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Ideal Paraconsistent Logics

Abstract. We define in precise terms the basic properties that an 'ideal propositional paraconsistent logic' is expected to have, and investigate the relations between them. This leads to a precise characterization of ideal propositional paraconsistent logics. We show that every three-valued paraconsistent logic which is contained in classical logic, and has a proper implication connective, is ideal. Then we show that for every $n > 2$ there exists an extensive family of ideal n-valued logics, each one of which is not equivalent to any k-valued logic with $k < n$.

Keywords: paraconsistent logics, ideal paraconsistency, many-valued logics

1. Introduction

Handling contradictory data is one of the most complex and important problems in reasoning under uncertainty. To handle inconsistent information one needs a logic that, unlike classical logic, allows contradictory yet non-trivial theories. Formally, this may be represented by $p, \neg p \not\vdash q$ (where p and q are propositional variables), meaning that a single contradiction should not imply every formula. Logics of this sort are called *paraconsistent* [17, 24]. The need for paraconsistent reasoning has been reinforced in recent years by practical considerations. For instance, many information systems are often contradictory due to their size and diversity. Unless inconsistency is maintained in a coherent way, query answering in such systems would be useless.

Nowadays there is a vast amount of different paraconsistent logics that have been suggested and investigated over the years (see, for instance, [9, 12, 13, 14]). It is not at all clear, however, how to choose among them. Thus, a natural question arises: what are the properties that an 'ideal' paraconsistent logic should satisfy? The standard general answer seems to be that an ideal paraconsistent logic should retain as much of classical logic as possible, while still allowing non-trivial inconsistent theories (see [15, 17]). But what does 'retaining as much of classical logic as possible' mean? A preliminary analysis shows that this involves the following three basic intuitive properties:

Containment in Classical Logic. As the general characterization given above to 'ideal paraconsistent logics' suggests, classical logic is usually taken as the reference logic for such logics. This means that while a

Studia Logica **0**: 1–30, 2011.

reasonable paraconsistent logic is necessarily more tolerant than classical logic (since it allows non-trivial contradictions), it should not validate any inference which classical logic forbids. In other words: it should be contained in classical logic.

Maximal Paraconsistency. The requirement from a paraconsistent logic **L** to "retain as much of classical logic as possible, while still allowing non-trivial inconsistent theories" has two different interpretations, corresponding to the two aspects of this demand:

Absolute maximal paraconsistency. Intuitively, this means that by trying to further extend **L** (without changing the language) we lose the property of paraconsistency.

Maximality relative to classical logic. Here the intuitive meaning is that **L** is so close to classical logic, that any attempt to further extend it should necessarily end up with classical logic.

Ideally, we would like of course an 'ideal paraconsistent logic' to have both types of maximality.

Reasonable language. Obviously, the language of a paraconsistent logic should have an official *negation* connective which is entitled to this name. This is insufficient, of course. Thus, in [3] we have shown that the three-valued logic whose only connective is Sette's negation [32], is maximally paraconsistent and it is obviously contained in classical logic. Still, nobody would take it as an 'ideal' paraconsistent logic, because its language is not sufficiently expressive. So an ideal paraconsistent logic should be in a language which is reasonably strong.

The three properties we expect an 'ideal paraconsistent logic' to have are all rather vague. Accordingly, our first goal in this paper is to define them in precise terms, and investigate the relations between them. This leads to a precise characterization of ideal paraconsistent logics. Our second goal is to examine which of the paraconsistent logics that have been studied in the literature are ideal. Our third and last goal is to provide a systematic way of constructing ideal paraconsistent logics.

The rest of this paper is divided to three parts: First, in the next section, we review the basic concepts underlying our investigations. Then, in Section 3, we define and investigate the properties that an ideal paraconsistent logic should have. Finally, Sections 4 and 5 are concerned with concrete ideal paraconsistent logics. The first of them is devoted to three-valued paraconsistent logics.

We show in it, for instance, that in the three-valued case, paraconsistent logics that are reasonably expressive and are contained in classical logic are already ideal. This includes all the 2^{20} three-valued paraconsistent logics shown in [3] to be maximally paraconsistent, including the 2^{13} three-valued logics of formal inconsistency (LFIs), shown in [15, 28] to be maximal relative to classical logic. In Section 5 we provide a systematic way of constructing ideal logics with any finite number of truth-values. We show that for every $n > 2$ there exists an extensive family of ideal n-valued logics, each one of which is not equivalent to any k-valued logic with $k < n$.[1]

2. Preliminaries

2.1. What is a Paraconsistent Logic?

In the sequel, $\mathcal{L}$ denotes a propositional language with a set $\mathcal{A}_{\mathcal{L}}$ of atomic formulas and a set $\mathcal{W}_{\mathcal{L}}$ of well-formed formulas. We denote the elements of $\mathcal{A}_{\mathcal{L}}$ by p, q, r (possibly with subscripted indexes), and the elements of $\mathcal{W}_{\mathcal{L}}$ by ψ, ϕ, σ. $\mathsf{Atoms}(\varphi)$ denotes the set of atomic formulas occurring in φ. Given a unary connective $\diamond$ of $\mathcal{L}$, we denote $\diamond^0\psi = \psi$ and $\diamond^i\psi = \diamond(\diamond^{i-1}\psi)$ (for $i \geq 1$). Sets of formulas in $\mathcal{W}_{\mathcal{L}}$ are called *theories* and are denoted by $\mathcal{T}$ or $\mathcal{S}$. We denote *finite* theories by Γ or Δ. Following the usual convention, we shall abbreviate $\mathcal{T} \cup \{\psi\}$ by $\mathcal{T}, \psi$. More generally, we shall write $\mathcal{T}, \mathcal{S}$ instead of $\mathcal{T} \cup \mathcal{S}$.

First, we define what is a 'logic'.

Definition 1. A (Tarskian) *consequence relation* for a language $\mathcal{L}$ (a tcr, for short) is a binary relation $\vdash$ between theories in $\mathcal{W}_{\mathcal{L}}$ and formulas in $\mathcal{W}_{\mathcal{L}}$, satisfying the following three conditions:

Reflexivity:	if $\psi \in \mathcal{T}$ then $\mathcal{T} \vdash \psi$.
Monotonicity:	if $\mathcal{T} \vdash \psi$ and $\mathcal{T} \subseteq \mathcal{T}'$, then $\mathcal{T}' \vdash \psi$.
Transitivity (cut) :	if $\mathcal{T} \vdash \psi$ and $\mathcal{T}', \psi \vdash \phi$ then $\mathcal{T}, \mathcal{T}' \vdash \phi$.

Let $\vdash$ be a tcr for $\mathcal{L}$.

- We say that $\vdash$ is *structural*, if for every uniform $\mathcal{L}$-substitution θ and every $\mathcal{T}$ and ψ, if $\mathcal{T} \vdash \psi$ then $\theta(\mathcal{T}) \vdash \theta(\psi)$.[2]

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[1] This paper is an extensively expanded version of [5].

[2] Where $\theta(\mathcal{T}) = \{\theta(\varphi) \mid \varphi \in \mathcal{T}\}$.

- We say that $\vdash$ is *non-trivial*, if there exist some non-empty theory $\mathcal{T}$ and some formula ψ such that $\mathcal{T} \not\vdash \psi$.
- We say that $\vdash$ is *finitary*, if for every theory $\mathcal{T}$ and every formula ψ such that $\mathcal{T} \vdash \psi$ there is a *finite* theory $\Gamma \subseteq \mathcal{T}$ such that $\Gamma \vdash \psi$.

DEFINITION 2. A (propositional) *logic* is a pair $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$, such that $\mathcal{L}$ is a propositional language, and $\vdash$ is a structural, non-trivial, and finitary consequence relation for $\mathcal{L}$.

NOTE 1. The conditions of being non-trivial and finitary are usually not required in the definitions of propositional logics. However, the first is convenient for excluding trivial logics (those in which every formula follows from every non-empty theory). The second is demanded since we believe that it is essential for practical reasoning, where a conclusion is always derived from a finite set of premises. In particular, every logic that has a decent proof system is finitary.

DEFINITION 3.

1. A logic $\mathbf{L}_1 = \langle \vdash_1, \mathcal{L} \rangle$ is an *extension* of a logic $\mathbf{L}_2 = \langle \vdash_2, \mathcal{L} \rangle$ (in the same language) if $\vdash_2 \subseteq \vdash_1$. We say that $\mathbf{L}_1$ is a *proper extension* of $\mathbf{L}_2$, if $\vdash_2 \subsetneq \vdash_1$.
2. A *rule* in a language $\mathcal{L}$ is a pair $\langle \Gamma, \psi \rangle$, where $\Gamma \cup \{\psi\}$ is a finite set of formulas in $\mathcal{L}$. We shall henceforth denote such a rule by Γ/ψ.
3. Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic. The *extension of* $\mathbf{L}$ *by a set of rules* S *for* $\mathcal{L}$, is the minimal extension $\mathbf{L}^* = \langle \vdash^*, \mathcal{L} \rangle$ of $\mathbf{L}$ such that $\Delta \vdash^* \varphi$ whenever $\Delta/\varphi \in S$. Extending $\mathbf{L}$ by an axiom schema φ means extending it by the rule $\emptyset/\psi$.

NOTE 2. It is easy to see that in Item 3 of Definition 3, $\vdash^*$ is the closure under cuts and weakenings of the set of all pairs $\theta(\Gamma)/\theta(\psi)$, where θ is a uniform $\mathcal{L}$-substitution, and either $\Gamma \vdash \psi$ or $\Gamma/\psi \in S$. This implies that an extension of a finitary tcr by a set of rules is finitary. This in turn implies that $\langle \mathcal{L}, \vdash^* \rangle$ is a logic (provided that $\vdash^*$ is non-trivial, which is guaranteed in all cases considered here). Moreover, we associate with a logic $\mathbf{L}$ the syntactic system which has as axioms all sequents of the form $\Gamma \Rightarrow \psi$, where Γ is finite and $\Gamma \vdash_{\mathbf{L}} \psi$, with cut as the only rule of inference. Adding ψ as an axiom schema to $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ means then just adding as new axioms all sequents of the form $\Gamma \Rightarrow \psi'$, where Γ is a finite theory and ψ' is some substitution instance of ψ.

We are now ready to define the notion of *paraconsistency* in precise terms:

DEFINITION 4. [17, 24] Let $\mathcal{L}$ be a language with a unary connective $\neg$. A logic $\langle \mathcal{L}, \vdash \rangle$ is called $\neg$-*paraconsistent*, if there are formulas ψ, ϕ in $\mathcal{W}_{\mathcal{L}}$, such that $\psi, \neg\psi \not\vdash \phi$.

NOTE 3. As $\vdash$ is structural, it is enough to require in Definition 4 that there are *atoms* p, q such that $p, \neg p \not\vdash q$. The definition above is adequate also for non-structural consequence relations.

In what follows, when it is clear from the context, we shall sometimes omit the '$\neg$' symbol and simply refer to paraconsistent logics.

2.2. Many-valued Matrices

The most standard semantic way of defining logics (and, in particular, paraconsistent ones) is by using the following type of structures (see, e.g., [23, 27, 35]).

DEFINITION 5. A (multi-valued) *matrix* for a language $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where

- $\mathcal{V}$ is a non-empty set of truth values,
- $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, called the *designated* elements of $\mathcal{V}$, and
- $\mathcal{O}$ includes an n-ary function $\widetilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \to \mathcal{V}$ for every n-ary connective $\diamond$ of $\mathcal{L}$.

In what follows, we shall denote by $\overline{\mathcal{D}}$ the elements in $\mathcal{V} \setminus \mathcal{D}$. The set $\mathcal{D}$ is used for defining satisfiability and validity, as defined below:

DEFINITION 6. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for $\mathcal{L}$.

- An $\mathcal{M}$-*valuation* for $\mathcal{L}$ is a function $\nu : \mathcal{W}_{\mathcal{L}} \to \mathcal{V}$ such that for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1, \ldots, \psi_n \in \mathcal{W}_{\mathcal{L}}$, $\nu(\diamond(\psi_1, \ldots, \psi_n)) = \widetilde{\diamond}_{\mathcal{M}}(\nu(\psi_1), \ldots, \nu(\psi_n))$. We denote the set of all the $\mathcal{M}$-valuations by $\Lambda_{\mathcal{M}}$.
- A valuation $\nu \in \Lambda_{\mathcal{M}}$ is an $\mathcal{M}$-*model* of a formula ψ (alternatively, ν $\mathcal{M}$-*satisfies* ψ), if it belongs to the set $mod_{\mathcal{M}}(\psi) = \{\nu \in \Lambda_{\mathcal{M}} \mid \nu(\psi) \in \mathcal{D}\}$. The $\mathcal{M}$-models of a theory $\mathcal{T}$ are the elements of the set $mod_{\mathcal{M}}(\mathcal{T}) = \cap_{\psi \in \mathcal{T}} mod_{\mathcal{M}}(\psi)$.

- A formula ψ is $\mathcal{M}$-*satisfiable* if $mod_{\mathcal{M}}(\psi) \neq \emptyset$. A theory $\mathcal{T}$ is $\mathcal{M}$-satisfiable if $mod_{\mathcal{M}}(\mathcal{T}) \neq \emptyset$.

In the sequel, we shall sometimes omit the prefix '$\mathcal{M}$' from the notions above. Also, when it is clear from the context, we shall omit the subscript '$\mathcal{M}$' in $\widetilde{\diamond}_{\mathcal{M}}$.

Definition 7. Given a matrix $\mathcal{M}$, the consequence relation $\vdash_{\mathcal{M}}$ that is *induced by* (or associated with) $\mathcal{M}$, is defined by: $\mathcal{T} \vdash_{\mathcal{M}} \psi$ if $mod_{\mathcal{M}}(\mathcal{T}) \subseteq mod_{\mathcal{M}}(\psi)$. We denote by $\mathbf{L}_{\mathcal{M}}$ the pair $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$, where $\mathcal{M}$ is a matrix for $\mathcal{L}$ and $\vdash_{\mathcal{M}}$ is the consequence relation induced by $\mathcal{M}$.

We say that a matrix $\mathcal{M}$ is *paraconsistent* if so is $\mathbf{L}_{\mathcal{M}}$. The following proposition has been proven in [33, 34]:

Proposition 1. *For every propositional language $\mathcal{L}$ and a finite matrix $\mathcal{M}$ for $\mathcal{L}$, $\mathbf{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ is a propositional logic.*[3] [4]

Example 1.

1. Propositional classical logic is induced by the two-valued matrix $\mathcal{M}_2 = \langle \{t, f\}, \{t\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}\} \rangle$ with the standard two-valued interpretations for $\vee$, $\wedge$ and $\neg$.

2. Priest's LP [30, 31] is induced by $\mathsf{LP} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}\} \rangle$, where $\vee$, $\wedge$ and $\neg$ have the standard Kleene's interpretations [26]:

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	$\top$
$\top$	t	$\top$	$\top$

$\tilde{\wedge}$	t	f	$\top$
t	t	f	$\top$
f	f	f	f
$\top$	$\top$	f	$\top$

$\tilde{\neg}$	
t	f
f	t
$\top$	$\top$

Proposition 2. *([30]) The tautologies of* LP *are the same as the classical ones:* $\vdash_{\mathsf{LP}} \psi$ *iff* $\vdash_{\mathcal{M}_2} \psi$*. Nevertheless,* LP *is paraconsistent, while* $\mathcal{M}_2$ *is not.*

3. What is an Ideal Paraconsistent Logic?

This section is devoted to a clarification of the intuitive demands from an ideal paraconsistent logic that were set forth in the introduction.

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[3]The non-trivial part in this result is that $\vdash_{\mathcal{M}}$ is finitary; It is easy to see that for *every* matrix $\mathcal{M}$ (not necessarily finite), $\vdash_{\mathcal{M}}$ is a structural and non-trivial tcr.

[4]This proposition was generalized to the case of finite non-deterministic matrices in [6].

3.1. Containment in Classical Logic

Containment in classical logic is a notion that is widely used in the literature (see, e.g., [15, 18, 25, 29]). Usually, a logic **L** has been defined (e.g. in [15]) to be 'contained in classical logic' if it has the same language as classical logic, and classical logic is an extension of **L** in this language. Unfortunately, although this definition seems very intuitive, it is in fact not well-defined, because it is not clear what is 'the language of classical logic' to which this 'definition' refers. For example: suppose someone uses the symbol "$\wedge$" for conjunction, while someone else uses "&" instead. Do they use the same language or not? What is more, someone may use $\neg$ and $\wedge$ as primitive connectives, another may use $\neg$ and $\vee$, and still another uses all of the three. Who of them uses "the language of classical logic"? Obviously, any choice would be too arbitrary, and the question whether a given logic is contained in classical logic should not depend on such arbitrary choices.

The next example demonstrates how serious the problem is.

Example 2. Consider the proof in [16] that the logic of formal inconsistency **LFI1** is maximally paraconsistent "relative to classical logic", where the language of classical logic is taken as $\{\wedge, \vee, \rightarrow, \neg\}$. This logic employs an additional unary connective $\bullet$, which is not definable by other connectives of the language. Thus, to be able to speak of maximal paraconsistency of **LFI1** relative to classical logic, the authors must *enrich* the language they call "the language of classical logic" with a corresponding connective $\bullet$, and supply an appropriate interpretation to it. The obtained logic, called **ECPL** ('extended propositional classical logic'), is a conservative extension of classical logic (that is, the tautologies of **ECPL** in the language of $\{\wedge, \vee, \rightarrow, \neg\}$ are exactly those of classical logic). Now, in [16] it is shown that the addition to **LFI1** of a tautology of **ECPL** that is not provable in **LFI1**, yields either **ECPL** or a trivial logic. Concluding (as is done in [16]) that this means that **LFI1** is maximal relative to classical logic is rather problematic, because this does not fit the given definition. What is more: a different choice of a two-valued interpretation for $\bullet$ (and nothing in the definition used in [16] forbids such a choice) would imply that *the same logic* is not even contained in classical logic!

Next we give an *exact* definition of what it means for a logic to be contained in classical logic.

Definition 8. Let $\mathcal{L}$ be a language with a unary connective $\neg$. A *bivalent $\neg$-interpretation for $\mathcal{L}$* is a function **F** that associates a two-valued truth-table with each connective of $\mathcal{L}$, such that $\mathbf{F}(\neg)$ is the classical truth table for negation. We denote by $\mathcal{M}_{\mathbf{F}}$ the two-valued matrix for $\mathcal{L}$ induced by $\mathbf{F}$.

DEFINITION 9. Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic for a language $\mathcal{L}$ with a unary connective $\neg$, let $\mathcal{M}$ be a matrix for $\mathcal{L}$, and let $\mathbf{F}$ be some bivalent $\neg$-interpretation for $\mathcal{L}$.

- $\mathbf{L}$ is $\mathbf{F}$*-contained in classical logic*, if for every $\varphi_1, \ldots, \varphi_n, \psi \in \mathcal{W}_{\mathcal{L}}$: if $\varphi_1, \ldots \varphi_n \vdash_{\mathbf{L}} \psi$ then $\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{M}_{\mathbf{F}}} \psi$.
- $\mathbf{L}$ is $\neg$*-contained in classical logic*, if it is $\mathbf{F}$-contained in it for some $\mathbf{F}$.
- $\mathcal{M}$ is $\mathbf{F}$-contained ($\neg$-contained) in classical logic if so is $\mathbf{L}_{\mathcal{M}}$.

PROPOSITION 3. *Let* $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ *be a matrix for a language with* $\neg$.

- *If* $\mathcal{M}$ *is* $\neg$*-contained in classical logic, then there is an element* $t \in \mathcal{D}$*, such that* $\neg t \notin \mathcal{D}$.
- $\mathcal{M}$ *is paraconsistent iff there is an element* $\top \in \mathcal{D}$*, such that* $\neg \top \in \mathcal{D}$.

Proof. Let $\mathbf{F}$ be a bivalent $\neg$-interpretation, such that $\mathbf{L}_{\mathcal{M}}$ is $\mathbf{F}$-contained in classical logic. Since $p \not\vdash_{\mathcal{M}_{\mathbf{F}}} \neg p$, also $p \not\vdash_{\mathcal{M}} \neg p$, and so there is some $t \in \mathcal{D}$, such that $\neg t \notin \mathcal{D}$. Since $\mathcal{M}$ is paraconsistent, $p, \neg p \not\vdash_{\mathcal{M}} q$, and so there is some $\top \in \mathcal{D}$, such that $\neg \top \in \mathcal{D}$. □

COROLLARY 1. *No two-valued matrix which is* $\neg$*-contained in classical logic can be paraconsistent.*

Proof. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a paraconsistent matrix $\neg$-contained in classical logic. By Proposition 3, $\mathcal{D}$ contains at least two truth-values. Since $\mathcal{D} \subset \mathcal{V}$, $\mathcal{V}$ must contain at least three truth-values. □

Corollary 1 implies that the semantics of a reasonable paraconsistent logic cannot be based on just the two classical truth-values t and f. Still, it will be convenient in what follows to concentrate on matrices which are (intuitively) obtained by adding to $\{t, f\}$ some 'abnormal' truth-values.

DEFINITION 10. A matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is called *proto-classical*, if there exists a *unique* element $a \in \mathcal{V}$, such that $a \in \mathcal{D}$ and $\neg a \notin \mathcal{D}$.

NOTATION 1. Given a proto-classical matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, we shall henceforth denote by t the element a of $\mathcal{V}$ such that $a \in \mathcal{D}$ while $\neg a \notin \mathcal{D}$, and by f the element $\neg t$ (so in proto-classical matrices: $t \in \mathcal{D}$, $f \notin \mathcal{D}$, and $f = \neg t$).

In the next proposition we introduce a particularly important class of paraconsistent matrices which are proto-classical:

PROPOSITION 4. *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a paraconsistent matrix which is $\neg$-contained in classical logic. If $|\mathcal{D}| = 2$ then $\mathcal{M}$ is a proto-classical matrix in which $\neg f = t$.*

Proof. From Proposition 3 it immediately follows that under the assumptions of the present proposition, $\mathcal{D} = \{t, \top\}$, where $\neg t \notin \mathcal{D}$, while $\neg \top \in \mathcal{D}$. Hence $\mathcal{M}$ is proto-classical.
Let $f = \neg t$. We have that $p, \neg\neg p \not\vdash_{\mathcal{M}} \neg\neg\neg p$, because $\mathcal{M}$ is $\neg$-contained in classical logic. Since $\neg \top \in \mathcal{D}$ and $\neg\neg\top \in \{\top, f\}$ (this is obvious when either $\neg \top = t$ or $\neg \top = \top$), a model in $\mathcal{M}$ of $\{p, \neg\neg p\}$ which is not a model of $\neg\neg\neg p$ should assign t to both p and $\neg\neg p$. It follows that $\neg f = \neg\neg t = t$. □

DEFINITION 11. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a proto-classical matrix for $\mathcal{L}$.

- An n-ary operation $\tilde{\diamond}$ of $\mathcal{M}$ is *classically closed*, if $\tilde{\diamond}(a_1, \ldots, a_n) \in \{t, f\}$ for every n-ary connective $\diamond$ of $\mathcal{L}$, and every $a_1, \ldots, a_n \in \{t, f\}$.
- $\mathcal{M}$ is *classically closed*, if all its operations are classically closed.
- $\mathcal{M}$ is *semi-classical* if it is classically closed and $\neg f = t$.

DEFINITION 12. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a semi-classical matrix for $\mathcal{L}$. The *bivalent $\neg$-interpretation* $\mathbf{F}_{\mathcal{M}}$ *induced by* $\mathcal{M}$ is defined by $\mathbf{F}_{\mathcal{M}}(\diamond) = \tilde{\diamond}_{\mathcal{M}}/\{t, f\}$, where $\tilde{\diamond}_{\mathcal{M}}/\{t, f\}$ is the reduction of $\tilde{\diamond}_{\mathcal{M}}$ to $\{t, f\}$.

PROPOSITION 5. *Every semi-classical matrix $\mathcal{M}$ for $\mathcal{L}$ is $\neg$-contained in classical logic. Moreover, $\mathbf{F}_{\mathcal{M}}$ is the unique bivalent $\neg$-interpretation, such that $\mathbf{L}_{\mathcal{M}}$ is $\mathbf{F}_{\mathcal{M}}$-contained in classical logic.*

Proof. Suppose that $\mathcal{M}$ is semi-classical. Let $\varphi_1, \ldots, \varphi_n, \psi \in \mathcal{W}_{\mathcal{L}}$, such that $\varphi_1, \ldots, \varphi_n \not\vdash_{\mathcal{M}_{\mathbf{F}_{\mathcal{M}}}} \psi$. Then there is some $\mathcal{M}_{\mathbf{F}_{\mathcal{M}}}$-valuation ν, such that $\nu(\varphi_i) = t$ for all $1 \leq i \leq n$ and $\nu(\psi) = f$. Since $\mathcal{M}$ is classically closed, ν is also an $\mathcal{M}$-valuation, and so $\varphi_1, \ldots, \varphi_n \not\vdash_{\mathcal{M}} \psi$. Hence, $\mathbf{L}_{\mathcal{M}}$ is $\mathbf{F}_{\mathcal{M}}$-contained in classical logic. Suppose for contradiction that there is some $\mathbf{F} \neq \mathbf{F}_{\mathcal{M}}$, such that $\mathbf{L}_{\mathcal{M}}$ is also $\mathbf{F}$-contained in classical logic. Then there is some n-ary connective $\diamond$ of $\mathcal{L}$, such that $\tilde{\diamond}/\{t, f\} = \mathbf{F}_{\mathcal{M}}(\diamond) \neq \mathbf{F}(\diamond)$. Hence, there are some $a_1, \ldots, a_n \in \{t, f\}$, such that $\tilde{\diamond}(a_1, \ldots, a_n) \neq \mathbf{F}(\diamond)(a_1, \ldots, a_n)$. Because $\mathbf{F}$ and $\mathbf{F}_{\mathcal{M}}$ are both bivalent $\neg$-interpretations, we may assume, without loss of generality, that $\mathbf{F}(\diamond)(a_1, \ldots, a_n) = t$ and $\tilde{\diamond}(a_1, \ldots, a_n) = f$. Next, for $i = 1, \ldots, n$ define $\varphi_i = p$ if $a_i = t$ and $\varphi_i = \neg p$ otherwise.

Since $\mathcal{M}$ is semi-classical, for every $a \in \mathcal{D}$ different from t it holds that $\neg a \in \mathcal{D}$. Hence, $p, \diamond(\varphi_1, \ldots, \varphi_n) \vdash_{\mathcal{M}} \neg p$, while $p, \diamond(\varphi_1, \ldots, \varphi_n) \nvdash_{\mathcal{M}_{\mathbf{F}}} \neg p$, in contradiction to the $\mathbf{F}$-containment of $\mathbf{L}_{\mathcal{M}}$ in classical logic. $\square$

Example 3. It is important to note that the condition of being classically closed cannot be dropped from the definition of a semi-classical matrix, even in case the matrix is paraconsistent. To show this, we present an example of a proto-classical matrix which is paraconsistent and $\neg$-contained in classical logic, satisfies the condition $\neg f = t$, and yet it is not classically closed. For this let $\mathcal{L} = \{\neg, \star\}$, and consider the matrix $\mathcal{M}_\star = \langle\{t, f, \top\}, \{t, \top\}, \mathcal{O}\rangle$ for $\mathcal{L}$, in which $\neg t = f$, $\neg f = t$, $\neg\top = \top$, and $\star t = \star f = \top, \star\top = t$. $\mathcal{M}_\star$ is obviously proto-classical with $\neg f = t$, paraconsistent, and not classically closed. It remains to show that it is $\neg$-contained in classical logic. Let $\mathbf{F}$ be the bivalent $\neg$-interpretation, in which the truth table of $\star$ consists only of t (that is, $\mathbf{F}(\star) = \lambda x.t$). We show that $\mathbf{L}_{\mathcal{M}_\star}$ is $\mathbf{F}$-contained in classical logic. Assume otherwise. Then there is a finite theory Γ and a formula ψ, such that $\Gamma \vdash_{\mathcal{M}_\star} \psi$ but $\Gamma \nvdash_{\mathcal{M}_{\mathbf{F}}} \psi$. Choose such Γ and ψ which are of minimal total length. Then the following holds:

- Γ does not contain a formula of the form $\neg^k\theta$ for $k > 1$. To see this, suppose for contradiction that $\neg^k\theta \in \Gamma$. Since $\neg^k\theta$ is equivalent to $\neg^{k-2}\theta$ in both $\mathcal{M}_\star$ and $\mathcal{M}_{\mathbf{F}}$, also $(\Gamma - \{\neg^k\theta\}) \cup \{\neg^{k-2}\theta\} \vdash_{\mathcal{M}_\star} \psi$ and $(\Gamma - \{\neg^k\theta\}) \cup \{\neg^{k-2}\theta\} \nvdash_{\mathcal{M}_{\mathbf{F}}} \psi$. This contradicts the minimality of Γ, ψ.
- Γ does not contain a formula of the form $\neg\star\theta$, since otherwise $\Gamma \vdash_{\mathcal{M}_{\mathbf{F}}} \phi$ for *every* ϕ, contradicting $\Gamma \nvdash_{\mathcal{M}_{\mathbf{F}}} \psi$.
- Γ contains no formula of the form $\star\theta$, since otherwise $\Gamma - \{\star\phi\} \vdash_{\mathcal{M}_\star} \psi$ and $\Gamma - \{\star\theta\} \nvdash_{\mathcal{M}_{\mathbf{F}}} \psi$, contradicting the minimality of $\mathcal{T}, \psi$.

Hence, it must be the case that $\Gamma \subseteq \{p, \neg p\}$. Since $\Gamma \nvdash_{\mathcal{M}_{\mathbf{F}}} \psi$, $\Gamma = \{p\}$ or $\Gamma = \{\neg p\}$. In addition, the following must hold for ψ:

- ψ is not of the form $\neg^k\phi$ for $k > 1$ (otherwise, like in the case with Γ, we could take $\neg^{k-2}\phi$ instead, contradicting the minimality of Γ, ψ).
- ψ is not of the form $\star\phi$ (since otherwise for any Δ, $\Delta \vdash_{\mathcal{M}_{\mathbf{F}}} \psi$).
- ψ is not a literal (since otherwise either $\Gamma \nvdash_{\mathcal{M}_\star} \psi$, or $\Gamma \vdash_{\mathcal{M}_{\mathbf{F}}} \psi$).

Hence, ψ must be of the form $\neg \star \phi$, and we may assume that ϕ is over $\{p\}$. Now, define for every formula θ over $\{p\}$ a unary truth-function $\mathbf{T}_\theta$ as follows:

$$\mathbf{T}_p(a) = a \quad \mathbf{T}_{\neg\phi}(a) = \tilde{\neg}(\mathbf{T}_\phi(a)) \quad \mathbf{T}_{\star\phi}(a) = \tilde{\star}(\mathbf{T}_\phi(a))$$

It is easy to show by induction on θ:

$$\text{For every } \mathcal{M}_\star\text{-valuation } \nu\text{: } \nu(\theta) = \mathbf{T}_\theta(\nu(p)). \tag{1}$$

We now prove the following claims:

$$\text{If } \mathbf{T}_\theta(\top) \neq \top \text{ then for } a \in \{t, f\},\ \mathbf{T}_\theta(a) = \top \tag{2}$$

$$\text{If } \mathbf{T}_\theta(\top) = \top \text{ then for } a \in \{t, f\},\ \mathbf{T}_\theta(\top) \neq \top \tag{3}$$

We prove (2) and (3) simultaneously by induction on θ. For $\theta = p$, $\mathbf{T}_p(\top) = \top$, and indeed for $a \in \{t, f\}$, $\mathbf{T}_p(a) = a \neq \top$. Now, let $\theta = \neg\delta$. Suppose that $\mathbf{T}_{\neg\delta}(\top) \neq \top$. Then also $\mathbf{T}_\delta(\top) \neq \top$, and by the induction hypothesis, for $a \in \{t, f\}$: $\mathbf{T}_\delta(a) = \top$, hence also $\mathbf{T}_{\neg\delta}(a) = \top$. Next, suppose that $\mathbf{T}_{\neg\delta}(\top) = \top$. Then also $\mathbf{T}_\delta(\top) = \top$, and by the induction hypothesis, for $a \in \{t, f\}$: $\mathbf{T}_\delta(a) \neq \top$, hence also $\mathbf{T}_{\neg\delta}(a) \neq \top$. Finally, let $\theta = \star\delta$. Suppose that $\mathbf{T}_{\star\delta}(\top) \neq \top$. Then $\mathbf{T}_\delta(\top) = \top$, and by the induction hypothesis, for $a \in \{t, f\}$: $\mathbf{T}_\delta(a) \neq \top$, hence also $\mathbf{T}_{\star\delta}(a) = \top$. Now, suppose that $\mathbf{T}_{\star\delta}(\top) = \top$. Then $\mathbf{T}_\delta(\top) \neq \top$, and by the induction hypothesis, for $a \in \{t, f\}$: $\mathbf{T}_\delta(a) = \top$, hence $\mathbf{T}_{\star\delta}(a) \neq \top$.

Finally, suppose without loss of generality that $\Gamma = \{p\}$. We show that $p \nvdash_{\mathcal{M}} \neg \star \phi$ for any ϕ over $\{p\}$. Let ϕ be such formula. If $\mathbf{T}_\phi(\top) = \top$, let ν be an $\mathcal{M}$-valuation such that $\nu(p) = \top$. Then by (1) above, $\nu(\phi) = \top$ and so $\nu(\neg \star \phi) = f$. Hence, $p \nvdash_{\mathcal{M}} \neg \star \phi$. Otherwise, $\mathbf{T}_\phi(\top) \neq \top$. By (2) above, $\mathbf{T}_\phi(t) = \top$. Let ν be an $\mathcal{M}$-valuation such that $\nu(p) = t$. Then by (1) above, $\nu(\phi) = \top$, and so again $\nu(\neg \star \phi) = f$ and $p \nvdash_{\mathcal{M}} \neg \star \phi$. Hence, in all cases $\Gamma \nvdash_{\mathcal{M}} \neg \star \phi$, in contradiction to our assumption that $\Gamma \vdash_{\mathcal{M}} \psi$. □

3.2. Reasonably Strong Languages

In order for a connective $\neg$ of a logic $\mathbf{L}$ to be entitled to the name "negation", it is necessary that $\mathbf{L}$ would be $\neg$-contained in classical logic. Hence, in what follows we concentrate only on such logics. This implies that in all of them $\neg$ has the following basic properties of a negation:

Proposition 6. *Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic that is $\neg$-contained in classical logic. Then for every formula ψ: $\psi \nvdash \neg\psi$ and $\neg\psi \nvdash \psi$.* [5]

As we said in the introduction, having a connective $\neg$ such that $\mathbf{L}$ is $\neg$-contained in classical logic is of course just the minimal demand from a

[5] A negation satisfying this property is called 'weak negation' in [5].

reasonable language for a paraconsistent logic. In addition to negation, a useful paraconsistent logic should provide other useful connectives, and an ideal one should in fact provide natural counterparts for all classical connectives (and because of the containment in classical logic, only for them). What is more: the language of any logic with the pretension of being 'ideal' should contain what is called in [1] 'the heart of logic': an *implication* connective which reflects the underlying consequence relation of that logic. Unlike [1], we believe that this means that such a connective should respect the full intuitionistic-classical deduction theorem. This would make it possible to directly reduce *all* inferences from premises in the logic to theoremhood in that logic. In addition, the presence of appropriate counterparts of the classical negation and implication already ensures the existence of a natural counterpart for every classical connective. These considerations lead to the following definition:

Definition 13.

1. A (primitive or defined) binary connective $\supset$ is a *proper implication* for a logic $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$, if the deduction theorem holds for $\supset$ and $\vdash$: For every theory $\mathcal{T}$ in $\mathcal{L}$, $\mathcal{T}, \psi \vdash \varphi$ iff $\mathcal{T} \vdash \psi \supset \varphi$.

2. A $\neg$-paraconsistent logic $\mathbf{L}$ is *normal*, if it is $\neg$-contained in classical logic and has a proper implication.

By the definition of a bivalent $\neg$-interpretation $\mathbf{F}$, $\mathbf{F}(\neg)$ is the classical truth-table. The next proposition shows that this is the case also for a proper implication.

Proposition 7. *Let* $\mathbf{L}$ *be a logic that is* $\mathbf{F}$*-contained in classical logic for some* $\mathbf{F}$*. If* $\supset$ *is a proper implication for* $\mathbf{L}$*, then* $\mathbf{F}(\supset)$ *is the classical interpretation for implication.*

Proof. Let $\mathbf{F}$ be a bivalent $\neg$-interpretation such that $\mathbf{L}$ is $\mathbf{F}$-contained in classical logic. Since $p \ \vdash_{\mathbf{L}} \ p$, also $\vdash_{\mathbf{L}} p \supset p$. Hence, $\vdash_{\mathcal{M}_{\mathbf{F}}} p \supset p$, and so $\mathcal{M}_{\mathbf{F}}$ must satisfy: $t \supset t = f \supset f = t$. Next, $p \vdash_{\mathbf{L}} q \supset q$, and since $\supset$ is a proper implication, $\vdash_{\mathbf{L}} p \supset (q \supset q)$. Hence also $\vdash_{\mathcal{M}_{\mathbf{F}}} p \supset (q \supset q)$, and so $\mathcal{M}_{\mathbf{F}}$ must satisfy: $f \supset t = t$. Finally, $p \supset q \vdash_{\mathbf{L}} p \supset q$, and since $\supset$ is a proper implication, $p \supset q, p \vdash_{\mathbf{L}} q$. Hence, also $q \supset q, p \vdash_{\mathcal{M}_{\mathbf{F}}} q$, and so $\mathcal{M}_{\mathbf{F}}$ must satisfy: $t \supset f = f$ (otherwise $\nu(p) = t, \nu(q) = f$ would be a counterexample). □

Note 4. The last proposition implies that for any normal logic $\mathbf{L}$ which is $\mathbf{F}$-contained in classical logic, $\mathbf{F}(\neg)$ and $\mathbf{F}(\supset)$ form a functionally complete set (i.e, any two-valued function is definable in terms of them). This shows the adequacy of the expressive power of normal logics.

Clearly, not all paraconsistent logics have a proper implication, even if they are $\neg$-contained in classical logic. The next proposition exhibits a particularly famous paraconsistent logic that lacks such an implication.

PROPOSITION 8. *Priest's three-valued logic LP [30, 31] does not have a proper implication.*

Proof. Recall that LP is induced by $\mathsf{LP} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}\} \rangle$, where the interpretations in this matrix of the primitive connectives are given in Example 1. Let $\leq_k$ be the partial order on $\{t, f, \top\}$, in which $\top$ is the $\leq_k$-maximal element and t, f are the (incomparable) $\leq_k$-minimal elements.[6] By induction on the structure of formulas in the language of $\{\neg, \wedge, \vee\}$, it is easy to verify that all the primitive or defined connectives in this language are $\leq_k$-monotonic: for every n-ary connective $\diamond$ and valuations ν, μ, if $\nu(\psi_i) \leq_k \mu(\psi_i)$ for every $1 \leq i \leq n$, then also $\nu(\diamond(\psi_1, \ldots, \psi_n)) \leq_k \mu(\diamond(\psi_1, \ldots, \psi_n))$. Now, suppose for contradiction that $\supset$ is a definable proper implication for LP. Then $\supset$ is $\leq_k$-monotonic. Since LP is semi-classical, $\tilde{\supset}(f, f) = t$ by Propositions 5 and 7. This and the $\leq_k$-monotonicity of $\supset$ imply that $\tilde{\supset}(\top, f) \in \mathcal{D}$. It follows that $p, p \supset q \not\vdash_{LP} q$ (because $\nu(p) = \top, \nu(q) = f$ provides a counterexample). This contradicts the fact that $\supset$ is a proper implication for LP (indeed, the fact that $p \supset q \vdash p \supset q$ implies that $p, p \supset q \vdash q$ whenever $\supset$ is a proper implication for $\vdash$). $\square$

3.3. Maximal Paraconsistency

3.3.1. Absolute Maximal Paraconsistency

The notion of *absolute* paraconsistency was first proposed in [5]. In contrast to the standard notions used in the literature (see the next subsubsection), this notion is not defined with respect to any particular logic.

DEFINITION 14. Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a $\neg$-paraconsistent logic

- We say that $\mathbf{L}$ is *maximally paraconsistent*, if every extension of $\mathbf{L}$ (in the sense of Definition 3) whose set of theorems *properly includes* that of $\mathbf{L}$, is not $\neg$-paraconsistent.
- A paraconsistent logic $\mathbf{L}$ is *strongly maximal*, if every proper extension of $\mathbf{L}$ (in the sense of Definition 3) is not $\neg$-paraconsistent.

[6] This order is known as the *knowledge* or the *information* order on $\{t, f, \top\}$; see [2, 11].

Clearly, strong maximality implies maximality. In [3] we gave an example that demonstrates that the converse does not hold.

It should be noted that one can easily construct a strongly maximal n-valued paraconsistent logic by considering a language $\mathcal{L}$ expressive enough to include a (primitive or defined) constant for each of the n values:

PROPOSITION 9. *Any logic $\mathbf{L}_{\mathcal{M}}$ of an n-valued matrix $\mathcal{M}$ for a language $\mathcal{L}$ in which all the n values are definable, is maximal in the strongest possible sense: it has no non-trivial extensions.*

Proof. Suppose that all the truth-values in $\mathcal{M}$ are definable in $\mathcal{L}$. So we may assume without loss of generality that for every truth-value of $\mathcal{M}$ there is a corresponding propositional constant in the language. By using these constants we can find for any rule which is not valid in $\mathcal{M}$ an instance $\psi_1, \ldots, \psi_n \vdash \varphi$, consisting of variable-free sentences, such that $\vdash_{\mathcal{M}} \psi_1, \ \ldots, \ \vdash_{\mathcal{M}} \psi_n$ and $\varphi \vdash_{\mathcal{M}} \sigma$ for every σ (the latter holds because φ has no model in $\mathcal{M}$). It follows that by adding such a rule to $\mathbf{L}_{\mathcal{M}}$ we can derive any formula in $\mathcal{W}_{\mathcal{L}}$. $\square$

COROLLARY 2. *Let $\mathbf{L}_{\mathcal{M}}$ be an n-valued paraconsistent logic, the language of which is functionally complete for $\mathcal{M}$.*[7] *Then $\mathbf{L}_{\mathcal{M}}$ is strongly maximal.*

Proposition 9 and Corollary 2 show the importance of demanding containment in classical logic, because the logics to which these results apply usually fail to have this property.

3.3.2. Maximality Relative to Classical Logic

The notion of maximality relative to classical logic is widely used in the literature (see, e.g., [15, 18, 25, 29]). However, as explained in Subsection 3.1, it has been based on a rather vague notion of containment in classical logic. Using Definition 9, we are now able to provide a more precise definition.

DEFINITION 15. Let $\mathbf{F}$ be a bivalent $\neg$-interpretation for a language $\mathcal{L}$ with a unary connective $\neg$.

- An $\mathcal{L}$-formula ψ is a *classical* $\mathbf{F}$*-tautology*, if ψ is satisfied by every two-valued valuation which respects all the truth-tables (of the form $\mathbf{F}(\diamond)$) that $\mathbf{F}$ assigns to the connectives of $\mathcal{L}$.

[7] I.e., every function from $\mathcal{V}^m$ to $\mathcal{V}$ is representable in the language.

- A logic $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ is $\mathbf{F}$-*complete*, if its set of theorems consists of all the classical $\mathbf{F}$-tautologies.
- $\mathbf{L}$ is $\mathbf{F}$-*maximal relative to classical logic*, if the following hold:
  - $\mathbf{L}$ is $\mathbf{F}$-contained in classical logic.
  - If ψ is a classical $\mathbf{F}$-tautology not provable in $\mathbf{L}$, then by adding ψ to $\mathbf{L}$ as a new axiom schema, an $\mathbf{F}$-complete logic is obtained.
- $\mathbf{L}$ is $\mathbf{F}$-*maximally paraconsistent relative to classical logic*, if it is $\neg$-paraconsistent and $\mathbf{F}$-maximal relative to classical logic.

DEFINITION 16. Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic for a language with a unary connective $\neg$. $\mathbf{L}$ is *maximally paraconsistent relative to classical logic* if there exists a bivalent $\neg$-interpretation $\mathbf{F}$ such that $\mathbf{L}$ is $\mathbf{F}$-maximally paraconsistent relative to classical logic.

EXAMPLE 4. Note that since Definition 16 is based only on extending the underlying set of *theorems* of a logic, *any* $\mathbf{F}$-complete paraconsistent logic is trivially $\mathbf{F}$-maximal with respect to classical logic. Here are two examples:

1. Define the logic $\mathbf{L_s}$ in some standard language $\mathcal{L}_{cl}$ of classical logic as follows: $\mathcal{T} \vdash_{\mathbf{L_s}} \psi$ if either $\psi \in \mathcal{T}$, or ψ is a classical $\mathcal{L}_{cl}$-tautology (it is easy to verify that $\mathbf{L_s}$ is indeed a logic). Then $\mathbf{L_s}$ is obviously $\neg$-contained in classical logic, $\neg$-paraconsistent (since $p, \neg p \nvdash_{\mathbf{L_s}} q$), and it is trivially maximal relative to classical logic. Hence, $\mathbf{L_s}$ is maximally paraconsistent relative to classical logic.

2. A less trivial example is given by Priest's paraconsistent logic LP (Example 1). This easily follows from Proposition 2.

A seemingly natural extension of the notion of maximal paraconsistency relative to classical logic would result if again we consider the *consequence relation* of the underlying logic, rather than just its set of theorems (analogously to the notion of strong maximality defined in the previous subsection):

DEFINITION 17. Let $\mathbf{F}$ be a bivalent $\neg$-interpretation for a language $\mathcal{L}$ with a unary connective $\neg$. A logic $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ is *strongly* $\mathbf{F}$-*maximal relative to classical logic*, if the following conditions hold:

- $\mathbf{L}$ is $\mathbf{F}$-contained in classical logic.
- Let Γ be a finite set of $\mathcal{L}$-formulas and ψ an $\mathcal{L}$-formula, such that $\Gamma \nvdash_{\mathbf{L}} \psi$, but $\Gamma \vdash_{\mathcal{M}_{\mathbf{F}}} \psi$. Then every extension of $\mathbf{L}$ by the rule Γ/ψ (in the sense of Definition 3) results in the logic $\mathbf{L_F} = \langle \mathcal{L}, \vdash_{\mathcal{M}_{\mathbf{F}}} \rangle$.

Definition 18. Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a paraconsistent logic for a language with a unary connective $\neg$. $\mathbf{L}$ is *strongly maximal relative to classical logic* if there exists a bivalent $\neg$-interpretation $\mathbf{F}$ such that $\mathbf{L}$ is strongly $\mathbf{F}$-maximal relative to classical logic.

Example 5. In [3] we have shown that LP is a strongly maximal paraconsistent logic (in the absolute sense of Definition 14), and in Example 4 we have shown that it is also maximal relative to classical logic. Nevertheless, LP is *not* strongly maximal relative to classical logic. To see this, assume otherwise. Then LP is strongly $\mathbf{F}$-maximal relative to classical logic for some bivalent $\neg$-interpretation $\mathbf{F}$. Since LP is semi-classical, by Proposition 5, $\mathbf{F} = \mathsf{LP}/\{t, f\}$, and so $\mathcal{M}_{\mathbf{F}}$ is the classical matrix. Now, let $\mathcal{T} \vdash_{LP+} \varphi$ if either $\mathcal{T} \vdash_{LP} \varphi$, or $\mathcal{T}$ is classically inconsistent. First, we show that $\langle \mathcal{L}, \vdash_{LP+} \rangle$ is a logic, where $\mathcal{L}$ is the language of LP. That $\vdash_{LP+}$ is finitary follows from the fact that so are LP and classical logic. Reflexivity and non-triviality of $\vdash_{LP+}$ follow from the reflexivity and non-triviality of $\vdash_{LP}$. For monotonicity, let $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{T} \vdash_{LP+} \psi$. If $\mathcal{T} \vdash_{LP} \psi$, then $\mathcal{T}' \vdash_{LP+} \psi$ by the monotonicity of $\vdash_{LP}$. Otherwise, $\mathcal{T}$ is classically inconsistent. Hence so is $\mathcal{T}'$, and so again $\mathcal{T}' \vdash_{LP+} \psi$. Finally, for transitivity, assume that $\mathcal{T} \vdash_{LP+} \varphi$ and $\mathcal{T}', \varphi \vdash_{LP+} \psi$. We show that $\mathcal{T}, \mathcal{T}' \vdash_{LP+} \psi$ by considering the possible cases:

1. $\mathcal{T} \vdash_{LP} \varphi$ and $\mathcal{T}', \varphi \vdash_{LP} \psi$. Then $\mathcal{T}, \mathcal{T}' \vdash_{LP+} \psi$ by transitivity of $\vdash_{LP}$.
2. $\mathcal{T} \vdash_{LP} \varphi$ and $\mathcal{T}' \cup \{\varphi\}$ is classically inconsistent. Since $\vdash_{LP} \subset \vdash_{CL}$ (where $\vdash_{CL}$ denotes the tcr of classical logic), $\mathcal{T} \vdash_{CL} \varphi$. This implies that $\mathcal{T} \cup \mathcal{T}'$ is classically inconsistent, and so $\mathcal{T} \cup \mathcal{T}' \vdash_{LP+} \psi$.
3. $\mathcal{T}$ is classically inconsistent. Then so is $\mathcal{T} \cup \mathcal{T}'$, and so $\mathcal{T} \cup \mathcal{T}' \vdash_{LP+} \psi$.

Next, we note that $\neg p, p \vee q \not\vdash_{LP} q$, and so also $\neg p, p \vee q \not\vdash_{LP+} q$. However, $\neg p, p \vee q \vdash_{\mathcal{M}_{\mathbf{F}}} q$ (recall that $\mathcal{M}_{\mathbf{F}}$ is the classical matrix). Hence, $\vdash_{LP+} \neq \vdash_{\mathcal{M}_{\mathbf{F}}}$. It follows that the extension of LP by the rule $\{p, \neg p\}/q$ (which is valid in $\mathcal{M}_{\mathbf{F}}$) is properly included in $\vdash_{\mathcal{M}_{\mathbf{F}}}$. On the other hand, this extension properly extends $\vdash_{LP}$, since $p, \neg p \not\vdash_{LP} q$. This contradicts our assumption about the strong $\mathbf{F}$-maximality of LP.

The last example indicates that strong maximality relative to classical logic might be a too strong demand. Indeed, we do not know if paraconsistent logics with this property actually exist. In any case, our next proposition shows that it is impossible to have it in reasonable languages. To show this, we need first the following lemma.

LEMMA 1. *Let* $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ *be a paraconsistent logic, and let* $\mathbf{F}$ *be a bivalent* $\neg$*-interpretation for* $\mathcal{L}$*. If* $\mathbf{L}$ *is strongly* $\mathbf{F}$*-maximal relative to classical logic then it is* $\mathbf{F}$*-complete.*

Proof. The rule $\{p, \neg p\}/q$ is obviously valid in $\mathcal{M}_{\mathbf{F}}$. Since $\mathbf{L}$ is paraconsistent, this rule is not valid in $\mathbf{L}$. Let $\mathbf{L}'$ be the extension of $\mathbf{L}$ by this rule. By the strong $\mathbf{F}$-maximality of $\mathbf{L}$ relative to classical logic, $\mathbf{L}' = \mathcal{M}_{\mathbf{F}}$, and so $\mathbf{L}'$ is $\mathbf{F}$-complete. It remains to show that $\mathbf{L}'$ has the same set of theorems (i.e., the same valid formulas) as $\mathbf{L}$. For this note that since $\mathbf{L}$ is $\neg$-contained in classical logic, there is no formula ψ such that both ψ and $\neg\psi$ are valid in $\mathbf{L}$. It follows that the rule $\{p, \neg p\}/q$ is *admissible* in $\mathbf{L}$. This easily entails that its addition to $\mathbf{L}$ indeed does not change the set of valid formulas (it changes only the consequence relation). □

PROPOSITION 10. *No paraconsistent normal logic is strongly maximal relative to classical logic.*

Proof. Suppose for contradiction that $\mathbf{F}$ is a $\neg$-interpretation for the language of a paraconsistent normal logic $\mathbf{L}$, and that $\mathbf{L}$ is strongly $\mathbf{F}$-maximal relative to classical logic. Then $\mathbf{L}$ is $\mathbf{F}$-contained in classical logic, and since $\supset$ is a proper implication for $\mathbf{L}$, by Proposition 7, $\mathbf{F}(\supset)$ is the classical implication. Hence, $\vdash_{\mathcal{M}_{\mathbf{F}}} p \supset (\neg p \supset q)$. By Lemma 1, this implies that $\vdash_{\mathbf{L}} p \supset (\neg p \supset q)$. But then $p, \neg p \vdash_{\mathbf{L}} q$, since $\supset$ is a proper implication for $\mathbf{L}$. This contradicts to the paraconsistency of $\mathbf{L}$. □

3.4. Ideal Paraconsistent Logics

Proposition 10 shows that there is no hope for achieving strong maximal paraconsistency relative to classical logic for paraconsistent logics in reasonable languages, expressive enough to capture proper implication. So in the case of maximality relative to classical logic we shall be satisfied with the weaker notion, that is based on extending the *set of theorems* of the underlying logic. The maximality of its full *consequence relation* will still be demanded in the case of absolute maximality. These considerations lead to the following definition of what we take to be an 'ideal paraconsistent logic':

DEFINITION 19. A $\neg$-paraconsistent logic $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ is called *ideal*, if it is normal (i.e., $\neg$-contained in classical logic and has a proper implication), maximally paraconsistent relative to classical logic, and strongly maximal.

4. Ideal Paraconsistent Three-Valued Logics

Three-valued matrices provide the most popular framework for reasoning with contradictory data. The major reason for this is that they provide the simplest semantic way of defining paraconsistent logics (cf. Corollary 1). In this section, we investigate ideal logics in this framework. We start by characterizing the three-valued paraconsistent matrices.

PROPOSITION 11. *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a three-valued paraconsistent matrix that is $\neg$-contained in classical logic. Then $\mathcal{M}$ is proto-classical, and it is isomorphic to a matrix $\mathcal{M}' = \langle \{t, f, \top\}, \{t, \top\}, \mathcal{O} \rangle$, in which $\neg t = f$, $\neg f = t$ and $\neg \top \in \{t, \top\}$.*

Proof. This is immediate from Propositions 3 and 4. □

In the rest of this section we assume that any matrix which satisfies the conditions of Proposition 11 has the form described there.

PROPOSITION 12. *Let $\mathcal{M}$ be a three-valued paraconsistent matrix such that $\mathbf{L}_{\mathcal{M}}$ is normal. Then $\mathcal{M}$ is semi-classical.*

Proof. Since $\mathbf{L}_{\mathcal{M}}$ is normal, it is $\mathbf{F}$-contained in classical logic for some bivalent $\neg$-interpretation $\mathbf{F}$. Hence, by Proposition 11, it remains to show that $\mathcal{M}$ is classically closed. For this define: $\psi \curlyvee \varphi = (\psi \supset \varphi) \supset \varphi$. From Proposition 7 it easily follows that $\mathbf{F}(\curlyvee)$ is the classical disjunction. Hence:

$$(\star) \quad \text{If } \mathcal{T} \vdash_{\mathcal{M}_\mathbf{F}} \psi \curlyvee \varphi \text{ and } \mathcal{T} \vdash_{\mathcal{M}_\mathbf{F}} \neg\psi \curlyvee \varphi \text{ then } \mathcal{T} \vdash_{\mathcal{M}_\mathbf{F}} \varphi.$$

Since $\supset$ is a proper implication for $\mathbf{L}_{\mathcal{M}}$, we also have:

$$(\star\star) \quad \psi \vdash_{\mathcal{M}} \psi \curlyvee \varphi \text{ and } \varphi \vdash_{\mathcal{M}} \psi \curlyvee \varphi.$$

Next, suppose for contradiction that there is some n-ary connective $\diamond$ and $a_1, \ldots, a_n \in \{t, f\}$, such that $\tilde{\diamond}(a_1, \ldots, a_n) \notin \{t, f\}$. Then $\tilde{\diamond}(a_1, \ldots, a_n) = \top$. For $i = 1, \ldots, n$ define: $\varphi_i = p_i$ if $a_i = t$ and $\varphi_i = \neg p_i$ if $a_i = f$. Let $\psi = \psi_1 \curlyvee \ldots \curlyvee \psi_n$, where $\psi_i = \neg p_i$ if $\varphi_i = p_i$, and $\psi_i = p_i$ if $\varphi_i = \neg p_i$. Then every $\mathcal{M}$-model ν of $\{\varphi_1, \ldots, \varphi_n\}$ is also an $\mathcal{M}$-model of $\diamond(p_1, \ldots, p_n) \curlyvee \psi$ and $\neg \diamond (p_1, \ldots, p_n) \curlyvee \psi$ (This is obvious by $(\star\star)$ in case $\nu(p_i) = \top$ for some i, and follows from $(\star\star)$ and our assumption about $\tilde{\diamond}(a_1, \ldots, a_n)$ otherwise). Hence, $\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{M}} \diamond(p_1, \ldots, p_n) \curlyvee \psi$, and $\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{M}} \neg\diamond(p_1, \ldots, p_n) \curlyvee \psi$. Since $\mathbf{L}_{\mathcal{M}}$ is $\mathbf{F}$-contained in classical logic, this in turn implies that $\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{M}_\mathbf{F}} \diamond(p_1, \ldots, p_n) \curlyvee \psi$, and $\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{M}_\mathbf{F}} \neg \diamond (p_1, \ldots, p_n) \curlyvee \psi$. Now, by $(\star)$ above, $\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{M}_\mathbf{F}} \psi$. By definition of ψ, this contradicts the fact that $\mathbf{F}(\curlyvee)$ is the classical disjunction. □

The next theorem establishes that for three-valued logics that are contained in classical logic, the absolute notion of strong maximality implies maximality relative to classical logic.

THEOREM 1. *Let $\mathcal{M}$ be a three-valued matrix that is $\neg$-contained in classical logic. If $\mathbf{L}_{\mathcal{M}}$ is a strongly maximal paraconsistent logic, then it is also maximally paraconsistent relative to classical logic.*

Proof. For the proof, we first need the following lemma.

LEMMA 2. *Let $\mathcal{M}$ be a paraconsistent three-valued matrix such that $\mathbf{L}_{\mathcal{M}}$ is strongly maximal. Suppose that there is some bivalent $\neg$-interpretation $\mathbf{F}$, such that $\mathbf{L}_{\mathcal{M}}$ is $\mathbf{F}$-contained in classical logic, but $\mathbf{L}_{\mathcal{M}}$ is not $\mathbf{F}$-maximal relative to classical logic. Then $\mathcal{M}$ is classically closed.*

Proof of the lemma. Suppose that $\mathbf{L}_{\mathcal{M}}$ is $\mathbf{F}$-contained in classical logic, but is not $\mathbf{F}$-maximal relative to classical logic. Then there is some classical $\mathbf{F}$-tautology ψ_0 not provable in $\mathbf{L}_{\mathcal{M}}$, such that adding it as an axiom to $\mathbf{L}_{\mathcal{M}}$ results in a logic $\mathbf{L}^*$ that is *not* $\mathbf{F}$-complete. Let σ be some classical $\mathbf{F}$-tautology not provable in $\mathbf{L}^*$. Let $\mathcal{S}^*$ be the set of all substitution instances of ψ_0. Then for every theory $\mathcal{T}$ we have that $\mathcal{T} \vdash_{\mathbf{L}^*} \phi$ iff $\mathcal{T}, \mathcal{S}^* \vdash_{\mathcal{M}} \phi$. Since $\mathbf{L}_{\mathcal{M}}$ is strongly maximal, this in particular entails:

$$\mathcal{S}^*, \varphi, \neg\varphi \vdash_{\mathcal{M}} \phi \text{ for every } \varphi, \phi. \tag{4}$$

Since $\nvdash_{\mathbf{L}^*} \sigma$, also $\mathcal{S}^* \nvdash_{\mathcal{M}} \sigma$. Hence, there is a valuation $\nu \in \Lambda_{\mathcal{M}}$ which is a model of $\mathcal{S}^*$, but $\nu(\sigma) = f$.

Next, we show that there is no formula ψ for which $\nu(\psi) = \top$. Assume for contradiction that this is not the case for some ψ. Since ν is a model of $\mathcal{S}^*$, it is also a model of $\mathcal{S}^* \cup \{\psi, \neg\psi\}$, and so it is a model of σ by (4) above. This contradicts the fact that $\nu(\sigma) = f$. It follows that $\nu(\psi) \in \{t, f\}$ for all ψ. We show that this implies that all operations of $\mathcal{M}$ are classically closed. Let $\diamond$ be some n-ary connective of $\mathcal{L}$ and let $a_1, \ldots, a_n \in \{t, f\}$. For $i = 1, \ldots, n$, define $\varphi_i = p_i$ if $\nu(p_i) = a_i$, and $\varphi_i = \neg p_i$ otherwise. Thus $\nu(\varphi_i) = a_i$, and $\tilde{\diamond}(a_1, \ldots, a_n) = \tilde{\diamond}(\nu(\varphi_1), \ldots, \nu(\varphi_n)) = \nu(\diamond(\varphi_1, \ldots, \varphi_n)) \in \{t, f\}$. $\square$

Back to the proof of Theorem 1. Let $\mathcal{M}$ be a paraconsistent three-valued matrix that is $\neg$-contained in classical logic, and such that $\mathbf{L}_{\mathcal{M}}$ is strongly maximal. Then in particular$\mathbf{L}_{\mathcal{M}}$ is $\mathbf{F}$-contained in classical logic for some $\mathbf{F}$. If $\mathbf{L}_{\mathcal{M}}$ is $\mathbf{F}$–maximal relative to classical logic, then we are done. Otherwise, $\mathcal{M}$ is semi-classical by Lemma 2. Hence, Proposition 5 implies that $\mathbf{F} = \mathbf{F}_{\mathcal{M}}$, and so $\mathbf{L}_{\mathcal{M}}$ is $\mathbf{F}_{\mathcal{M}}$-contained in classical logic. We end by showing that $\mathbf{L}_{\mathcal{M}}$

is $\mathbf{F}_{\mathcal{M}}$-maximal relative to classical logic. The proof of this is very similar to the proof of Lemma 2: Let ψ' be a classical $\mathbf{F}_{\mathcal{M}}$-tautology not provable in $\mathbf{L}_{\mathcal{M}}$ and let $\mathcal{S}'^{*}$ be the set of all of its substitution instances. Let $\mathbf{L}'^{*}$ be the logic obtained by adding ψ' as a new axiom to $\mathbf{L}_{\mathcal{M}}$. Then for every theory $\mathcal{T}$ we have that $\mathcal{T} \vdash_{\mathbf{L}'^{*}} \phi$ iff $\mathcal{T}, \mathcal{S}'^{*} \vdash_{\mathcal{M}} \phi$. In particular, since $\mathcal{M}$ is strongly maximal, Condition (4) holds for $\mathcal{S}'^{*}$. Suppose for contradiction that there is some classical $\mathbf{F}_{\mathcal{M}}$-tautology σ not provable in $\mathbf{L}'^{*}$. Since $\nvdash_{\mathbf{L}'^{*}} \sigma$, also $\mathcal{S}'^{*} \nvdash_{\mathcal{M}} \sigma$. Hence, there is a valuation $\nu \in \Lambda_{\mathcal{M}}$ which is a model of $\mathcal{S}'^{*}$, but $\nu(\sigma) = f$. If there is some ψ, such that $\nu(\psi) = \top$, then since ν is a model of $\mathcal{S}'^{*}$, it is also a model of $\mathcal{S}'^{*} \cup \{\psi, \neg\psi\}$, and so by (4) it is a model of σ, in contradiction to the fact that $\nu(\sigma) = f$. Otherwise, $\nu(\psi) \in \{t, f\}$ for all ψ, and so ν is an $\mathcal{M}_{\mathbf{F}_{\mathcal{M}}}$-valuation, which assigns f to σ. This contradicts the fact that $\vdash_{\mathcal{M}_{\mathbf{F}_{\mathcal{M}}}} \sigma$. Hence, all classical $\mathbf{F}_{\mathcal{M}}$-tautologies are provable in $\mathbf{L}'^{*}$, and so $\mathbf{L}_{\mathcal{M}}$ is $\mathbf{F}_{\mathcal{M}}$-maximal relative to classical logic. $\square$

Now we are ready to prove one of the major results of this paper:

THEOREM 2. *Every normal three-valued paraconsistent logic is ideal.*

Proof. By Theorem 1, it suffices to show that every normal three-valued paraconsistent logic is strongly maximal. So let $\mathcal{M}$ be a three-valued paraconsistent matrix for a language $\mathcal{L}$ with a proper implication, which is $\neg$-contained in classical logic. Let $\langle \mathcal{L}, \vdash \rangle$ be a proper extension of $\mathbf{L}_{\mathcal{M}}$ by some set of rules. Then there is a finite theory Γ and a formula ψ in $\mathcal{L}$, such that $\Gamma \vdash \psi$ but $\Gamma \nvdash_{\mathcal{M}} \psi$. In particular, there is a valuation $\nu \in mod_{\mathcal{M}}(\Gamma)$ such that $\nu(\psi) = f$. Consider the substitution θ, defined for every $p \in \mathsf{Atoms}(\Gamma \cup \{\psi\})$ by:

$$\theta(p) = \begin{cases} q_0 & \text{if } \nu(p) = t, \\ \neg q_0 & \text{if } \nu(p) = f, \\ p_0 & \text{if } \nu(p) = \top, \end{cases}$$

where p_0 and q_0 are two different atoms in $\mathcal{L}$. Note that $\theta(\Gamma)$ and $\theta(\psi)$ contain (at most) the variables p_0, q_0, and that for every valuation $\mu \in \Lambda_{\mathcal{M}}$ where $\mu(p_0) = \top$ and $\mu(q_0) = t$ it holds that $\mu(\theta(\phi)) = \nu(\phi)$ for every formula ϕ such that $\mathsf{Atoms}(\{\phi\}) \subseteq \mathsf{Atoms}(\Gamma \cup \{\psi\})$. Thus:

($\star$) *Any* $\mu \in \Lambda_{\mathcal{M}}$ such that $\mu(p_0) = \top$, $\mu(q_0) = t$ is an $\mathcal{M}$-model of $\theta(\Gamma)$ that does not $\mathcal{M}$-satisfy $\theta(\psi)$.

Now, consider the following two cases:

Case I. There is a formula $\phi(p, q)$ such that for every $\mu \in \Lambda_{\mathcal{M}}$, $\mu(\phi) \neq \top$ if $\mu(p) = \mu(q) = \top$.

In this case, let $\mathsf{tt} = \phi(p_0, p_0) \supset \phi(p_0, p_0)$. Note that $\mu(\mathsf{tt}) = t$ for every $\mu \in \Lambda_{\mathcal{M}}$ such that $\mu(p_0) = \top$. Now, as $\vdash$ is structural, $\Gamma \vdash \psi$ implies that

$$\theta(\Gamma)\ [\mathsf{tt}/q_0] \vdash \theta(\psi)\ [\mathsf{tt}/q_0]. \tag{5}$$

Also, by the property of tt and by ($\star$), any $\mu \in \Lambda_{\mathcal{M}}$ for which $\mu(p_0) = \top$ is a model of $\theta(\Gamma)\ [\mathsf{tt}/q_0]$ but does not $\mathcal{M}$-satisfy $\theta(\psi)\ [\mathsf{tt}/q_0]$. Thus,

- $p_0, \neg p_0 \vdash_{\mathcal{M}} \theta(\gamma)\,[\mathsf{tt}/q_0]$ for every $\gamma \in \Gamma$. As $\langle \mathcal{L}, \vdash \rangle$ is stronger than $\mathbf{L}_{\mathcal{M}}$, this implies that

$$p_0, \neg p_0 \vdash \theta(\gamma)\ [\mathsf{tt}/q_0] \text{ for every } \gamma \in \Gamma. \tag{6}$$

- The set $\{p_0,\ \neg p_0,\ \theta(\psi)[\mathsf{tt}/q_0]\}$ is not $\mathcal{M}$-satisfiable. This implies:

$$p_0, \neg p_0, \theta(\psi)\ [\mathsf{tt}/q_0] \vdash_{\mathcal{M}} q_0$$

  Again, as $\langle \mathcal{L}, \vdash \rangle$ is stronger than $\mathbf{L}_{\mathcal{M}}$, we have that

$$p_0,\ \neg p_0,\ \theta(\psi)\ [\mathsf{tt}/q_0] \vdash q_0. \tag{7}$$

By (5)–(7) $p_0, \neg p_0 \vdash q_0$. Thus $\langle \mathcal{L}, \vdash \rangle$ is not $\neg$-paraconsistent.

Case II. For every formula ϕ in p, q and for every $\mu \in \Lambda_{\mathcal{M}}$, if $\mu(p) = \mu(q) = \top$ then $\mu(\phi) = \top$.
Again, as $\vdash$ is structural, and since $\Gamma \vdash \psi$,

$$\theta(\Gamma)\ [q_0 \supset q_0/q_0] \vdash \theta(\psi)\ [q_0 \supset q_0/q_0]. \tag{8}$$

In addition, ($\star$) above entails that any valuation $\mu \in \Lambda_{\mathcal{M}}$ such that $\mu(p_0) = \top$ and $\mu(q_0) \in \{t, f\}$ is a model of $\theta(\Gamma)\ [q_0 \supset q_0/q_0]$ which is not a model of $\theta(\psi)\ [q_0 \supset q_0/q_0]$. Thus, the only $\mathcal{M}$-model of $\{p_0, \neg p_0, \theta(\psi)\ [q_0 \supset q_0/q_0]\}$ is the one in which both of p_0 and q_0 are assigned the value $\top$. It follows that $p_0, \neg p_0, \theta(\psi)\ [q_0 \supset q_0/q_0] \vdash_{\mathcal{M}} q_0$. Thus,

$$p_0, \neg p_0, \theta(\psi)\ [q_0 \supset q_0/q_0] \vdash q_0. \tag{9}$$

By using ($\star$) again (for $\mu(q_0) \in \{t, f\}$) and the condition of case II (for $\mu(q_0) = \top$), we have:

$$p_0, \neg p_0 \vdash \theta(\gamma)\ [q_0 \supset q_0/q_0] \text{ for every } \gamma \in \Gamma. \tag{10}$$

Again, by (8)–(10) above, we have that $p_0, \neg p_0 \vdash q_0$, and so $\langle \mathcal{L}, \vdash \rangle$ is not $\neg$-paraconsistent in this case either. $\square$

Since an ideal paraconsistent logic is in particular normal, by Theorem 2 we have:

COROLLARY 3. *A three-valued paraconsistent logic is ideal iff it is normal.*

EXAMPLE 6. Sette's logic P_1 [32] (and all of its fragments containing Sette's negation), the logic PAC [8, 4], J_3 [18], and the 2^{20} three-valued logics considered in [3] (including the 2^{13} LFIs from [15]), are all ideal paraconsistent logics.[8]

NOTE 5. There are exactly sixteen possible proper implications in a three-valued paraconsistent matrix which is $\neg$-contained in classical logic. They are given in the table below (where we denote by '$x \wr y$' that x and y are two optional values):

$\tilde{\supset}$	t	f	$\top$
t	t	f	$t \wr \top$
f	t	t	$t \wr \top$
$\top$	$t \wr \top$	f	$t \wr \top$

It is easy to check that these are all proper implications. To see that they are indeed the only possible options, let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a normal three-valued logic. By Proposition 7, $t \tilde{\supset} t = f \tilde{\supset} f = f \tilde{\supset} f = t$ and $t \tilde{\supset} f = f$. Also, since $\psi \supset \phi \vdash \psi \supset \phi$, we have that $\psi, \psi \supset \phi \vdash \phi$. Thus, $\top \tilde{\supset} f \notin \mathcal{D}$ (otherwise, $\nu(\psi) = \top$, $\nu(\phi) = f$ is a counter-example), and so $\top \tilde{\supset} f = f$. Finally, since $\psi, \phi \vdash \psi$, we have that $\psi \vdash \phi \supset \psi$, thus $\phi \supset \psi \in \mathcal{D}$ whenever $\psi \in \mathcal{D}$. This implies that $\top \tilde{\supset} t \in \{t, \top\}$, and $x \tilde{\supset} \top \in \{t, \top\}$ for every $x \in \{t, f, \top\}$.[9]

Now, since $\tilde{\neg} t = f$, $\tilde{\neg} f = t$ and $\tilde{\neg} \top \in \{t, \top\}$ (Proposition 11), this gives us 32 different[10] three-valued normal logics for the language of $\{\neg, \supset\}$. By Theorem 2, they are all ideal.

NOTE 6. Although all of the ideal logics described above are different, they share some common core that ensures their being ideal. This core can be naturally captured using the tool of *non-deterministic matrices* (Nmatrices), introduced in [6] (and already used in [3] to characterize the core of strong maximality of three-valued paraconsistent logics. The definitions of the various notions used in this note can be found there[11]). Nmatrices are a generalization of the standard semantics of matrices, obtained by relaxing the principle of truth-functionality: the truth-value of a compound formula

[8] In contrast, Priest's LP (see [31] and Example 1) is *not* ideal, since by Proposition 8 it lacks a proper implication, and so it is not normal.

[9] These are exactly the 16 implication connectives of the 8Kb LFIs, shown in [15] to be maximally paraconsistent relative to classical logic.

[10] This is tedious, but not difficult, to show.

[11] See [7] for a detailed presentation of Nmatrices.

is chosen non-deterministically from some set of options. Now, the set of 32 different three-valued normal logics described above is exactly the set of logics induced by the possible determinizations of the following three-valued Nmatrix $\mathcal{M}_I$ (for $\mathcal{L} = \{\neg, \supset\}$):

	$\tilde{\neg}$
t	$\{f\}$
f	$\{t\}$
$\top$	$\{t, \top\}$

$\tilde{\supset}$	t	f	$\top$
t	$\{t\}$	$\{f\}$	$\{t, \top\}$
f	$\{t\}$	$\{t\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$

The paraconsistent Nmatrix $\mathcal{M}_I$ represents therefore the "essence" of what makes the logics in this family ideal paraconsistent logics. Accordingly, for selecting a paraconsistent logic for some application one may start with $\mathbf{L}_{\mathcal{M}_I}$ as a natural basis. The choice of what other logical principles to adopt would depend then on considerations peculiar to the application at hand.

More generally, call a paraconsistent Nmatrix $\mathcal{M}$ *pre-ideal* if for every determinization $\mathcal{M}_d$ of $\mathcal{M}$, $\mathbf{L}_{\mathcal{M}_d}$ is an ideal paraconsistent logic. Pre-ideal Nmatrices thus provide a compact representation of ideal paraconsistent logics, up to the point in which choices based on other considerations should be made. By the above, $\mathcal{M}_I$ is an example of such a pre-ideal paraconsistent Nmatrix. What is more: $\supset$ obviously remains a proper implication in any logic which is induced by some three-valued refinement of some extension of $\mathcal{M}_I$ by new (perhaps non-deterministic) three-valued connectives. Hence, by Theorem 2, any such extension which has only classically closed determinizations is also pre-ideal. A particularly important such a pre-ideal paraconsistent Nmatrix is the Nmatrix $\mathcal{M}_{\mathsf{8Kb}}$ for $\{\neg, \supset, \vee, \wedge\}$ (defined in [3]), which underlies exactly (the $\circ$-free fragments of) the Marcos-Carnielli 2^{13} (ideal) paraconsistent logics mentioned above. This Nmatrix is the extension of $\mathcal{M}_I$ by the following interpretations of $\wedge$ and $\vee$:

$\tilde{\wedge}$	t	f	$\top$
t	$\{t\}$	$\{f\}$	$\{t, \top\}$
f	$\{f\}$	$\{f\}$	$\{f\}$
$\top$	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$

$\tilde{\vee}$	t	f	$\top$
t	$\{t\}$	$\{t\}$	$\{t, \top\}$
f	$\{t\}$	$\{f\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$

As observed in [3], a strongly sound and complete axiomatization for this logic can be obtained by adding to $C_{\min}$ [15] the following $\circ$-free counterparts of the **(a)**-axioms of da Costa [17]:

$$(\mathbf{a}_{\wedge})^* \quad \neg(\psi \wedge \varphi) \supset (\neg\psi \vee \neg\varphi)$$

$$(\mathbf{a}_{\vee})^* \quad \neg(\psi \vee \varphi) \supset \big((\neg\psi \wedge \neg\varphi) \vee (\neg\psi \wedge \psi) \vee (\neg\varphi \wedge \varphi)\big)$$

$$(\mathbf{a}_{\supset})^* \quad \neg(\psi \supset \varphi) \supset \big((\psi \wedge \neg\varphi) \vee (\neg\psi \wedge \psi) \vee (\neg\varphi \wedge \varphi)\big)$$

5. The Finite-Valued Case

The discussion in the previous section raises the question whether all the ideal logics are three-valued. In this section we show that this is far from being the case. In fact, we show that for every $n > 2$ there is an extensive family of n-valued ideal logics, each of which is not equivalent to any k-valued logic with $k < n$.

PROPOSITION 13. *Let $\mathcal{M}$ be a semi-classical, $\neg$-paraconsistent matrix for a language $\mathcal{L}$ which includes a unary connective $\diamond$ such that for some $n > 2$ the following conditions are satisfied:*

1. $p, \neg p \vdash_{\mathcal{M}} \diamond^{n-2} p$,
2. $p, \neg p, \diamond^k p \vdash_{\mathcal{M}} q$, *for* $1 \leq k \leq n-3$,
3. $p, \neg p, \neg\diamond^k p \vdash_{\mathcal{M}} q$, *for* $1 \leq k \leq n-3$.

Then $\mathcal{M}$ has at least n elements, including at least $n-2$ non-designated elements.

Proof. By Proposition 3, there should be at least one element $t \in \mathcal{D}$, such that $f = \tilde{\neg} t \notin \mathcal{D}$ and at least one element $\top \in \mathcal{D}$, such that $\tilde{\neg}\top \in \mathcal{D}$. Let $\bot_k = \tilde{\diamond}^k \top$ for every $1 \leq k \leq n-3$. Then $\bot_k \in \overline{\mathcal{D}}$ for $1 \leq k \leq n-3$ (I.e, $\bot_k \in \mathcal{V} \setminus \mathcal{D}$; otherwise $p, \neg p, \diamond^k p \not\vdash_{\mathcal{M}} q$). Moreover, $\bot_1, \ldots, \bot_{n-3}$ are different from each other, because otherwise we would get that $\tilde{\diamond}^i \top \in \overline{\mathcal{D}}$ for every $i > 0$, and this contradicts the condition that $p, \neg p \vdash_{\mathcal{M}} \diamond^{n-2} p$. It follows that $t, \top, \bot_1, \ldots, \bot_{n-3}$ are all different from each other. Now, by Condition (3) above, for every $1 \leq k \leq n-3$ we have that $\tilde{\neg}\bot_k \in \overline{\mathcal{D}}$ (otherwise the valuation $\mu(p) = \top$ would contradict this rule). On the other hand, since $\mathcal{M}$ is semi-classical, $\tilde{\neg} f \in \mathcal{D}$. Hence, f is different from $\bot_1, \ldots, \bot_{n-3}$. Obviously, f is also different from t and $\top$ (since it is in $\overline{\mathcal{D}}$). It follows that $t, \top, f, \bot_1, \ldots, \bot_{n-3}$ are all different from each other. □

Now we can construct the promised family of ideal n-valued logics:

THEOREM 3. *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an n-valued matrix for a language containing the unary connectives $\neg$ and $\diamond$, and the binary connective $\supset$. Suppose that $n > 3$, and that the following conditions hold in $\mathcal{M}$:*

1. $\mathcal{V} = \{t, f, \top, \bot_1, \ldots, \bot_{n-3}\}$ *and* $\mathcal{D} = \{t, \top\}$,
2. $\tilde{\neg} t = f, \tilde{\neg} f = t$, *and* $\tilde{\neg} x = x$ *otherwise,*

3. *$\tilde{\diamond} t = f$, $\tilde{\diamond} f = t$, $\tilde{\diamond}\top = \bot_1$, $\tilde{\diamond}\bot_i = \bot_{i+1}$ for $i < n-3$, and $\tilde{\diamond}\bot_{n-3} = \top$,*

4. *$a \tilde{\supset} b = t$ if $a \notin \mathcal{D}$ and $a \tilde{\supset} b = b$ otherwise,*

5. *For every other n-ary connective $\star$ of $\mathcal{L}$, $\tilde{\star}$ is $\{t, f\}$-closed.*

Then $\mathbf{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ is an ideal n-valued paraconsistent logic that is not equivalent to any k-valued logic with $k < n$.

Proof. It can be easily checked that $\mathcal{M}$ satisfies all the conditions of Proposition 13. Thus $\vdash_{\mathcal{M}} \neq \vdash_{\mathcal{M}'}$ for every matrix $\mathcal{M}'$ with less than n elements.

Next, we note that for any $\mathcal{M}$-valuation ν, $\nu(\diamond p \supset p \supset p) = t$. Hence we may assume that $\mathcal{L}$ includes propositional constants f and t such that $\nu(\mathsf{f}) = f$ and $\nu(\mathsf{t}) = t$ for any $\mathcal{M}$-valuation ν.

We divide the rest of the proof to several lemmas, showing that $\mathbf{L}_{\mathcal{M}}$ satisfies all the properties of an ideal paraconsistent logic.

Lemma 3. *$\mathbf{L}_{\mathcal{M}}$ is a normal $\neg$-paraconsistent logic.*

Proof. Clearly, $\mathcal{M}$ is $\neg$-paraconsistent and semi-classical. Hence, by Proposition 5, $\mathbf{L}_{\mathcal{M}}$ is $\neg$-contained in classical logic. It is also easy to verify that the classical deduction theorem obtains for $\supset$ and $\vdash_M$. Hence, $\supset$ is a proper implication, and so $\mathbf{L}_{\mathcal{M}}$ is normal. □

Lemma 4. *$\mathcal{M}$ is strongly maximal.*

Proof. Note first that for any $a \in \mathcal{V} \setminus \{t, f\}$ there is $0 \leq j_a \leq n-2$, such that a valuation μ is a model in $\mathcal{M}$ of $\{\diamond^{j_a} p, \neg\diamond^{j_a} p\}$ iff $\mu(p) = a$ ($j_\top = 0$ or $j_\top = n-2$, and $j_{\bot_i} = n-2-i$). Let $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ be any proper extension of $\mathbf{L}_{\mathcal{M}}$. Then there are some $\psi_1, \ldots, \psi_k$ and φ, such that $\psi_1, \ldots, \psi_k \vdash_{\mathbf{L}} \varphi$, but $\psi_1, \ldots, \psi_k \not\vdash_{\mathcal{M}} \varphi$. From the latter it follows that there is a valuation μ, such that $\mu(\psi_i) \in \mathcal{D}$ for every $1 \leq i \leq k$, and $\mu(\varphi) \in \overline{\mathcal{D}}$. Let $p_1, \ldots, p_m$ be the atoms occurring in $\{\psi_1, \ldots, \psi_k, \varphi\}$. Since we can substitute the propositional constant f for any p such that $\mu(p) = f$, and t for any p such that $\mu(p) = t$, we may assume that $\mu(p)$ is in $\mathcal{V} \setminus \{t, f\}$ for any atom p. Accordingly, let $j_i = j_{\mu(p_i)}$ for $1 \leq i \leq m$. By the observations above, μ is the only model of the set $\Psi = \bigcup_{1 \leq i \leq m} \{\diamond^{j_i} p_i\ ,\ \neg\diamond^{j_i} p_i\}$. It follows that $\Psi \vdash_{\mathcal{M}} \psi_i$ for every $1 \leq i \leq k$, and $\Psi \cup \{\varphi\} \vdash_{\mathcal{M}} q$ (where q is a new variable). Hence $\Psi \vdash_{\mathbf{L}} q$. Now, by substituting $\diamond^{n-j_i-2} p$ for p_i (where p is different from q), we can unify Ψ to $\{\diamond^{n-2} p, \neg\diamond^{n-2} p\}$. But in $\mathbf{L}_{\mathcal{M}}$ both elements of this set follow from $\{p, \neg p\}$. Thus, $p, \neg p \vdash_{\mathbf{L}} q$, and so $\mathbf{L}$ is not paraconsistent. □

Lemma 5. *$\mathcal{M}$ is maximally $\neg$-paraconsistent relative to classical logic.*

Proof. Let φ be a formula that is not $\mathcal{M}$-valid, and let $\mathcal{S}$ be the set of instances of φ. Suppose for contradiction that there is a classical tautology θ such that $\mathcal{S} \not\vdash_{\mathcal{M}} \theta$. Let $\mathcal{T}$ be a maximal theory extending $\mathcal{S}$, such that $\mathcal{T} \not\vdash_{\mathcal{M}} \theta$. Then for every formula ψ, either $\psi \in \mathcal{T}$, or $\mathcal{T}, \psi \vdash_{\mathcal{M}} \theta$ and so $\mathcal{T} \vdash_{\mathcal{M}} \psi \supset \theta$. Obviously, $\mathcal{T} \vdash_{\mathcal{M}} \psi$ iff $\psi \in \mathcal{T}$.

Next, for any truth value $a \in \mathcal{V}$ and formula $\psi \in \mathcal{W}_{\mathcal{L}}$, define formulas $\phi_1^a(\psi)$ and $\phi_2^a(\psi)$ as follows:

$$
\begin{array}{lll}
\phi_1^t(\psi) = \psi & \phi_2^t(\psi) = \neg \diamond \psi & \\
\phi_1^f(\psi) = \neg\psi & \phi_2^f(\psi) = \diamond\psi & \\
\phi_1^\top(\psi) = \psi & \phi_2^\top(\psi) = \neg\psi & \\
\phi_1^{\perp_i}(\psi) = \diamond^{n-2-i}\psi & \phi_2^{\perp_i}(\psi) = \neg\phi_1^{\perp_i}(\psi) & \text{for } i = 1, \ldots, .n-3
\end{array}
$$

It is easy to check that the following holds:

($\star$) For any $a \in \mathcal{V}$, $\psi \in \mathcal{W}_{\mathcal{L}}$, and an $\mathcal{M}$-valuation ν, $\nu(\psi) = a$ iff ν satisfies both $\phi_1^a(\psi)$ and $\phi_2^a(\psi)$.

Now, define a valuation ν by: $\nu(\psi) = a$ if $\phi_1^a(\psi) \in \mathcal{T}$ and $\phi_2^a(\psi) \in \mathcal{T}$. We show the following facts:

1. ν *is well-defined:* This follows from the following two facts:

   (a) There must be an $a \in \mathcal{V}$ such that $\mu(\psi) = a$.
   Indeed, let $\phi_3^a(\psi) = \phi_1^a(\psi) \supset (\phi_2^a(\psi) \supset \theta)$. By ($\star$), every $\mathcal{M}$-valuation satisfies $\phi_1^a(\psi)$ and $\phi_2^a(\psi)$ for some $a \in \mathcal{V}$. Hence

   $$\{\phi_3^a(\psi) \mid a \in \mathcal{V}\} \vdash_{\mathcal{M}} \theta$$

   Now, if $\mu(\psi) \neq a$ for some $a \in \mathcal{V}$, then $\phi_1^a(\psi) \notin \mathcal{T}$ or $\phi_2^a(\psi) \notin \mathcal{T}$, so $\mathcal{T} \cup \{\phi_1^a(\psi), \phi_2^a(\psi)\}$ is a proper extension of $\mathcal{T}$. Hence, $\mathcal{T} \cup \{\phi_1^a(\psi), \phi_2^a(\psi)\} \vdash_{\mathcal{M}} \theta$, and so $\mathcal{T} \vdash_{\mathcal{M}} \phi_3^a(\psi)$ by the deduction theorem. Thus, if $\mu(\psi) \neq a$ for every $a \in \mathcal{V}$ we get that $\mathcal{T} \vdash_{\mathcal{M}} \phi_3^a(\psi)$ for every $a \in \mathcal{V}$, and so $\mathcal{T} \vdash_{\mathcal{M}} \theta$. A contradiction.

   (b) If $a \neq b$, it is not possible that both $\mu(\psi) = a$ and $\mu(\psi) = b$.
   Otherwise, $\phi_1^a(\psi), \phi_2^a(\psi), \phi_1^b(\psi), \phi_2^b(\psi)$ are all in $\mathcal{T}$. But by ($\star$), the set $\{\phi_1^a(\psi), \phi_2^a(\psi), \phi_1^b(\psi), \phi_2^b(\psi\}$ is not $\mathcal{M}$-satisfiable in case $a \neq b$, and so $\phi_1^a(\psi), \phi_2^a(\psi), \phi_1^b(\psi), \phi_2^b(\psi) \vdash_{\mathcal{M}} \theta$. A contradiction.

2. ν *is a legal valuation:*
   Let $\psi = \diamond(\psi_1, \ldots, \psi_n)$. Suppose that $\nu(\psi_i) = a_i$ for $i = 1, \ldots, n$ and

$\tilde{\diamond}(a_1, ..., a_n) = b$. Then for every i both $\phi_1^{a_i}(\psi_i) \in \mathcal{T}$ and $\phi_2^{a_i}(\psi_i) \in \mathcal{T}$. Now by $(\star)$, for $j = 1, 2$,

$$\bigcup_{i=1}^{n} \{\phi_1^{a_i}(\psi_i), \phi_2^{a_i}(\psi_i)\} \vdash_{\mathcal{M}} \phi_j^b(\psi).$$

It follows that $\phi_j^b(\psi) \in \mathcal{T}$ for $j = 1, 2$. Hence, by the definition of ν, $\nu(\psi) = b$, that is, $\nu(\diamond(\psi_1, \ldots, \psi_n)) = \tilde{\diamond}(a_1, ..., a_n)$, as required.

3. *ν is a model of $\mathcal{T}$ which is not a model of θ:*

   Let $\psi \in \mathcal{T}$. If $\neg\psi \in \mathcal{T}$ then $\phi_1^{\top}(\psi) \in \mathcal{T}$ and $\phi_2^{\top}(\psi) \in \mathcal{T}$, thus $\nu(\psi) = \top \in \mathcal{D}$. Otherwise, $\neg\psi \notin \mathcal{T}$, and so $\neg\psi \supset \theta \in \mathcal{T}$. Since $\psi, \neg\psi \supset \theta, \neg \diamond \psi \supset \theta \vdash_{\mathcal{M}} \theta$, this implies that $\neg \diamond \psi \supset \theta \notin \mathcal{T}$, and so $\neg \diamond \psi \in \mathcal{T}$. It follows that in this case $\phi_1^t(\psi) \in \mathcal{T}$ and $\phi_2^t(\psi) \in \mathcal{T}$, thus again $\nu(\psi) = t \in \mathcal{D}$.

   Clearly, ν cannot be a model of θ, since if $\nu(\theta) \in \{t, \top\}$, then in particular $\phi_1^t(\theta) \in \mathcal{T}$ or $\phi_1^{\top}(\theta) \in \mathcal{T}$. In either case $\theta \in \mathcal{T}$, a contradiction to $\mathcal{T} \not\vdash_{\mathcal{M}} \theta$.

4. *ν is a classical valuation:*

   We show that ν is into $\{t, f\}$. Assume for contradiction that there are $a \notin \{t, f\}$ and ψ_a such that $\nu(\psi_a) = a$. It is easy to see that this implies that for every b there is a sentence ψ_b such that $\nu(\psi_b) = b$ (Indeed, since $\nu(\theta) \notin \mathcal{D}$, $\nu(\theta \supset \theta) = t$ and $\nu(\neg(\theta \supset \theta)) = f$, thus $\psi_t = \theta \supset \theta$ and $\psi_f = \neg(\theta \supset \theta)$. For $b \notin \{t, f\}$, ψ_b may be taken as a sentence of the form $\diamond^k \psi_a$, where k is such that $\tilde{\diamond}^k a = b$). Since φ is not valid in $\mathcal{M}$, there is an $\mathcal{M}$-valuation μ such that $\mu(\varphi) \notin \mathcal{D}$. Assume that $\mathsf{Atoms}(\varphi) = \{q_1, \ldots, q_k\}$ and that $\mu(q_i) = b_i$ for $i = 1, \ldots, k$. Let $\psi = \varphi\{\psi_{b_i}/q_i\}$. Then $\nu(\psi) = \mu(\varphi) \notin \mathcal{D}$. On the other hand, $\psi \in \mathcal{S} \subseteq \mathcal{T}$. Hence $\nu(\psi) \in \mathcal{D}$, since ν is a model of $\mathcal{T}$. A contradiction.

Now, since θ is a classical tautology and ν is a classical valuation, necessarily $\nu(\theta) = t$, but this contradicts the fact that ν is not a model of θ. This concludes the proof of Lemma 5. □

Theorem 3 now follows from the last three lemmas. □

EXAMPLE 7. Let $\mathcal{M}_4 = \langle \{t, f, \top.\bot\}, \{t, \top\}, \mathcal{O} \rangle$ be the four-valued matrix for the language which consists of an implication connective $\supset$, defined by:

$$a \mathbin{\tilde{\supset}} b = t \text{ if } a \in \{f, \bot\} \text{ and } a \mathbin{\tilde{\supset}} b = b \text{ if } a \in \{t, \top\},$$

and the following two unary connectives:

1. The usual negation $\neg$ of Dunn and Belnap [10, 11, 19], defined by: $\tilde{\neg} t = f$, $\tilde{\neg} f = t$, $\tilde{\neg} \top = \top$ and $\tilde{\neg} \bot = \bot$.
2. Fitting's conflation $-$ [20], defined by: $\tilde{-} t = t$, $\tilde{-} f = f$, $\tilde{-} \top = \bot$ and $\tilde{-} \bot = \top$.

It is easy to verify that by defining $\diamond\psi = \neg - \psi$, we turn $\mathcal{M}_4$ into a matrix for which Theorem 3 is applicable. It follows that $\mathbf{L}_{\mathcal{M}_4}$ is an ideal four-valued paraconsistent logic, which is equivalent to no three-valued logic. Moreover, this remains the case for any extension of this logic by classically closed connectives. In particular, Theorem 3 applies to the logic of the bilattice [21, 22] $\mathcal{FOUR}$, obtained from $\mathcal{M}_4$ by the addition of the standard Dunn-Belnap four-valued conjunction and disjunction (defined by $a \tilde{\vee} b = sup_{\leq_t}\{a, b\}$, and $a \tilde{\wedge} b = inf_{\leq_t}\{a, b\}$, where $\leq_t$ is the partial order on $\mathcal{FOUR}$ defined by: $f \leq_t \top, \bot \leq_t t$). In [2] it was shown that this ideal paraconsistent logic provides a very natural and convenient framework for reasoning with uncertain information, and that it has a corresponding cut-free, sound and complete Gentzen-type proof system (as well as a sound and complete Hilbert-type proof system).

Note 7. All the logics introduced in Theorem 3, including that of Example 7, have the further important property that $\neg\neg\psi$ is equivalent in them to ψ (in the strongest possible sense: each of them can be substituted for the other in any context). The same is true for the ideal logics that are induced by semi-classical three-valued matrices in which $\tilde{\neg}\tilde{\neg}\top = \top$.

Acknowledgement

This research was supported by The Israel Science Foundation (grant No 280-10).

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