Renewable Composite Quantile Method and Algorithm for Nonparametric Models with Streaming Data

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Abstract

We are interested in renewable estimations and algorithms for nonparametric models with streaming data. In our method, the nonparametric function of interest is expressed through a functional depending on a weight function and a conditional distribution function (CDF). The CDF is estimated by renewable kernel estimations combined with function interpolations, based on which we propose the method of renewable weighted composite quantile regression (WCQR). Then we fully use the model structure and obtain new selectors for the weight function, such that the WCQR can achieve asymptotic unbiasness when estimating specific functions in the model. We also propose practical bandwidth selectors for streaming data and find the optimal weight function minimizing the asymptotic variance. The asymptotical results show that our estimator is almost equivalent to the oracle estimator obtained from the entire data together. Besides, our method also enjoys adaptiveness to error distributions, robustness to outliers, and efficiency in both estimation and computation. Simulation studies and real data analyses further confirm our theoretical findings.

Keywords: Renewable algorithm, Streaming Data, Composite quantile regression, Nonparametric regression, Polynomial interpolation

1 Introduction

1.1 Problem Setup and Challenges

In this paper, we are interested in nonparametric regression problems for massive data taking the form of streaming data. Specifically, the considered nonparametric model is that

$$Y = m(X) + \sigma(X)\varepsilon, \tag{1}$$

where Y and X are supposed respectively to be scalar response variable and covariate for simplicity; ε is the random error independent of X, and the distribution of ε is unknown and satisfies $\mathbb{E}[\varepsilon] = 0$ and $\operatorname{Var}[\varepsilon] = 1$; $m : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to [0, \infty)$ are both unknown functions. The considered streaming data consist of a series of cross-sectional data chunks $\mathcal{D}_t =$ $\{(X_{tj}, Y_{tj}) : 1 \leq j \leq n_t\}$ for $t = 1, 2, \cdots$, where all (X_{tj}, Y_{tj}) are independent and identically distributed (i.i.d.) observations of (X, Y). In our setting, the data chunks $\mathcal{D}_1, \mathcal{D}_2, \cdots$, are not available simultaneously, but arrive sequentially one after another.

As we know, most conventional statistic algorithms are designed under the premise that the full data can be fitted on the computer memory simultaneously. However, such a premise is no longer true for streaming data mentioned above. To deal with streaming data, the online-updating (or renewable) algorithms are widely considered. For example, at the time t, one has obtained a summary statistic T_t of the historical data $\bigcup_{s \leq t} \mathcal{D}_s$. Then as the new data chunk \mathcal{D}_{t+1} arrives, T_t is updated to T_{t+1} by incremental computation without accessing the historical raw data, i.e., $T_{t+1} = \mathcal{R}(T_t; \mathcal{D}_{t+1})$ with \mathcal{R} a function independent with $\bigcup_{s < t} \mathcal{D}_s$. When modified into the above renewable form, the statistics may lose desirable statistical properties, which brings new challenges in designing statistical algorithms for online-updating.

The first challenge arises from the data partitioning. As we know, nonparametric methods inevitably suffer from estimation bias. The bias can not be reduced by simply averaging the local estimators from each data chunk, which essentially prevents the renewable estimator from achieving the standard statistical convergence rate. Hence when designing algorithms for streaming data, it is crucial to sufficiently reduce the estimation bias.

The second challenge lies in the potentially poor quality of steaming data. Outliers and fattailed features are more likely to hide in these massive raw data. And even worse, it is quite hard to detect or address them, because the relevant procedures usually involve reusing the historical raw data. Thus it has a significant value for renewable algorithms that the obtained estimator is robust to outliers or adaptive to fat-tailed features.

The third challenge is caused from the exploding data size. The streaming data source usually generates extremely large amounts of raw data in a short period of time. To deal with such a rapid data stream, the updating algorithm should be implemented efficiently.

1.2 Existing Works and Motivations

There have been many works developed for streaming data. The existing online-updating

methods can be classified into the following categories. In some restrictive cases, the estimator has a closed-form expression and the value can be exactly obtained by some recursive updating operations, see, e.g., Schifano et al. (2016); Bucak and Gunsel (2009); Nion and Sidiropoulos (2009), etc. However, it is more often the case that the estimator has no closed-form expression, then iterative algorithms of online-updating are often used to approximate the value of the estimator, see, e.g., Robbins and Monro (1951); Toulis et al. (2014); Moroshko et al. (2015); Chen et al. (2019), etc. Additionally, several online cumulative frameworks are proposed for likelihood, estimating equations and so on, see, e.g., Luo and Song (2020); Lin et al. (2020); Wang et al. (2022), etc. And there are also some works based on the deep learning techniques, see, Ashfahani and Pratama (2019); Das et al. (2019); Pratama et al. (2019), to name a few.

In the first scenario, the obtained estimator enjoys exactly the same statistical properties as that of the oracle estimator obtained by using the offline methods together with the full dataset. However, such a result deeply relies on the closed-form expression of the estimator, which is unavailable for most robust estimators including the quantile estimators. Without the closed-form expression, the differentiability condition of the objective functions is required to achieve the oracle property, see, e.g., Luo and Song (2020) and Lin et al. (2020). However, most robust objective functions (e.g., quantile based objective functions) are not differentiable.

In this paper, we will address the above issues by proposing a new method for stream data, where the renewable algorithm does not rely on the estimator's closed-form expression, but the obtained estimator still achieves the standard statistical convergence rate. Meanwhile, we also focus on some aspects of aforementioned three challenges and pursue that the proposed method enjoys robustness to outliers or adaptiveness to various error distributions, and the updating algorithm is simple such that it can be implemented efficiently.

1.3 Contributions and Article Frame

In this paper, a renewable composite quantile method and algorithm are proposed to estimate

the nonparametric functions in the model (1) with streaming data.

Inspired by L-Estimation (see, e.g., Koenker and Portnoy, 1987; Portnoy and Koenker, 1989; Boente and Fraiman, 1994), we express the nonparametric function through a functional instead of a closed-form expression. Here the functional takes the form of an integral depending on a weight function and a conditional distribution function (CDF) of Y. Then the renewable estimation is attained by two steps:

- 1. *Numerical Approximation:* The CDF in the functional is approximated by function interpolations. Then the nonparametric function can be approximately expressed by a finite number of function values of the CDF.
- 2. Statistical Approximation: The aforementioned function values of CDF are estimated by kernel estimators, which have closedfrom expressions and can be exactly obtained through recursive updating algorithms.

By combining the above numerical and statistical approximations, we finally propose our renewable weighted composite quantile regression (WCQR) method for streaming data.

By the renewable WCQR, the functions $m(\cdot)$ and $\sigma(\cdot)$ can be estimated by correctly selecting the weight function. Specifically speaking, we fully use the structure of the model (1) and obtain new selection criterions for the weight functions, under which the renewable estimator can estimate $m(\cdot)$ or $\sigma(\cdot)$ asymptotically unbiased. Further, we deduce the asymptotic distributions of the proposed estimators. Based on this, a practical bandwidth selector is proposed for the onlineupdating estimator, and the optimal weight function is also obtained by minimizing the asymptotic variance under the constraint of the above selection criterions. Finally, our theoretical findings are demonstrated by simulation studies and real data analyses.

Compared with the competitors, our method has the following main virtues:

1) Oracle comparability. Through numerical approximations, our WCQR estimator is assembled from some renewable statistics exactly obtained via online-updating. Thanks to this, not only the algorithm gets rid of any restriction on the chunk size or chunk number of the streaming data, but also the obtained estimator enjoys almost the same asymptotic properties as that of the oracle estimator obtained on the full data set.

- 2) *Robustness.* Benefit from robust feature of the quantile estimation, when the model has a symmetric error distribution, our regression method enjoys robustness compared with the common methods such as ordinary least squares.
- 3) Model adaptiveness. With model-based weight functions, our estimation method is adaptive to symmetric or asymmetric models. Different from existing methods, the selection criterions of the weight functions are directly established on the structure of the nonparametric models instead of the information of the errors. Thus in our method, the weight selection does not rely on any pilot estimations for the error distributions and the model adaptiveness can be preserved in case of streaming data.
- 4) Estimation efficiency. Under the above weight criterions, we find the optimal weight function, under which our renewable estimator enjoys minimized asymptotic variance in estimating m(x) and $\sigma(x)$.
- 5) Computational efficiency. Thanks to the closedform expression of the kernel estimators, our algorithm is quite simple in updating procedures with no need for solving any optimization problems or nonlinear equations. This feature is particularly desirable for rapid data stream.

The paper is then organized in the following way. In Section 2, we give some preliminaries above the existing composite quantile estimations and the L-Estimation. In Section 3, the main idea of our methodologies is introduced in detail, including the computation of the renewable WCQR estimator, the selection of the weight functions, and some specific estimators and detailed renewable algorithms for estimating m(x) and $\sigma(x)$. In Section 4, the asymptotic properties of the proposed method are established; the selector of online-updating bandwidth is proposed and the optimal selection of the weight function is discussed. Section 5 contains comprehensive simulation studies and real data analyses to further demonstrate the desirable performance of the proposed estimators and algorithms. Some algorithms, lemmas, and technical proofs are deferred to the supplementary material.

1.4 Notations

For a random variable Z, denote by $f_Z(\cdot)$ and $F_Z(\cdot)$ the probability density function (PDF) and the distribution function of Z, respectively. Denote by $Q_{Y|x}(\tau)$ the conditional 100 τ % quantile of Y given $X = x \in \mathbb{R}$ and $\tau \in (0, 1)$, i.e. $Q_{Y|x}(\tau) = \inf \{t : \mathbb{P}(Y \le t | X = x) \ge \tau\}$. Denote by $F_{Y|x}(\cdot)$ the conditional distribution function (CDF) of Y given X = x, i.e., $F_{Y|x}(y) =$ $\mathbb{P}(Y < y | X = x)$. Denote by $f_{Y|x}(\cdot)$ the conditional PDF of Y given X = x. For $a, b \in \mathbb{R}$, denote $a \wedge b = \min \{a, b\}$. For indexes i and j, denote by δ_{ij} the kronecker symbol, i.e., $\delta_{ij} = 1$ if i = j and $\delta_{ij} = 0$ if $i \neq j$. Denote by $I_d(\cdot)$ the identity function, i.e., $I_d(y) = y$. Denote by $I(\cdot)$ the indicative function, i.e., if the proposition \mathcal{P} is true, $I(\mathcal{P}) =$ 1, and otherwise, $I(\mathcal{P}) = 0$. For a function f: $\mathbb{R} \to \mathbb{R}$, denote by Supp $\{f(\cdot)\}$ the support set of f; for $I_0 \subset \mathbb{R}$ and $1 \leq p < \infty$, denote $||f||_{p,I_0} =$ $(\int_{I_0} |f(y)|^p dy)^{1/p}$ and $||f||_{\infty,I_0} = \sup_{y \in I_0} |f(y)|$. For a finite point set $G \subset \mathbb{R}$, denote $\Delta(G) =$ $\max_{y_i \in G} (\min_{y_i \in G \setminus \{y_i\}} |y_i - y_j|)$, i.e., the maximum spacing between any two adjacent points in G; denote by #G the number of elements in G, and denote by $\min G$ (resp. $\max G$) the minimum (resp. maximum) of G. For two sets G_1 and G_2 , denote $G_1 \setminus G_2 = \{y : y \in G_1, y \notin G_2\}$. For a set $E \subset \mathbb{R}$ and a function $f : \mathbb{R} \to \mathbb{R}$, denote f(E) = $\{f(x): x \in E\}$; denote $I(E) = [\inf E, \sup E]$ the closed interval generated by E.

2 Preliminary

In this section, we will briefly review some fundamental works, from which we obtain inspirations for our method.

2.1 Composite Quantile Regression and L-Estimation

To estimate the regression function m(x), the well-known composite quantile regression (CQR) estimator (see, e.g., Zou and Yuan, 2008; Kai et al., 2010) takes the form of

$$\widehat{m}_{\rm cqr}(x) = \frac{1}{q} \sum_{i=1}^{q} \widehat{Q}_{Y|x}(\tau_i), \qquad (2)$$

where $\widehat{Q}_{Y|x}(\tau_i)$ are some consistent estimators of the quantile $Q_{Y|x}(\tau_i)$, and τ_i are quantile levels chosen as $\tau_i = i/(q+1)$. When the PDF of ε is symmetric around 0, one can expect $m(x) = 1/q \sum_{i=1}^{q} Q_{Y|x}(\tau_i)$ implying that $\widehat{m}_{cqr}(x)$ is an asymptotic unbiased estimator of m(x). However, when the error is general, the aforementioned equality is not necessarily true and the naive CQR estimator may suffer from non-negligible bias. To address this issue, Sun et al. (2013) extended the CQR estimator to the WCQR estimator in from of

$$\widehat{m}_{\mathrm{wcqr}}(x) = \sum_{i=1}^{q} \omega_i \widehat{Q}_{Y|x}(\tau_i), \qquad (3)$$

where ω_i are weights selected to satisfy $\sum_{i=1}^{q} \omega_i = 1$ and $\sum_{i=1}^{q} \omega_i F_{\varepsilon}^{-1}(\tau_i) = 0$, such that the equality $m(x) = \sum_{i=1}^{q} \omega_i Q_{Y|x}(\tau_i)$ is established and meanwhile, the non-negligible bias of $\widehat{m}_{wcqr}(x)$ is eliminated.

The CQR and WCQR estimators can be expressed as L-estimators (see, e.g., Gutenbrunner and Jurečková, 1992; Koenker and Zhao, 1994) taking the form

$$\widehat{m}_{\mathrm{L}}(x;\,\nu) = \int_{[0,1]} \widehat{Q}_{Y|x}(\tau) \,\mathrm{d}\nu(\tau)\,,\qquad(4)$$

with ν a measure on [0, 1]. To cover the CQR (resp. WCQR) estimators, the measure of L-estimators can be particularly chosen as $\nu = 1/q \sum_{i=1}^{q} \delta_{\tau_i}$ (resp. $\nu = \sum_{i=1}^{q} \omega_i \delta_{\tau_i}$) with δ_{τ_i} a unit mass on τ_i . Additionally, when the measure ν is well chosen based on the error distribution in (1), the L-estimator can efficiently (or robustly) estimate m(x) and $\sigma(x)$ (see, e.g., Serfling, 1980; Koenker and Portnoy, 1987; Portnoy and Koenker, 1989; Koenker, 2005, etc).

In this paper, we still call $\hat{m}_{\rm L}(x; \nu)$ the WCQR estimator to highlight the weighted composite of quantile estimators.

2.2 Local Polynomial Interpolation

We will briefly review the main idea of local polynomial interpolation (LPI), which is a fundamental tool for numerical approximations and will be applied in our method. For more details about the interpolation techniques and their applications to stochastic computing, readers may refer to Gautschi (2012); Sauer (2011); Burden et al. (2015); Zhao et al. (2006, 2014); Fu et al. (2017), etc. Suppose that $(y_1, f_1), \dots, (y_q, f_q)$ are given data point generated from an unknown objective function $f : \mathbb{R} \to \mathbb{R}$, i.e., $f_i = f(y_i)$ for $i = 1, \dots, q$. Denote the node set $G_x = \{y_i\}_{i=1}^q$ and the elements y_i are called the interpolation nodes.

The so-called LPI aims at finding a piecewise polynomial function passing through all the given data points of $f(\cdot)$. To achieve this, for $1 \leq l \leq q$ and $y \in \mathbb{R}$, we denote by $\mathcal{N}_l(y, G_x)$ the set consisting of the *l* nearest interpolation nodes around y, or strictly speaking, $\mathcal{N}_l(y, G_x) = G_0$ with G_0 the unique set satisfying:

- a) $\#G_0 = l, G_0 \subset G_x;$
- b) for any $y' \in G_0$ and $y'' \in G_x \setminus G_0$, it holds that $|y - y'| \leq |y - y''|$, and whenever the equality holds, y' < y''.

Then we define the lth-degree LPI basis functions as

$$L(y, y_{i}; l, G_{x}) = \ell_{i}(y) I(y_{i} \in \mathcal{N}_{l+1}(y, G_{x})) \quad (5)$$

for $y_i \in G_x$ with $\ell_i(\cdot)$ the *l*th-degree Lagrange interpolating polynomials defined as

$$\ell_{i}\left(y\right) = \frac{\prod_{y_{j} \in \mathcal{N}_{l+1}\left(y, G_{x}\right) \setminus \left\{y_{i}\right\}}\left(y - y_{j}\right)}{\prod_{y_{j} \in \mathcal{N}_{l+1}\left(y, G_{x}\right) \setminus \left\{y_{i}\right\}}\left(y_{i} - y_{j}\right)}$$

Using the basis functions, the lth-degree LPI function is constructed as

$$\mathcal{I}_{l}f(y) = \sum_{y_{i}\in G_{x}} f(y_{i}) L(y, y_{i}; l, G_{x}).$$
(6)

Here and in the following context, we formally use \mathcal{I}_l to denote the *l*th-degree LPI operator in the sense that $\mathcal{I}_l f(\cdot)$ is a *l*th-degree LPI function obtained by (6).

It is easy to verify that $L(y_i, y_j; l, G_x) = \delta_{ij}$, implying that $\mathcal{I}_l f(\cdot)$ indeed passes through the points $\{(y_i, f_i)\}_{i=1}^q$. The accuracy of the LPI function approximating the objective function can be guaranteed by the following standard result (see, e.g., Theorem 3.3 of Burden et al. 2015 or Theorem 3.3 of Sauer 2011):

Lemma 1 (Burden et al. 2015; Sauer 2011) Suppose that $f(\cdot)$ has continuous derivatives up to order l + 1,

and $\mathcal{I}_l f(\cdot)$ is given in (6). Then it holds that

$$f(y) = \mathcal{I}_l f(y) + \frac{f^{(l+1)}(c)}{(l+1)!} \prod_{y_i \in \mathcal{N}_{l+1}(y, G_x)} (y - y_i),$$

where c is a number depends on y and lies in $[\min \overline{\mathcal{N}}_{l+1}(y), \max \overline{\mathcal{N}}_{l+1}(y)]$ with $\overline{\mathcal{N}}_{l+1}(y) = \mathcal{N}_{l+1}(y, G_x) \cup \{y\}.$

3 Methodology

In this section, we aim at proposing a renewable WCQR estimation to estimate m(x) and $\sigma(x)$ for $x \in I_*$ with I_* a bounded interval on \mathbb{R} .

3.1 Renewable WCQR Estimation

We start form a special case of (4), where the measure ν exists a density function $J(\cdot)$ on [0, 1], and the WCQR estimator $\hat{m}_{\rm L}(x; \nu)$ can be expressed as

$$\widehat{r}(x; J) = \int_{[0,1]} J(\tau) \,\widehat{Q}_{Y|x}(\tau) \,\mathrm{d}\tau.$$
(7)

By selecting appropriate $J(\cdot)$, the estimator $\hat{r}(x; J)$ is able to estimate a kind of parameters that can expressed as

$$r(x; J) = \int_{[0,1]} J(\tau) Q_{Y|x}(\tau) d\tau.$$
 (8)

To obtain $\hat{r}(x; J)$, the conventional approach is to minimize an L_1 -norm loss function characterized by the check function $\rho_{\tau}(u) =$ $u(\tau - I(u \leq 0))$. However, solving such a nonsmooth minimization problem is quite difficult when the data are of the form of streaming data sets. To address this issue, we manage to avoid estimating the quantiles in (3). Thus we introduce the substitution: $y = \hat{Q}_{Y|x}(\tau)$, i.e., $\tau = \hat{F}_{Y|x}(y)$ and rewrite (7) into

$$\widehat{r}(x; J) = \int_{\mathbb{R}} y J\left(\widehat{F}_{Y|x}\left(y\right)\right) d\widehat{F}_{Y|x}\left(y\right), \qquad (9)$$

where $\widehat{F}_{Y|x}(y)$ is an estimator of the CDF $F_{Y|x}(y)$.

Now the the key problem is to obtain a renewable estimation of $F_{Y|x}(\cdot)$. To this end, we approximate $F_{Y|x}(\cdot)$ by its LPI function given by

$$\mathcal{I}_{l}F_{Y|x}(y) = \sum_{y_{i}\in G_{x}} F_{Y|x}(y_{i}) L(y, y_{i}; l, G_{x}), \quad (10)$$

where $G_x = \{y_i\}_{i=1}^q$ is the set of interpolation nodes chosen for $x \in I_*$ and $L(\cdot, y_i; l, G_x)$ are PLI basis functions defined in (5). Then the unknown values $F_{Y|x}(y_i)$ can be estimated by their empirical analogues obtained from the streaming data sets $\mathcal{D}_1, \mathcal{D}_2, \cdots$, i.e.,

$$\widehat{F}_{Y|x,t}(y_i) = \frac{\widehat{S}_{Y|x,t}(y_i)}{\widehat{f}_{X,t}(x)} \text{ for } y_i \in G_x, \quad (11)$$

where $\widehat{f}_{X,t}(x)$ and $\widehat{S}_{Y|x,t}(y_i)$ are renewable statistics obtained by

$$N_{t} = N_{t-1} + n_{t},$$

$$\hat{f}_{X,t}(x) = \frac{N_{t-1}}{N_{t}} \hat{f}_{X,t-1}(x) + \frac{1}{N_{t}} \sum_{j=1}^{n_{t}} K_{h_{t}}(X_{tj} - x), \quad (12)$$

$$\widehat{S}_{Y|x,t}(y_i) = \frac{N_{t-1}}{N_t} \widehat{S}_{Y|x,t-1}(y_i) + \frac{1}{N_t} \sum_{j=1}^{n_t} I(Y_{tj} < y_i) K_{h_t}(X_{tj} - x)$$
(13)

with initial values $N_0 = \hat{f}_{X,0}(x) = \hat{S}_{Y|x,0}(y_i) = 0$ and the bandwidths $h_t > 0$. Based on (10), we plug in the estimators $\hat{F}_{Y|x,t}(y_i)$ of $F_{Y|x}(y_i)$ and obtain the interpolated empirical CDF as follows

$$\mathcal{I}_{l}\widehat{F}_{Y|x,t}\left(y\right) = \sum_{y_{i}\in G_{x}}\widehat{F}_{Y|x,t}\left(y_{i}\right)L\left(y,y_{i};\ l,G_{x}\right).$$
(14)

By (9) with $\mathcal{I}_{l}\widehat{F}_{Y|x,t}(y)$ in place of $\widehat{F}_{Y|x}(y)$, we can obtain the renewable WCQR estimator as

$$\widetilde{r}_t(x; J) = \int_{\mathbb{R}} y J\left(\mathcal{I}_l \widehat{F}_{Y|x,t}(y)\right) d\mathcal{I}_l \widehat{F}_{Y|x,t}(y) \,.$$
(15)

Since the expression of $\mathcal{I}_l \widehat{F}_{Y|x,t}(\cdot)$ is known, the integral in (15) can be accurately approximated by numerical integrations, e.g., the well-known trapezoidal rule, Simpson rule and Romberg integration, etc (see, e.g., Section 3 of Gautschi 2012).

By applying LPI on the x-axis, we can extend the pointwise estimator $\tilde{r}_t(x; J)$ to estimate the function $r(\cdot; J)$ on the entire interval $x \in I_*$. Specifically, we can approximate $r(\cdot; J)$ by its LPI function, i.e.,

$$\mathcal{I}_{l}r(x; J) = \sum_{x_{i} \in G_{*}} r(x_{i}; J)L(x, x_{i}; l, G_{*}), \quad (16)$$

where $G_* = \{x_i\}_{i=1}^{\bar{q}}$ is a set of grid points introduced on the interval I_* , and typically x_i can be chosen as equal-spaced grid points on I_* . With $r(x_i; J)$ estimated by $\tilde{r}_t(x_i; J)$, we can obtain the renewable interpolated WCQR estimator as

$$\mathcal{I}_{l}\widetilde{r}_{t}(x; J) = \sum_{x_{i} \in G} \widetilde{r}_{t}(x_{i}; J)L(x, x_{i}; l, G_{*}) \quad (17)$$

for $x \in I_*$. In Section 3.3, we will see that $\mathcal{I}_l \tilde{r}_t(\cdot; J)$ plays a role in selecting the weight function $J(\cdot)$ for streaming data.

For the restriction on the weight function in (15), we have the following remark.

Remark 1 By Lemma 1, we can conclude that

$$\|\mathcal{I}_{l}F_{Y|x} - F_{Y|x}\|_{\infty, I(G_{x})}$$

$$\leq \|F_{Y|x}^{(l+1)}\|_{\infty, I(G_{x})} |\Delta(G_{x})|^{l+1}$$

with $I(G_x) = [\min G_x, \max G_x]$. Thus if $F_{Y|x}(\cdot)$ is sufficiently smooth and the nodes are dense enough, the error caused from LPI can be negligible on $I(G_x)$. However, when y moves away from the interval $I(G_x)$, the LPI approximation cannot guarantee its accuracy. Fortunately, we can select appropriate $J(\cdot)$ to suppress the error of LPI when y lies outside $I(G_x)$. Specifically, for the renewable WCQR estimator $\tilde{r}_t(x; J)$, we can select $J(\cdot)$ satisfying

$$\operatorname{Supp}\left\{J\left(\cdot\right)\right\} \subset \mathcal{I}_{l}\widehat{F}_{Y|x,t}^{-1}\left(I\left(G_{x}\right)\right).$$
(18)

And for the interpolated WCQR estimator $\mathcal{I}_l \tilde{r}_t(\cdot; J)$, we can select $J(\cdot)$ satisfying (18) for all $x \in G_*$ with G_* satisfying $I(G_*) \supset I_*$. Given $I(G_x)$ wide enough, the above restrictions are mild in robust estimations, because $\text{Supp} \{J(\cdot)\} \subset (0,1)$ is a natural condition to guarantee the robustness of $\hat{r}(x; J)$ (see, e.g., Section 8.1.3 of Serfling, 1980).

In the remainder of this section, we mainly discuss the selection of the weight function $J(\cdot)$, which plays a key role in reducing the estimation bias and variance of our renewable WCQR estimator.

3.2 Model-Based Weight Selections

To estimate m(x) (resp. $\sigma(x)$) by renewable WCQR estimation, we should select appropriate

weight functions $J(\cdot)$ to establish the equality r(x; J) = m(x) (resp. $\sigma(x)$). To this end, recalling (8) and using the relation $Q_{Y|x}(\tau) = m(x) + \sigma(x)F_{\varepsilon}^{-1}(\tau)$ obtained from the model (1), we can deduce

$$r(x; J) = m(x) \int_{[0,1]} J(\tau) d\tau + \sigma(x) \int_{[0,1]} J(\tau) F_{\varepsilon}^{-1}(\tau) d\tau,$$
(19)

which yields the conditions for estimating m(x):

$$C_{m1} : \int_{[0,1]} J(\tau) d\tau = 1,$$

$$C_{m2} : \int_{[0,1]} J(\tau) F_{\varepsilon}^{-1}(\tau) d\tau = 0,$$
(20)

and the conditions for estimating $\sigma(x)$:

$$C_{\sigma 1} : \int_{[0,1]} J(\tau) d\tau = 0,$$

$$C_{\sigma 2} : \int_{[0,1]} J(\tau) F_{\varepsilon}^{-1}(\tau) d\tau = 1.$$
(21)

Among the above conditions, C_{m1} and $C_{\sigma 1}$ can be easily satisfied. However, C_{m2} and $C_{\sigma 2}$ are related to the unknown quantile function $F_{\varepsilon}^{-1}(\cdot)$. Unless the error distribution is known to be symmetric, it is quite difficult to choose $J(\cdot)$ in an renewable manner; see the following remark.

Remark 2 In the existing works, e.g., Sun et al. (2013); Lin et al. (2019); Jiang et al. (2016), the condition C_{m2} is fulfilled empirically by replacing the unknown function $F_{\varepsilon}^{-1}(\cdot)$ with its estimator $\hat{F}_{\varepsilon}^{-1}(\cdot)$. Here $\hat{F}_{\varepsilon}^{-1}(\cdot)$ is the sample quantile function of the "pseudo" samples $\hat{\varepsilon}_{ti} = (Y_{ti} - \hat{m}(X_{ti}))/\hat{\sigma}(X_{ti})$, where $\hat{m}(\cdot)$ and $\hat{\sigma}(\cdot)$ are some pilot estimators of $m(\cdot)$ and $\sigma(\cdot)$. The generation of pseudo samples involves reusing the historical raw data and requires sophisticated computations, which are hardly to be implemented for streaming data.

To avoid the problem mentioned above, we fully use the structure of the Model (1) and obtain the following important lemma, which gives an alternative way to fulfill the conditions C_{m2} and $C_{\sigma 2}$.

Lemma 2 Let $W : \mathbb{R} \to \mathbb{R}$ be a function satisfying $\mathbb{E}[W(X)\sigma(X)] > 0$, then the following results hold: *i*) if C_{m1} holds, then

$$\mathbb{E}\left[W\left(X\right)\sigma\left(X\right)\right]\int_{[0,1]} J(\tau)F_{\varepsilon}^{-1}\left(\tau\right)d\tau$$

$$=\mathbb{E}\left[W(X)\left(r\left(X;\ J\right)-Y\right)\right];$$
ii) if $C_{\sigma 1}$ *holds, then*

$$\mathbb{E}\left[W(X)r^{2}\left(X;\ J\right)\right]\left(\int_{[0,1]} J(\tau)F_{\varepsilon}^{-1}\left(\tau\right)d\tau\right)^{2}$$

$$=\mathbb{E}\left[W\left(X\right)\left(Y^{2}-m^{2}\left(X\right)\right)\right].$$

Lemma 2 suggests the following alternative conditions on the weight function:

$$C'_{m2} : \mathbb{E} \left[W(X)r(X; J) \right] = \mathbb{E} \left[W(X)Y \right], \quad (22)$$

$$C'_{\sigma 2} : \mathbb{E} \left[W(X)r^2(X; J) \right]$$

$$= \mathbb{E}\left[W(X)\left(Y^2 - m^2\left(X\right)\right)\right].$$
(23)

Lemma 2 also shows that $(C_{m1}, C_{m2}) \Leftrightarrow (C_{m1}, C'_{m2})$ and $(C_{\sigma 1}, C_{\sigma 2}) \Leftrightarrow (C_{\sigma 1}, C'_{\sigma 2})$, i.e., the condition C_{m2} (resp. $C_{\sigma 2}$) can be equivalently replaced with C'_{m2} (resp. $C'_{\sigma 2}$). The following remark shows the advantage of C'_{m2} over C_{m2} .

Remark 3 As stated in Remark 2, to fulfill C_{m2} , the existing methods in Sun et al. (2013); Lin et al. (2019); Jiang et al. (2016) rely on the estimation of the inverse of the CDF of ε , which is unavailable directly from the sample set of (X, Y). Instead of directly related to the distribution of the error, the condition C'_{m2} only relies on the two expectations in (22), which can be directly estimated by the sample of (X, Y), and the estimation has the convergence rate of parametric estimation. Moreover, in the next subsection, we will see that C'_{m2} can be easily expressed in a renewable form.

Although the condition $C'_{\sigma 2}$ in (23) relies on the unknown regression function $m(\cdot)$, it will be shown in the next subsection that the unknown $m(\cdot)$ does not bring any essential difficulties to our renewable estimation.

3.3 Specific Renewable WCQR Estimators and Algorithms

To construct specific $J(\cdot)$ satisfying the conditions in the last subsection, we can obtain specific WCQR estimators for m(x) and $\sigma(x)$. Throughout this subsection, we assume that the function $W(\cdot)$ in (22) and (23) satisfies Supp $\{W(\cdot)\} \subset I_*$, under which we can avoid estimating r(x; J) with x outside of I_* . The simplest example of $W(\cdot)$ is that $W(x) = I(x \in I_*)$.

3.3.1 Estimators for the Conditional Mean

We first consider a simple case, where the model (1) is symmetric in the sense that the PDF of ε is symmetric around 0. For symmetric models, (C_{m1}, C_{m2}) can be easily fulfilled whenever $J(\cdot)$ is normalized and symmetric around 1/2. A representative example is the α -trimmed weight function:

$$J_{m,0.5}(\tau) = 0.5L_{\alpha}(\tau) + 0.5U_{\alpha}(\tau), \qquad (24)$$

where $L_{\alpha}(\tau) = (0.5 - \alpha)^{-1} I (\alpha \leq \tau \leq 0.5)$ and $U_{\alpha}(\tau) = (0.5 - \alpha)^{-1} I (0.5 < \tau \leq 1 - \alpha)$ with $\alpha \in (0, 0.5)$ selected according to Remark 1. With the weight function $J_{m,0.5}(\cdot)$, the WCQR estimator $\tilde{r}_t(x; J_{m,0.5})$ given by (15) is a renewable version of the naive local α -trimmed mean (NTM) (see, e.g., Bednar and Watt, 1984; Boente and Fraiman, 1994), and in the following, we will call $\tilde{r}_t(x; J_{m,0.5})$ the renewable NTM or the NTM if there is no confusion.

The following remark shows that the NTM is robust to outliers but not adaptive to symmetric or asymmetric error distributions.

Remark 4 The prototype of $\tilde{r}_t(x; J_{m,0.5})$ is the α trimmed mean, which has been singled out by several prominent authors as the quintessential robust estimator of location (see, e.g., Bickel and Lehmann, 1975; Stigler, 1977; Koenker, 2005). And we can expect that $\tilde{r}_t(x; J_{m,0.5})$ enjoys robustness comparable to the α -trimmed mean. However, since $\tilde{r}_t(x; J_{m,0.5})$ actually estimate the location $r(x; J_{m,0.5})$ rather than the conditional mean m(x), the estimation consistency deeply relies on the symmetry of the error to guarantee the conditions (C_{m1}, C_{m2}) and the equality $r(x; J_{m,0.5}) = m(x)$. When the error distribution is asymmetric, the estimator $\tilde{r}_t(x; J_{m,0.5})$ will suffer from a non-negligible bias caused from $r(x; J_{m,0.5}) \neq m(x)$.

To address the issue mentioned in Remark 4, we should modify the α -trimmed weight function $J_{m,0.5}(\cdot)$, such that the conditions (C_{m1}, C_{m2}) can be fulfilled for general error distributions. Thus we generalize $J_{m,0.5}(\cdot)$ into

$$J_{m,w}(\tau) = \omega L_{\alpha}(\tau) + (1-\omega) U_{\alpha}(\tau), \qquad (25)$$

where $w \in \mathbb{R}$ is a parameter selected to satisfy the alternative condition C'_{m2} , i.e.,

$$w = \frac{E_{WY} - E_{WU}}{E_{WL} - E_{WU}} \tag{26}$$

with

$$E_{WY} = \mathbb{E} \left[W(X)Y \right],$$

$$E_{WL} = \int_{I_*} W(x) r(x; L_{\alpha}) f_X(x) dx,$$

$$E_{WU} = \int_{I_*} W(x) r(x; U_{\alpha}) f_X(x) dx.$$

Since $J_{m,w}(\cdot)$ satisfies (C_{m1}, C'_{m2}) implying that $r(x; J_{m,w}) = m(x)$, i.e., the non-negligible bias is corrected. In the following text, we will call $\tilde{r}_t(x; J_{m,w})$ the renewable bias-corrected local α -trimmed mean (BCTM) or the BCTM if there is no confusion.

Remark 5 The BCTM $\hat{r}(x; J_{m,w})$ is an extension of the NTM $\hat{r}(x; J_{m,0.5})$. Actually, when the model (1) is symmetric, by straightforward calculation, we have w = 1/2, in which case $\hat{r}(x; J_{m,w})$ is identical to $\hat{r}(x; J_{m,0.5})$.

To obtain the BCTM for streaming data, we should construct a renewable estimator for the unknown w. Based on the expressions in (26) and the plug-in estimators, w can be estimated by

$$\widehat{w}_t = \frac{\widehat{E}_{WY,t} - \widehat{E}_{WU,t}}{\widehat{E}_{WL,t} - \widehat{E}_{WU,t}}$$
(27)

where

$$\begin{aligned} \widehat{E}_{WY,t} &= \frac{N_{t-1}}{N_t} \widehat{E}_{WY,t-1} + \frac{1}{N_t} \sum_{j=1}^{n_t} W(X_{tj}) Y_{tj} \\ \widehat{E}_{WL,t} &= \int_{I_*} W(x) \mathcal{I}_l \widetilde{r}_t(x; L_\alpha) \mathcal{I}_l \widehat{f}_{X,t}(x) \, \mathrm{d}x, \\ \widehat{E}_{WU,t} &= \int_{I_*} W(x) \mathcal{I}_l \widetilde{r}_t(x; U_\alpha) \mathcal{I}_l \widehat{f}_{X,t}(x) \, \mathrm{d}x \end{aligned}$$

with the initial value $\widehat{E}_{WY,0} = 0$. Here $\mathcal{I}_l \widetilde{r}_t (\cdot ; L_\alpha)$ and $\mathcal{I}_l \widetilde{r}_t (\cdot ; U_\alpha)$ are the interpolated WCQR estimator given in (17), and $\mathcal{I}_l f_X (\cdot)$ is the interpolated empirical PDF given by

$$\mathcal{I}_{l}f_{X}\left(\cdot\right) = \sum_{x_{i}\in G_{*}}\widehat{f}_{X,t}\left(x_{i}\right)L\left(\cdot,x_{i};\ l,G_{*}\right) \quad (28)$$

with $\hat{f}_{X,t}(x_i)$ the renewable statistics obtained by (12) and G_* the node set introduced in (16).

If we omit the error caused by LPI approximation, \hat{w}_t is a consistent estimator of w, and we known from the Slutsky's Lemma that the estimator $\tilde{r}_t(x; J_{m,\hat{w}_t})$ has the same asymptotic distribution as that of $\tilde{r}_t(x; J_{m,w})$.

In the following remark, we show the pros and cons of the BCTM compared with the NTM.

Remark 6 Compared with the NTM, the first advantage of the BCTM is that the estimation consistency is based on the structure of the model instead of the symmetry of the error. Thus the BCTM is adaptive to symmetric or asymmetric error distributions. However this adaptiveness comes with a little costs in robustness, when estimating the parameter ω . Actually, it is easy to check that the estimators $\widehat{E}_{W1,t}$ and $\widehat{E}_{W2,t}$ have bounded influence functions, and thus both of them are robust to outliers. However, the estimator $E_{WY,t}$ is somewhat non-robust. Fortunately, the unknown $E_{WY,t}$ is a scalar parameter rather than a general function, thus the estimator $\widehat{E}_{WY,t}$ can achieve $\sqrt{N_t}$ -consistency, which is faster than the optimal convergence rate of nonparametric regression estimation. Hence the final nonparametric estimator $\widetilde{r}_t(x; J_{m,\widehat{w}_t})$ is less susceptible to the weak robustness of $\widehat{E}_{WY,t}$. This point will be further demonstrated in our numerical studies in Section 5.

In Algorithm 1, we present the detail procedures to obtain the renewable BCTM for the estimation of m(x) with x belonging to some gird points in I_1 .

3.3.2 Estimators for the Conditional Variance

To estimate $\sigma(x)$, the first condition $C_{\sigma 1}$ can be fulfilled by taking $J(\cdot)$ antisymmetry around 1/2. Then parallel to the NTM, we introduce the antisymmetry α -trimmed weight function:

$$J_{\sigma,1}(\tau) = \left(-L_{\alpha}\left(\tau\right) + U_{\alpha}\left(\tau\right)\right). \tag{29}$$

From (19) and (29), we can see that

$$r(x; J_{\sigma,1}) = c_0 \sigma(x)$$
 for $x \in I_*$

with $c_0 = \int_{[0,1]} J_{\sigma,1}(\tau) F_{\varepsilon}^{-1}(\tau) d\tau$ a constant independent with x. Thus the renewable WCQR estimator $\tilde{r}_t(x; J_{\sigma,1})$ is qualified to estimate $\sigma(x)$ consistently up to scale; that is,

$$\tilde{r}_t\left(x; \ J_{\sigma,1}\right) = c_0 \sigma(x) + O\left(\left|\Delta\left(G_x\right)\right|^{l+1}\right) + o_p\left(1\right).$$

In the following text, we call $\tilde{r}_t(x; J_{\sigma,1})$ the naive α -trimmed standard derivation (NTSD), since it is modified from the NTM and is used to estimate the conditional standard derivation.

If we aim higher and want to estimate $\sigma(x)$ consistently, we should rescale the weight function such that $c_0 = 1$, or equivalently, $C_{\sigma 2}$ holds. To this end, we extend $J_{\sigma,1}(\cdot)$ into

$$J_{\sigma,\theta}(\tau) = \theta J_{\sigma,1}(\tau) \tag{30}$$

where θ is a scale parameter selected to satisfying the alternative $C'_{\sigma 2}$, i.e.,

$$\theta^2 = \frac{\mathbb{E}\left[W(X)\left(Y^2 - m^2\left(X\right)\right)\right]}{\mathbb{E}\left[W(X)r^2\left(X; \ J_{\sigma,1}\right)\right]}.$$
 (31)

The main issue is to obtain a renewable estimation of the unknown θ . Similar to the idea introduced in Section 3.3.1, we can estimate θ by

$$\widehat{\theta}_t = \sqrt{\frac{\widehat{E}_{WY^2,t} - \widehat{E}_{Wm^2,t}}{\widehat{E}_{Wr^2,t}}}$$
(32)

where

$$\begin{split} \widehat{E}_{WY^{2},t} &= \frac{N_{t-1}}{N_{t}} \widehat{E}_{WY^{2},t-1} + \frac{1}{N_{t}} \sum_{j=1}^{n_{t}} W\left(X_{tj}\right) Y_{tj}^{2}, \\ \widehat{E}_{Wm^{2},t} &= \int_{I_{*}} W\left(x\right) \widetilde{m}_{t}^{2}(x) \mathcal{I}_{l} \widehat{f}_{X,t}\left(x\right) \mathrm{d}x, \\ \widehat{E}_{Wr^{2},t} &= \int_{I_{*}} W\left(x\right) \left(\mathcal{I}_{l} \widetilde{r}_{t}\right)^{2} \left(x; \ J_{\sigma,1}\right) \mathcal{I}_{l} \widehat{f}_{X,t}\left(x\right) \mathrm{d}x \end{split}$$

Algorithm 1 Renewable BCTM for estimating $\{m(x_i)\}_{i=1}^{\bar{q}}$ with x_i gird points on \mathbb{R}

Input Set of grid points $G_* = \{x_i\}_{i=1}^{\overline{q}}$, kernel function $K(\cdot)$, node sets G_{x_i} for $x_i \in G_*$, PLI degree l1: Set initial values $N_0 = \widehat{E}_{WY,0} = \widehat{f}_{X,0}(x_i) = \widehat{S}_{Y|x,0}(y_{ij}) = 0$ for $y_{ij} \in G_{x_i}$ and $x_i \in G_*$ 2: for $t = 1, 2, \cdots$ do Obtain the *t*-th data chunk $\mathcal{D}_t = \{X_{tj}, Y_{tj}\}_{j=1}^{n_t}$ 3: Select the *t*-th bandwidth h_t 4: $N_t = N_{t-1} + n_t$ 5: $\widehat{E}_{WY,t} = N_{t-1}/N_t \widehat{E}_{WY,t-1} + 1/N_t \sum_{i=1}^{n_t} W(X_{ti}) Y_{ti}$ 6: for $x_i \in G_*$ do 7: $\hat{f}_{X,t}(x_i) = N_{t-1}/N_t \hat{f}_{X,t-1}(x_i) + 1/N_t \sum_{j=1}^{n_t} K_{h_t}(X_{tj} - x_i), \text{ where } K_h(\cdot) = 1/hK(\cdot/h)$ 8: $\widehat{S}_{Y|x_i,t}(y_{ij}) = N_{t-1}/N_t \widehat{S}_{Y|x_i,t-1}(y_{ij}) + 1/N_t \sum_{j=1}^{n_t} I(Y_{tj} < y_{ij}) K_{h_t}(X_{tj} - x_i) \text{ for } y_{ij} \in G_{x_i}$ 9: end for 10: if the estimators for $\{m(x_i)\}_{x_i \in G_*}$ are needed then 11:Define the function $\mathcal{I}_l \widehat{f}_{X,t}(\cdot) = \sum_{x_i \in G_*} \widehat{f}_{X,t}(x_i) L(\cdot, x_i; l, G_*)$ 12:Define the function $\mathcal{I}_{l}\widehat{F}_{Y|x_{i},t}\left(\cdot\right) = \sum_{y_{ij}\in G_{x_{i}}}\widehat{S}_{Y|x_{i},t}\left(y_{ij}\right)/\widehat{f}_{X,t}\left(x_{i}\right)L\left(\cdot,y_{ij};\ l,G_{x_{i}}\right)$ for $x_{i}\in G_{*}$ 13:for $J \in \{L_{\alpha}, U_{\alpha}\}$ do 14:
$$\begin{split} \widetilde{r}_{t}(x_{i}; J) &= \int_{\mathbb{R}} y J(\mathcal{I}_{l} \widehat{F}_{Y|x_{i}, t}(y)) \mathrm{d}\mathcal{I}_{l} \widehat{F}_{Y|x_{i}, t}(y) \text{ for } x_{i} \in G_{*} \\ \text{Define the function } \mathcal{I}_{l} \widetilde{r}_{t}(\cdot; J) &= \sum_{x_{i} \in G_{*}} \widetilde{r}_{t}(x_{i}; J) L(\cdot, x_{i}; l, G_{*}) \end{split}$$
15:16:end for 17: $\widehat{E}_{WL,t} = \int_{L} W(x) \mathcal{I}_{l} \widetilde{r}_{t}(x; L_{\alpha}) \mathcal{I}_{l} \widehat{f}_{X,t}(x) \,\mathrm{d}x$ 18: $\widehat{E}_{WU,t} = \int_{I} W(x) \mathcal{I}_{l} \widetilde{r}_{t}(x; U_{\alpha}) \mathcal{I}_{l} \widehat{f}_{X,t}(x) dx$ 19: $\widehat{w}_{t} = \left(\widehat{E}_{WY,t} - \widehat{E}_{WU,t}\right) / \left(\widehat{E}_{WL,t} - \widehat{E}_{WU,t}\right)$ 20: $\widetilde{r}_t(x_i; J_{m,\widehat{w}_t}) = \widehat{w}_t \widetilde{r}_t(x_i; L_{\alpha}) + (1 - \widehat{w}_t) \widetilde{r}_t(x_i; U_{\alpha}) \text{ for } x_i \in G_*$ 21:Output $\{\widetilde{r}_t(x_i; J_{m,\widehat{w}_t})\}_{x_i \in G_*}$ as the estimators for $\{m(x_i)\}_{x_i \in G_*}$ 22: end if 23:24: end for

with the initial value $\widehat{E}_{WY^2,0} = 0$. Here $\mathcal{I}_{l}\widetilde{r}_{t}(\cdot; J_{\sigma,1})$ and $\mathcal{I}_{l}\widehat{f}_{X,t}(\cdot)$ are renewable estimator obtained by (17) and (28), respectively, and $\widetilde{m}_{t}(\cdot)$ is an renewable estimator of $m(\cdot)$, e.g.,

$$\widetilde{m}_t(\cdot) = \mathcal{I}_l \widetilde{r}_t(\cdot ; J_{m,w}).$$

with w = 0.5 for symmetric models and $w = \hat{w}_t$ given in (27) for asymmetric models.

Based on (15) with $J_{\sigma,\hat{\theta}_t}(\cdot)$ in place of $J(\cdot)$, we can obtain the renewable estimator $\tilde{r}_t(x; J_{\sigma,\hat{\theta}_t})$ for the conditional standard deviation $\sigma(x)$. In the following text, we will call $\tilde{r}_t(x; J_{\sigma,\hat{\theta}_t})$ the rescaled α -trimmed standard derivation (RTSD).

In the following remark, we make a comparison between the NTSD and the RTSD.

Remark 7 The RTSD $\tilde{r}_t(x; J_{\sigma,\hat{\theta}_t})$ contains more information about $\sigma(x)$, since it can identify the constant c_0 , which is unrevealed by the NTSD $\tilde{r}_t(x; J_{\sigma,1})$. In

another aspect, the NTSD enjoys desirable robustness comparable to the classic α -trimmed mean. While the RTSD depends on the plug-in estimator $\hat{E}_{WY^2,t}$, and similar to the discussions in Remark 6, $\hat{E}_{WY^2,t}$ is somewhat non-robust but enjoys the convergence rate of parametric estimation.

In the supplementary material, we present a complete algorithm to obtain renewable WCQR estimators for m(x) and $\sigma(x)$ from asymmetric models.

In the following remark, we show the desirable computation efficiency of our algorithms in dealing with rapid data steams.

Remark 8 Algorithm 1 mainly consists of two parts: the updating part (Lines 5 - 10) where the cumulative statistics are updated, and the estimation part (Lines 12 - 21) where the WCQR estimator is computed by calculating integrals. Notice that the updating part is relatively simple without solving any nonlinear equations. And at each updating step, the computations among each $x_* \in G_*$ can be implemented parallelly. Benefit from this, the updating part can be implemented fast enough to catch up with the rapid data steam. The estimation part seems to be computationally intensive. Fortunately, this part is implemented only when one needs the current value of the WCQR estimator. Thus it would not cause trouble in computation speed even when the data stream is rapid. The above desirable feature is also enjoyed by Algorithm A.1 in the supplementary material.

4 Theoretical Analyses

In this section, we will deduce the asymptotic distribution of the renewable WCQR estimator, based on which we will propose renewable bandwidth selectors and obtain the optimal weight functions by minimizing the asymptotic variance.

4.1 Asymptotic Distributions

For theoretical analyses of the estimator $\tilde{r}_t(x; J)$, we introduce the following standard assumptions.

Assumption 1 The functions $f_X(\cdot)$, $m(\cdot)$ and $\sigma(\cdot)$ have continuous derivatives up to order 4 on I_* . On a open set containing Supp $\{J(\cdot)\}$, the function $F_{\varepsilon}(\cdot)$ has continuous derivatives up to order max $\{l + 1, 4\}$, where l is the degree of PLI given in (10). The function $f_X(\cdot)$ admits a positive lower bounded on I_* .

Assumption 2 The kernel function K(u) is symmetric and compactly supported, and satisfies that $\int_{\mathbb{R}} K(u) du = 1, \ k_{4,1} = \int_{\mathbb{R}} u^4 K(u) du < \infty$ and $k_{0,2} = \int_{\mathbb{R}} K^2(u) du < \infty$.

Assumption 3 The bandwidth h_t and smoothing parameter b_t satisfies that as $t \to \infty$,

$$h_t = o(1) \quad and \quad \sum_{s=1}^t \frac{n_s}{h_s} \left(\sum_{s=1}^t n_s h_s^4 \right)^{-2} = o(1), \quad (33)$$

$$\frac{1}{N_t} \sum_{s=1}^{s} \frac{n_s}{N_t h_s} = o(1).$$
(34)

Assumption 4 The L-score function $J(\cdot)$ is piecewise continuously differentiable. There exist constants 0 < $\underline{\tau} < \overline{\tau} < 1 \text{ such that } \operatorname{Supp} \{J(\cdot)\} \subset [\underline{\tau}, \overline{\tau}]. \text{ The node set} \\ G_x \text{ satisfies } [\underline{\tau}, \overline{\tau}] \subset (\min F_{Y|x} (G_x), \max F_{Y|x} (G_x)).$

In Assumptions 1 and 2, all the conditions on $f_X(\cdot)$, $m(\cdot)$, $\sigma(\cdot)$ and $K(\cdot)$ are standard for nonparametric regressions, and the condition on $F_{\varepsilon}(\cdot)$ is required by the LPI approximation to guarantee its accuracy. Among the conditions in Assumption 3, the first one in (33) is required for a vanishing estimation bias; the second one in (33) is used to simplify the bias terms in the asymptotic results; the last condition (34) is necessary for a bounded estimation variance, and it is fulfilled whenever $N_t \min_{1 \le s \le t} h_s \to \infty$. Assumption 4 is used to bound the remainder term in the asymptotic expansion of $\tilde{r}_t(x; J)$; the restrictions on Supp $\{J(\cdot)\}$ and G_x are also required by the LPI approximation and in practical applications, it can be fulfilled empirically; see Remark 1.

The following theorem gives the asymptotic properties of the interpolated empirical CDF $\mathcal{I}_l \widehat{F}_{Y|x,t}(\cdot)$.

Theorem 3 Under Assumptions 1 - 3, it holds for $y \in \mathbb{R}$ that

$$\begin{cases} \frac{1}{N_t} \sum_{s=1}^t \frac{n_s}{N_t h_s} \end{cases}^{-1/2} \left\{ \mathcal{I}_l \widehat{F}_{Y|x,t} \left(y \right) - F_{Y|x} \left(y \right) \\ -R_{\mathcal{I}_l,F,x} \left(y \right) - \mathcal{I}_l B_{F,x} \left(y \right) \sum_{s=1}^t \frac{n_s h_s^2}{N_t} \right\} \\ \stackrel{\mathrm{d}}{\to} N \left(0, \mathcal{I}_l^2 C_{F,x} \left(y, y \right) \right), \end{cases}$$

where

$$R_{\mathcal{I}_{l},F,x}(y) = -\frac{F_{Y|x}^{(l+1)}(y)(c)}{(l+1)!} \prod_{y_{i} \in \mathcal{N}_{l+1}(y,G_{x})} (y-y_{i}),$$
$$\mathcal{I}_{l}B_{F,x}(y) = \sum_{y_{i} \in G_{x}} B_{F,x}(y_{i}) L(y,y_{i}; l,G_{x}),$$
$$\mathcal{I}_{l}^{2}C_{F,x}(z_{1},z_{2}) = \sum_{y_{i},y_{j} \in G_{x}} L(z_{1},y_{i}; l,G_{x}) \times L(z_{2},y_{j}; l,G_{x}) C_{F,x}(y_{i},y_{j})$$

with

$$B_{F,x}(y_i) = \frac{k_{2,1}}{2} \left\{ \partial_x^2 F_{Y|x}(y_i) + 2\partial_x F_{Y|x}(y_i) \frac{f'_X(x)}{f_X(x)} \right\}, \quad (35)$$
$$C_{F,x}(y_i, y_j) = \frac{k_{0,2}}{f_X(x)} \left\{ F_{Y|x}(y_i \wedge y_j) \right\}$$

$$-F_{Y|x}\left(y_{i}\right)F_{Y|x}\left(y_{j}\right)\right\}$$
(36)

and $c \in [\min \overline{\mathcal{N}}_{l+1}(y), \max \overline{\mathcal{N}}_{l+1}(y)]$ with $\overline{\mathcal{N}}_{l+1}(y)$ given in Lemma 1.

Based on Theorem 3, we can conclude that

$$\begin{aligned} \left\| \mathcal{I}_{l} \widehat{F}_{Y|x,t} - F_{Y|x} \right\|_{2,I(G_{x})} &= O(\left| \Delta \left(G_{x} \right) \right|^{l+1}) \\ + O\left(\sum_{s=1}^{t} \frac{n_{s} h_{s}^{2}}{N_{t}} \right) + O_{p}\left(\sqrt{\frac{1}{N_{t}} \sum_{s=1}^{t} \frac{n_{s}}{N_{t} h_{s}}} \right). \end{aligned}$$
(37)

From (37) we can see that the error of our interpolated empirical CDF consists of two parts: the first part depending on $\Delta(G_x)$ is a numerical error caused from the LPI approximation, and the second part depending on the bandwidths consists of statistical errors corresponding to the bias and variance of kernel estimations. Because the statistical errors have a convergence rate the same with that of the oracle empirical CDF obtained on the imaginary full data set $\cup_{s \leq t} \mathcal{D}_t$, and the numerical error is usually negligible compared with the statistical ones (see the following Remark 9), it can be expected that our estimator enjoys a performance almost as well as the oracle estimator.

Remark 9 In practical applications, the numerical error in (37) is usually much smaller than the remaining statistical errors. Actually, when the nodes in G_x are uniformly spaced, the numerical error is of order $O((\#G_x)^{-(l+1)})$, which can be reduced significantly by increasing the number $\#G_x$ of nodes and the degrees l of LPI, whenever the computation resources permit. Unlike the numerical one, the statistical errors deeply rely on the number N_t of samples, and the convergence rate is relatively slow (no more than $O(N_t^{-2/5}))$.

Based on Theorem 3, we have the following main result on the asymptotic property of the estimator $\tilde{r}_t(x; J)$.

Theorem 4 Under Assumptions 1 - 4, it holds that

$$\left\{\frac{1}{N_t}\sum_{s=1}^t \frac{n_s}{N_t h_s}\right\}^{-1/2} \left\{\widetilde{r}_t(x; J) - r(x; J) - B_{\mathcal{I}_l,m,x}\sum_{s=1}^t \frac{n_s h_s^2}{N_t} - R_{\mathcal{I}_l,m,x}\right\} \stackrel{\mathrm{d}}{\to} N\left(0, \Sigma_{\mathcal{I}_l,m,x}\right).$$
(38)

where

$$B_{\mathcal{I}_{l},m,x} = -\int_{\mathbb{R}} J\left(F_{Y|x}\left(y\right)\right) \mathcal{I}_{l}B_{F,x}\left(y\right) dy,$$

$$R_{\mathcal{I}_{l},m,x} = -\int_{\mathbb{R}} J\left(F_{Y|x}\left(y\right)\right) R_{\mathcal{I}_{l},F,x}\left(y\right) dy$$

$$+ O\left(|\Delta\left(G_{x}\right)|^{2l+2}\right),$$

$$\Sigma_{\mathcal{I}_{l},m,x} = \int_{\mathbb{R}} \int_{\mathbb{R}} C_{\mathcal{I}_{l},J,x}\left(z_{1},z_{2}\right) dz_{1} dz_{2}$$

with

$$C_{\mathcal{I}_l,J,x}\left(z_1,z_2\right) = J\left(F_{Y|x}\left(z_1\right)\right) J\left(F_{Y|x}\left(z_2\right)\right)$$
$$\times \mathcal{I}_l^2 C_{F,x}\left(z_1,z_2\right)$$

and $\mathcal{I}_{l}B_{F,x}(y)$, $R_{\mathcal{I}_{l},F,x}(y)$ and $\mathcal{I}_{l}^{2}C_{F,x}(z_{1},z_{2})$ given in Theorem 3.

Theorem 4 shows that the LPI operator \mathcal{I}_l has influence on the asymptotic bias and variance of $\tilde{r}_t(x; J)$, while its influence is no more than the order of numerical errors. The following corollary states the details.

Corollary 5 Under the conditions of Theorem 4, it holds that

$$B_{\mathcal{I}_l,m,x} = B_{m,x} + O\left(\left|\Delta\left(G_x\right)\right|^{l+1}\right),$$

$$R_{\mathcal{I}_l,m,x} = O\left(\left|\Delta\left(G_x\right)\right|^{l+1}\right)$$

$$\Sigma_{\mathcal{I}_l,m,x} = \Sigma_{m,x} + O\left(\left|\Delta\left(G_x\right)\right|^{l+1}\right),$$

where the dominant terms are

$$B_{m,x} = -\int_{\mathbb{R}} J\left(F_{Y|x}\left(y\right)\right) B_{F,x}\left(y\right) dy$$
$$\Sigma_{m,x} = \frac{k_{0,2}\sigma^{2}(x)}{f_{X}(x)} \int_{[\underline{\tau},\overline{\tau}]} \int_{[\underline{\tau},\overline{\tau}]} S_{J}\left(\tau_{1},\tau_{2}\right) d\tau_{1} d\tau_{2}$$

with

$$S_J(\tau_1, \tau_2) = \frac{(\tau_1 \wedge \tau_2 - \tau_1 \tau_2) J(\tau_1) J(\tau_2)}{f_{\varepsilon} \left(F_{\varepsilon}^{-1}(\tau_1)\right) f_{\varepsilon} \left(F_{\varepsilon}^{-1}(\tau_2)\right)}.$$
 (39)

Combining Theorem 4 and Corollary 5, we can find that the error between $\tilde{r}_t(x; J)$ and $\tilde{r}_t(x; J)$ has a convergence rate the same with the error given in (37), i.e.,

$$\widetilde{r}_t(x; J) - r(x; J) = O(|\Delta(G_x)|^{l+1}) + O\left(\sum_{s=1}^t \frac{n_s h_s^2}{N_t}\right) + O_p\left(\sqrt{\frac{1}{N_t} \sum_{s=1}^t \frac{n_s}{N_t h_s}}\right),$$
(40)

where the first term is the numerical error caused from LPI, and the remaining two terms are statistical errors caused from kernel estimations. By the discussion in Remark 9, the numerical error is usually negligible compared with the statistical ones. Moreover, the convergence rates of the statistical errors are the same with that of the oracle estimator $\hat{r}(x; J)$ obtained on the imaginary full data set $\bigcup_{s \leq t} \mathcal{D}_t$. Theoretically, our renewable WCQR estimator behaves almost as well as the oracle estimator.

Based on the relation (19) and the results in Lemma 2 and Theorem 4, we can immediately obtain the following corollary, which gives asymptotic properties of the estimators introduced in Section 3.3.

Corollary 6 Given $J(\cdot)$ satisfying (C_{m1}, C_{m2}) or (C_{m1}, C'_{m2}) (resp. $(C_{\sigma 1}, C_{\sigma 2})$ or $(C_{\sigma 1}, C'_{\sigma 2})$), the result (38) in Theorem 4 holds with r(x; J) replaced with m(x) (resp. $\sigma(x)$).

4.2 The Selection of Bandwidths

For implementing bandwidth selection, we need to calculate the asymptotic mean square error (AMSE) and the asymptotic mean integrated squared error (AMISE) between $\tilde{r}_t(x; J)$ and r(x; J). It follows from Theorem 4 and Corollary 5 that the asymptotic bias and variance of $\tilde{r}_t(x; J)$ can be given by

$$\operatorname{Bias}\left\{\widetilde{r}_{t}(x; J)\right\} = B_{m,x} \sum_{s=1}^{t} \frac{n_{s}}{N_{t}} h_{s}^{2}$$
$$+ o\left(\sum_{s=1}^{t} \frac{n_{s}}{N_{t}} h_{s}^{2}\right) + O(\left|\Delta\left(G_{x}\right)\right|^{l+1}),$$
$$\operatorname{Var}\left\{\widetilde{r}_{t}(x; J)\right\} = \frac{1}{N_{t}} \sum_{s=1}^{t} \frac{n_{s}}{N_{t} h_{s}} \Sigma_{m,x}$$

$$+\frac{1}{N_{t}}\sum_{s=1}^{t}\frac{n_{s}}{N_{t}h_{s}}\left(o\left(1\right)+O(\left|\Delta\left(G_{x}\right)\right|^{l+1})\right).$$

By Remark 9, it is reasonably to assume the numerical error is negligible compared with the statistical ones, thus we can omit the terms $O(|\Delta (G_x)|^{l+1})$ and other high order terms in the asymptotic bias and variance. Then the AMSE and AMISE can be respectively defined as

AMSE {
$$\widetilde{r}_t(x; J)$$
} = $\mathcal{E}_t(x; H_t)$,
AMISE { $\widetilde{r}_t(x; J)$ } = $\int_{I_*} \mathcal{E}_t(x; H_t) \widetilde{W}(x) dx$,

where $W(\cdot)$ is a weight function defined on I_* ; $H_t = (h_1, \cdots, h_t)^\top$ is a vector consisting of the used bandwidth components h_s up to the *t*-th data chunk and $\mathcal{E}_t(x; \cdot)$ is an error function defined by

$$\mathcal{E}_t (x; H_t) = \left(B_{m,x} \sum_{s=1}^t \frac{n_s h_s^2}{N_t} \right)^2 + \frac{1}{N_t} \sum_{s=1}^t \frac{n_s}{N_t h_s} \Sigma_{m,x}$$

$$(41)$$

with the asymptotic bias $B_{m,x}$ and the asymptotic variance $\Sigma_{m,x}$ given in Corollary 5.

4.2.1 Theoretically optimal bandwidths

We first consider an ideal situation where the streaming data is finite with the terminal time T and the cumulative number N_T known throughout the updating procedures $t = 1, \dots, T$. In this case, the theoretical optimal variable bandwidths $\tilde{H}_T^*(x) = (\tilde{h}_1^*(x), \dots, \tilde{h}_T^*(x))^{\top}$ and the optimal constant bandwidth $\tilde{H}_T^* = (\tilde{h}_1^*, \dots, \tilde{h}_T^*)^{\top}$ can be obtained conventionally by minimizing the terminal AMSE and AMISE, respectively, i.e.,

$$\begin{split} \tilde{H}_T^*\left(x\right) &= \mathop{\arg\min}_{H_T \in [0,\infty)^T} \mathcal{E}_T\left(x; \ H_T\right), \\ \tilde{H}_T^* &= \mathop{\arg\min}_{H_T \in [0,\infty)^T} \int_{I_*} \mathcal{E}_T\left(x; \ H_T\right) \widetilde{W}(x) \mathrm{d}x. \end{split}$$

A straightforward calculation leads to

$$\tilde{h}_{t}^{*}(x) = (C_{h}(x))^{1/5} N_{T}^{-1/5}, \ \tilde{h}_{t}^{*} = (C_{h})^{1/5} N_{T}^{-1/5}$$
(42)

for $t = 1, \dots, T$, where

$$C_{h}(x) = \frac{\sum_{m,x}}{4(B_{m,x})^{2}},$$
$$C_{h} = \frac{\int_{I_{*}} \sum_{m,x} \widetilde{W}(x) dx}{4 \int_{I_{*}} (B_{m,x})^{2} \widetilde{W}(x) dx}$$

The following remark shows the standard statistical convergence rate of $\tilde{r}_T(x; J)$ under the optimal bandwidths.

Remark 10 Recalling the error expressions in (37) and (40), we conclude that when the bandwidths are the optimal ones given in (42), the statistical errors of $\mathcal{I}_l \widehat{F}_{Y|x,T}(\cdot)$ and $\widetilde{r}_T(x; J)$ both enjoy the optimal convergence rate of $O_p\left(N_T^{-2/5}\right)$. We should remark that such a standard convergence rate is obtained on the streaming data sets without any restrictions on the chunk size or chunk number.

4.2.2 Practical sub-optimal bandwidths

It is usually closer to the real condition that the streaming data are endless or the terminal cumulative number N_T is unpredictable. In such a case, the theoretical optimal bandwidths in (42) are impractical, and we have to find sub-optimal bandwidths that do not rely on the information of future streaming data sets. To this end, we introduce the following lemma, which reveals the structure of the error function given in (41).

Lemma 7 The function \mathcal{E}_t defined in (41) has the decomposition:

$$\mathcal{E}_t (x; H_t) = \left(\frac{N_{t-1}}{N_t}\right)^2 \mathcal{E}_{t-1} (x; H_{t-1}) + \left(\frac{n_t}{N_t}\right)^2 E_t (x, h_t; H_{t-1})$$

with $E_t(x, h_t; H_{t-1})$ defined by

$$E_t(x, h_t; H_{t-1}) = h_t^4 (B_{m,x})^2 + \frac{1}{n_t h_t} \Sigma_{m,x} + 2h_t^2 \sum_{s=1}^{t-1} \frac{n_s h_s^2}{n_t} (B_{m,x})^2$$

From Lemma 7, we can see that at t-th updating procedure, the AMSE \mathcal{E}_t can be expressed as a weighted sum of two parts: the first one \mathcal{E}_{t-1} is the error of the last updating step, which has a fixed value at the current updating procedure; the second part $E_t(x, h_t; H_{t-1})$ relies on h_t and H_{t-1} , where H_{t-1} has been given in the previous updating procedure and only h_t is need to be determined. Thus the optimal value of h_t should minimize $E_t(x, h_t; H_{t-1})$ after the value of H_{t-1} is given. We thus define the sub-optimal variable and constant bandwidths respectively by

$$\begin{split} h_t^*\left(x\right) &= \mathop{\arg\min}_{h_t \in [0,\infty)} E_t\left(x,h_t; \ H_{t-1}^*\left(x\right)\right), \\ h_t^* &= \mathop{\arg\min}_{h_t \in [0,\infty)} \int_{I_*} E_t\left(x,h_t; \ H_{t-1}^*\right) \widetilde{W}(x) \mathrm{d}x, \end{split}$$

where $H_{t-1}^*(x) = (h_1^*(x), \dots, h_{t-1}^*(x))^\top$ and $H_{t-1}^* = (h_1^*, \dots, h_{t-1}^*)^\top$ are the sub-optimal bandwidths selected for the first t-1 data chunks. By straightforward calculation, the sub-optimal bandwidths can be obtained by solving the following equations:

$$h_t^*(x) = \{C_h(x)\}^{1/3} \left(\sum_{s=1}^{t-1} n_s (h_s^*(x))^2 + n_t (h_t^*(x))^2\right)^{-1/3},$$
(43)

$$h_t^* = C_h^{1/3} \left(\sum_{s=1}^{t-1} n_s \left(h_s^* \right)^2 + n_t \left(h_t^* \right)^2 \right)^{-1/3}.$$
 (44)

The sub-optimal bandwidths in (43) and (44) are given by non-linear equations, which are not convenient for practical applications, especially in the scenario of the fast online-updating. To address this issue, we use $h_{s-1}^*(x)$ and h_{s-1}^* to approximate the unknown $h_s^*(x)$ and h_s^* on the right side of (43) and (44), respectively, which lead to the renewable bandwidths that

$$\widehat{h}_{t}(x) = \{C_{h}(x)\}^{1/3} \{S_{h,t}(x)\}^{-1/3},
\widehat{h}_{t} = C_{h}^{1/3} S_{h,t}^{-1/3}$$
(45)

for $t = 2, \cdots, T$, where

$$S_{h,t}(x) = S_{h,t-1}(x) + n_t \left(\hat{h}_{t-1}^*(x)\right)^2,$$

$$S_{h,t-1} = S_{h,t-1} + n_t \left(\hat{h}_{t-1}^*\right)^2$$

with the initial values given by $S_{h,1}(x) = S_{h,1} = 0$, $\tilde{h}_1^*(x) = (C_h(x))^{1/5} n_1^{-1/5}$ and $\tilde{h}_t^* = (C_h)^{1/5} n_1^{-1/5}$.

The optimal and sub-optimal bandwidths in (42) and (45) all depend on unknown parameters. In practical applications, the unknown parameters can be estimated by cross validations on a validation data set. More details will be discussed in Section 5.

4.3 The Optimal Weight Functions

Although the renewable WCQR estimators introduced in Section 3.3 can estimate m(x) and $\sigma(x)$ at a convergence rate comparable to the oracle estimators, their weight functions are generally not optimal in terms of estimation variance. Based on the asymptotic distribution in Theorem 4, we can go a step further to find the optimal weight function in the sense the associated renewable estimator estimate m(x) or $\sigma(x)$ with a minimized asymptotic variance.

To this end, we introduce the set $\mathbb{J}_m(\underline{\tau},\overline{\tau})$ (resp. $\mathbb{J}_{\sigma}(\underline{\tau},\overline{\tau})$) consisting of all the weight functions $J(\cdot)$: $[0,1] \to \mathbb{R}$ satisfying the constrains (C_{m1}, C_{m2}) (resp. $(C_{\sigma 1}, C_{\sigma 2})$) and being square integrable with a support in $[\underline{\tau},\overline{\tau}] \subset (0,1)$. Here $\underline{\tau}$ and $\overline{\tau}$ are pre-given parameters introduced in Assumption 4. By Theorem 4 and Corollary 5, the variance of $\hat{r}(x; J)$ can be expressed as

$$\left(\frac{1}{N_t}\sum_{s=1}^t \frac{n_s}{N_t h_s}\right)^{-1} \operatorname{Var}\left\{\tilde{m}(x; J)\right\}$$
$$= \frac{k_{0,2}\sigma^2(x)}{f_X(x)} V(J) + O\left(|\Delta(G_x)|^{l+1}\right) + o(1)$$

with $V(\cdot)$ a quadratic functional given by

$$V(J) = \int_{[\underline{\tau},\overline{\tau}]} \int_{[\underline{\tau},\overline{\tau}]} S_J(\tau_1,\tau_2) \,\mathrm{d}\tau_1 \mathrm{d}\tau_2.$$
(46)

For $z(\cdot) = m(\cdot)$ or $\sigma(\cdot)$, the optimal weight function $J_z^*(\cdot)$ for estimating z(x) is given by the following functional minimization problem:

$$J_{z}^{*}\left(\cdot\right) = \operatorname*{arg\,min}_{J \in \mathbb{J}_{z}(\underline{\tau}, \overline{\tau})} V\left(J\right) \text{ for } z = m, \sigma.$$
(47)

By tools of variational analysis, we can obtain the closed-form expression of $J_z^*(\cdot)$ as in the following theorem and corollary.

Theorem 8 Given $f_{\varepsilon}(\cdot)$ twice differentiable, the optimal weight function in (47) can be expressed by

$$J_{z}^{*}\left(\tau\right) = I\left(\underline{\tau} \leq \tau \leq \overline{\tau}\right) \left(C_{1}\Psi_{1}\left(\tau\right) + C_{2}\Psi_{2}\left(\tau\right)\right), \quad (48)$$

where $\Psi_i(\cdot)$, i = 1, 2, are two basis function given by

$$\Psi_{i}(\tau) = -\left\{ \left(\delta_{1i} + \delta_{2i} \mathbf{I}_{\mathrm{d}}\right) \log f_{\varepsilon} \right\}^{\prime\prime} \left(F_{\varepsilon}^{-1}(\tau)\right) \quad (49)$$

with I_d the identity function, i.e., $I_d(y) = y$ for $y \in \mathbb{R}$; the coefficients C_1 and C_2 are coefficients selected to satisfy the corresponding conditions in (20) or (21) *i.e.*,

$$\begin{array}{ll} if \ z=m, \ C_i=\frac{\delta_{1i}A_{12}-\delta_{2i}A_{11}}{A_{01}A_{12}-A_{02}A_{11}}, & i=1,2,\\ if \ z=\sigma, \ C_i=\frac{\delta_{2i}A_{01}-\delta_{1i}A_{02}}{A_{01}A_{12}-A_{02}A_{11}}, & i=1,2, \end{array}$$

with

$$A_{kj} = \int_{[\underline{\tau},\overline{\tau}]} \left(F_{\varepsilon}^{-1}(\tau) \right)^{k} \Psi_{j}(\tau) \, \mathrm{d}\tau,$$

for k = 0, 1, and j = 1, 2.

The following remark shows that Theorem 8 is an extension of existing results about the optimal weight functions in L-Estimation.

Remark 11 Theorem 8 allows us to find the optimal weight function under the constraint $\operatorname{Supp} \{J(\cdot)\} \subset [\underline{\tau}, \overline{\tau}]$. This constraint is artificially introduced to control the error caused from LPI (see Remark 1). If we consider a special case of Theorem 8, where $[\underline{\tau}, \overline{\tau}] =$ [0, 1], i.e, the constraint on $\operatorname{Supp} \{J(\cdot)\}$ is removed, then the optimal weight function $J_m^*(\cdot)$ (resp. $J_{\sigma}^*(\cdot)$) degenerates into $\Psi_1(\cdot)$ (resp. $\Psi_2(\cdot)$), which is just the optimal score functions for m(x) (reps. $\sigma(x)$) given in Portnoy and Koenker (1989); Koenker (2005).

The optimal weight function in (48) is deeply related to the distribution function and the PDF of the error, which can be difficult to estimate especially in the case of streaming data. In the following corollary, we manage to express J_z^* (·) using the CDF $F_{Y|x}$ (·).

Corollary 9 Under Assumption 1, the optimal weight function in (48) can be expressed as

$$J_{z}^{*}(\tau) = I\left(\underline{\tau} \leq \tau \leq \overline{\tau}\right) \left(C_{1}\Psi_{1}(\tau) + C_{2}\Psi_{2}(\tau)\right), \quad (50)$$

where $\tilde{\Psi}_{i}(\cdot)$, i = 1, 2, are basis functions given by $\tilde{\Psi}_{i}(\tau) = -\int_{I_{*}} \{(\delta_{1i} + \delta_{2i}\mathbf{I}_{d})\log F'_{Y|x}\}''\left(F^{-1}_{Y|x}(\tau)\right) \mathrm{d}x$

and the coefficients \tilde{C}_1 and \tilde{C}_2 are selected to fulfill the corresponding conditions (C_{z1}, C'_{z2}) , i.e.,

$$\begin{array}{l} \mbox{if } z=m, \\ \left\{ \begin{split} \tilde{C}_1 &= \frac{d_0 \tilde{A}_{12} - d_1 \tilde{A}_{02}}{\tilde{A}_{01} \tilde{A}_{12} - \tilde{A}_{02} \tilde{A}_{11}}, \\ \\ \tilde{C}_2 &= \frac{-d_0 \tilde{A}_{11} + d_1 \tilde{A}_{01}}{\tilde{A}_{01} \tilde{A}_{12} - \tilde{A}_{02} \tilde{A}_{11}}, \\ \\ \mbox{if } z=\sigma, \end{array} \right. \\ \left\{ \begin{split} \tilde{C}_1 \tilde{A}_{01} + \tilde{C}_2 \tilde{A}_{02} &= 0, \\ \\ \tilde{C}_1^2 \tilde{D}_{1,1} + 2 \tilde{C}_1 \tilde{C}_2 \tilde{D}_{1,2} + \tilde{C}_2^2 \tilde{D}_{2,2} &= d_3^2, \end{split} \right. \end{array}$$

with $d_0 = 1$, $d_1 = \mathbb{E}[W(X)Y]$ and

$$\begin{split} d_3^2 &= \mathbb{E}\left[W(X)(Y^2 - m^2\left(X\right))\right],\\ \tilde{A}_{kj} &= \int_{I_*} \int_{\mathbb{R}} \left\{yW\left(x\right)f_X\left(x\right)\right\}^k \\ &\quad \times \tilde{\Psi}_j(F_{Y|x}\left(y\right)) \mathrm{d}F_{Y|x}\left(y\right)\mathrm{d}x,\\ for \ k &= 0,1 \ and \ j = 1,2. \end{split}$$

Corollary 9 suggests a renewable estimation method for the optimal weight function based on the interpolated empirical CDF given in (14). Specifically speaking, the estimator of J_z^* (·) can be given by (50) with the unknown $F_{Y|x}$ (·) and its derivatives replaced by $\mathcal{I}_l \hat{F}_{Y|x,t}$ (·) and its corresponding derivatives, and the involved constants \tilde{C}_i can be estimated by renewable procedures similar to the ones introduced in (27) and (32).

5 Numerical Experiments

In this section, we conduct simulation studies and real data analyses to verify performance of various estimators involved in this paper.

In the numerical experiments, all the involved kernel functions are taken as the Epanechnikov kernel, i.e., $K(z) = \max\{0, 3/4(1-z^2)\}$. To implement the renewable WCQR estimation, 3rd-degree LPI is used to obtain the interpolated empirical CDF. To obtain the LPI nodes, the set G_* is formed by 100 grid points evenly distributed over the intervals I_* . And for each x_i in G_* , the set G_{x_i} consists of the points y_{ij} given by

$$y_{ij} = \underset{y \in \mathbb{R}}{\operatorname{arg\,min}} \sum_{q: X_q \in \mathcal{N}_k(x_i, \mathcal{D}_{\operatorname{val}})} \rho_{\tau_j} \left(Y_q - y \right)$$

for $\tau_j = j/100$ and $j = 1, \dots, 99$. Here \mathcal{D}_{val} is a validation data set containing the first 2000 samples collected from the data stream. And the function \mathcal{N}_k is defined in Section 2.2 with kempirically selected as max $\{0.1 \# \mathcal{D}_{\text{val}}, \# G_{x_i}\}$.

For implementations, we need to artificially determine a terminal time T for the streaming data sets. Depending on whether or not the cumulative number N_T is supposed to be known, we consider two bandwidth selectors: the "oracle" optimal constant bandwidth \tilde{h}_t^* from (42) and the renewable constant bandwidth \hat{h}_t from (42) and the renewable constant C_h in (42) and (45). The unknown constant C_h in (42) and (45) is replaced by its estimator \hat{C}_h obtained by 10-fold Cross-Validation on the validation data set \mathcal{D}_{val} .

We mainly discuss the performance of the following three proposed renewable estimators:

- \hat{r}_{ntm} : It is the renewable NTM $\tilde{r}_T(x; J_{m,0.5})$ given by (15) with the weight function $J_{m,0.5}$ given in (25) with $\alpha = 0.1$; the bandwidth is selected as the renewable \hat{h}_t given in (45) with C_h replaced by \hat{C}_h .
- \hat{r}_{bctm} : It is the renewable BCTM $\tilde{r}_T(x; J_{m,\hat{w}_T})$ given by (15) with the weight function J_{m,\hat{w}_T} given in (25) with $\alpha = 0.1$ and \hat{w}_T obtained by (27); the bandwidth selector is the same with that of \hat{r}_{ntm} .
- \hat{r}_{rtsd} : It is the renewable RTSD $\tilde{r}_T(x; J_{\sigma,\hat{\theta}_T})$ given by (15) with the weight function $J_{\sigma,\hat{\theta}_T}$ given in (30) with $\alpha = 0.1$ and $\hat{\theta}_T$ obtained by (32); the bandwidth selector is the same with that of \hat{r}_{ntm} .

For comparison, we introduce the following benchmark estimators:

- $\hat{r}_{ntm}^*, \hat{r}_{bctm}^*$ and \hat{r}_{rtsd}^* : They are the oracle counterparts of $\hat{r}_{ntm}, \hat{r}_{bctm}$ and \hat{r}_{rtsd} , respectively, i.e., all of them are computed on the full data $\cup_{t \leq T} \mathcal{D}_t$ with the bandwidth selected as the estimated oracle optimal constant bandwidths, i.e., $h = \hat{C}_h N_T^{-1/5}$.
- \hat{r}_{ntm}^{a} , \hat{r}_{bctm}^{a} and \hat{r}_{rtsd}^{a} : They are the simpleaverage counterparts of \hat{r}_{ntm} , \hat{r}_{bctm} and \hat{r}_{rtsd} , respectively, which are obtained by simply averaging the corresponding local estimators computed on the data chunks $\mathcal{D}_{1}, \dots, \mathcal{D}_{T}$. For example, $\hat{r}_{ntm}^{a}(x) = 1/T \sum_{t=1}^{T} \hat{r}_{ntm}^{[t]}(x)$, where $\hat{r}_{ntm}^{[t]}$ is the analogue of \hat{r}_{ntm}^{*} computed on \mathcal{D}_{t} .

Here all the bandwidths are selected as the estimated local optimal constant bandwidths as $h_t = \hat{C}_h n_t^{-1/5}$.

• \hat{r}_{nw}^* and \hat{r}_{nwsd}^* : They are the oracle Nadaraya-Watson (NW) estimators for m(x) and $\sigma(x)$, respectively, which are computed on the full data $\cup_{t < T} \mathcal{D}_t$, i.e.,

$$\hat{r}_{nw}^{*}(x) = \frac{\sum_{t=1}^{T} \sum_{j=1}^{n_{t}} Y_{tj} K_{h} (X_{tj} - x)}{\sum_{t=1}^{T} \sum_{j=1}^{n_{t}} K_{h} (X_{tj} - x)},$$

and

$$=\frac{\left(\widehat{r}_{\text{nwsd}}^{*}\right)^{2}(x)}{\sum_{t=1}^{T}\sum_{j=1}^{n_{t}}\left(Y_{tj}-\widehat{r}_{\text{nw}}^{*}(x)\right)^{2}K_{h}\left(X_{tj}-x\right)}{\sum_{t=1}^{T}\sum_{j=1}^{n_{t}}K_{h}\left(X_{tj}-x\right)},$$

where the bandwidth is selected as the oracle optimal constant bandwidths, i.e., $h = C'_h N_T^{-1/5}$ with the constant C'_h estimated by 10-fold Cross-Validation on \mathcal{D}_{val} .

5.1 Simulation Studies

In the simulation studies, we will consider various experiment conditions, such as homoscedastic or heteroscedastic models, and symmetric or asymmetric errors. To this end, we consider the following two models:

Model1 :
$$Y = \sin(2X) + 2 \exp(-16X^2) + 0.5\varepsilon$$

with $X \sim N(0,1)$, $I_* = [-1.5, 1.5]$,
Model2 : $Y = X \sin(2\pi X) + (2 + \cos(2\pi X))\varepsilon$
with $X \sim U(0,1)$, $I_* = [0,1]$,

where Model 1 is homoscedastic and adopted from Fan and Gijbels (1992), and Model 2 is heteroscedastic and adopted from Kai et al. (2010) We consider various kinds of distributions of ε . Moreover, we also use the mixtures of two error distributions to model so-called contaminated data. Specifically, a mixture distribution is chosen as $0.95F_{\varepsilon}+0.05F_{\lambda\varepsilon}$ with a multiplying factor λ , where F_{ε} is the distribution function of ε . If without special statement, all the distributions of ε involved in the simulations are centralized. Based on the combinations of different models and error distributions, we consider the following four examples:

- Example 1a: the Model 1 with various symmetric error distributions;
- Example 1b: the Model 1 with various asymmetric error distributions;
- Example 2a: the Model 2 with various symmetric error distributions;
- Example 2b: the Model 2 with various asymmetric error distributions.

To model the streaming data, we generate the full data of size N_T from the considered models and equally divide the full data into T data chunks.

For each estimation, the number of replications in the simulation is designed as 200. The performance of any estimator $\hat{g}(\cdot)$ of a function $g(\cdot)$ is evaluated by the average squared errors (ASEs) defined by

ASE
$$(\hat{g}) = \frac{1}{\#G_*} \sum_{x_i \in G_*} |\hat{g}(x_i) - g(x_i)|^2.$$

To compare the performance of two estimators \hat{g}_1 and \hat{g}_2 , we use the ratio of average squared errors (RASEs):

RASE
$$(\hat{g}_1, \hat{g}_2) = \frac{ASE(\hat{g}_1)}{ASE(\hat{g}_2)}$$

5.1.1 Scenario 1: Streaming Data with varying Chunk Size

We first discuss the influence of data partitioning on our WCQR estimators. To this end, we fix the full sample size $N_T = 10^5$ and successively take the chunk number $T = 10, 10^2, 10^3, 10^4$, resulting the chunk sizes $n_t = 10^4, 10^3, 10^2, 10$, respectively. The multiplying factor is $\lambda = 1$, i.e., the streaming data is not contaminated. Then we test the WCQR estimators in Example 1a and 1b with the associated simple-average estimators and oracle estimators used as benchmarks. The relevant RASEs are reported in Table 1.

From Table 1, we can see that the renewable WCQR estimators \hat{r}_{ntm} , \hat{r}_{bctm} and \hat{r}_{rtsd} perform well with all the associated RASEs not only insensitive to chunk size n_t but also closed to 1. On the contrary, the simple-average estimators \hat{r}_{ntm}^a , \hat{r}_{bctm}^a and \hat{r}_{rtsd}^a is susceptible to the chunk size. For small chunk sizes, i.e., $n_t \leq 10^3$, their performances are significantly inferior than that of the oracle estimators. In the extreme case $n_t = 10$, the renewable WCQR estimators still work well,

Example	Error distribution	Indicator	$n_t \equiv 10^4$		$n_t \equiv 10^3$		$n_t \equiv 10^2$		$n_t \equiv$	$n_t \equiv 10$	
1			mean	std.	mean	std.	mean	std.	mean	std.	
1a	N(0, 1)	$\text{RASE}(\hat{r}_{\text{ntm}}^*, \hat{r}_{\text{ntm}})$	0.9043	0.0304	0.9940	0.0466	1.0248	0.0401	0.9189	0.0416	
		$\text{RASE}(\widehat{r}_{\text{ntm}}^*, \widehat{r}_{\text{ntm}}^{\text{a}})$	0.8989	0.0498	0.1525	0.0342	0.0007	0.0000	-	-	
		$RASE(\hat{r}_{bctm}^*, \hat{r}_{bctm})$	0.8901	0.0218	0.9404	0.0477	1.0069	0.0416	0.9225	0.0420	
		$\text{RASE}(\widehat{r}_{\text{bctm}}^{*},\widehat{r}_{\text{bctm}}^{\text{a}})$	0.7856	0.0253	0.4210	0.0176	0.0017	0.0009	_	—	
1b	Pareto(3)	$RASE(\hat{r}_{hctm}^*, \hat{r}_{bctm})$	1.0606	0.0401	1.0257	0.0433	1.0307	0.0473	1.0216	0.0421	
		$RASE(\hat{r}_{bctm}^*, \hat{r}_{bctm}^a)$	0.6277	0.0548	0.4033	0.0731	0.0016	0.0004	-	-	
		$RASE(\hat{r}_{rtsd}^*, \hat{r}_{rtsd})$	1.0002	0.0004	1.0005	0.0004	1.0004	0.0004	1.0004	0.0004	
		$\text{RASE}(\widehat{r}_{\text{rtsd}}^*, \widehat{r}_{\text{rtsd}}^{\text{a}})$	0.5221	0.0471	0.1362	0.0288	0.0013	0.0012	-	-	

Table 1 The performance comparison between the renewable estimators and their oracle counterparts under streaming data with various chunk sizes, the full sample size is $N_T = 10^5$, the multiplying factor of contaminated data is $\lambda = 1$.

Table 2 The performance comparison between the oracle NW estimators and the renewable WCQR estimators under various models with contaminated streaming data, the full sample size is $N_T = 10^5$ and the chunk size is $n_t \equiv 1000$.

Example	Error distribution	λ	$\text{RASE}(\widehat{r}^*_{\text{nw}}, \widehat{r}_{\text{ntm}})$		$RASE(\hat{r_1})$	$\hat{r}_{ m bctm}, \hat{r}_{ m bctm})$	$\text{RASE}(\widehat{r}^*_{\text{nwsd}},\widehat{r}_{\text{rtsd}})$		
			mean	std.	mean	std.	mean	std.	
1a	N(0, 1)	1	0.9558	0.0528	0.9527	0.0512	0.7368	0.0765	
		3	1.1806	0.1248	1.1830	0.1231	0.9939	0.0352	
		5	1.6637	0.2761	1.6278	0.2609	1.0059	0.0227	
		10	2.7210	0.5874	3.1802	0.6356	1.0139	0.0158	
1b	F(10, 6)	1	0.0182	0.0048	1.1897	0.3326	1.1172	0.1190	
		3	0.0219	0.0072	1.5075	0.4427	1.0719	0.0994	
		5	0.0289	0.0089	1.7405	0.4581	1.0700	0.1068	
		10	0.0653	0.0199	4.1594	1.2333	1.0752	0.0829	
2a	N(0, 1)	1	0.8586	0.1025	0.8786	0.1127	0.8108	0.1005	
		3	1.1278	0.0948	1.1822	0.0922	0.9417	0.0182	
		5	1.3174	0.1746	1.8534	0.2941	1.0508	0.0135	
		10	3.2014	0.7741	2.6362	0.6085	1.0620	0.0067	
2b	F(10, 6)	1	0.0237	0.0072	1.1536	0.2070	0.9569	0.0395	
	(-) -)	3	0.0279	0.0072	1.1630	0.2657	0.9941	0.0354	
		5	0.0389	0.0182	1.6001	0.6350	1.0865	0.0362	
		10	0.0683	0.0238	2.2746	0.7554	1.0768	0.0231	

but the simple-average estimators can not give results because the data chunk is too small to compute the local estimators. These imply that our renewable algorithm enjoys desirable performance robust to the chunk size of the streaming data, and the obtained renewable WCQR estimators are comparable to the oracle ones.

5.1.2 Scenario 2: Models with Contaminated Streaming Data

We turn to test the robustness and model adaptiveness of our WCQR estimators. The benchmark estimators are the oracle NW estimators. We successively take various multiplying factors $\lambda = 1, 3, 5, 10$ for contaminated streaming data. Since all the involved estimators are impervious to the data partitioning, we only consider a fixed chunk size $n_t = 100$ with the full data size $N_T = 10^5$. Then we test the estimators in the four examples and report the relevant RASEs in Table 2.

For the results in Table 2, we have the following discussions:

(i) In Examples 1a and 2a, the model is symmetric and the two estimators \hat{r}_{ntm} and \hat{r}_{bctm} show

Table 3	3 The	: performa	ance comparison	between th	e oracle NW	estimators	s and the	renewable	WCQR est	imators	under
various	models	s and erro	or distributions,	the full san	ple size is Λ	$T_T = 10^5, t$	he chunk	size is n_t	$\equiv 1000$ and	the mul	tiplying
factor o	f conta	aminated	data is $\lambda = 1$.								

Example	Error distribution	$\text{RASE}(\widehat{r}_{\text{nw}}^*,\widehat{r}_{\text{ntm}})$		$RASE(\hat{r}_{1})$	$\hat{r}_{ m bctm}, \hat{r}_{ m bctm})$	$\text{RASE}(\widehat{r}^*_{\text{nwsd}},\widehat{r}_{\text{rtsd}})$		
I I		mean	std.	mean	std.	mean	std.	
1a	Standard Laplace $t(3)$	$1.3054 \\ 1.5500$	$0.1579 \\ 0.2860$	$1.2949 \\ 1.5221$	$0.1558 \\ 0.2757$	$0.9860 \\ 1.0369$	$0.0145 \\ 0.0801$	
1b	F(4, 6) Pareto(3)	$\begin{array}{c} 0.0114 \\ 0.0151 \end{array}$	$0.0036 \\ 0.0063$	$1.2578 \\ 1.2186$	$0.3805 \\ 0.3968$	$1.0196 \\ 1.0700$	$0.0720 \\ 0.1414$	
2a	Standard Laplace $t(3)$	$1.3050 \\ 1.6960$	$0.1538 \\ 0.2882$	$1.2607 \\ 1.6480$	$0.1442 \\ 0.2751$	$0.9280 \\ 0.9598$	$0.0081 \\ 0.0331$	
2b	F(10, 6) Lognorm $(0, 1)$	$0.0237 \\ 0.0137$	$0.0072 \\ 0.0036$	$1.1536 \\ 1.1069$	$0.2070 \\ 0.2369$	$0.9569 \\ 0.9527$	$0.0395 \\ 0.0092$	

similar behaviors. Specifically, when $\gamma = 1$, i.e., the error is normal without contaminations, both of them are slightly inferior than the oracle NW estimator \hat{r}_{nw}^* . However, when $\lambda > 1$, i.e., the streaming data are contaminated, both of them outperform \hat{r}_{nw}^* . Moreover, the values of RASE $(\hat{r}_{nw}^*, \hat{r}_{ntm})$ and RASE $(\hat{r}_{nw}^*, \hat{r}_{bctm})$ increase as λ is increasing, which suggests that compare with the NW estimator, the NTM and BCTM are more robust to data contaminations. We also notice that even if λ is large, there is no obvious gap between the performance of \hat{r}_{ntm} and \hat{r}_{bctm} . This justifies our claim in Remark 6 that the robustness of the BCTM is not susceptible to the non-robust estimation of E_{WY} .

- (iii) We focus on the results of \hat{r}_{ntm} and \hat{r}_{bctm} in Examples 1b and 2b, where the model is asymmetric. Contrary to the case of symmetric models, the behaviors between the NTM and the BCTM are quite different. All the values of RASE ($\hat{r}_{nw}^*, \hat{r}_{bctm}$) are closed to zero, indicating that the NTM is far inferior to the NW estimator and it can be inconsistent for asymmetric models. While, the values of RASE ($\hat{r}_{nw}^*, \hat{r}_{bctm}$) suggest that the BCTM still works well and even outperforms the NW estimators when the data is contaminated. The above results show that under the weight selection criterions in Section 3.2, our renewable WCQR estimator is adaptive to symmetric and asymmetric models.
- (iv) We turn to discuss the results of the RTSD \hat{r}_{rtsd} in the four examples. As λ is increasing, the

RASEs between \hat{r}_{nwsd} and \hat{r}_{rtsd} show an increasing trend with the values uniformly larger than 1 when $\lambda > 3$. This means that our estimator \hat{r}_{rtsd} is more robust than \hat{r}_{nwsd} and enjoy more advantages when the streaming data are contaminated.

5.1.3 Scenario 3: Models with Nonnormal Error Distributions

We focus on the performance of our WCQR estimators for non-normal error distributions. We still consider a fixed chunk size $n_t = 100$ with the full data size $N_T = 10^5$. The multiplying factor is $\lambda = 1$, i.e., there is no contaminated data. We use the the NW estimators as a benchmark, and test the WCQR estimators in the four examples with nonnormal error distributions. The relevant RASEs are reported in Table 3.

From Table 3, we have the following finds:

- (i) For symmetric models, i.e. Examples 1a and 2a, both of the NTM \hat{r}_{ntm} and the BCTM \hat{r}_{bctm} show advantage over the NW estimator \hat{r}_{nw}^* , which suggests that our WCQR estimators can be more efficient than the NW estimators when the error is nonnormal. The phenomenon is in line with the feature of CQR method in (Zou and Yuan, 2008; Kai et al., 2010).
- (ii) For asymmetric models, i.e., Examples 1b and 2b, benefit from the model adaptiveness mentioned in the last scenario, $\hat{r}_{\rm bctm}$ maintains its advantage over $\hat{r}_{\rm nw}^*$. And as expected, the NTM $\hat{r}_{\rm ntm}$ does not work because of the nonnegligible bias arising in asymmetric models.

Table 4 The relations studied in Real Data Example

Example	Х	Υ	$\#\mathcal{D}_{\mathrm{train}}$	$\# \mathcal{D}_{\mathrm{test}}$
3a	DEWP	O3	$305572 \\ 311146$	101808
3b	WSPM	PM10		103146

(iii) The results of RASE $(\hat{r}_{nwsd}^*, \hat{r}_{rtsd})$ show that the RTSD \hat{r}_{rtsd} seem not superior than \hat{r}_{nwsd}^* . This is not surprising, because the weight function $J_{\sigma,\hat{\theta}_T}$ used by the RTSD is generally not optimal in terms of estimation variance. Moreover, we also note that \hat{r}_{rtsd} is a renewable estimator obtained from streaming data, but \hat{r}_{nwsd}^* is an oracle estimator directly computed on the full data set.

5.2 Real Data Example

For case study, we apply our method to the Beijing Multi-Site Air-Quality Data set from the UCI machine learning repository ¹. This data set consists of hourly data about 6 main air pollutants and 6 relevant meteorological variables collected from 12 nationally-controlled air-quality monitoring sites in Beijing, China. The observational data cover the time period from March 1st, 2013 to February 28th, 2017, and the 420768 observed values of each variable. Our goal is to fit the relationship between the main air pollutants and the relevant meteorological variables in the dataset.

From the data set, we select two pairs of air pollutants and meteorological variables as the response variable Y and the covariate X in model (1), and then we obtain two examples listed in Table 4. In Table 4, the terms DEWP, O3, WSPM and PM10 are abbreviations to dew point temperature (degree Celsius), O3 concentration (ug/m3̂), wind speed (m/s) and PM10 concentration (ug/m3̂), respectively.

To test the performance of the involved estimators, we drop the data that suffer from data missing and then divide the remainder data set into training set $\mathcal{D}_{\text{train}}$ and testing set $\mathcal{D}_{\text{test}}$. The set $\mathcal{D}_{\text{train}}$ consists of the data collected before March 1st, 2017, which are used to obtain the involved estimators. And the set $\mathcal{D}_{\text{test}}$ consists of the data collected after March 1st, 2017, which are used to test the involved estimators. The number of samples in $\mathcal{D}_{\text{train}}$ and $\mathcal{D}_{\text{test}}$ are listed in Table 5. To model the data stream, the data in $\mathcal{D}_{\text{train}}$ are revealed to the algorithm chronologically by chunks. According to the reality, we consider three different sizes of data chunks, where the data chunks are respectively formed by monthly, daily and hourly data in $\mathcal{D}_{\text{train}}$. To simulate the data contamination, we randomly select 5% samples (X_{ti}, Y_{ti}) from $\mathcal{D}_{\text{train}}$ and replace by $(X_{ti}, Y_{ti} + \eta_{ti})$, where η_{ti} are random numbers sampled from $N(0, \gamma^2 \hat{\sigma}^2)$ with $\hat{\sigma}$ the sample standard deviation of all Y_{ti} in $\mathcal{D}_{\text{train}}$.

For an estimator $\widehat{g}(\cdot)$, its prediction accuracy is described by the root mean square error (RMSE) and the mean absolute error (MAE) on $\mathcal{D}_{\text{test}}$, namely

$$\operatorname{RMSE}(\widehat{g}) = \sqrt{\frac{1}{n_{\text{test}}} \sum_{(X_i, Y_i) \in \mathcal{D}_{\text{test}}} (Y_i - \widehat{g}(X_i))^2},$$
$$\operatorname{MAE}(\widehat{g}) = \frac{1}{n_{\text{test}}} \sum_{(X_i, Y_i) \in \mathcal{D}_{\text{test}}} |Y_i - \widehat{g}(X_i)|,$$

where n_{test} is the number of observations in $\mathcal{D}_{\text{test}}$.

The RMSEs and MAEs of the involved estimators are reported in Tables 5 and 6, respectively. From Tables 5 and 6, we have the following findings:

- (i) In most cases, the renewable estimators \hat{r}_{ntm} and \hat{r}_{bctm} show performance impervious to the data partitioning. Specifically, their RMSE and MAE are insensitive to the data chunk levels and are almost the same with that of the oracle estimators. On the contrary, the simple-average estimators \hat{r}_{bctm}^{a} is susceptible to the data chunk levels, and their performance deteriorates significantly when the data chunks becomes smaller. This shows that our renewable WCQR estimation can overcome the challenge arising from the data partitioning, and our renewable estimators enjoy asymptotic properties comparable to that of the oracle estimator obtained on the full data set.
- (ii) In all cases, the errors of the renewable NTM \hat{r}_{ntm} is insensitive to γ . For most cases with $\gamma > 0$, i.e., the data is contaminated, \hat{r}_{ntm} is superior than \hat{r}_{nw}^* in both terms of RMSE and MAE. Moreover, this advantage is enlarged when γ is relatively large. In general, the above

¹https://archive-beta.ics.uci.edu/ml/datasets/ beijing+multi+site+air+quality+data

Example	γ	Data chunk levels	$\widehat{r}_{\mathrm{ntm}}^*$	$\widehat{r}_{\mathrm{ntm}}$	$\widehat{r}_{\mathrm{ntm}}^{\mathrm{a}}$	$\widehat{r}^*_{ m bctm}$	$\widehat{r}_{ m bctm}$	$\widehat{r}_{ m bctm}^{ m a}$	$\widehat{r}^*_{ m nw}$
3a	0	Monthly	53.168	53.163	58.669	52.747	52.740	60.817	52.694
		Daily		53.197	65.282		52.779	63.218	
		Hourly		53.165	66.347		52.751	63.286	
	200	Monthly	52 019	52.004	50.257	52 911	52 106	61 012	55 910
	200	Deile	55.018	53.004	09.207 65.205	35.211	53.190	61.915	55.210
		Dany		55.005	00.090		00.207 59.541	01.279	
		Houriy		52.919	00.042		32.341	04.832	
	300	Monthly	53.145	53.130	60.294	53.892	53.874	62.241	54.038
		Daily		53.017	66.648		53.889	66.288	
		Hourly		52.928	65.717		53.880	59.579	
	500	Monthly	52974	52 960	62.377	55 559	55 545	65 395	57 476
	000	Daily	02:01 1	52 931	66 749	00.000	55 507	63 527	011110
		Hourly		52.835	67.130		55 535	68 755	
		noung		02.000	011100		00.000	001100	
	800	Monthly	53.003	52.990	65.289	59.933	59.920	61.698	60.573
		Daily		53.024	70.791		59.980	71.584	
		Hourly		52.984	69.802		59.143	65.848	
3h	0	Monthly	91 956	91 963	95 182	91 346	91.350	95 737	91 412
0.0	0	Daily	011000	91.987	106.143	011010	91.382	106 577	011112
		Hourly		91.956	126.516		91.377	116.189	
	200	Monthly	91.825	91.818	95.501	91.839	91.819	97.605	93.098
		Daily		91.846	106.158		91.913	100.807	
		Hourly		91.826	125.766		100.772	166.941	
	300	Monthly	91.847	91.838	94.179	92.560	92.657	98.384	94.072
		Daily		91.832	158.011		92.571	141.565	
		Hourly		91.842	172.405		92.165	159.528	
	500	Monthly	01 099	01 995	06.005	04.976	04 280	05.005	102 724
	500	Doily	91.025	91.620	90.905 159 407	94.270	94.209	90.000	105.754
		Hourly		01 821	160.080		94.009	140.770	
		nouny		91.001	100.069		92.009	102.401	
	800	Monthly	91.775	91.778	98.658	99.216	99.162	100.999	106.553
		Daily		91.863	162.436		99.334	153.601	
		Hourly		91.835	185.107		99.321	166.387	

Table 5 The RMSEs of various estimators in fitting the test set from the Beijing Multi-Site Air-Quality Data set

results suggest that our renewable WCQR estimation can achieve robustness for contaminated streaming data.

(iii) The behavior of the renewable BCTM $\hat{r}_{\rm bctm}$ is between that of $\hat{r}_{\rm ntm}$ and $\hat{r}_{\rm nw}^*$. Specifically speaking, when $\gamma = 0$, i.e., there is no contaminated data, $\hat{r}_{\rm bctm}$ provides RMSE and MAE quite close to that of $\hat{r}_{\rm nw}^*$, and both of them are superior than the NTM in terms of RMSE. While, as γ is increasing, the RMSE of $\hat{r}_{\rm bctm}$ is enlarged but still less than the one of $\hat{r}_{\rm nw}^*$, and moreover, its MAE does not increase significantly. The above results confirm our claim in Remark 6 that our renewable BCTM can consistently estimate the conditional mean and enjoy robustness to some extent. In summary, by comprehensively investigating the numerical results in various experiment conditions, we can conclude the desirable performance of our estimation method and algorithms.

Supplementary information. The supplementary material contains a detailed algorithm for renewable WCQR estimation, the relevant lemmas and technical proofs for the theoretical results.

Declarations

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Example	γ	Data chunk levels	$\widehat{r}^*_{\rm ntm}$	$\widehat{r}_{\mathrm{ntm}}$	$\widehat{r}_{ m ntm}^{ m a}$	$\widehat{r}^*_{\rm bctm}$	$\widehat{r}_{ m bctm}$	$\widehat{r}_{\rm bctm}^{\rm a}$	$\widehat{r}^*_{\rm nw}$
3a	0	Monthly	39.721	39.720	42.291	40.397	40.393	39.931	40.515
		Daily		39.735	45.079		40.404	40.979	
		Hourly		39.725	45.711		40.409	40.294	
	200	Monthly	39.791	39.782	42.368	39.669	39.660	42.149	41.601
		Daily		39.811	45.264		39.654	40.132	
		Hourly		39.719	45.662		39.628	39.970	
	300	Monthly	39.917	39.907	43.394	39.638	39.628	42.826	40.059
		Daily		39.791	45.927		39.534	42.064	
		Hourly		39.718	45.382		39.470	42.296	
	500	Monthly	39.762	39.753	44.338	39.621	39.612	40.789	41.603
		Daily		39.746	63.445		39.624	52.976	
		Hourly		39.668	56.339		39.582	48.944	
	800	Monthly	39.783	39.775	50.630	41.116	41.107	47.656	43.676
		Daily		39.810	53.428		41.167	47.270	
		Hourly		39.782	56.896		40.822	47.200	
3b	0	Monthly	63.040	63.024	73.760	65.046	65.123	71.739	65.654
		Daily		63.135	88.917		65.045	79.515	
		Hourly		63.072	111.921		65.098	93.365	
	200	Monthly	63.281	63.295	74.404	63.243	63.273	73.788	64.896
		Daily		63.319	88.801		63.257	80.436	
		Hourly		63.296	111.117		82.052	97.375	
	300	Monthly	63.319	63.334	71.934	62.643	62.540	73.134	64.627
		Daily		63.307	86.940		62.544	77.583	
		Hourly		63.326	107.375		63.419	92.971	
	500	Monthly	63.293	63.305	76.692	62.139	62.139	71.840	71.727
		Daily		63.287	81.489		62.164	72.000	
		Hourly		63.351	99.660		66.735	85.378	
	800	Monthly	63.234	63.225	107.322	63.344	63.321	97.311	68.131
		Daily		63.297	115.269		63.427	99.265	
		Hourly		63.341	103.093		63.445	91.261	

Table 6 The MAEs of various estimators in fitting the data in the test set from the Beijing Multi-Site Air-Quality Data set

Competing Interests. The authors have no competing interests to declare that are relevant to the content of this article.

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