mathbf{x}_{n-1}) &= \mathbb{E}[\mathbf{f}_n(\mathbf{X}_n) | \mathbf{X}_{n-1} = \mathbf{x}_{n-1}] \\ &= \mathbb{E}\left[m^{N_1} f_n(\mathbf{X}_n) \middle| \mathbf{X}_{n-1} = \mathbf{x}_{n-1}\right] = \frac{m^{N_1} Q_n f_n(\mathbf{x}_{n-1})}{m^{N_1} g_{n-1}(\mathbf{x}_{n-1})}, \end{aligned}$$

which implies

$$\begin{aligned} Q_n \mathbf{f}_n(\mathbf{x}_{n-1}) &= g_{n-1}(\mathbf{x}_{n-1}) M_n \mathbf{f}_n(\mathbf{x}_{n-1}) \\ &= m^{N_1} g_{n-1}(\mathbf{x}_{n-1}) \times \frac{m^{N_1} Q_n f_n(\mathbf{x}_{n-1})}{m^{N_1} g_{n-1}(\mathbf{x}_{n-1})}, \end{aligned}$$

showing (68). The proof of (69) follows by an induction since

$$Q_{p,n} \mathbf{f}_n(\mathbf{x}_p) = Q_{p,n-1} Q_n \mathbf{f}_n(\mathbf{x}_p).$$

Proof of Theorem 5

Asymptotic bias behavior: For any fixed N_1 , the asymptotic bias behavior of $\eta_n^{N_2}(\mathbf{f}_n)$ is given for any $\mathbf{f}_n \in \mathcal{B}_b(\mathbb{E}_n)$ by applying Theorem 4 to the island particle model in the bootstrap case:

$$\lim_{N_2 \rightarrow \infty} N_2 \mathbb{E} \left[\eta_n^{N_2}(\mathbf{f}_n) - \eta_n(\mathbf{f}_n) \right] = - \sum_{p=0}^n \eta_p [P_{p,n}(1) P_{p,n}(\mathbf{f}_n - \eta_n(\mathbf{f}_n))].$$

For linear functions \mathbf{f}_n of the form $\mathbf{f}_n = m^{N_1} f_n$ where $f_n \in \mathcal{B}_b(\mathbb{E}_n)$, Lemma 3 states that

$$\begin{aligned} P_{p,n}(\mathbf{f}_n - \eta_n(\mathbf{f}_n))(\mathbf{X}_p) &= m^{N_1} P_{p,n}(f_n - \eta_n(f_n))(\mathbf{X}_p) \\ &= \eta_p^{N_1}(P_{p,n}(f_n - \eta_n(f_n))), \quad (71) \end{aligned}$$

and

$$P_{p,n}(1)(\mathbf{X}_p) = m^{N_1} P_{p,n}(1)(\mathbf{X}_p) = \eta_p^{N_1}(P_{p,n}(1)). \quad (72)$$

Therefore, we get

$$\begin{aligned}
\eta_p [\mathbf{P}_{p,n}(1) \mathbf{P}_{p,n}(\mathbf{f}_n - \eta_n(\mathbf{f}_n))] &\stackrel{(1)}{=} \frac{\boldsymbol{\gamma}_p [\mathbf{P}_{p,n}(1) \mathbf{P}_{p,n}(\mathbf{f}_n - \eta_n(\mathbf{f}_n))]}{\boldsymbol{\gamma}_p(1)} \quad (73) \\
&\stackrel{(2)}{=} \frac{\mathbb{E} \left[\mathbf{P}_{p,n}(1)(\mathbf{X}_p) \mathbf{P}_{p,n}(\mathbf{f}_n - \eta_n(\mathbf{f}_n))(\mathbf{X}_p) \prod_{\ell=0}^{p-1} \mathbf{g}_\ell(\mathbf{X}_\ell) \right]}{\boldsymbol{\gamma}_p(1)} \\
&\stackrel{(3)}{=} \frac{\mathbb{E} \left[\eta_p^{N_1}(P_{p,n}(f_n - \eta_n(f_n))) \eta_p^{N_1}(P_{p,n}(1)) \prod_{\ell=0}^{p-1} \eta_\ell^{N_1}(g_\ell) \right]}{\boldsymbol{\gamma}_p(1)},
\end{aligned}$$

where (1) is simply the definition (15) of η_p , (2) stems from the definition (16) of $\boldsymbol{\gamma}_p$, and (3) follows from Theorem 1, the definition (13) of $(\mathbf{g}_\ell)_{\ell \geq 0}$ and equations (71) and (72). As $\eta_p(P_{p,n}(f_n - \eta_n(f_n))) = 0$ we can apply Lemma 2 and

$$\begin{aligned}
&\lim_{N_1 \rightarrow \infty} N_1 \eta_p [\mathbf{P}_{p,n}(1) \mathbf{P}_{p,n}(\mathbf{f}_n - \eta_n(\mathbf{f}_n))] \\
&= \sum_{\ell=0}^p \mathbb{E} [W_\ell(P_{\ell,p}(P_{p,n}(1) - \eta_p P_{p,n}(1))) W_\ell(P_{\ell,p} P_{p,n}(f_n - \eta_n(f_n)))] \\
&= \sum_{\ell=0}^p \mathbb{E} [W_\ell(P_{\ell,n}(1) - P_{\ell,p}(1)) W_\ell(P_{\ell,n}(f_n - \eta_n(f_n)))] ,
\end{aligned}$$

from which we conclude that

$$\begin{aligned}
&\lim_{N_1 \rightarrow \infty} \lim_{N_2 \rightarrow \infty} N_1 N_2 \mathbb{E} [\boldsymbol{\eta}_n^{N_2}(\mathbf{f}_n) - \eta_n(\mathbf{f}_n)] \\
&= - \sum_{p=0}^n \sum_{\ell=0}^p \mathbb{E} [W_\ell(P_{\ell,n}(1) - P_{\ell,p}(1)) W_\ell(P_{\ell,n}(f_n - \eta_n(f_n)))] \\
&= - \sum_{\ell=0}^n \sum_{p=\ell}^n \mathbb{E} [W_\ell(P_{\ell,n}(1) - P_{\ell,p}(1)) W_\ell(P_{\ell,n}(f_n - \eta_n(f_n)))] \\
&= B_n(f_n) + \tilde{B}_n(f_n) ,
\end{aligned}$$

where $B_n(f_n)$ is defined in (38) and $\tilde{B}_n(f_n)$ is given in (42).

Asymptotic variance behavior: For any fixed N_1 , the asymptotic variance behavior of $\boldsymbol{\eta}_n^{N_2}(\mathbf{f}_n)$ is given for any $\mathbf{f}_n \in \mathcal{B}_b(\mathbb{E}_n)$ by applying Theorem 4 to the

island particle model in the bootstrap case:

$$\lim_{N_2 \rightarrow \infty} N_2 \text{Var} \left(\boldsymbol{\eta}_n^{N_2}(\mathbf{f}_n) \right) = \sum_{p=0}^n \boldsymbol{\eta}_p \left[\mathbf{P}_{p,n} (\mathbf{f}_n - \boldsymbol{\eta}_n(\mathbf{f}_n))^2 \right].$$

For linear functions \mathbf{f}_n of the form $\mathbf{f}_n = m^{N_1} f_n$ where $f_n \in \mathcal{B}_b(\mathbb{E}_n)$, using the same steps as in (73), we get

$$\boldsymbol{\eta}_p \left[\mathbf{P}_{p,n} (\mathbf{f}_n - \boldsymbol{\eta}_n(\mathbf{f}_n))^2 \right] = \frac{\mathbb{E} \left[\eta_p^{N_1} (P_{p,n} (f_n - \eta_n(f_n)))^2 \prod_{\ell=0}^{p-1} \eta_\ell^{N_1} (g_\ell) \right]}{\gamma_p(1)}.$$

As $\eta_p(P_{p,n} (f_n - \eta_n(f_n))) = 0$ we can apply Lemma 2 and

$$\begin{aligned} \lim_{N_1 \rightarrow \infty} N_1 \boldsymbol{\eta}_p \left[\mathbf{P}_{p,n} (\mathbf{f}_n - \boldsymbol{\eta}_n(\mathbf{f}_n))^2 \right] &= \sum_{\ell=0}^p \mathbb{E} \left[W_\ell (P_{\ell,p} P_{p,n} (f_n - \eta_n(f_n)))^2 \right] \\ &= \sum_{\ell=0}^p \mathbb{E} \left[W_\ell (P_{\ell,n} (f_n - \eta_n(f_n)))^2 \right], \end{aligned}$$

from which we conclude that

$$\begin{aligned} \lim_{N_1 \rightarrow \infty} \lim_{N_2 \rightarrow \infty} N_1 N_2 \text{Var} \left(\boldsymbol{\eta}_n^{N_2}(\mathbf{f}_n) \right) &= \sum_{p=0}^n \sum_{\ell=0}^p \mathbb{E} \left[W_\ell (P_{\ell,n} (f_n - \eta_n(f_n)))^2 \right] \\ &= \sum_{\ell=0}^n \sum_{p=\ell}^n \mathbb{E} \left[W_\ell (P_{\ell,n} (f_n - \eta_n(f_n)))^2 \right] = V_n(f_n) + \tilde{V}_n(f_n), \end{aligned}$$

where $V_n(f_n)$ is defined in (39) and $\tilde{V}_n(f_n)$ is given in (43).

6.4 Proof of Theorem 7

Lemma 4 *Let ϵ_n be a nonnegative constant such that $\epsilon_n g_n \in [0, 1]$. Then*

$$\Psi_n(\mu_n) = \mu_n S_{n, \mu_n},$$

where S_{n, μ_n} is defined in (46).

Proof By (46) and (6) we have for any $A_n \in \mathcal{E}_n$

$$\begin{aligned}
\mu_n S_{n,\mu_n}(A_n) &= \int \mu_n(dx_n) S_{n,\mu_n}(x_n, A_n) \\
&= \int \mu_n(dx_n) [\epsilon_n g_n(x_n) \delta_{x_n}(A_n) + (1 - \epsilon_n g_n(x_n)) \Psi_n(\mu_n)(A_n)] \\
&= \epsilon_n \int_{A_n} \mu_n(dx_n) g_n(x_n) + (1 - \epsilon_n \mu_n(g_n)) \Psi_n(\mu_n)(A_n) \\
&= \epsilon_n \mu_n(g_n) \Psi_n(\mu_n)(A_n) + (1 - \epsilon_n \mu_n(g_n)) \Psi_n(\mu_n)(A_n) = \Psi_n(\mu_n)(A_n) . \square
\end{aligned}$$

Let $\mathcal{F}_n^{N_1}$ be the increasing filtration associated to the particle evolution $\mathcal{F}_n^{N_1} \stackrel{\text{def}}{=} \sigma(\mathbf{X}_p, 0 \leq p \leq n)$. As in the proofs of Theorem 1 and Theorem 9, the only point is to prove that

$$\mathbb{E} \left[\eta_p^{N_1}(f_p) \middle| \mathcal{F}_{p-1}^{N_1} \right] = \frac{\eta_{p-1}^{N_1}(Q_p f_p)}{\eta_{p-1}^{N_1}(g_{p-1})} ,$$

where Q_p is defined in (22). Or,

$$\begin{aligned}
\mathbb{E} \left[\eta_p^{N_1}(f_p) \middle| \mathcal{F}_{p-1}^{N_1} \right] &= \frac{1}{N_1} \sum_{i=1}^{N_1} \mathbb{E} \left[f_p(X_p^i) \middle| \mathcal{F}_{p-1}^{N_1} \right] = \frac{1}{N_1} \sum_{i=1}^{N_1} \mathbf{M}_p(f_p)(X_{p-1}^i) \\
&= \eta_{p-1}^{N_1} \mathbf{M}_p(f_p) = \eta_{p-1}^{N_1} S_{p-1, \eta_{p-1}^{N_1}} M_p(f_p) = \Psi_{p-1}(\eta_{p-1}^{N_1}) M_p(f_p) = \frac{\eta_{p-1}^{N_1}(Q_p f_p)}{\eta_{p-1}^{N_1}(g_{p-1})} ,
\end{aligned} \tag{74}$$

using respectively (11), (47), Lemma 4 and (6).

6.5 Proof of Proposition 1

For the ϵ -interaction bootstrap, the sequence $(W_p^{N_1})_{1 \leq p \leq n}$ converges in law, as N_1 tends to infinity, to a sequence of n independent centered Gaussian random fields $(W_p)_{0 \leq p \leq n}$ with variance given by

$$\begin{aligned}
\mathbb{E} \left[W_p(f_p)^2 \right] &= \eta_{p-1} S_{p-1, \eta_{p-1}} M_p f_p^2 - \eta_{p-1} \left[(S_{p-1, \eta_{p-1}} M_p f_p)^2 \right] \\
&= \Psi_{p-1}(\eta_{p-1})(M_p f_p^2) - \eta_{p-1} \left[(S_{p-1, \eta_{p-1}} M_p f_p)^2 \right] ,
\end{aligned}$$

thanks to Lemma 4.

In the special case $\epsilon_p = 0$ (the bootstrap case), the function $S_{p,\eta_p}g_p$ is constant and equal to $\Psi_p(\eta_p)(g_p)$ and the variance for the bootstrap is just

$$\Psi_{p-1}(\eta_{p-1})(M_p f_p^2) - (\Psi_{p-1}(\eta_{p-1})M_p f_p)^2$$

Therefore, the variance of the ϵ -interaction bootstrap may be decomposed as follows

$$\begin{aligned} \mathbb{E} \left[W_p(f_p)^2 \right] &= \left(\Psi_{p-1}(\eta_{p-1})(M_p f_p^2) - (\Psi_{p-1}(\eta_{p-1})M_p f_p)^2 \right) \\ &\quad - \left(\eta_{p-1} \left[(S_{p-1,\eta_{p-1}}M_p f_p)^2 \right] - (\Psi_{p-1}(\eta_{p-1})M_p f_p)^2 \right) . \end{aligned}$$

Observing,

$$\begin{aligned} \eta_{p-1} \left[(S_{p-1,\eta_{p-1}}M_p f_p)^2 \right] - (\Psi_{p-1}(\eta_{p-1})M_p f_p)^2 \\ = \eta_{p-1} \left([S_{p-1,\eta_{p-1}}M_p f_p - \Psi_{p-1}(\eta_{p-1})(M_p f_p)]^2 \right) \geq 0 , \end{aligned}$$

allows to conclude.

6.6 Proof of Theorem 9

Using (55), (57), (58), and for \mathbf{f}_n such that $\mathbf{f}_n(\mathbf{X}_n) = \left(\sum_{i=1}^{N_1} w_n^i \right)^{-1} \sum_{i=1}^{N_1} w_n^i f_n(X_n^i) = m^{N_1} f_n(\mathbf{X}_n) = \eta_n^{N_1}(f_n)$, we get

$$\boldsymbol{\gamma}_n(\mathbf{f}_n) \stackrel{\text{def}}{=} \mathbb{E} \left[\mathbf{f}_n(\mathbf{X}_n) \prod_{0 \leq p < n} \mathbf{g}_p(\mathbf{X}_p) \right] = \mathbb{E} \left[\eta_n^{N_1}(f_n) \prod_{0 \leq p < n} \eta_p^{N_1}(g_p) \right] .$$

By (56), it suffices to prove that $\mathbb{E} \left[\gamma_n^{N_1}(f_n) \right] = \gamma_n(f_n)$. We define by $\mathcal{F}_n^{N_1}$ the increasing filtration associated to the particle evolution $\mathcal{F}_n^{N_1} \stackrel{\text{def}}{=} \sigma(\mathbf{X}_p, 0 \leq p \leq n)$.

We will show that for any $p > 0$ and $f_p \in \mathcal{B}_b(\mathbb{E}_p)$, we have $\mathbb{E} \left[\eta_p^{N_1}(f_p) \middle| \mathcal{F}_{p-1}^{N_1} \right] =$

$\eta_{p-1}^{N_1}(Q_p f_p)/\eta_{p-1}^{N_1}(g_{p-1})$, where Q_p is defined in (22). Indeed, by the definitions (53) of M_p and (55) of $\eta_p^{N_1}$,

$$\begin{aligned} \mathbb{E} \left[\eta_p^{N_1}(f_p) \middle| \mathcal{F}_{p-1}^{N_1} \right] &= \sum_{i=1}^{N_1} \frac{\omega_p^i}{\sum_{j=1}^{N_1} \omega_p^j} \mathbb{E} \left[f_p(X_p^i) \middle| \mathbf{X}_{p-1} \right] \\ &= \mathbf{1}_{\Theta_{p-1,\alpha}}(\mathbf{X}_{p-1}) \left[\frac{\sum_{i=1}^{N_1} \omega_{p-1}^i g_{p-1}(X_{p-1}^i) M_p f_p(X_{p-1}^i)}{\sum_{i=1}^{N_1} \omega_{p-1}^i g_{p-1}(X_{p-1}^i)} \right] \\ &\quad + \mathbf{1}_{\Theta_{p-1,\alpha}^c}(\mathbf{X}_{p-1}) \left[\frac{1}{N_1} \sum_{i=1}^{N_1} \frac{\sum_{j=1}^{N_1} \omega_{p-1}^j g_{p-1}(X_{p-1}^j) M_p f_p(X_{p-1}^j)}{\sum_{j=1}^{N_1} \omega_{p-1}^j g_{p-1}(X_{p-1}^j)} \right] \\ &= \frac{\eta_{p-1}^{N_1}(Q_p f_p)}{\eta_{p-1}^{N_1}(g_{p-1})}. \end{aligned}$$

The proof follows exactly along the same lines as Theorem 1. By iterating this step we get

$$\begin{aligned} \mathbb{E} \left[\gamma_n^{N_1}(f_n) \right] &= \mathbb{E} \left[\eta_0^{N_1}(Q_1 \cdots Q_n f_n) \right] = \mathbb{E} \left[Q_1 \cdots Q_n f_n(X_0^1) \right] \\ &= \gamma_0 Q_1 \cdots Q_n f_n = \gamma_n(f_n). \end{aligned}$$

As the reader may have noticed, this unbiased property doesn't depend on the definition of the sets $\Theta_{p,\alpha}$ defining the resampling times. From this observation, we underline that Theorem 9 is also true for more general classes of resampling time criterion.

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References

1. O. Cappé and E. Moulines. On the use of particle filtering for maximum likelihood parameter estimation. In *European Signal Processing Conference (EUSIPCO)*, Antalya, Turkey, September 2005.
2. N. Chopin. A sequential particle filter method for static models. *Biometrika*, 89:539–552, 2002.
3. N. Chopin, P. Jacob, and O. Papaspiliopoulos. Smc2: A sequential monte carlo algorithm with particle markov chain monte carlo updates. *J.R. Stat. Soc. B.* (to appear 2013).
4. P. Del Moral. *Feynman-Kac Formulae. Genealogical and Interacting Particle Systems with Applications*. Springer, 2004.
5. P. Del Moral, A. Doucet, and A. Jasra. On adaptive resampling strategies for sequential Monte Carlo methods. *Bernoulli*, 18(1):252–278, 2012.
6. P. Del Moral, P. Hu, and L. Wu. On the concentration properties of interacting particle processes. *Foundations and Trends in Machine Learning*, 3(3-4):225–289, 2012.
7. R. Douc and E. Moulines. Limit theorems for weighted samples with applications to sequential Monte Carlo methods. *Ann. Statist.*, 36(5):2344–2376, 2008.
8. A. Doucet, N. De Freitas, and N. Gordon, editors. *Sequential Monte Carlo Methods in Practice*. Springer, New York, 2001.
9. Garland Durham and John Geweke. Massively parallel sequential monte carlo for bayesian inference. *Manuscript*, URL http://www.censoc.uts.edu.au/pdfs/geweke_papers/gp_working_9.pdf. Nalan Bastürk, Lennart Hoogerheide, Anne Opschoor, Herman K. van Dijk, 29, 2011.
10. J. Liu and R. Chen. Blind deconvolution via sequential imputations. *J. Am. Statist. Assoc.*, 90(420):567–576, 1995.
11. J.S. Liu. *Monte Carlo Strategies in Scientific Computing*. Springer, New York, 2001.

Algorithm 2 ESS within ϵ_p -bootstrap interaction for $\epsilon_p = \left(\text{essup}_{\eta_p^{N_1}}(g_p)\right)^{-1}$

1: Initialization:

2: **for** i from 1 to N_2 **do**

3: Set $\omega_0^i = \left(\omega_0^{i,j}\right)_{j=1}^{N_1} = (1, \dots, 1)$.

4: Sample $\mathbf{X}_0^i = \left(X_0^{i,j}\right)_{j=1}^{N_1}$ independently distributed according to η_0 .

5: **end for**

6: **for** p from 0 to $n-1$ **do**

7: Island selection step:

8: **for** i from 1 to N_2 **do**

– With probability $g_p(\mathbf{X}_p^i) / \max_{1 \leq k \leq N_2} g_p(\mathbf{X}_p^k)$, set $I_p^i = i$.

– With probability $1 - g_p(\mathbf{X}_p^i) / \max_{1 \leq k \leq N_2} g_p(\mathbf{X}_p^k)$, sample I_p^i multinomially with probability proportional to $\{g_p(\mathbf{X}_p^l) / \sum_{k=1}^{N_2} g_p(\mathbf{X}_p^k)\}_{l=1}^{N_2}$.

9: **end for**

10: Island mutation step:

11: **for** i from 1 to N_2 **do**

12: Particle selection and weight updating within each island:

13: Set $N_1^{\text{eff}} = \left(\sum_{j=1}^{N_1} \omega_p^{i,j} g_p(X_p^{I_p^i,j})\right)^2 / \sum_{j=1}^{N_1} \left(\omega_p^{i,j} g_p(X_p^{I_p^i,j})\right)^2$.

14: **if** $N_1^{\text{eff}} \geq \alpha_{\text{Particles}} N_1$ **then**

15: For $1 \leq j \leq N_1$, set $\omega_{p+1}^{i,j} = \omega_p^{i,j} g_p(X_p^{I_p^i,j})$.

16: Set $\mathbf{J}_p^i = (J_p^{i,j})_{j=1}^{N_1} = (1, 2, \dots, N_1)$.

17: **else**

18: Set $\omega_{p+1}^i = \left(\omega_{p+1}^{i,j}\right)_{j=1}^{N_1} = (1, \dots, 1)$.

19: Sample $\mathbf{J}_p^i = (J_p^{i,j})_{j=1}^{N_1}$ multinomially with probability proportional to $\left(\omega_p^{i,j} g_p(X_p^{I_p^i,j})\right)_{j=1}^{N_1}$.

20: **end if**

21: Particle mutation:

22: For $1 \leq j \leq N_1$, sample independently $X_{p+1}^{i,j}$ according to $M_{p+1}(X_p^{I_p^i, L_p^{i,j}}, \cdot)$, where

$$L_p^{i,j} = J_p^{i,j}.$$

23: **end for**

24: **end for**

25: Approximate $\eta_n(f_n)$ by $\frac{1}{N_2 \sum_{j=1}^{N_1} \omega_n^{i,j}} \sum_{i=1}^{N_2} \sum_{j=1}^{N_1} \omega_n^{i,j} f_n(X_n^{i,j})$.

Algorithm 3 ESS within ESS island filter

1: Initialization:

2: Set $\boldsymbol{\Omega}_0 = (\Omega_0^i)_{i=1}^{N_2} = (1, \dots, 1)$.

3: **for** i from 1 to N_2 **do**

4: Set $\boldsymbol{\omega}_0^i = (\omega_0^{i,j})_{j=1}^{N_1} = (1, \dots, 1)$.

5: Sample $\mathbf{X}_0^i = (X_0^{i,j})_{j=1}^{N_1}$ independently distributed according to η_0 .

6: **end for**

7: **for** p from 0 to $n - 1$ **do**

8: Island selection step and weight updating:

9: Set $N_2^{\text{eff}} = \left(\sum_{i=1}^{N_2} \Omega_p^i g_p(\mathbf{X}_p^i) \right)^2 / \sum_{i=1}^{N_2} (\Omega_p^i g_p(\mathbf{X}_p^i))^2$.

10: **if** $N_2^{\text{eff}} \geq \alpha_{\text{Islands}} N_2$ **then**

11: For $1 \leq i \leq N_2$, set $\Omega_{p+1}^i = \Omega_p^i g_p(\mathbf{X}_p^i)$.

12: Set $\mathbf{I}_p = (I_p^i)_{i=1}^{N_2} = (1, 2, \dots, N_2)$.

13: **else**

14: Set $\boldsymbol{\Omega}_{p+1} = (\Omega_{p+1}^i)_{i=1}^{N_2} = (1, \dots, 1)$.

15: Sample $\mathbf{I}_p = (I_p^i)_{i=1}^{N_2}$ multinomially with probability proportional to $(\Omega_p^i g_p(\mathbf{X}_p^i, \boldsymbol{\omega}_p^i))_{i=1}^{N_2}$.

16: **end if**

17: Island mutation step:

18: **for** i from 1 to N_2 **do**

19: Particle selection and weight updating within each island:

20: Set $N_1^{\text{eff}} = \left(\sum_{j=1}^{N_1} \omega_p^{i,j} g_p(X_p^{i,j}) \right)^2 / \sum_{j=1}^{N_1} \left(\omega_p^{i,j} g_p(X_p^{i,j}) \right)^2$.

21: **if** $N_1^{\text{eff}} \geq \alpha_{\text{Particles}} N_1$ **then**

22: For $1 \leq j \leq N_1$, set $\omega_{p+1}^{i,j} = \omega_p^{i,j} g_p(X_p^{i,j})$.

23: Set $\mathbf{J}_p^i = (J_p^{i,j})_{j=1}^{N_1} = (1, 2, \dots, N_1)$.

24: **else**

25: Set $\boldsymbol{\omega}_{p+1}^i = (\omega_{p+1}^{i,j})_{j=1}^{N_1} = (1, \dots, 1)$.

26: Sample $\mathbf{J}_p^i = (J_p^{i,j})_{j=1}^{N_1}$ multinomially with probability proportional to $(\omega_p^{i,j} g_p(X_p^{i,j}))_{j=1}^{N_1}$.

27: **end if**

28: Particle mutation:

29: For $1 \leq j \leq N_1$, sample independently $X_{p+1}^{i,j}$ according to $M_{p+1}(X_p^{i,j}, L_p^{i,j}, \cdot)$, where $L_p^{i,j} = J_p^{i,j}$.

30: **end for**

31: **end for**

32: Approximate $\eta_n(f_n)$ by $\frac{1}{\sum_{i=1}^{N_2} \Omega_n^i} \sum_{i=1}^{N_2} \frac{\Omega_n^i}{\sum_{j=1}^{N_1} \omega_n^{i,j}} \sum_{j=1}^{N_1} \omega_n^{i,j} f_n(X_n^{i,j})$.
