A DUALITY FOR THE ALGEBRAS OF A ŁUKASIEWICZ $n+1$-VALUED MODAL SYSTEM

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Abstract. In this paper, we develop a duality for the varieties of a Łukasiewicz $n+1$-valued modal system. This duality is an extension of Stone duality for modal algebras. Some logical consequences (such as completeness results, correspondence theory. . .) are the derived and we propose some ideas for future research.

1. Introduction

When one looks backwards in the history of modern logic, one can notice that many-valued logics and modal logics arose approximatively at the same time. These two approaches of the science or reasoning are indeed two ways to free oneself of the rigid frame proposed by the classical two-valued propositional calculus. On the one hand, with many-valued logics as defined by Łukasiewicz (see [25] or [26] for an English translation, [27] and for a complete study [15], [33] or [34]), the logician can choose the truth values of his propositions in a set with more than two elements. On the other hand, by enriching the language with some modalities, modal logics allow the propositions to be possible but not true for example. Surprisingly enough, it is in order to interpret these new modalities that Łukasiewicz firstly introduced his many-valued logics (see chapter 21 in [15] for precisions). From then on, these two types of logics have been very successful among the community of logicians, mathematicians and computer scientists (through temporal logic, dynamic logic, expert systems. . .).

In both cases, the algebraic study of these logics were very grateful. On the one hand, C.C. Chang introduced the theory of MV-algebras in 1958 in order to obtain an algebraic proof of the completeness of the Łukasiewicz infinite-valued logic. Then, the class of MV-algebras was studied by the algebraists as an extension of the variety of Boolean algebras gifted with lots of interesting properties (see [7] for a survey on the theory of MV-algebras).

On the other hand, apart from the algebraic semantics (they were firstly introduced in [29] for the extensions of **S4**; see [3] or [1] for a survey), one has to wait the work of Kripke in 1963 (see [22]) to obtain a significant completness result for modal logics. Kripke's semantic, also called *possible worlds' semantic*, gives a very intuitive interpretation of modal propositions: a proposition is necessarily true in a world α if it is true in any world accessible to α.

This idea gave a new boost to the theory of modal logics with a wide range of so called *relational semantics* (see [3] and [1] for a survey) and to their algebraic aspects. Is is mainly a fact of the connection discovered between the relational and algebraic semantics through the theory of Boolean algebras with operators (see [19], [20] for the beginning of the theory or [3] for a survey), which are the Lindenbaum-Tarski algebras of modal logics. One of the contributions of the study of the algebras of modal logics is a great simplification of the proof of the completeness of these logics, thanks to the concept of canonical model (see [3]). Another useful tool to connect modal models and modal algebras is an extension of Stone duality to the class of Boolean algebras with

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operators (see [17], [35] or [3]). With this duality, one can see any Boolean algebra with operators as the dual algebra of its "dual frame" and each normal modal logic is axiomatized by the logic of the "dual frame" of its LIDENBAUM-TARSKI algebra. The more recent works [23] and [24] give a connection between Boolean algebras with operators, STONE duality, the theory of coalgebras and modal logics. The paper [36] shows that even for very basic notions such as semi-irreducibility, STONE duality can remain an enriching tool. All these works, together with the papers [13] and [14] about canonical extension, prove that lots of interesting results can still be found in the field.

Obviously, the idea of merging modal logic and many-valued logic rapidly appeared attractive, and some mathematicians could not resist the temptation to create their own modal many-valued logic (see [16], [12], [10], [11], [32] and [18]). For some of these works, the will is clearly to save KRIPKE's semantic. Indeed, this semantic can easily be extended to the many-valued realm by considering that the worlds (or the accessibility relation) are many-valued. A concept of canonical model can even be considered (see [18]).

Moreover, for some of these attempts, a completeness result can be obtained. So, with all the applications of the corresponding theories in the two-valued modal logic in view, the development of the algebraic aspects of these logics (through a theory of modal many-valued algebras), and the construction of a duality similar to the STONE duality for Boolean algebras with operators are very important works to accomplish for the future of these logics. For example, a duality would be considered as a bridge between modal many-valued algebras (and the algebraic semantics) and many-valued relational models.

We propose to do this job in this paper for the $n+1$-valued modal logics introduced in [18] (which generalize the logics of [32]). We first recall briefly the $n+1$-valued modal system defined in [18]. Then, we introduce the corresponding varieties of the many-valued modal algebras and study their very first properties.

Finally, we construct a duality for the varieties of $n+1$-valued modal algebras. As for the Boolean case (where the duality for Boolean algebras with operators is constructed over STONE's duality), we ride on an existing duality for the $n+1$-valued MV-algebras for this construction. By considering $n = 1$ in this duality, we can recover STONE duality for modal algebras. Then, some consequences are derived. Among them, we obviously find that the class of the dual structures forms a very adequate semantic for the $n+1$-valued modal logics (i.e. each $n+1$-valued modal logic L is complete with respect to one of these topological structures). We also obtain a characterization of the Boolean operators on the algebra of the idempotent elements of an $n+1$-valued algebra A that can be extended to an operator on A. We eventually develop the very first examples of correspondance theory, compute coproducts in the dual category and expose some ideas that we should explore in the future.

2. A ŁUKASIEWICZ $n+1$-VALUED MODAL SYSTEM

One should admit that the success of modal logics among mathematicians, logicians and computer scientists is mainly a fact of the existence of relational semantics for these logics. Indeed, these semantics are very intuitive and more attracting than the algebraic ones.

The definition of a KRIPKE model can easily be extended to a many-valued realm. Among the possibilities that can be explored for such a generalization, one of the more natural ones is to allow the worlds to be $[0,1]$-valued (i.e. the interpretations of the propositional variables are set in the real unit interval $[0,1]$) and to reason, in each world, in a ŁUKASIEWICZ fashion (i.e. the interpretations of the variables are extended to formulas by the way of ŁUKASIEWICZ T-norm $x \to y = \min(1, 1+y-x)$ and negation $\neg x = 1-x$). The value of a formula $\Box\phi$ in the world α is then defined by the infemum of the values of ϕ in each world accessible to α. This idea is made precise by the following definition. We use **Prop** to denote a countable set of propositional variables and Form to denote the set of well formed formulas over the language $\mathcal{L}_{\mathcal{MMV}} = \{\to, \neg, \Box\}$

whose propositional variables are set in **Prop** (these formulas are defined in the obvious way).

Definition 2.1. A frame is a pair (M, R) where M is a non empty set and R is a binary relation on M. A *many-valued* KRIPKE *model* (M, R, v) is given by a frame (M, R) and a map $v : M \times \mathbf{Prop} \to [0, 1]$.

The map v can be extended inductively to a valuation $v : M \times \mathsf{Form} \to [0, 1]$ which assigns a truth value to each formula in each world by the way of the following inductive rules:

- $v(\phi \to \psi) = \min(1 + v(\psi) - v(\phi), 1)$,
- $v(\neg\phi) = 1 - v(\phi)$,
- $v(\Box\phi) = \bigwedge_{w \in Rv} w(\phi)$.

A formula ϕ is *true* in the world α if $v(\alpha, \phi) = 1$. If ϕ is true in each world of the model M, then ϕ is *valid* in M. A *tautology* is a formula ϕ which is true in every many-valued KRIPKE model.

If n is a positive integer, an $n+1$-*valued* KRIPKE *model* is a many-valued KRIPKE model (M, R, v) such that v ranges in $\text{Ł}_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$.

The paper [18] is dedicated to the investigation of the logic of these many-valued models. The authors give an axiomatization of a normal many-valued modal logic which admits the preceding models as a sound semantic. Moreover, they also obtain a completeness result for the corresponding normal $n+1$-valued modal logic and the $n+1$-valued KRIPKE frames. In order to convince the reader that the varieties of algebras for which we obtain a duality in this paper are in fact the classes of algebras of these $n+1$-valued modal logics, we give the axiomatization of these logics (which should be compared with the definition 3.1 of a modal operator). Recall that we define $\phi \oplus \psi$ by $\neg\phi \to \psi$ and $\phi \odot \psi$ by $\neg(\neg\phi \oplus \neg\psi)$ so that $\oplus$ and $\odot$ can be added to the language $\mathcal{L}_{MMV}$ without lost of generality. Conversely, $x \to y$ can be defined by $\neg x \oplus y$.

Definition 2.2. A *many-valued modal logic* is a set of formulas $\mathsf{L} \subseteq \mathsf{Form}$ which is closed under *modus ponens*, substitution and the necessiation rule RN (from ϕ infer $\Box\phi$) and which contains

- an axiomatic base of the ŁUKASIEWICZ logic,
- the axiom **K** of modal logic: $\Box(p \to q) \to (\Box p \to \Box q)$,
- the axioms $\Box(p \oplus p) \leftrightarrow \Box p \oplus \Box p$ and $\Box(p \odot p) \leftrightarrow \Box p \odot \Box p$.

As usual, the smallest of these logics is denoted by K. If in addition the logic L contains an axiomatic base of the $n+1$-valued ŁUKASIEWICZ logic, then L is said to be an $n+1$-*valued modal logic*. The smallest $n+1$-valued modal logic is denoted by K_n. Note that the logic K_n is a generalization of a the logic defined in [32].

In [18], the authors prove the following completeness result. It is important to note that this result can also be obtained as a consequence of proposition 4.13 of this paper.

Proposition 2.3. *A formula ϕ is in K_n if and only if ϕ is valid in every $n+1$-valued* KRIPKE *model.*

3. MODAL OPERATOR ON AN MV-ALGEBRA

We should first introduce the reader very briefly to the theory of MV-algebras. MV-algebras were introduced in 1958 by C.C. CHANG (see [4] and [5]) as a many-valued counterpart of Boolean algebras. They indeed capture the properties of the ŁUKASIEWICZ many-valued logics in the language of universal algebra.

An MV-algebra can be viewed as an algebra $A = \langle A; \oplus, \odot, \neg, 0, 1\rangle$ of type (2,2,1,0,0) such that $\langle A; \oplus, 0\rangle$ is an Abelian monoid and which satisfies the following identities: $\neg\neg x = x$, $x \oplus 1 = 1$,

$\neg 0 = 1$, $x \odot y = \neg(\neg x \oplus \neg y)$, $(x \odot \neg y) \oplus y = (y \odot \neg x) \oplus x$. The relation $\leq$ defined on every MV-algebra by $x \leq y$ if $x \to y = 1$ is a bounded distributive lattice order. A good survey of the theory of MV-algebras can be found in [7]. The most important MV-algebra is the algebra $\langle [0,1], \oplus, \odot, \neg, 0, 1\rangle$ defined on the real unit interval $[0,1]$ by $x \oplus y = \min(x+y, 1)$ and $\neg x = 1 - x$. The subalgebra $Ł_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ of $[0,1]$ defined for every positive integer n also plays a key role in the study of many-valued logics, since it is the set of the truth values of the Lucasiewicz $n+1$-valued logic. Note that the variety of Boolean algebras is exactly the variety of idempotent MV-algebras (i.e. the MV-algebras A such that $A \models x \oplus x = 1$), or equivalently the variety generated by $Ł_1$.

With the help of MV-algebras, C.C. Chang proved the completeness of the Łukasiewicz infinite-valued calculus by proving that the variety $\mathcal{MV}$ of MV-algebras is the variety generated by the MV-algebra $[0,1]$. Similarly, a formula ϕ is a theorem of the Łukasiewicz $n+1$-valued logic if and only if it is an $Ł_n$-tautology or equivalently if it is valid in every algebra of the variety $\mathbb{HSP}(Ł_n)$. The latter variety, that we denote by $\mathcal{MV}_n$, can so be viewed as the variety of the algebras of the Łukasiewicz $n+1$-valued logics.

In this section we define the variety of many-valued modal algebras, which appear as the algebras of the many-valued modal logics of section 2. In the next section, we construct dualities for some of its subvarieties (namely the varieties of the "$n+1$-valued modal algebras").

Definition 3.1. A map $\Box : \underline{\underline{A}} \to \underline{\underline{A}}$ on an MV-algebra $\underline{\underline{A}}$ is a *modal operator* (or simply an *operator*) if it fulfills the following three conditions:

(MO1) the map $\Box$ is conormal: $\underline{\underline{A}} \models \Box 1 = 1$,
(MO2) the map $\Box$ satisfies the axiom **K** of modal logic: $\underline{\underline{A}} \models \Box(x \to y) \to (\Box x \to \Box y) = 1$,
(MO3) $\underline{\underline{A}} \models \Box(x \oplus x) = \Box x \oplus \Box x$ and $\underline{\underline{A}} \models \Box(x \odot x) = \Box x \odot \Box x$.

We denote by $\mathcal{MMV}$ the variety of *MV-algebras with an operator* (or simply of *modal MV-algebras*), i.e. the variety of algebras $\underline{A} = \langle A, \oplus, \odot, \neg, \Box, 0, 1\rangle$ of type $(2,2,1,1,0,0)$ such that $\underline{\underline{A}} = \langle A, \oplus, \odot, \neg, 0, 1\rangle$ is an MV-algebra and $\Box$ is an operator on $\underline{\underline{A}}$. For convenience, the $\mathcal{MMV}$-algebra $\langle A, \oplus, \odot, \neg, \Box, 0, 1\rangle$ is usually denoted by $\underline{A}$, and we use $\underline{\underline{A}}$ to denote its reduct in $\mathcal{MV}$. Validity of $\mathcal{L}_{MMV}$-formulas in $\mathcal{MMV}$-algebras is defined in the obvious way. The subvariety of $\mathcal{MMV}$ formed by the algebras $\langle \underline{\underline{A}}, \Box\rangle$ with $\underline{\underline{A}}$ in $\mathcal{MV}_n$ is denoted by $\mathcal{MMV}_n$.

Note that (MO2) is equivalent to $\underline{\underline{A}} \models \Box(x \to y) \leq \Box x \to \Box y$ and implies the monotonicity of $\Box$. The only "original" axioms are the two axioms of (MO3). But they are not exotic since they boil down to a tautological identity in the case of $\underline{\underline{A}}$ being a Boolean algebra. We refer the reader to [18] for details about the role of these axioms in the development of the logics K and K_n.

We claim throughout this paper that $\mathcal{MMV}_n$ is the variety of the algebras of the $n+1$-valued modal logics. This statement is made precise in the following proposition.

Proposition 3.2. *A formula ϕ is in K_n if and only if it is valid in every $\mathcal{MMV}_n$-algebra.*

Proof. The proof follows directly from the fact the Lindenbaum-Tarski algebra of K_n is an $\mathcal{MMV}_n$-algebra. $\Box$

We now give a link between operators on an MV-algebra $\underline{\underline{A}}$ and modal operators of Boolean algebras on the algebra $\mathfrak{B}(\underline{\underline{A}}) = \{x \in A \mid x \oplus x = x\}$ of idempotent elements of $\underline{\underline{A}}$.

Proposition 3.3. *If $\underline{A}$ is an $\mathcal{MMV}$-algebra then $\langle \mathfrak{B}(\underline{\underline{A}}), \Box|_{\mathfrak{B}(\underline{\underline{A}})}\rangle$ is a Boolean algebra with a modal operator.*

Proof. From the equation $\Box(x \oplus x) = \Box x \oplus \Box x$, we infer directly that $\Box|_{\mathfrak{B}(\underline{\underline{A}})}$ is valued in $\mathfrak{B}(\underline{\underline{A}})$.

Moreover, it is well known that the axiom **K** is equivalent to $\Box(x \wedge y) = \Box x \wedge \Box y$ in the case of Boolean algebras. $\Box$

Another problem that we should consider is the characterization of the operators that are just homomorphisms of MV-algebras.

Proposition 3.4. *If $\underline{A} = \langle \underline{\underline{A}}, \Box \rangle$ is an $\mathcal{MMV}$-algebra, then $\Box$ is an $\mathcal{MV}$-homomorphism on $\underline{\underline{A}}$ if and only if $\Box$ is normal (i.e. $\underline{A} \models \Box 0 = 0$) and $\underline{A} \models \Box(x \oplus y) = \Box x \oplus \Box y$.*

Proof. The necessity of the proposition is obvious. Let us prove its sufficiency. Since $\Box(x \oplus y) = \Box x \oplus \Box y$ for every x and y in A, we just have to prove that $\underline{A} \models \Box(\neg x) = \neg \Box x$.

The inequality $\Box(\neg x) \leq \neg \Box x$ follows from the normality of $\Box$ and the axiom **K**.

To obtain the other inequality, we note that we have successively

$$\neg \Box x \leq \Box \neg x \Leftrightarrow \neg \Box x \to \Box \neg x = 1 \Leftrightarrow \Box \neg x \oplus \Box x = 1 \Leftrightarrow \Box(x \oplus \neg x) = 1,$$

since $\Box$ is an operator which preserves $\oplus$. We can then conclude using the conormality of the operator $\Box$. $\Box$

The fact that many-valued modal algebras satisfy the axiom **K** of modal logic has a trivial (but very important for the construction of a duality) consequence on the behavior of $\Box$ with respect to filters. Let us recall that a filter of the $\mathcal{MV}$-algebra $\underline{\underline{A}}$ is a subset F of A such that $1 \in F$ and $y \in F$ whenever $\{x, x \to y\} \subseteq F$.

Proposition 3.5. *If $\underline{A}$ is an $\mathcal{MMV}$-algebra and if F is a filter of $\underline{\underline{A}}$ then $\Box^{-1}(F)$ is also a filter of $\underline{\underline{A}}$.*

Proof. The proof follows directly from the conormality of $\Box$ and the axiom (K). $\Box$

Examples 3.6. Here are some basic examples of operators.

1. The identity map is an operator on every MV-algebra $\underline{\underline{A}}$.
2. The map $\mathbf{1} : \underline{\underline{A}} \to \underline{\underline{A}} : a \mapsto 1$ is an operator on every MV-algebra $\underline{\underline{A}}$.
3. $\Box : \underline{\underline{Ł}}_6 \times \underline{\underline{Ł}}_6 \to \underline{\underline{Ł}}_6 \times \underline{\underline{Ł}}_6 : (x, y) \mapsto (\min(x, y), \min(x, y))$ is an operator on $\underline{\underline{Ł}}_6 \times \underline{\underline{Ł}}_6$.
4. If $\mathcal{C}$ denotes Chang's MV-algebra and if k is a positive integer, the the map $\Box_k : x \mapsto k.x$ is an operator on $\mathcal{C}$.

Let us remark that we can mimic the idea of the third example to construct an operator on $\underline{\underline{A}}^k$ for every positive integer k and every MV-chain $\underline{\underline{A}}$.

Moreover, even if the class of simple MV-algebras (which is exactly $\mathbb{IS}([0, 1])$) is much richer that the class of simple Boolean algebras (which only contains the two element Boolean algebra) one does not have more freedom to construct operators on simple MV-algebras than on simple Boolean algebras. To prove this statement, we need the following definition.

Definition 3.7. If r is in $\mathbb{Q} \cap [0, 1]$, we denote by τ_r a composition of the terms $x \oplus x$ and $x \odot x$ such that $\tau_r(x) = 0$ for every $x \in [0, r[$ and $\tau_r(x) > 0$ for every $x \in [r, 1]$. A proof of the existence of such terms can be found in [32] for example. Furthermore, we can always choose τ_r such that $\tau_r(x) = 1$ for every $x \in Ł_n \cap [r, 1]$ (but this choice is not independant of n).

Since an operator satisfies the two axioms of (MO3), we have $\underline{A} \models \Box \tau_r(x) = \tau_r(\Box x)$ for every $\mathcal{MMV}$-algebra $\underline{A}$ and every $r \in \mathbb{Q} \cap [0, 1]$. As we shall see in the sequel, these terms play a key role in the construction of a duality for $\mathcal{MMV}_n$.

Proposition 3.8. *Is $\underline{\underline{A}}$ is a simple MV-algebra and if $\Box$ is an operator on $\underline{\underline{A}}$, then either $\Box$ is the identity map, or $\Box$ is the constant map* $\mathbf{1}$.

Proof. If $\Box 0 = 1$, then $\Box x = 1$ for all $x \in A$ by monotonicity of $\Box$. Otherwise, $\Box 0 = 0$ and it follows that if x is an element of A, $\Box \tau_r(x) = 0$ for every r in $\mathbb{Q} \cap]x, 1]$. Thus, $\tau_r(\Box x) = 0$, which means that $\Box x < r$, for every r in $\mathbb{Q} \cap]x, 1]$. Similarly, $\tau_s(x) > 0$ for every $s \in [0, x]$ and by monotonicity, $\Box \tau_s(x) = \tau_s(\Box x) > 0$, which is equivalent to $\Box x \geq s$. We can eventually conclude that $\Box x = x$ for every x in A. $\Box$

To close this brief description of $\mathcal{MMV}$, note that we can obtain with the help of the completeness results 2.3 and 3.2 that every $\mathcal{MMV}_n$ algebra satisfies $\Box(x \wedge y) = \Box x \wedge \Box y$, but the formula $\Box(x \odot y) = \Box x \odot \Box y$ is not valid in $\mathcal{MMV}_n$ (it is really easy to build a counter example using the operator defined in the third example of 3.6).

4. A duality for the category of the $n+1$-valued modal algebras

4.1. **A natural duality for the algebras of Łukasiewicz $n+1$-valued logic.** It is well known that a strong natural duality (in the sense of Davey and Werner in [8]) can be constructed for each of the varieties $\mathcal{MV}_n = \mathbb{HSP}(\text{Ł}_n) = \mathbb{ISP}(\text{Ł}_n)$. The existence of these natural dualities is a consequence of the semi-primality of Ł_n. These dualities, from which we can recover the Stone duality for Boolean algebras by considering $n = 1$, are a good starting point for the construction of a duality for the classes of the algebras of the $n+1$-valued modal logic. Indeed, in the classical two-valued case, the dual of a Boolean algebra with a modal operator B is obtained by adding a structure (a binary relation R) to the Stone dual of the Boolean reduct of B. It is the idea we propose to follow throughout this paper: the dual of an $\mathcal{MMV}_n$ algebra $\underline{A}$ will be obtained by adding a structure to the dual of the reduct $\underline{\underline{A}}$ of $\underline{A}$ in $\mathcal{MV}_n$.

To help the reader, we recall the basic facts about the natural duality for $\mathcal{MV}_n$. We use standard notations of the theory of category and natural duality. Hence, we denote our algebras by underlined Roman capital letters and our topological structures by "undertilded" Roman capital letters.

We denote by $\underline{\underline{\text{Ł}}}_n$ the MV-subalgebra $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ of $[0, 1]$. We define $\underset{\approx}{\text{Ł}}_n$ as the topological structure

$$\underset{\approx}{\text{Ł}}_n = \langle \text{Ł}_n; \{\text{Ł}_m \mid m \in \operatorname{div}(n)\}, \tau \rangle,$$

where τ is the discrete topology, $\operatorname{div}(n)$ is the set of the positive divisors of n and Ł_m (with $m \in \operatorname{div}(n)$) is a distinguished (closed) subspace of $\langle \text{Ł}_n, \tau \rangle$ (which can also be viewed as an unary relation on Ł_n). We denote by $\mathcal{MV}_n$ the category whose objects are the members of the variety $\mathbb{HSP}(\underline{\underline{\text{Ł}}}_n)$ and whose morphisms are the $\mathcal{L}_{MV}$-homomorphisms. Finally, we denote by $\mathcal{X}_n$ the category whose objects are the members of the topological quasi-variety $\mathbb{IS}_c\mathbb{P}(\underset{\approx}{\text{Ł}}_n)$ (i.e. the topological structures which are isomorphic to a closed substructure of a power of $\underset{\approx}{\text{Ł}}_n$) and whose morphisms are the continuous maps $\phi : \underset{\approx}{X} \to \underset{\approx}{Y}$ such that $\phi(r_{\underset{\approx}{m}}^{X}) \subseteq r_{\underset{\approx}{m}}^{Y}$. If $\underline{\underline{A}}$ is an $\mathcal{MV}_n$-algebra, the set $\mathcal{MV}_n(\underline{\underline{A}}, \underline{\underline{\text{Ł}}}_n)$ is viewed as a substructure of $\underset{\approx}{\text{Ł}}_n^A$ and is so equipped with the topology induced by $\underset{\approx}{\text{Ł}}_n^A$. So, if $[a : \frac{i}{n}]$ denotes the subspace $\{u \in \mathcal{MV}_n(\underline{\underline{A}}, \underline{\underline{\text{Ł}}}_n) \mid u(a) = \frac{i}{n}\}$ whenever $a \in A$ and $i \in \{0, \ldots, n\}$, then $\{[a : \frac{i}{n}] \mid a \in A \text{ and } i \in \{0, \ldots, n\}\}$ is a clopen subbasis of the topology of $\mathcal{MV}_n(\underline{\underline{A}}, \underline{\underline{\text{Ł}}}_n)$. Note that it is also the case of $\{[b : 1] \mid b \in \mathfrak{B}(\underline{\underline{A}})\}$.

With these notations, the results about natural duality on $\mathcal{MV}_n$ can be briefly summarized by the following proposition.

Proposition 4.1. *Let us denote by* D_n *and* E_n *the functors*

$$\mathsf{D}_n : \mathcal{MV}_n \to \mathcal{X}_n : \begin{cases} \underline{\underline{A}} \in \mathcal{MV}_n \mapsto \mathsf{D}_n(\underline{\underline{A}}) = \mathcal{MV}_n(\underline{\underline{A}}, \underline{\underline{L}}_n) \\ f \in \mathcal{MV}_n(\underline{\underline{A}}, \underline{\underline{B}}) \mapsto \mathsf{D}_n(f) \in \mathcal{X}_n(\mathsf{D}_n(\underline{\underline{B}}), \mathsf{D}_n(\underline{\underline{A}})), \end{cases}$$

where $\mathsf{D}_n(f)(u) = u \circ f$ *for all* $u \in \mathsf{D}_n(\underline{\underline{B}})$, *and*

$$\mathsf{E}_n : \mathcal{X}_n \to \mathcal{MV}_n : \begin{cases} \underset{\approx}{X} \in \mathcal{X}_n \mapsto \mathsf{E}_n(\underset{\approx}{X}) = \mathcal{X}_n(\underset{\approx}{X}, \underset{\approx}{L}_n) \\ \psi \in \mathcal{X}_n(\underset{\approx}{X}, \underset{\approx}{Y}) \mapsto \mathsf{E}_n(\psi) \in \mathcal{MV}_n(\mathsf{E}_n(\underset{\approx}{Y}), \mathsf{E}_n(\underset{\approx}{X})), \end{cases}$$

where $\mathsf{E}_n(\psi)(\alpha) = \alpha \circ \psi$ *for all* $\alpha \in \mathsf{E}_n(\underset{\approx}{Y})$.

The functors D_n *and* E_n *define a strong natural duality between the category* $\mathcal{MV}_n$ *and* $\mathcal{X}_n$. *Thus, these two functors map embeddings onto surjective morphisms and conversely.*

Since this duality is natural, the canonical isomorphism between an $\mathcal{MV}_n$-algebra $\underline{\underline{A}}$ and its bidual $\mathsf{E}_n\mathsf{D}_n(\underline{\underline{A}})$ is the evaluation map

$$e_{\underline{\underline{A}}} : \underline{\underline{A}} \to \mathsf{E}_n\mathsf{D}_n(\underline{\underline{A}}) : a \mapsto e_{\underline{\underline{A}}}(a) : u \mapsto u(a),$$

and if $\underset{\approx}{X}$ is an object of $\mathcal{X}_n$, the map

$$\epsilon_{\underset{\approx}{X}} : \underset{\approx}{X} \to \mathsf{D}_n\mathsf{E}_n(\underset{\approx}{X}) : u \mapsto \epsilon_{\underset{\approx}{X}}(u) : \alpha \mapsto \alpha(u)$$

is the canonical $\mathcal{X}_n$-isomorphism between $\underset{\approx}{X}$ and $\mathsf{D}_n\mathsf{E}_n(\underset{\approx}{X})$. Note that we can see the map

$$e_{\underline{\underline{A}}} : \underline{\underline{A}} \to \prod_{u \in \mathsf{D}_n(\underline{\underline{A}})} u(\underline{\underline{A}}) : a \mapsto (u(a))_{u \in \mathsf{D}_n(\underline{\underline{A}})}$$

as a Boolean representation of $\underline{\underline{A}}$ by its simple quotients (see [2] for the definition of a Boolean representation of an algebra).

In the sequel, we denote by X the underlying topological space of the member $\underset{\approx}{X}$ of $\mathcal{X}_n$. Note that if $\underline{\underline{A}}$ is an algebra of $\mathcal{MV}_n$, then the underlying topological space of $\mathsf{D}_n(\underline{\underline{A}})$ is homeomorphic to the STONE dual of the Boolean algebra $\mathfrak{B}(\underline{\underline{A}}) = \{x \in \underline{\underline{A}} \mid x \oplus x = x\}$ of the idempotent elements of $\underline{\underline{A}}$.

Let us recall that the objects of $\mathcal{X}_n$ are exactly the topological structures

$$\underset{\approx}{X} = \langle X; \{r^{\underset{\approx}{X}}_{\underset{\approx}{m}} \mid m \in \operatorname{div}(n)\}, \tau\rangle,$$

where

(X1) $\langle X, \tau\rangle$ is a STONE space (i.e. τ is a compact HAUSDORFF zero-dimensional topology);
(X2) $r^{\underset{\approx}{X}}_{\underset{\approx}{m}}$ is a closed subspace of X for every $m \in \operatorname{div}(n)$;
(X3) we have $r^{\underset{\approx}{X}}_{\underset{\approx}{n}} = X$ and $r^{\underset{\approx}{X}}_{\underset{\approx}{m}} \cap r^{\underset{\approx}{X}}_{\underset{\approx}{k}} = r^{\underset{\approx}{X}}_{\underset{\approx}{\gcd(m,k)}}$ for all m and k in $\operatorname{div}(n)$.

Finally, note that some authors have considered other types of dualities for classes of MV-algebras. For instance, see [28] for a PRIESTLEY style duality for $\mathcal{MV}$ and [6] for a duality for the class of locally finite MV-algebras.

4.2. **A duality for $\mathcal{MMV}_n$.** The idea is to mimic the construction that is done when one wants to dualize an operator on a Boolean algebra: an operator on an $\mathcal{MV}_n$-algebra $\underline{\underline{A}}$ is translated into a binary relation on the dual of $\underline{\underline{A}}$.

Of course, as in the Boolean case, not every binary relation R on $\mathsf{D}_n(\underline{\underline{A}})$ is the dual of an operator on $\underline{\underline{A}}$. We should characterize these dual relations R in terms of conditions involving the topology, but also the structure defined by the relations r_m on $\mathsf{D}_n(\underline{\underline{A}})$.

4.2.1. *Dualization of objects.* We give here the construction and the characterization of the dual of the $\mathcal{MMV}_n$-algebras.

Definition 4.2. If $\underline{\underline{A}}$ is an $\mathcal{MMV}_n$-algebra, we define the binary relation $R_\Box^{\mathsf{D}_n(\underline{\underline{A}})}$ on $\mathsf{D}_n(\underline{\underline{A}})$ by

$$(u,v) \in R_\Box^{\mathsf{D}_n(\underline{\underline{A}})} \quad \text{if} \quad \forall\, x \in \underline{\underline{A}}\ (u(\Box x) = 1 \Rightarrow v(x) = 1).$$

As usual, for the sake of readability, we prefer to forget in our notations the dependence of the previous definition on $\underline{\underline{A}}$ and $\Box$ and simply denote by R the relation $R_\Box^{\mathsf{D}_n(\underline{\underline{A}})}$. Moreover, if X is a subset of $\mathsf{D}_n(\underline{\underline{A}})$, we denote by $R(X)$ (resp. $R^{-1}(X)$) the set $R(X) = \{v \in \mathsf{D}_n(\underline{\underline{A}}) \mid \exists\, u \in X\ (u,v) \in R\}$ (resp. the set $R^{-1}(X) = \{u \in \mathsf{D}_n(\underline{\underline{A}}) \mid \exists\, v \in X\ (u,v) \in R\}$).

This relation is the only information needed to recover the operator on the bidual of $\underline{\underline{A}}$. The proof of this statement strongly relies on the terms τ_r of Definition 3.7 and on the two equations of (MO3).

Proposition 4.3. *If $\underline{\underline{A}}$ is an $\mathcal{MMV}_n$-algebra, then, for every $a \in A$ and every $u \in \mathsf{D}_n(\underline{\underline{A}})$,*

$$u(\Box a) = \bigwedge_{v \in Ru} v(a).$$

Proof. We first prove that $u(\Box a) \leq v(a)$ for every $v \in Ru$. Otherwise, there is a $v \in Ru$ and an $i \in \{0,\dots,n\}$ such that $v(a) < \frac{i}{n} = u(\Box a)$. It follows that $v(\tau_{i/n}(x)) = \tau_{i/n}(v(x)) = 0$ and that $u(\Box\tau_{i/n}(x)) = \tau_{i/n}(u(\Box x)) = 1$. Since $(u,v) \in R$, the latter identity implies that $v(\tau_{i/n}(x)) = 1$, which is is a contradiction.

Now, suppose that we can find $j \in \{1,\dots,n\}$ such that

$$u(\Box a) < \frac{j}{n} = \bigwedge_{v \in Ru} v(a).$$

We then obtain that $u(\Box\tau_{j/n}(a)) = 0$ and $v(\tau_{j/n}(a)) = 1$ for all $v \in Ru$.

But, since $\underline{\underline{A}}/\Box^{-1}(u^{-1}(1))$ is a member of $\mathbb{ISP}(\underline{\underline{\text{Ł}}}_n)$, we can easily construct an $\mathcal{MV}_n$-homomorphism $w : \underline{\underline{A}} \to \underline{\underline{\text{Ł}}}_n$ such that $(u,w) \in R$ and $w(\tau_{j/n}(a)) \neq 1$. $\Box$

Let us now study the properties of these relations. In the following proposition, as we shall see in the proof, the many-valued nature of $\underline{\underline{A}}$ is entirely captured in the third statement (the two first ones are just about properties of $\mathfrak{B}(\underline{\underline{A}})$). We denote by $\mathsf{Spec}(B)$ the STONE dual of a Boolean algebra B ($\mathsf{Spec}(B)$ is seen in the natural duality shape as the set of the characters $\chi : B \to 2$ where 2 is the two element Boolean algebra).

Proposition 4.4. *If $\underline{\underline{A}}$ is an $\mathcal{MMV}_n$-algebra, then*

(1) *the relation R is closed in $\mathsf{D}_n(\underline{\underline{A}}) \times \mathsf{D}_n(\underline{\underline{A}})$;*
(2) *$R^{-1}(\omega)$ is a clopen subset for every clopen subset ω of $\mathsf{D}_n(\underline{\underline{A}})$;*
(3) *$R(r_m^{\mathsf{D}_n(\underline{\underline{A}})}) \subseteq r_m^{\mathsf{D}_n(\underline{\underline{A}})}$ for all $m \in \operatorname{div}(n)$.*

Proof. It is well known that the map $\psi : \mathsf{D}_n(\underline{\underline{A}}) \to \mathsf{Spec}(\mathfrak{B}(\underline{\underline{A}})) : u \mapsto u|_{\mathfrak{B}(\underline{\underline{A}})}$ is a homeomorphism (see [30]). It is now not hard to prove using the term τ_1 of Definition 3.7 that $(u,v) \in R$ if and only if $(\psi(u),\psi(v))$ belongs to the dual relation of the operator $\Box|_{\mathfrak{B}(\underline{\underline{A}})}$ under the STONE duality for Boolean algebras with an operator. Thus, if we forget the structure of $\mathsf{D}_n(\underline{\underline{A}})$ and consider it simply as a topological space, then the relation R on $\mathsf{D}_n(\underline{\underline{A}})$ appears as the dual of a Boolean operator and, as such, has to satisfy the two first properties.

Let us prove the third item. Suppose that there is an $m \in \mathrm{div}(n)$ and a $u \in r_m^{\mathsf{D}_n(\underline{\underline{A}})}$ such that $Ru \setminus r_m^{\mathsf{D}_n(\underline{\underline{A}})} \neq \emptyset$. Now, define m' by

$$\frac{1}{m'} = \inf\{v(x) \mid v \in Ru \setminus r_m^{\mathsf{D}_n(\underline{\underline{A}})}, x \in A \text{ et } v(x) \neq 0\}.$$

Obviously, the integer m' is not a divisor of m and we can find a $v \in Ru \setminus r_m^{\mathsf{D}_n(\underline{\underline{A}})}$ and a a in A such that $v(a) = \frac{1}{m'}$. Then, we can consider a clopen subset $\Omega \subseteq \mathsf{D}_n(\underline{\underline{A}}) \setminus r_m^{\mathsf{D}_n(\underline{\underline{A}})}$ containing v. Thus, by considering $\underline{\underline{A}}$ as a Boolean representation on its simple quotients, we can construct the element

$$b = a|_\Omega \cup 1|_{\mathsf{D}_n(\underline{\underline{A}})\setminus\Omega}$$

which belongs to A. It follows that

$$u(\Box b) = \bigwedge_{w\in Ru} w(b) = \bigwedge_{w\in Ru\cap\Omega} w(a) = v(a) = \frac{1}{m'}$$

which is a contradiction since $u \in r_m^{\mathsf{D}_n(\underline{\underline{A}})}$. □

As we shall now see, the three conditions which appear in the preceding proposition characterize the binary relations R on $\mathsf{D}_n(\underline{\underline{A}})$ which are the dual relations of an operator on $\underline{\underline{A}}$.

Definitions 4.5. A *modal relation* on a structure $\underset{\approx}{X}$ of $\mathcal{X}_n$ is a binary relation R on X such that

(**R1**) R is a closed subspace of $X \times X$ (i.e. Ru is closed for all $u \in X$);
(**R2**) $R^{-1}(\omega)$ is a clopen susbet for every clopen subset ω of X;
(**R3**) $R(r_m^{\underset{\approx}{X}}) \subseteq r_m^{\underset{\approx}{X}}$ for every divisor m of n.

We denote by $\mathcal{MX}_n$ the class formed by the structures $\underset{\sim}{X} = \langle \underset{\approx}{X}, R\rangle$ where $\underset{\approx}{X}$ is a member of $\mathcal{X}_n$ and R is a modal relation on $\underset{\approx}{X}$.

Proposition 4.6. *If $\underset{\sim}{X}$ is an $\mathcal{MX}_n$-structure, then the operation $\Box_R$ defined on $\mathsf{E}_n(\underset{\approx}{X})$ by*

$$(\Box_R\alpha)(u) = \wedge_{v\in Ru}\alpha(v)$$

for all $u \in X$ is an operator on $\mathsf{E}_n(\underset{\approx}{X})$.

Proof. We first note that

$$(\Box_R\alpha)(u) = \tfrac{i}{n} \quad\Leftrightarrow\quad \begin{cases} u \in R^{-1}((\tau_{(i+1)/n}(\alpha))^{-1}(0)) \cap X \setminus R^{-1}((\tau_{i/n}(\alpha))^{-1}(0)) & \text{if } i < n \\ u \in X \setminus R^{-1}((\tau_1(\alpha))^{-1}(0)) & \text{if } i = n. \end{cases}$$

Now, the continuity of $\Box_R\alpha$ is a direct consequence of the axiom (**R2**). Finally, with the help of (**R3**), we obtain that $\Box_R\alpha$ is a member of $\mathsf{E}_n(\underset{\approx}{X})$.

Let us now prove that $\Box_R$ is a modal operator on $\mathsf{E}_n(\underset{\approx}{X})$. The equation $\Box_R 1 = 1$ is trivially satisfied. Then, if τ is one of the terms $x \oplus x$ or $x \odot x$ and if α is a member of $\mathsf{E}_n(\underset{\approx}{X})$, it holds

$$\tau(\Box_R\alpha) = \Box_R\tau(\alpha)$$

if and only if

$$\tau(\bigwedge_{v\in Ru} \alpha(v)) = \bigwedge_{v\in Ru} \tau(\alpha)(v)$$

for all $u \in X$. The latter identity is obtained by continuity of τ and by compactness of Ru.

We eventually have to show that $\Box_R$ satisfies the axiom **K**. If α and β are two members of $\mathsf{E}_n(\underset{\approx}{X})$ and if $u \in X$, we obtain successively

$$\begin{aligned} & (\Box_R(\alpha \to \beta) \to (\Box_R\alpha \to \Box_R\beta))(u) = 1 \\ \Leftrightarrow\quad & (\Box_R(\alpha \to \beta)(u)) \leq (\Box_R\alpha)(u) \to (\Box_R\beta)(u) \\ \Leftrightarrow\quad & \textstyle\bigwedge_{v\in Ru}(\beta(v) \oplus \neg\alpha(v)) \leq (\bigwedge_{v\in Ru}\beta(v)) \oplus \neg(\bigwedge_{v'\in Ru}\alpha(v')) \\ \Leftrightarrow\quad & \textstyle\bigwedge_{v\in Ru}(\beta(v) \oplus \neg\alpha(v)) \leq \bigwedge_{v\in Ru}(\beta(v) \oplus \neg\bigwedge_{v'\in Ru}\alpha(v')), \end{aligned}$$

and we conclude easily. □

Proposition 4.7. *If $\underset{\sim}{X}$ is an $\mathcal{MX}_n$-structure, if $\Box_R$ is the operator on $\mathsf{E}_n(\underset{\approx}{X})$ defined in Proposition 4.6 and if $R_{\Box_R}$ is the modal relation defined on $\mathsf{D}_n\mathsf{E}_n(\underset{\approx}{X})$ in Proposition 4.3, then the relations R and $R_{\Box_R}$ coincide up to the canonical $\mathcal{X}_n$-isomorphism $\epsilon_{\underset{\approx}{X}}$.*

Proof. It follows directly from the definition of $\Box_R$ that if $(u, v) \in R$ then $(\epsilon_{\underset{\approx}{X}}(u), \epsilon_{\underset{\approx}{X}}(v)) \in R_{\Box_R}$.

Conversely, suppose that u and v are two elements of X such that $(\epsilon_{\underset{\approx}{X}}(u), \epsilon_{\underset{\approx}{X}}(v)) \in R_{\Box_R}$. It means, by definition of $\Box_R$, that

$$\forall\ \alpha \in \mathsf{E}_n(\underset{\approx}{X})\ (Ru \subseteq \alpha^{-1}(1) \Rightarrow v \in \alpha^{-1}(1)).$$

Let us now assume that $(u, v) \notin R$. Since R is a closed relation, there is a clopen subset ω of X such that $(u, v) \in \omega \subseteq (X \times X) \setminus R$. Equivalently, we can find two idempotent elements α and β of $\mathsf{E}_n(\underset{\approx}{X})$ such that

$$(u, v) \in \alpha^{-1}(0) \times \beta^{-1}(0) \subseteq (X \times X) \setminus R.$$

Thus, it follows that

$$Ru \subseteq X \setminus \beta^{-1}(0) = \beta^{-1}(1),$$

so that v belongs to $\beta^{-1}(1)$, a contradiction. □

4.2.2. *Dualization of morphisms.* We now obtain the dual version of the $\mathcal{MMV}_n$-morphisms. We first prove that the dual of an $\mathcal{MMV}_n$-morphism $f : \underline{A} \to \underline{B}$ is an $\mathcal{X}_n$-morphism which is also a bounded morphism for R. So, the following proposition should not astonish the reader who is used with the theory of duality for Boolean algebras with operators (in this proposition, we see R as a map $u \mapsto Ru$).

Proposition 4.8. *If $f : \underline{A} \to \underline{B}$ is an $\mathcal{MMV}_n$-homomorphism between the two $\mathcal{MMV}_n$-algebras $\underline{A}$ and $\underline{B}$, then $\mathsf{D}_n(f)$ is an $\mathcal{X}_n$-morphism which satisfies*

$$\mathsf{D}_n(f) \circ R = R \circ \mathsf{D}_n(f).$$

Proof. Let us choose an element u of $\mathsf{D}_n(\underline{\underline{B}})$. The inclusion of $\mathsf{D}_n(f)(Ru)$ into $R(\mathsf{D}_n(f)(u))$ is easily obtained. Let us now prove that $R(\mathsf{D}_n(f)(u)) \subseteq \mathsf{D}_n(f)(Ru)$. If v' is an element of $R(\mathsf{D}_n(f)(u))$, then we obtain by definition that

$$\forall\ x \in A\ \ (u(\Box f(x)) = 1 \Rightarrow v'(x) = 1).$$

But, since $\mathsf{D}_n(f)(Ru)$ is a closed subspace of $\mathsf{D}_n(\underline{\underline{A}})$, it is sufficient to prove that every neighborhood of v' meets $\mathsf{D}_n(f)(Ru)$.

So, let us pick a $b \in \mathfrak{B}(\underline{\underline{A}})$ such that $v' \in [b : 0]$. We can construct an element v of Ru such that $v(f(b)) \neq 0$. Such a v is for example given by $v = \pi \circ w$ where π is the quotient map $\pi : \underline{\underline{B}} \to \underline{\underline{B}}/\Box^{-1}(u^{-1}(1))$ and $w : \underline{\underline{B}}/\Box^{-1}(u^{-1}(1)) \to \underline{\underline{\mathrm{L}}}_n$ is an MV-homomorphism which satisfies $w(\pi(f(b))) = 0$. □

As we shall see, we have exactly obtained in the preceding proposition the dual notion of $\mathcal{MMV}_n$-homomorphism.

Proposition 4.9. *If $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are two $\mathcal{MX}_n$-structures and if $\phi : \underset{\sim}{X} \to \underset{\sim}{Y}$ is an $\mathcal{X}_n$-morphism which satisfies*

$$\phi \circ R = R \circ \phi,$$

then $\mathsf{E}_n(\phi)$ is an $\mathcal{MMV}_n$-homomorphism.

Proof. It is a direct application of the definitions. □

4.2.3. *A categorical duality for $\mathcal{MMV}_n$.* We gather the preceding results in order to construct a duality for the category $\mathcal{MMV}_n$. Let us define the dual category.

Definition 4.10. The category $\mathcal{MX}_n$ is the category

- whose objects are the structures $\underset{\sim}{X} = \langle \underset{\approx}{X}, R_{\underset{\approx}{X}} \rangle$ where
  - $\underset{\approx}{X}$ is an $\mathcal{X}_n$-structure,
  - $R_{\underset{\approx}{X}}$ is a modal relation on $\mathcal{X}_n$;
- whose morphisms are the maps $\psi : \underset{\sim}{X} \to \underset{\sim}{Y}$ such that
  - ψ is an $\mathcal{X}_n$-morphism,
  - $\psi \circ R_{\underset{\approx}{X}} = R_{\underset{\approx}{Y}} \circ \psi$.

As usual, if $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are two objects of $\mathcal{MX}_n$, we denote by $\mathcal{MX}_n(\underset{\sim}{X}, \underset{\sim}{Y})$ the class of the $\mathcal{MX}_n$-morphisms from $\underset{\sim}{X}$ to $\underset{\sim}{Y}$.

Our previous developments can be bringed together in the following duality theorem.

Theorem 4.11 (Duality for $\mathcal{MMV}_n$)**.** *Let us denote by $\mathsf{D}_n^* : \mathcal{MMV}_n \to \mathcal{MX}_n$ the functor defined by*

$$\mathsf{D}_n^* : \begin{cases} \underline{\underline{A}} \mapsto \langle \mathsf{D}_n(\underline{\underline{A}}), R_{\Box_{\underline{\underline{A}}}}^{\mathsf{D}_n(\underline{\underline{A}})} \rangle \\ f \in \mathcal{MMV}_n(\underline{\underline{A}}, \underline{\underline{B}}) \mapsto \mathsf{D}_n(f) \in \mathcal{MX}_n(\mathsf{D}_n^*(\underline{\underline{B}}), \mathsf{D}_n^*(\underline{\underline{A}})) \end{cases}$$

where $R_{\Box_{\underline{\underline{A}}}}^{\mathsf{D}_n(\underline{\underline{A}})}$ is the relation defined in Proposition 4.3.

Let us also denote by $\mathsf{E}_n^* : \mathcal{MX}_n \to \mathcal{MMV}_n$ the functor defined by

$$\mathsf{E}_n^* : \begin{cases} \underset{\sim}{X} \mapsto \langle \mathsf{E}_n(\underset{\approx}{X}), \Box_{R_{\underset{\approx}{X}}} \rangle \\ \psi \in \mathcal{MX}_n(\underset{\sim}{X}, \underset{\sim}{Y}) \mapsto \mathsf{E}_n(\psi) \in \mathcal{MMV}_n(\mathsf{E}_n^*(\underset{\sim}{Y}), \mathsf{E}_n^*(\underset{\sim}{X})) \end{cases}$$

where $\Box_{R_{\underset{\approx}{X}}}$ is the operator defined in Proposition 4.6.

Then the functors D_n^* and E_n^* define a categorical duality between $\mathcal{MMV}_n$ and $\mathcal{MX}_n$.

First note that this duality is not a natural duality. It would indeed implies that the duality for Boolean algebras with a modal operator (which is exactly the content of Theorem 4.11 if we set $n = 1$) is natural.

The reader who wishes to illustrate this duality can easily construct the dual structures of the $\mathcal{MMV}_n$-algebras of the Examples 3.6.

As a first consequence of this duality, we obtain directly that if $\Box_{\mathfrak{B}(\underline{\underline{A}})}$ is a Boolean operator on $\mathfrak{B}(\underline{\underline{A}})$, then there is at most one extension of $\Box_{\mathfrak{B}(\underline{\underline{A}})}$ to an operator on $\underline{\underline{A}}$. Indeed, the dual of $\Box_{\mathfrak{B}(\underline{\underline{A}})}$ under STONE duality is a binary relation on the underlying topological space of $\mathsf{D}_n(\underline{\underline{A}})$ (see the proof of Proposition 4.4) which in turns is or is not a modal relation on $\mathsf{D}_n(\underline{\underline{A}})$ (according to (**R3**) of Definition 4.5). We turn this piece of argument into a useful criterion.

Proposition 4.12. *If $\underline{\underline{A}}$ is an $\mathcal{MV}_n$-algebra and if $\Box_{\mathfrak{B}(\underline{\underline{A}})}$ is a Boolean operator on $\mathfrak{B}(\underline{\underline{A}})$, then there is an operator on $\underline{\underline{A}}$ whose restriction to $\mathfrak{B}(\underline{\underline{A}})$ coincides with $\Box_{\mathfrak{B}(\underline{\underline{A}})}$ if and only if the relation R defined on $\mathsf{D}_n(\underline{\underline{A}})$ by application of* STONE *duality to $\langle \mathfrak{B}(\underline{\underline{A}}), \Box_{\mathfrak{B}(\underline{\underline{A}})} \rangle$ is a modal relation on $\mathsf{D}_n(\underline{\underline{A}})$. In this case, this operator is unique.*

As another consequence, we obviously find that the structures of $\mathcal{MX}_n$ form an adequate semantic for the $n+1$-valued ŁUKASIEWICZ modal logics. It means that we can give the following flavor of completeness to the duality for $\mathcal{MMV}_n$. Note that a valuation on a structure $\underset{\sim}{X}$ of $\mathcal{MX}_n$ is a map $v : X \times \mathbf{Prop} \to \text{Ł}_n$ such that $v(\cdot, p)$ is continuous and maps r_m^X into Ł_m for every $p \in \mathbf{Prop}$ and every $m \in \operatorname{div}(n)$. These valuations are extended to formulas in the obvious way.

Proposition 4.13. *A formula ϕ can be obtained from the hypotheses Γ as a theorem of the $n+1$-valued* Łukasiewicz*' modal system if and only if it is valid in every $\mathcal{MX}_n$-structure in which the formulas of Γ are valid.*

Since $\mathcal{MMV}_n$ is the variety of algebras of the $n+1$-valued Łukasiewicz modal logics, it is quite natural to wonder if the classical first order properties of frames which admit a characterization by a modal formula under Stone duality for Boolean algebras with a modal operator still admit this translation under our new duality. As a matter of fact, since these properties are properties of frames and do not involve the many-valueness in any way, we can easily obtain the following proposition.

Proposition 4.14. *If $\underline{A}$ is an $\mathcal{MMV}_n$-algebra, then*

1. *the equation $\Box(x \oplus y) = \Box x \oplus \Box y$ and $\Box 0 = 0$ are simultaneously satisfied in $\underline{A}$ if and only if $R_\Box^{\mathsf{D}_n(\underline{A})}$ is an $\mathcal{X}_n$-morphism;*
2. *the equation $\Box x \to \Box\Box x = 1$ is satisfied in $\underline{A}$ if and only if $R_\Box^{\mathsf{D}_n(\underline{A})}$ is transitive;*
3. *the operator $\Box$ is an interior operator if and only if $R_\Box^{\mathsf{D}_n(\underline{A})}$ is a preorder.*

Furthermore, because of the many-valueness, we can consider some new types of first order properties for the relations R, namely the properties that express how the relation R behave with respect to the relations r_m. If these properties admit a characterization by modal formulas, these formulas should be many-valued in essence. We give two easy examples of this new type of correspondence theory.

Proposition 4.15. *If $\underline{A}$ is an $\mathcal{MMV}_n$-algebra, then*

- $\underline{A} \models \Diamond p \vee \Diamond\neg p$ *if and only if* $\mathsf{D}^*(\underline{A}) \models \forall u \; Ru \cap r_1^{\mathsf{D}^*(\underline{A})} \neq \emptyset$,
- $\underline{A} \models \Box(p \vee \neg p)$ *if and only if* $\mathsf{D}^*(\underline{A}) \models \forall u \; Ru \subseteq r_1^{\mathsf{D}^*(\underline{A})}$,

Note that, roughly speaking, the first proprety expresses that every world is connected to a Boolean world and the second one that every world is exclusively connected to Boolean worlds. Obviously, these correspondence results can be adapted if we change "Boolean world" by "m-valued world" with $m \in \mathrm{div}(n)$.

4.2.4. *Computing the coproducts in $\mathcal{MX}_n$.* Coproducts of dual structures are classical constructions that one computes when one wants to obtain new members of the dual category. For example, the job has been done in [21] for the dual categories of Boolean algebras with operators and has been considered in [31] for the members of $\mathcal{X}_n$. The problems in these constructions arise mainly from topology: when one computes non finite coproducts, one has to pay attention to preserve compacity and to conserve closed relations in order to stay in the category. The idea is to base the coproducts of the structures $(\underset{\sim}{X}_i)_{i\in I}$ on the Stone-Čech compactification of the topological sum of the topological spaces X_i $(i \in I)$.

In fact, we can carefully merge the results of [21] and [31] to obtain the construction of the coproducts in $\mathcal{MX}_n$. The crucial point is to take care that the condition (**R3**) of Definition 4.5 is still satisfied in the compactification.

Let us recall the construction of the Stone-Čech compactification of a completely regular topological space X. We denote by $C(X)$ the set of the continuous maps from X to $[0,1]$. Then, the evaluation map $e : X \to [0,1]^{C(X)}$ defined by $(e(x))_f = f(x)$ is continuous and is a homeomorphism from X to $e(X)$. If $\beta(X)$ denotes the closure of $e(X)$ in $[0,1]^{C(X)}$ then $(e, \beta(X))$ is the Stone-Čech *compactification* of X. We set the notation $\overline{Y}$ aside to denote the closure in $\beta(X)$ of a subset Y of $\beta(X)$ and we identify X and $e(X)$ in $\beta(X)$. Finally, note that the coproduct of the Stone spaces X_i $(i \in I)$ in the category of Stone spaces with continuous maps is given by the STONE-ČECH compactification of the topological sum of the X_i. The set X_i can always be considered as being pairwise disjoint (otherwise we can replace X_i by $\{(x,i) \mid x \in X_i\}$ for all $i \in I$ with the obvious topology).

Note that the clopen subsets of $\beta(X)$ are exactly the $\overline{\Omega}$ where Ω is a clopen subset of X and that $\beta(X) \setminus \overline{F} = (X \setminus F)^-$ for every closed subspace F of X.

We provide the proofs of the following two lemmas even if they are part of folkore and can be found in [21], since the cited paper is not easily accessible.

Lemme 4.16. *If X is a topological space whose set of clopen subsets is a base of the topology and if R is a closed binary relation on X, then $R^{-1}(K)$ is a closed subspace of X for every compact subspace K of X.*

Proof. The proof is obtained thanks to a standard compacity argument. □

Lemme 4.17. *Assume that I is a non empty set, that $(X_i)_{i\in I}$ is a family of* STONE *spaces and that $\beta(X)$ is the* STONE-ČECH *compactification of the topological sum X of the X_i.*

1. *If F and F' are two disjoint closed subspaces of X, then $\overline{F}$ and $\overline{F'}$ are disjoint in $\beta(X)$.*
2. *If F and F' are two closed subspaces of X, then $(F \cap F')^- = \overline{F} \cap \overline{F'}$ in $\beta(X)$.*
3. *If R is a closed binary relation on X and if $\overline{R}$ denotes its closure in $\beta(X)$ then $\overline{R}^{-1}(\overline{\Omega}) = (R^{-1}(\Omega))^-$ for every clopen subset Ω of X.*

Proof. (1) Let us denote by Ω_i a clopen subset of X_i such that $X_i \cap F \subseteq \Omega_i$ and $X_i \cap F' = \emptyset$ for all $i \in I$ and by Ω the open set $\cup_{i\in I}\Omega_i$. Thus, $\overline{F} \subseteq \overline{\Omega}$ and $\overline{F'} \subseteq (X \setminus \Omega)^- = X \setminus \overline{\Omega}$ since Ω_i is a zero-set in X_i for all $i \in I$.

(2) We prove the non trivial inclusion: let x be an element of $\overline{F} \cap \overline{F'}$ and Ω a clopen subset of X such that $x \in \overline{\Omega}$. We prove that $(\overline{\Omega} \cap F) \cap (\overline{\Omega} \cap F') \neq \emptyset$. Otherwise, it follows by (1) that $(\Omega \cap F) \cap (\Omega \cap F') = \emptyset$. But, since $\Omega \cap F$ is a closed subspace of X,

$$x \in \overline{\Omega} \cap \overline{F} \subseteq (\overline{\Omega} \cap F)^- = (\Omega \cap F)^-,$$

and we obtain similarly that $x \in (\Omega \cap F')^-$.

(3) The inclusion $(R^{-1}(\Omega))^- \subseteq \overline{R}^{-1}(\overline{\Omega})$ follows directly from Lemma 4.16. For the other inclusion, let x be an element of $\overline{R}^{-1}(\overline{\Omega})$ and U be a neighborhood of x in $\beta(X)$. Then, there is a y in $\overline{\Omega}$ such that $(x,y) \in \overline{R}$. So, the set $U \times \overline{\Omega}$ is a neighborhood of (x,y) in $\beta(X) \times \beta(X)$ and there are two elements t and z of X such that $(t,z) \in ((U \cap X) \times \Omega) \cap R$. Thus, $U \cap R^{-1}(\Omega) \neq \emptyset$. □

The idea to use STONE-ČECH compactification to compute coproducts in $\mathcal{X}_n$ can be traced back to [31]. We extend this construction to the category $\mathcal{MX}_n$.

Proposition 4.18. *If I is an non empty set and $(\underset{\sim}{X}_i = \langle \underset{\approx}{X}_i, R_i\rangle)_{i\in I}$ is a family of $\mathcal{MX}_n$-structures (that we consider pairwise disjoint), then the structure*

$$\langle \beta(X), \{(r_m^X)^- : m \in \mathrm{div}(n)\}, \overline{R}, \tau\rangle$$

where

- *X is the topological sum of the X_i ($i \in I$),*
- *$\beta(X)$ is the* STONE-ČECH *compactification of X and X is identified to $e(X)$ in $\beta(X)$,*
- *r_m^X is the union of the subspaces $r_m^{X_i}$ ($i \in I$), for every $m \in \mathrm{div}(n)$*
- *R is the union of the relations R_i ($i \in I$),*

is the coproduct in $\mathcal{MX}_n$ of the $(\underset{\sim}{X}_i)_{i\in I}$ ($i \in I$).

Proof. We first have to prove that the proposed structure is an object of $\mathcal{MX}_n$. First, it is clear that its underlying topological space is a Stone space. The identities $r_m^{\beta(X)} \cap r_{m'}^{\beta(X)} = r_{gcd(m,m')}^{\beta(X)}$ are obtained as a consequence of item (2) of Lemma 4.17.

We now prove that $\overline{R}$ is a modal relation on $\beta(X)$. First of all, the third item of Lemma 4.17 implies that $\overline{R}^{-1}(U)$ is clopen subset for every clopen subset U of $\beta(X)$. To prove that $\overline{R}(r_m^{\beta(X)}) \subseteq r_m^{\beta(X)}$, we proceed *ad absurdum.* Assume that $(x, y) \in \overline{R}$ with $x \in r_m^{\beta(X)}$ but $y \in \beta(X) \setminus r_m^{\beta(X)}$. Let us consider a clopen subset Ω of X such that $y \in \overline{\Omega} \subseteq \beta(X) \setminus r_m^{\beta(X)}$. Then, thanks to item (3) of Lemma 4.17, we obtain that x belongs to $\overline{R}^{-1}(\overline{\Omega}) = \overline{R^{-1}(\Omega)}$. We thus can find a $t \in r_m^X \cap \overline{R^{-1}(\Omega)} = r_m^X \cap R^{-1}(\Omega)$. Finally, it means that there is a $z \in \Omega$ such that $(t, z) \in R$, which is a contradiction since $\Omega \subseteq X \setminus r_m^X$. We so have proved that $\overline{R}$ satisfies condition (**R3**) of definition 4.5, and have finished to prove that the proposed structure belongs to $\mathcal{MX}_n$.

Now, let us prove that we have computed the coproduct of the $\underset{\sim}{X}_i$. We denote by $\sigma_i : X_i \to X$ the inclusion map of X_i into $\beta(X)$ for every $i \in I$. These maps are obviously $\mathcal{MX}_n$-morphisms (use the fact that $\overline{R}(u) = R(u)$ for every $u \in X$). Then, suppose that $f_i : \underset{\sim}{X}_i \to \underset{\sim}{Y}$ is an $\mathcal{MX}_n$-morphism valued in an $\mathcal{MX}_n$-structure $\underset{\sim}{Y}$ for every $i \in I$. Since $\beta(X)$ is the coproduct of the topological spaces X_i, there is a unique continuous map $f : \beta(X) \to Y$ such that $f \circ \sigma_i = f_i$ for every $i \in I$. It is so sufficient to prove that $f(\overline{r_m^X}) \subseteq r_m^{\underset{\sim}{Y}}$ for every divisor m of n. First, assume that $y \in Y \setminus r_m^{\underset{\sim}{Y}}$ and denote by Ω a clopen subset of Y such that $y \in \Omega$ and $\Omega \cap r_m^{\underset{\sim}{Y}} = \emptyset$. It follows that $f^{-1}(y) \subseteq f^{-1}(\Omega) \subseteq \beta(X) \setminus r_m^X$. Thus, $f^{-1}(y) \subseteq \beta(X) \setminus \overline{r_m^X}$ which proves that f is an $\mathcal{X}_n$-morphism. Finally, we prove that f is a bounded morphism, i.e. that $f(\overline{R}(u)) = R^{\underset{\sim}{Y}}(f(u))$ for every u in $\beta(X)$. The inclusion from left to right is easily obtained. We proceed with the other inclusion. First suppose that $x \in R^{\underset{\sim}{Y}}(f(u))$. It suffices to show that every clopen neighborhood V of x meets $f(\overline{R}(u))$ since the latter is closed. Let Ω be any clopen subset of X such that $\overline{\Omega}$ contains u. It then follows that

$$R^{\underset{\sim}{Y}}(f(\overline{\Omega})) = R^{\underset{\sim}{Y}}(\overline{f(\Omega)}) \subseteq \overline{R^{\underset{\sim}{Y}}(f(\Omega))} = \overline{f(R^{\underset{\sim}{X}}(\Omega))} = f(\overline{R(\Omega)}) = f(\overline{R}(\overline{\Omega})).$$

Hence, since $x \in R^{\underset{\sim}{Y}}(f(u)) \subseteq f(\overline{R}(\overline{\Omega}))$, the intersection $\overline{R}^{-1}(f^{-1}(V)) \cap \overline{\Omega}$ is not empty. We obtain that u belongs to $\overline{R}^{-1}(f^{-1}(V))$ since this subspace is closed in $\beta(X)$. It means that V contains an element of $f(\overline{R}(u))$ and so that $x \in \overline{f(\overline{R}(u))} = f(\overline{R}(u))$. $\square$

5. Conclusion

In this paper, we have obtained a duality for the category $\mathcal{MMV}_n$, which is the category of the algebras of a very natural Łukasiewicz $n+1$-valued modal system. As a particular case of this duality, we can recover the classical Stone duality for Boolean algebras with a modal operator. Among the consequences of this duality, one can find a completeness result for every $n+1$-valued modal logic: a logic L is complete with respect to the $\mathcal{MX}_n$-structures in which the formulas of L are valid. We now present a few ideas for future research.

First of all, as it has been recently done for Boolean algebras with operators in [36], we could use this duality to characterize the subdirectly irreducible $\mathcal{MMV}_n$-algebras. We could also try to describe the finitely generated $\mathcal{MMV}_n$-algebras, following the work of [9] for Heyting algebras.

An other task should be to place this work in the coalgebraic setting. Indeed, colagebras are a very natural language for the study of transition systems, and it appears clearly that the structures of $\mathcal{MX}_n$ could be described as coalgebras over the base category of Stone spaces. This would provide an interesting illustration of the works [24] and [23] in which the authors use the Vietoris topology to construct coalgebras on Stone spaces. It would prove that these constructions can also be used to describe a new range of transition systems which are not Boolean in essence, and would provide us with a rich language to study the consequences of this duality.

Moreover, this duality suggests us to use a new type of structures as a semantic for the $n+1$-valued ŁUKASIEWICZ modal system. Indeed, a frame $\mathfrak{F} = \langle M, R\rangle$ becomes an $n+1$-valued model by the addition of a valuation $v : M \times \mathbf{Prop} \to Ł_n$. So, the set of truth values in a world α of the model is determined by the valuation. But, as suggested by this duality, we can define an alternative type of (first order) structures for which the set of truth values in a world α is known *a priori*, independently of any valuation. Such an $n+1$-*frame* $\langle M, R, \{r_m \mid m \in \mathrm{div}(n)\}\rangle$ would be given by a non empty set M, a binary relation R on M and a subset r_m of M for every divisor m of n. The members of r_m are the worlds of M that can interpret the truth values of the formulas only in $Ł_m$. It means in fact that we restrict the set of possible valuations on such structures: a valuation $v : M \times \mathsf{Form} \to Ł_n$ has to satisfy $v(\phi, \alpha) \in Ł_m$ if $\alpha \in r_m$. In view, such a structure should satisfy $R(r_m) \subseteq r_m$ for every $m \in \mathrm{div}(n)$.

Since this new class of structures is obtained from the class of frames by restricting the possible valuations for the frames, we should be able to derive some new completeness results: there should be a wide range of logics L which are complete with respect to a class of $n+1$-frames but which are not complete with respect to any class of KRIPKE frames. As an exemple of this fact, let us quote the logic $\mathcal{MMV}_n + \Box(p \vee \neg p) + \Box\Box p + \neg(\Diamond p \wedge \Diamond\neg p)$ (see [18] for developments). It would be interesting to explore these completness results or to develop a new correspondence theory between $\mathcal{L}_{MMV}$-propositions and first order formulas on $n+1$-frames (the very first illustration of this theory are given in proposition 4.15).

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