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On Some Compatible Operations on Heyting Algebras

Abstract. We study some operations that may be defined using the minimum operator in the context of a Heyting algebra. Our motivation comes from the fact that 1) already known compatible operations, such as the successor by Kuznetsov, the minimum dense by Smetanich and the operation G by Gabbay may be defined in this way, though almost never explicitly noted in the literature; 2) defining operations in this way is equivalent, from a logical point of view, to two clauses, one corresponding to an introduction rule and the other to an elimination rule, thus providing a manageable way to deal with these operations. Our main result is negative: all operations that arise turn out to be Heyting terms or the mentioned already known operations or operations interdefinable with them. However, it should be noted that some of the operations that arise may exist even if the known operations do not. We also study the extension of Priestley duality to Heyting algebras enriched with the new operations.

Keywords: Intuitionistic logic, Heyting algebra, compatible operation.

Introduction

In this paper we study certain operations on Heyting algebras that correspond to new connectives in intuitionistic logic. For material on Heyting algebras the reader may see [1]. The study of new connectives was started by Novikov in the 1950s and a first example was found by Smetanich (see e.g. [16]) in 1960. It was a unary connective but it is easily seen that it is essentially a constant (see [17]). Later on, in 1977, Gabbay developed his own approach and found a unary connective we will call G (see [11]). For a comparison between Novikov's and Gabbay's approach the reader may see [18]. In 1978, Kuznetsov (see [14]), interested in building an intuitionistic version of the provability logic **GL** of Gödel and Löb, found a unary connective S known later as the successor in [3], where it is noted that in Heyting chains where the corresponding algebraic operation exists, it is the successor with respect to the order. Kuznetsov provides axioms and also, from an algebraic point of view, defines S using the minimum operator. In 2001, Caicedo and Cignoli undertook a general approach considering compatible

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operations (see [3]). A function $f : H^n \to H$ (where H is a Heyting algebra) is called compatible if and only if it is compatible with all the congruence relations of H (such a function is compatible with a congruence relation θ of H if and only if $x_i \theta y_i$, for $i = 1, \ldots, n$ implies that $f(x_1, \ldots x_n)\theta f(y_1, \ldots y_n)$). It can easily be seen (see [3, Lemma 2.1]) that, in the unary case, f is a compatible function of H iff $x \leftrightarrow y \leq f(x) \leftrightarrow f(y)$, for all x, y in H. The algebraic concept of compatibility means that in the logic resulting from intuitionistic logic by adding formulas in the usual way and axioms corresponding to the new say unary connective k it is possible to derive what is sometimes called the *uniqueness axiom*: $(\varphi \leftrightarrow \psi) \to (k\varphi \leftrightarrow k\psi)$. All the extensions of intuitionistic logic resulting from the given examples of new connectives satisfy this axiom, which appeared explicitly in the already cited [16], [17] and [18] as a condition in Novikov's approach.

We said that Kuznetsov defined the mentioned operation S using the minimum operator. It may be seen that also both the Smetanich connective and Gabbay's connective G may be defined in this way (though not easily seen in the case of G). In [3] it is also proved that both G and the unary connective γ, a variant of the Smetanich constant introduced by Caicedo and Cignoli, may be defined using S. One aim of this paper is to show that any operation defined by means of certain minimization scheme may be defined using the successor. A second aim is to extend Priestley duality between bounded distributive lattices and topological spaces to Heyting algebras with the new operations.

We study some unary operations $f(x)$ defined as $min\{y : Rxy\}$. More precisely, in Section 1 we consider all the operations that come from the schema $g(x) = min\{y : (y \to x) \circ tx \leq y\}$, where $\circ \in \{\wedge, \vee, \to, \leftarrow\}$ ($x \leftarrow y := y \to x$) and t is a Heyting term, distinguishing the cases that are not Heyting terms. In Section 2 we study whether the mentioned cases are polynomials. In sections 3 and 4 we consider, respectively, logical and topological aspects of the mentioned operations. In Section 5, we study the interdefinability of the operations that appear in Section 1.

1. Operations given by the min Operator

We are interested in examining the operations that arise from the schema

$$g(x) = min\{y : (y \to x) \circ tx \leq y\},$$

where $\circ \in \{\wedge, \vee, \to, \leftarrow\}$ ($x \leftarrow y := y \to x$) and t is a Heyting term. It will be enough to consider the terms given by the Rieger-Nishimura lattice.

This schema is motivated by the operation S, known as the successor and introduced in [14] (see also [3],[4], [5] and [7]), that may be defined as $min\{y : y \to x \leq y\}$ and may also be obtained from the given schema with $\circ = \wedge$ substituting either 1 or any term greater or equal to $\neg x \vee \neg\neg x$ for tx, also with $\circ = \vee$ substituting $\perp$ or x for tx, and also with $\circ = \leftarrow$ substituting any term greater or equal to $\neg x \vee \neg\neg x$ for tx. Note that S may be characterized by equations, e.g. $Sx \to x \leq Sx$ and $Sx \leq y \vee (y \to x)$.

In some cases the resulting operations may be given by Heyting terms:

LEMMA 1. *The operations corresponding in the given schema to $\circ = \wedge$ and $tx = \perp, x$ or $\circ = \vee$ and $\neg x \vee \neg\neg x \leq tx$ or all cases of $\circ = \to$ or $\circ = \leftarrow$ and $tx = \perp, x$ may be given by the Heyting terms that appear in the corresponding row and column of the table in the following page.*

PROOF. The proof in the case of $\circ = \to$ with $tx = x \vee \neg x$ may be seen in [6]. We just consider the cases $\circ = \vee, \to$ with $\neg x \vee \neg\neg x \leq tx$. For the first case, it is enough to see that a) $(tx \to x) \vee tx \leq tx$ and b) if $(y \to x) \vee tx \leq y$, then $tx \leq y$. But b) is immediate. Let us see a): it is enough to prove that $tx \to x \leq \neg\neg x$, because $\neg\neg x \leq \neg x \vee \neg\neg x \leq tx$. But $(tx \to x) \wedge \neg x \leq 0$, because $(tx \to x) \wedge \neg x \leq \neg x \vee \neg\neg x \leq tx$ and $(tx \to x) \wedge \neg x \leq \neg tx$. For the second case, it is enough to prove that c) $(1 \to x) \to tx \leq 1$ and d) if $(y \to x) \to tx \leq y$, then $1 \leq y$. But c) is immediate. Let us see d): suppose $(y \to x) \to tx \leq y$. As $\neg x \vee \neg\neg x \leq tx$, it follows that $(y \to x) \to (\neg x \vee \neg\neg x) \leq (y \to x) \to tx$. Then, using the supposition, we have that (F) $(y \to x) \to (\neg x \vee \neg\neg x) \leq y$. Then, as $\neg x \leq (y \to x) \to (\neg x \vee \neg\neg x)$, we have that $\neg x \leq y$. Then $(y \to x) \wedge \neg x \leq y$, and as $(y \to x) \wedge \neg x \leq y \to x$, it follows that $(y \to x) \wedge \neg x \leq x$. So, $y \to x \leq \neg x \to x = \neg\neg x \leq \neg x \vee \neg\neg x$. But then $1 \leq (y \to x) \to (\neg x \vee \neg\neg x)$. Then, using (F), we have that $1 \leq y$. ■

It can be easily seen that the operations corresponding to the other cases, i.e. the cases not mentioned in the previous lemma, are not Heyting terms, because they do not exist e.g. in the real interval $[0, 1]$. We will be interested in these operations that are not Heyting terms.

In what follows we consider the cases that are not equal to Heyting terms nor equal to S.

The cases $\circ = \wedge$ with $tx = x \vee \neg x$, $\neg\neg x \to x$ both provide an operation that already occurs in the literature with the name γ (see [3]).

The case $\circ = \wedge$ with $tx = \neg\neg x$ also provides an operation that already occurs in the literature: it is an operation called G in [3] that appears for the first time in [11], but from a logical point of view.

The case $\circ = \wedge$ with $tx = \neg x$ provides an operation we call h.

The case $\circ = \vee$ with $tx = \neg\neg x$ and the cases $\circ = \leftarrow$ with $tx = \neg x$, $x \vee \neg x$, $\neg\neg x \to x$ provide an operation we call T.

The case $\circ = \vee$ with $tx = \neg x$, $x \vee \neg x$ provides an operation we call Q.

The case $\circ = \vee$ with $tx = \neg\neg x \to x$ and the case $\circ = \leftarrow$ with $tx = \neg\neg x$ provide an operation we call Y.

All the cases considered are resumed in the following table:

tx	$\wedge$	$\vee$	$\to$	$\leftarrow$
$\bot$	$\bot$	Sx	$\bot$	1
x	x	Sx	x	1
$\neg x$	hx	Qx	$\neg x$	Tx
$\neg\neg x$	Gx	Tx	$\neg\neg x$	Yx
$x \vee \neg x$	γx	Qx	$\neg\neg x \to x$	Tx
$\neg\neg x \to x$	γx	Yx	$\neg\neg x \to x$	Tx
$\neg x \vee \neg\neg x \leq tx$	Sx	tx	1	Sx

Note that in the given table the functions appearing in the last two columns are either functions expressible by Heyting terms or functions that already appear in one of the first two columns. So, in the following proposition it is not necessary to consider the conditional.

Proposition 1. *Let $g(x) = min\{y : tx \circ (y \to x) \leq y\}$, where $\circ \in \{\wedge, \vee\}$. Then g may be characterized by equations, g is compatible and g exists if S exists with $g(x) = Sx \circ tx$.*

Proof. 1) To see that g is characterized by equations consider

a) $tx \circ (g(x) \to x) \leq g(x)$,

b) $g(x) \leq z \vee (tx \circ (z \to x))$.

2) To see that g is compatible, using b) with $z = g(y)$ we have that

$$(x \leftrightarrow y) \wedge g(x) \leq (x \leftrightarrow y) \wedge (g(y) \vee (tx \circ (g(y) \to x))). \tag{I}$$

Now, using the fact that the unary function $H(w) = g(y) \vee (tw \circ (g(y) \to w))$ is compatible and [3, Lemma 2.1], we have that

$$x \leftrightarrow y \leq H(x) \leftrightarrow H(y).$$

So, in particular, and also using a) we have that

$$(x \leftrightarrow y) \wedge (g(y) \vee (tx \circ (g(y) \to x))) \leq g(y) \vee (ty \circ (g(y) \to y)) = g(y). \tag{II}$$

Using (I) and (II) it follows that $(x \leftrightarrow y) \wedge g(x) \leq g(y)$. Thus, we have that $x \leftrightarrow y \leq g(x) \leftrightarrow g(y)$. Therefore, again by [3, Lemma 2.1], it follows that the function g is compatible.

3) Let us suppose that S exists. Firstly, it is the case that $Sx \circ tx \in \{y : tx \circ (y \to x) \leq y\}$ because having that $Sx \to x \leq Sx$ it follows that $tx \circ ((Sx \circ tx) \to x) \leq Sx \circ tx$. Secondly, let us take an y such that $tx \circ (y \to x) \leq y$. Then, using that $Sx \leq y \vee (y \to x)$, it follows that $Sx \circ tx \leq y$. ■

In the context of a Heyting algebra the new operations may be defined with the help of the successor in the following way:

$$\begin{array}{ll} hx = Sx \wedge \neg x, & Gx = Sx \wedge \neg\neg x, \\ \gamma x = Sx \wedge (x \vee \neg x), & Qx = Sx \vee \neg x, \\ Tx = Sx \vee \neg\neg x = S(\neg\neg x), & Yx = Sx \vee (\neg\neg x \to x) = S(\neg\neg x \to x). \end{array}$$

But the reciprocal is not true: there are Heyting algebras where the new operations exist but the successor does not. Let $H_{\omega+1}$ be the totally-ordered Heyting algebra resulting from the addition of a new element α to the set ω of natural numbers in such a way that $n < \alpha$, for each $n \in \omega$, and let $H_{\omega+2}$ be the Heyting algebra resulting from the addition of a new element β to $H_{\omega+1}$ in such a way that $\alpha < \beta$. In the Heyting algebra $(H_{\omega+2})^{op}$ (that is, $H_{\omega+2}$ with the inverse order) the functions h, γ and T exist but the successor does not, and in the Heyting algebra $(H_{\omega+1})^{op}$ the functions G, Q and Y exist but the successor does not.

2. Polynomiality

The notion of polynomial used here is simply that from universal algebra: polynomials are functions arising from constant functions and the identity function by means of the Heyting operations. The simplest examples of compatible functions in a Heyting algebra H are the polynomial functions of H, in particular, all constant functions.

Let us see that

Proposition 2. *If H is a Heyting algebra where h exists, then h is polynomial in H.*

Proof. Let us see that $hx = h0 \wedge \neg x$. Firstly, our goal is to see that $((h0 \wedge \neg x) \to x) \wedge \neg x \leq h0 \wedge \neg x$. But $((h0 \wedge \neg x) \to x) \wedge (h0 \wedge \neg x) \leq 0$. So, $((h0 \wedge \neg x) \to x) \wedge \neg x \leq \neg h0 = \neg 0 \wedge \neg h0 \leq h0$. Secondly, suppose that

$(y \to x) \wedge \neg x \leq y$. Let us see that $h0 \wedge \neg x \leq y$. But $h0 \leq y \vee (\neg x \wedge \neg y)$, $y \leq y$ and $\neg x \wedge \neg y \leq (y \to x) \wedge \neg x \leq y$. So, $h0 \leq y$. ■

It can similarly be seen that γ and T are polynomial in a Heyting algebra where they exist, seeing respectively that $\gamma x = \gamma 0 \vee x$ and $Tx = T0 \vee \neg\neg x$.

In the remainder of this section we study the affine completeness of Heyting algebras enriched with S, G, Q and Y, respectively. Remember that an algebra H is affine complete if any compatible function of H is given by a polynomial of H. It is known that boolean algebras and finite Heyting algebras are affine complete (see [12], [15] and Cor. 3.6.1 of [13]).

We write F for any of the following four operations: S, G, Q or Y. Note that in $H_{\omega+2}$, if $x \neq 0$, then $Fx = Sx$. Then we have the following

Lemma 2. *Let $N_0 = \omega - \{0\}$. If p is a polynomial of $(H_{\omega+2}, F)$, then there exist n and $x_0 \in N_0$ such that $p(x) = S^{(n)}(x)$, for every $x_0 \leq x \in N_0$ (where $S^{(0)}(x) = x$) or there exists $x_0 \in N_0$ such that $p(x) = a$, for every $x_0 \leq x \in N_0$, where $a \in H_{\omega+2}$.*

Proof. The proof is by induction on the complexity of the polynomials of $(H_{\omega+2}, F)$. The basic step holds. Let p be a polynomial of complexity $m + 1$ and suppose that the property holds for polynomials of complexity less than $m + 1$. If x_0, x_1 and $a \in \omega$, then we define $x_2 = max\{x_0, x_1\}$ and $x_3 = max\{x_0, x_1, a+1\}$. Let q and r be polynomials of complexity less than $m + 1$. We have the following cases:

(i) Let $p(x) = q(x) \wedge r(x)$. Let $q(x) = S^{(n)}(x)$, for every $x_0 \leq x \in N_0$ and $r(x) = S^{(p)}(x)$, for every $x_1 \leq x \in N_0$. Then $p(x) = S^{(l)}(x)$, for every $x_2 \leq x \in N_0$, with $l = min\ \{n, p\}$. Let $q(x) = S^{(n)}(x)$, for every $x_0 \leq x \in N_0$, and $r(x) = a$, for every $x_1 \leq x \in N_0$. If $a \in \omega$, then we have that $p(x) = a$, for every $x_3 \leq x \in N_0$. If $a \notin \omega$, then $p(x) = S^{(n)}(x)$, for every $x_2 \leq x \in N_0$. Let $q(x) = a$, for every $x_0 \leq x \in N_0$, and $r(x) = b$, for every $x_1 \leq x \in N_0$. Then $p(x) = c$, for every $x_2 \leq x \in N_0$, with $c = a \wedge b$.

(ii) Let $p(x) = q(x) \vee r(x)$. This is similar to (i).

(iii) Let $p(x) = q(x) \to r(x)$. Let $q(x) = S^{(n)}(x)$, for every $x_0 \leq x \in N_0$, and $r(x) = S^{(p)}(x)$, for every $x_1 \leq x \in N_0$. If $n \leq p$, then $p(x) = \beta$, for every $x_2 \leq x \in N_0$. If $n > p$, then $p(x) = S^{(p)}(x)$, for every $x_2 \leq x \in N_0$. Let $q(x) = S^{(n)}(x)$, for every $x_0 \leq x \in N_0$, and $r(x) = a$, for every $x_1 \leq x \in N_0$. If $a \in \omega$, then $p(x) = a$, for every $x_3 \leq x \in N_0$. If $a \notin \omega$, then $p(x) = \beta$, for every $x_2 \leq x \in N_0$. Let $q(x) = a$, for every $x_0 \leq x \in N_0$, and $r(x) = S^{(n)}(x)$, for every $x_1 \leq x \in N_0$. If $a \in \omega$, then $p(x) = \beta$, for every $x_3 \leq x \in N_0$. If $a \notin \omega$, then $p(x) = S^{(n)}(x)$, for every $x_1 \leq x \in \omega$. Let $q(x) = a$, for every

$x_0 \leq x \in N_0$, and $r(x) = b$, for every $x_0 \leq x \in N_0$. If $a \leq b$, then $p(x) = \beta$, for every $x_2 \leq x \in N_0$. If $a > b$, then $p(x) = b$, for every $x_2 \leq x \in N_0$.

(iv) Let $p(x) = S(q(x))$. If $q(x) = S^{(n)}(x)$, for every $x_0 \leq x \in N_0$, then $p(x) = S^{(n+1)}(x)$, for every $x_0 \leq x \in N_0$. If $q(x) = a$, for every $x_0 \leq x \in N_0$, then $p(x) = a + 1$, for every $x_0 \leq x \in N_0$. ■

In the next proposition we use the compatible function $f : H_{\omega+2} \to H_{\omega+2}$ defined by:

$$f(x) = \begin{cases} \alpha & \text{if } x \text{ is even or } x = \alpha, \\ \beta & \text{if } x \text{ is odd or } x = \beta. \end{cases}$$

PROPOSITION 3. *The function f is not a polynomial of $(H_{\omega+2}, F)$. In particular, $(H_{\omega+2}, F)$ is not affine complete.*

PROOF. It follows from the definition of f and Lemma 2. ■

3. Logical Approach

Let us now consider our new operations from a logical point of view. Let $I + k$ be the axiomatic system obtained extending 1) the usual language of intuitionistic logic I with a new unary connective k and formulas $k\varphi$, for every formula φ and 2) any axiomatic system of I with the following axioms (*modus ponens* (MP) remains the only rule):

kI: $(F\varphi \circ (k\varphi \to \varphi)) \to k\varphi$,
kE: $k\varphi \to (\psi \vee (F\varphi \circ (\psi \to \varphi)))$,

where $F\varphi$ is the formula corresponding to the term tx and $\circ$ is either $\wedge$ or $\vee$. The symbols "I" and "E" abbreviate the words "Introduction" and "Elimination", which seem appropriate here due to the similarity with the usual Introduction and Elimination rules in Natural Deduction. For example, in the case of G we get the axioms:

GI: $(\neg\neg\varphi \wedge (G\varphi \to \varphi)) \to G\varphi$,
GE: $G\varphi \to (\psi \vee (\neg\neg\varphi \wedge (\psi \to \varphi)))$.

Note that in this way we provide a two formula axiomatization instead of the original five axioms given by Gabbay.

These extended systems will enjoy the Deduction Theorem, because MP remains the only rule. Moreover, the connective k may be seen to be *univocal* in the sense that $\vdash_{I+k+k'} k\varphi \leftrightarrow k'\varphi$, where $I + k + k'$ is the logic resulting from the extension of the language of $I+k$ with the connective k' of the same arity as k providing formulas in the usual way and duplicating the axioms for k in $I+k$ with axioms for k'. To see that the connective k is univocal just consider the following derivation (the other conditional is analogous):

1.	$k\varphi \to (k'\varphi \vee (F\varphi \circ (k'\varphi \to \varphi)))$	kE
2.	$(F\varphi \circ (k'\varphi \to \varphi)) \to k'\varphi$	$k'I$
3.	$k\varphi \to k'\varphi$	$1, 2, I$.

Note that the first two properties of Proposition 1 also follow from the just given fact and [3, Cor. 4.3].

It may be seen, using the customary algebraic consequence relation, that $I+k$ is sound and complete w.r.t. the variety of Heyting algebras enriched with k. Soundness follows easily by induction. In order to prove strong completeness either use the univocity of k and [3, Theorems 4.1 and 4.2] or check routinely that $\vdash_{I+k} (\varphi \leftrightarrow \psi) \to (k\varphi \leftrightarrow k\psi)$ and use [3, Theorem 4.1].

Moreover, the following holds:

PROPOSITION 4. *The logics $I+k$ are* conservative*, that is, if $\Gamma \vdash_{I+k} \varphi$, then $\Gamma \vdash_I \varphi$, for φ in the language of I.*

PROOF. Due to the Deduction Theorem it is enough to prove for φ in the language of I that if $\vdash_{I+k} \varphi$, then $\vdash_I \varphi$. Let us assume that $\vdash_{I+k} \varphi$. Let g be the (algebraic) operation corresponding to the connective k. Then $\varphi = 1$ holds in every Heyting algebra where g exists. But g exists in all finite Heyting algebras, because S exists in every finite Heyting algebra (see [7, Proposition 4]) and we have seen in Proposition 1 that if S exists, then also g exists. Now, using the finite model property of intuitionistic logic it follows that $\vdash_I \varphi$. ∎

Note that the properties of univocity and conservative extension are also considered e.g. in [2].

4. Topological Approach

For a partially ordered set $(X, \leq)$ and $Y \subseteq X$, we recall that the downset of Y is the set

$$(Y] = \{x \in X : x \leq y, \text{ for some } y \in Y\}.$$

We also recall that an Esakia space is a Priestley space $(X, \leq)$ such that if U is a clopen in X, then $(U]$ is clopen. Alternatively, the Priestley space $(X, \leq)$ is an Esakia space if for every open set U, then $(U]$ is open. Let $g : (X, \leq) \to (Y, \leq)$ be a morphism of posets. We say that g is a p-morphism if for every $x \in X$ and $z \in Y$ such that $f(x) \leq z$, there is $y \in X$ with $x \leq y$ and $g(y) = z$. A morphism of Esakia spaces is a continuous p-morphism. The category of Esakia spaces $\mathcal{E}$ has as objects Esakia spaces and as arrows morphisms of Esakia spaces. We use $\mathcal{H}$ for the category of Heyting algebras.

It is known (see [8]) that the categories $\mathcal{H}$ and $\mathcal{E}$ are dually equivalent:

$$\mathbf{X} : \mathcal{H} \leftrightarrows \mathcal{E} : \mathbf{D}.$$

This fact, known as Esakia duality, is an extremely useful tool in giving dual descriptions of algebraic concepts important for the study of Heyting algebras.

By Esakia duality, each Heyting algebra H gives rise to the Esakia space $(\mathbf{X}(H), \subseteq)$, where $\mathbf{X}(H)$ is the set of prime filters of H, and the topology on $\mathbf{X}(H)$ is given by the following basis $\{\varphi_H(a) \cap (\varphi_H(b))^c : a, b \in H\}$, where

$$\varphi_H(a) = \{P \in \mathbf{X}(H) : a \in P\}.$$

Given Heyting algebras H and K and a Heyting algebra homomorphism $f : H \to K$, we have that the map $\mathbf{X}(f) : \mathbf{X}(K) \to \mathbf{X}(H)$ given by $\mathbf{X}(f)(P) = f^{-1}(P)$ is an Esakia morphism. Conversely, each Esakia space $(X, \leq)$ gives rise to the Heyting algebra $\mathbf{D}(X) = (D(X), \cap, \cup, \to, \emptyset, X)$, where $D(X)$ is the set of clopen upsets of X, and for every U, V clopen upsets of X, we have that the implication is given by $V \to U = (V \cap U^c]^c$. If $(X, \leq)$ and $(Y, \leq)$ are Esakia spaces and $g : (X, \leq) \to (Y, \leq)$ is an Esakia morphism, then the map $\mathbf{D}(g) : \mathbf{D}(Y) \to \mathbf{D}(X)$ given by $\mathbf{D}(g)(U) = g^{-1}(U)$ is a Heyting algebra homomorphism.

Given a Heyting algebra H, the map $\varphi_H : H \to \mathbf{D}(\mathbf{X}(H))$ establishes the desired isomorphism of Heyting algebras. Given an Esakia space $(X, \leq)$, the map $\epsilon_X : X \to \mathbf{X}(\mathbf{D}(X))$, given by

$$\epsilon_X = \{U \in \mathbf{D}(X) : x \in U\},$$

establishes the desired isomorphism of Esakia spaces.

Let $(X \leq)$ be a poset. If $A \subseteq X$, we write A_M for the set of maximal elements of A.

In [4] there are equivalences for the categories of Heyting algebras with S, γ and G; in particular, it was proved that in the clopen upsets of the respective topological spaces we have that

$$S(U) = U \cup (U^c)_M, \;\; \gamma(U) = U \cup X_M, \;\; G(U) = U \cup [\neg\neg U \cap (U^c)_M]. \quad \text{(III)}$$

Here $\neg U = U \to \emptyset$. In this section we give categorical equivalences for the categories of Heyting algebras with Q, T and Y. For that we consider Heyting algebras where the following unary function exists:

$$f_t(x) = min\{y \in H : (y \to x) \vee tx \leq y\},$$

where t is a Heyting term. The function f_t can be characterized by the following equations (see proof of Proposition 1):

(ft1) $(f_t(x) \to x) \vee tx \leq f_t(x)$,

(ft2) $f_t(x) \leq y \vee (y \to x) \vee tx$.

Let $t\mathcal{H}$ be the category whose objects, to be called *t-algebras*, are algebras (H, f_t), with $H \in \mathcal{H}$, and whose morphisms are morphisms of $\mathcal{H}$ that commute with f_t. An Esakia space $(X, \leq)$ is a *t-space* if, for every $U \in \mathbf{D}(X)$, the set $(U^c)_M \cup (U^c \cap tU)$ is clopen. Note that it is immediate that $(X, \leq)$ is a t-space if and only if it is an Esakia space such that, for every $U \in \mathbf{D}(X)$, the set $U \cup (U^c)_M \cup tU$ is clopen (note that this set is also an upset). Let $(X, \leq)$ and $(Y, \leq)$ be t-spaces and $g : (X, \leq) \to (Y, \leq)$ is a morphism in $\mathcal{E}$. We say that g is a *t-morphism* if, for every $U \in \mathbf{D}(Y)$, it holds that

$$g^{-1}(U) \cup g^{-1}[(U^c)_M] \cup g^{-1}(tU) = g^{-1}(U) \cup [g^{-1}(U^c)]_M \cup g^{-1}(tU).$$

We denote by $t\mathcal{E}$ the category whose objects are t-spaces and whose morphisms are t-morphisms. In what follows we will see that there is a dual categorical equivalence between $t\mathcal{H}$ and $t\mathcal{E}$.

There are several well known results about Esakia duality (see [8], [9] and [10]). We will frequently use the following: if $(X, \leq)$ is an Esakia space and V is a closed subset of X, then, for every $x \in V$, there is a $v \in V_M$ such that $x \leq v$. In general, the set of maximal elements of a closed subset of an Esakia space is closed, but it is not necessarily open.

LEMMA 3. *If $(X, \leq) \in t\mathcal{E}$, then there exists f_t in $\boldsymbol{D}(X)$ and we have that*

$$f_t(U) = U \cup (U^c)_M \cup tU.$$

PROOF. For every $U \in \mathbf{D}(X)$ define the set

$$E_U = \{V \in \mathbf{D}(X) : (V \to U) \cup tU \subseteq V\}.$$

Firstly, we prove that $U \cup (U^c)_M \cup tU \in E_U$. It is equivalent to see that

$$((U^c)_M \cup (tU \cap U^c)]^c \cup tU \subseteq U \cup (U^c)_M \cup tU.$$

Let $x \in ((U^c)_M \cup (tU \cap U^c)]^c$ and suppose that $x \notin U$. Using that U^c is closed we have that there is a $y \in (U^c)_M$ such that $x \leq y$, so $x \in ((U^c)_M \cup (tU \cap U^c)]$, a contradiction. Then $((U^c)_M \cup (tU \cap U^c)]^c \cup tU \subseteq U \cup (U^c)_M \cup tU$. Secondly, let $V \in E_U$. Let us see that $U \cup (U^c)_M \cup tU \subseteq V$. Let $x \in U \cup (U^c)_M \cup tU$. Suppose that $x \notin V$. Then, $x \notin tU$ and $x \notin V \to U$. Thus, there is $y \in V \cap U^c$ such that $x \leq y$. In particular, $x \in U^c$ (because U is an upset), so $x \in (U^c)_M$. Thus, $x = y$, so $y \notin V$, a contradiction. ■

Note that if $g : (X, \leq) \to (Y, \leq)$ is a function between posets and $U \subseteq Y$, then

$$g^{-1}(U) \cup g^{-1}[(U^c)_M] \cup g^{-1}(tU) = g^{-1}(U \cup (U^c)_M \cup tU). \qquad \text{(IV)}$$

Thus, the previous lemma allows us to obtain the following

LEMMA 4. *If $g \in t\mathcal{E}$, then $\boldsymbol{D}(g) \in t\mathcal{H}$.*

From the two previous lemmas it follows that $\mathbf{D}$ is a functor from $t\mathcal{E}$ to $t\mathcal{H}$.

If $H \in \mathcal{H}$ and $A \subseteq H$, we write $F(A)$ for the filter generated by A.

LEMMA 5. *Let $H \in \mathcal{H}$. If the function f_t exists, then for every $x \in H$ it holds that*

$$\varphi_H(f_t(x)) = \varphi_H(x) \cup (\varphi_H^c(x))_M \cup \varphi_H(tx).$$

In particular, $(\boldsymbol{X}(H), \subseteq) \in t\mathcal{E}$.

PROOF. Let $P \in \varphi_H(f_t(x))$. Suppose that $x \notin P$, $t(x) \notin P$ and let $Q \in \mathbf{X}(H)$ such that $P \subseteq Q$ and $x \notin Q$. In what follows we prove that $P = Q$. Let $y \in Q$ and suppose that $y \notin P$, so by equation $(ft2)$ we have that $y \to x \in P$. Thus $y \to x \in Q$, so $x \in Q$ (because $y \in Q$), a contradiction. Thus $P = Q$, so $\varphi_H(f_t(x)) \subseteq \varphi_H(x) \cup (\varphi_H^c(x))_M \cup \varphi_H(tx)$.

Conversely, let $P \in \varphi_H(x) \cup (\varphi_H^c(x))_M \cup \varphi_H(tx)$. If $x \in P$ or $tx \in P$, then by equation $(ft1)$ we have that $f_t(x) \in P$, so let us consider the case that $P \in (\varphi_H^c(x))_M$. Suppose that $f_t(x) \notin P$. Now consider the filter $F = F(P \cup \{f_t(x)\})$. By equation $(ft1)$ and the fact that $f_t(x) \notin P$ we have that $x \notin F$, so by the Prime Filter Theorem there is a $Q \in \mathbf{X}(H)$ such that $P \subseteq F \subseteq Q$ and $x \notin Q$; in particular, we have that $P = F = Q$, so $f_t(x) \in P$, a contradiction. Therefore, $\varphi_H(x) \cup (\varphi_H^c(x))_M \cup \varphi_H(tx) \subseteq \varphi_H(f_t(x))$. ■

REMARK 1. *If $(H, f_t) \in t\mathcal{H}$, then φ_H is an isomorphism in $t\mathcal{H}$. This holds because the function f_t is characterized by equations.*

LEMMA 6. *If $h : (H, f_t) \to (K, f_t) \in t\mathcal{H}$, then $\boldsymbol{X}(h) \in t\mathcal{E}$.*

Proof. It follows that $\mathbf{D}(\mathbf{X}(h)) = \varphi_G h \varphi_H^{-1}$ because $\mathcal{H}$ and $\mathcal{E}$ are dually equivalent. Also, we have that for every $U \in \mathbf{D}(\mathbf{X}(H))$ there is an $x \in H$ such that $U = \varphi_H(x)$. Then by Remark 1 it follows that

$$\mathbf{X}(h)^{-1}(f_t(U)) = \varphi_G h f_t(x) = f_t \varphi_G h(x), \tag{V}$$

$$f_t(\mathbf{X}(h)^{-1}(U)) = f_t(\varphi_G h \varphi_H^{-1}(U)) = f_t \varphi_G h(x). \tag{VI}$$

From (V) and (VI) it follows that $\mathbf{X}(h)^{-1}(f_t(U)) = f_t(\mathbf{X}(h)^{-1}(U))$. Then by equation (IV) we conclude that $\mathbf{X}(h) \in t\mathcal{E}$. ■

From the two previous lemmas it follows that $\mathbf{X}$ is a functor from $t\mathcal{E}$ to $t\mathcal{H}$.

Lemma 7. *If $(X, \leq) \in t\mathcal{E}$, then $\epsilon_X \in t\mathcal{E}$. In particular, ϵ_X is an isomorphism in $t\mathcal{E}$.*

Proof. It is enough to see that for every $V \in \mathbf{X}(\mathbf{D}(X))$ it holds that

$$[\epsilon_X^{-1}(V^c)]_M = \epsilon_X^{-1}[(V^c)_M].$$

Let $x \in [\epsilon_X^{-1}(V^c)]_M$, so $\epsilon_X(x) \in V^c$. Let $\epsilon_X(x) \leq z$ with $z \in V^c$. Using that ϵ_X is an isomorphism of lattices it follows that there is a y such that $\epsilon_X(y) = z$ with $x \leq y$. In particular, $\epsilon_X(y) \in V^c$ and $x \leq y$. Thus $x = y$, so $\epsilon_X(x) = \epsilon_X(y) = z$, so $x \in \epsilon_X^{-1}[(V^c)_M]$. Conversely, let $x \in \epsilon_X^{-1}[(V^c)_M]$, so $\epsilon_X(x) \in (V^c)_M$. Let z be such that $x \leq z$ with $z \in \epsilon_X^{-1}(V^c)$. Therefore, $\epsilon_X(z) \in V^c$. The function ϵ_X preserves order, so $\epsilon_X(x) \leq \epsilon_X(z)$. As $\epsilon_X(z) \in V^c$, we have that $\epsilon_X(x) = \epsilon_X(z)$. However, ϵ_X is an injective function. Thus $x = z$ and then $x \in [\epsilon_X^{-1}(V^c)]_M$. ■

The following theorem follows from the previous results.

Theorem 1. *There is a dual categorical equivalence between $t\mathcal{H}$ and $t\mathcal{E}$.*

The following lemma is well known and will be used in the next section.

Lemma 8. *Let $(X, \leq)$ be an Esakia space. Then, for every $U \in \boldsymbol{D}(X)$, it holds that*

$$(U^c)_M \subseteq \neg U \cup \neg\neg U.$$

5. Interdefinability

In this section we show how to interdefine the operations introduced in Section 1. Firstly, we consider the polynomial operations and show that they are interdefinable. Secondly, we consider the non-polynomial operations G, Q and Y and prove algebraically the particular case that if G exists, then Q also exists with $Qx = Gx \vee \neg x$. Thirdly, we use results of Section 4 to simplify the just mentioned algebraic proof for similar cases. Finally, we show that S, though impossible to be defined using just one of the other operators, may be defined taking two of them, one polynomial and the other not so.

Regarding the polynomial operations the following holds: if either h, γ or T exist, then also the other exist respectively with

$$\begin{array}{lll} \gamma x = h0 \vee x, & hx = \gamma 0 \wedge \neg x, & hx = T0 \wedge \neg x, \\ Tx = h0 \vee \neg\neg x, & Tx = \gamma 0 \vee \neg\neg x, & \gamma x = T0 \vee x. \end{array}$$

The proof of this fact is similar to the proof of Proposition 2. Note that $\gamma = h0 = \gamma 0 = T0$.

Let us now consider the cases of the non-polynomial operations G, Q and Y. They are all interdefinable. The following is one case which we prove algebraically.

Proposition 5. *If G exists, then also Q exists with $Qx = Gx \vee \neg x$.*

Proof. Firstly, let us see that $Gx \vee \neg x \in \{y : (y \to x) \vee \neg x \leq y\}$, i.e. that $((Gx \vee \neg x) \to x) \vee \neg x \leq Gx \vee \neg x$. But $\neg x \leq Gx \vee \neg x$. Now we have to see that $(Gx \vee \neg x) \to x \leq Gx \vee \neg x$. In fact, we can prove that $(Gx \vee \neg x) \to x \leq Gx$. Having that $(Gx \to x) \wedge \neg\neg x \leq Gx$, it is enough to get a) $(Gx \vee \neg x) \to x \leq Gx \to x$ and b) $(Gx \vee \neg x) \to x \leq \neg\neg x$. For a) just consider that $Gx \leq Gx \vee \neg x$. For b), it is the case that $(Gx \vee \neg x) \to x \leq \neg x \to \neg(Gx \vee \neg x) = \neg x \to (\neg Gx \wedge \neg\neg x) \leq \neg x \to \neg\neg x = \neg\neg x$.

Secondly, let us see that if $(y \to x) \vee \neg x \leq y$, then $Gx \vee \neg x \leq y$. Using the hypothesis $(y \to x) \vee \neg x \leq y$, it is enough to see that a) $Gx \leq y$ and b) $\neg x \leq y$. Part b) is immediate because $\neg x \leq (y \to x) \vee \neg x \leq y$. For a), consider that from the definition of G it follows that $Gx \leq y \vee ((y \to x) \wedge \neg\neg x)$. It is obvious that $y \leq y$. Then, it is enough to see that $(y \to x) \wedge \neg\neg x \leq y$. But $(y \to x) \wedge \neg\neg x \leq y \to x \leq (y \to x) \vee \neg x \leq y$. ■

The previous proposition appears as the first line in the second column of the following cases. In general, if either G, Q or Y exist, then also the others exist, respectively with

$$Gx = Qx \wedge \neg\neg x, \quad Qx = Gx \vee \neg x, \quad Yx = Gx \vee (\neg\neg x \to x),$$
$$Gx = Yx \wedge \neg\neg x, \quad Qx = (Yx \wedge \neg\neg x) \vee \neg x, \quad Yx = Qx \vee (\neg\neg x \to x).$$

In order to prove these equations use Theorem 1 and the third equation of (III). The two cases of the first column and the last case of the third column are consequences of an easy computation and the other cases follow from Lemma 8.

However, taking one polynomial and one non-polynomial operation, S may be defined. The following precise fact may be seen algebraically: if either h and G, γ and G, or T and Q exist, then also S exists, respectively with

$$Sx = hx \vee Gx,$$
$$Sx = \gamma x \vee Gx,$$
$$Sx = Tx \wedge Qx.$$

The second equation already appears in [3]. The third also follows from Theorem 1 and Lemma 8.

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