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Frontal operators in weak Heyting algebras

Abstract. In this paper we shall introduce the variety **FWHA** of frontal weak Heyting algebras as a generalization of the frontal Heyting algebras introduced by Leo Esakia in [10]. A frontal operator in a weak Heyting algebra A is an expansive operator τ preserving finite meets which also satisfies the equation $\tau(a) \leq b \vee (b \to a)$, for all $a, b \in A$. These operators were studied from an algebraic, logical and topological point of view by Leo Esakia in [10]. We will study frontal operators in weak Heyting algebras and we will consider two examples of them. We will give a Priestley duality for the category of frontal weak Heyting algebras in terms of relational spaces $\langle X, \leq, T, R\rangle$ where $\langle X, \leq, T\rangle$ is a *WH*-space [6], and R is an additional binary relation used to interpret the modal operator. We will also study the *WH*-algebras with successor and the *WH*-algebras with gamma. For these varieties we will give two topological dualities. The first one is based on the representation given for the frontal weak Heyting algebras. The second one is based on certain particular classes of *WH*-spaces.

Keywords: modal operators, frontal operators, weak Heyting algebras, Priestley duality

1. Introduction

In [10] L. Esakia defines the modalized Heyting calculus mHC, which consists of an augmentation of the Heyting propositional calculus by a modal operator. The algebraic models of mHC are Heyting algebras with a unary operator (called *frontal operator*) subject to additional identities. These algebras are called *frontal Heyting algebras*. Frontal operators in Heyting algebras were studied in [3], [10] and [16]. They are always compatible operations, but not necessarily new or implicit in the sense of [2]. Classical examples of new implicit frontal operators are the functions γ (Example 3.1 of [2]), and the successor (Example 5.2 of [2]).

On the other hand, the variety of *weak Heyting algebras* was introduced in [6] (under the name of weakly Heyting algebras or *WH*-algebras), as the algebraic counterpart of the least subintuitionistic logic wK considered in [5]. A *WH*-algebra is a bounded distributive lattice with a binary operation $\to$ with the properties of the strict implication in the modal logic K. Heyting algebras are examples of *WH*-algebras. Other examples of *WH*-algebras that appear in the literature are the Basic algebras introduced by M. Ardeshir and W. Ruitenburg in [1], and the subresiduated lattices of G. Epstein and A. Horn in [9]. Each one of the varieties of *WH*-algebras studied in [6] corresponds to two propositional logics wK_σ and sK_σ defined in [5]. The logics wK_σ and sK_σ are the strict implication fragments of the local and global consequence relations defined by means of Kripke models (see [5]), respectively. In this paper we introduce the frontal WH-algebras as a generalization of the frontal Heyting algebras.

The paper is organized as follows. In Section 2, we recall the concepts and basic results of the Priestley duality for WH-algebras and for the variety of bounded distributive lattices with a modal operator. Also, we give definitions and useful notations we need in the paper. In Section 3 we define the variety of frontal WH-algebras which are a generalization of frontal Heyting algebras. We give and study two examples of them, which will be called WH-algebras with successor, or SWH-algebras, and WH-algebras with gamma, or γWH-algebras. In Section 4 we give a representation and a topological duality for the category of frontal WH-algebras based on the duality for the WH-algebras (see [6]) and the duality for modal lattices (see [13], [7], or [8]). We define the frontal WH-spaces as structures $\langle X, \leq, T, R\rangle$ where $\langle X, \leq, T\rangle$ is a WH-space, $\langle X, \leq, R\rangle$ is a modal Priestley space and certain conditions are satisfied that connect the relations T, R and $\leq$. From this duality we obtain a duality similar to the one given in [3] for the category of frontal Heyting algebras (see also [16]). In Section 5 we study two equivalent representations for the WH-algebras with successor. These algebras are a generalization of the KM-algebras studied by L. Esakia in [10]. The first representation is based on the frontal WH-spaces previously studied, i.e., the operator is interpreted by means of the relation R in the standard way. The other representation is based on a particular class of WH-spaces, and in this case the modal operator is interpreted by means of the relations $\leq$ and T. We prove that these two representations are isomorphic. In Section 6 we study the representation for the variety of γWH-algebras. In this case we also give two representations. In Section 7 we give some remarks about the relation between frontal operators in WH-algebras and frontal operators in Heyting algebras.

2. Preliminaries

If X is a set, then the power set of X will be denoted by $\mathcal{P}(X)$. If A is a distributive lattice, then $\mathrm{Fi}(A)$ and $\mathrm{Id}(A)$ will respectively denote the family of filters of A and the family of ideals of A. The filter (ideal) generated by a subset $X \subseteq A$ will be denoted by $\mathrm{F}(X)$ ($\mathrm{I}(X)$). We will write $\uparrow a$ ($\downarrow a$) to refer to the filter (ideal) generated by $\{a\}$. The family of the prime filters of A is denoted by $X(A)$. Given a bounded distributive lattice A, let $\varphi\colon A \to \mathcal{P}(X(A))$ be the Stone map defined by $\varphi(a) = \{P \in X(A) : a \in P\}$, for each $a \in A$. The family $\varphi[A] = \{\varphi(a) : a \in A\}$ is closed under unions, intersections, and contains $\emptyset$ and A; it is therefore a bounded distributive lattice.

Given a poset $\langle X, \leq \rangle$, a set $Y \subseteq X$ is said to be *upward closed* (or *upset*) if it is closed under $\leq$, that is if for every $x \in Y$ and every $y \in X$, if $x \leq y$ then $y \in Y$. The set of all *upward closed sets* of X will be denoted by $\mathcal{P}_u(X)$. The set complement of a subset $Y \subseteq X$ will be denoted by Y^c or $X - Y$. For each $Y \subseteq X$, the upset (downset) generated by Y is $[Y) = \{x \in X \mid \exists y \in Y (y \leq x)\}$ $((Y] = \{x \in X \mid \exists y \in Y (x \leq y)\})$. If $Y = \{y\}$, then we will write $[y)$ and $(y]$ instead of $[\{y\})$ and $(\{y\}]$, respectively. A *totally order-disconnected topological space* is a triple $\langle X, \leq \rangle = \langle X, \leq, \mathcal{T} \rangle$, where $\langle X, \leq \rangle$ is a poset, $\langle X, \mathcal{T} \rangle$ is a topological space and given $x, y \in X$ such that $x \nleq y$ there is a clopen upset U such that $x \in U$ and $y \notin U$. A *Priestley space* is a compact totally order-disconnected topological space. A morphism between Priestley spaces is a continuous and monotone function between them. If $\langle X, \leq \rangle$ is a Priestley space, the family of all clopen upsets of $\langle X, \leq \rangle$ is denoted by $D(X)$, and it is well known that it is a bounded distributive lattice.

The Priestley space of a bounded distributive lattice A is the triple $\langle X(A), \subseteq, \mathcal{T}_A \rangle$, where $\mathcal{T}_A$ is the topology generated by taking as a subbase the family $\{\varphi(a) : a \in A\} \cup \{\varphi(a)^c : a \in A\}$. It is well know that $A \cong D(X(A))$. For more details on Priestley spaces see [17].

A *weak Heyting algebra,* or *WH-algebra*, is an algebra $\langle A, \vee, \wedge, \rightarrow, 0, 1 \rangle$, where $\langle A, \vee, \wedge, 0, 1 \rangle$ is a bounded distributive lattice and $\rightarrow : A \times A \rightarrow A$ is a map such that for all $a, b, c \in A$,

1. $(a \rightarrow b) \wedge (a \rightarrow c) = a \rightarrow (b \wedge c)$,
2. $(a \rightarrow c) \wedge (b \rightarrow c) = (a \vee b) \rightarrow c$,
3. $(a \rightarrow b) \wedge (b \rightarrow c) \leq a \rightarrow c$,
4. $a \rightarrow a = 1$.

From this definition it is immediate that the class of *WH*-algebras is a variety. If A is a *WH*-algebra and $a, b, c \in A$, then by Proposition 3.2 of [5] the following fact holds: if $a \leq b$, then $c \rightarrow a \leq c \rightarrow b$, $b \rightarrow c \leq a \rightarrow c$ and $a \rightarrow b = 1$. A *weak Heyting homomorphism* between two *WH*-algebras A and B is a bounded lattice homomorphism $h : A \rightarrow B$ such that $h(a \rightarrow b) = h(a) \rightarrow h(b)$, for all $a, b \in A$. We denote by $\boldsymbol{WH}$ the category that has weak Heyting algebras as objects and weak Heyting homomorphisms as arrows.

Recall that a *Heyting algebra* is an algebra $\langle A, \vee, \wedge, \rightarrow, 0, 1\rangle$, with an additional binary operation $\rightarrow\colon A \times A \rightarrow A$ satisfying $a \wedge b \leq c$ iff $a \leq b \rightarrow c$ for all $a, b, c \in A$. Heyting algebras are examples of *WH*-algebras. By the results given in [5] a *WH*-algebra A is a Heyting algebra iff A satisfies the additional inequalities (I) $a \leq 1 \rightarrow a$, and (R) $a \wedge (a \rightarrow b) \leq b$, for every $a, b \in A$.

Let A be a *WH*-algebra. We define the relation $T_{\rightarrow}$ on $\mathrm{Fi}(A)$ by:

$$(F, G) \in T_{\rightarrow} \text{ iff } (\forall a, b \in A)(((a \rightarrow b \in F \ \&\ a \in G) \Longrightarrow b \in G). \tag{2.1}$$

Let F be a filter of A. We define the operator $D_F\colon \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ by

$$D_F(X) = \left\{ b \in A : \exists Y \subseteq X \text{ finite such that } (\bigwedge Y \rightarrow b) \in F \right\},$$

where $\bigwedge Y$ is the infimum of Y, so if Y is empty, $\bigwedge Y = 1$. In Proposition 3.4 of [5] it was proved that for every $X \subseteq A$ the set $D_F(X)$ is a filter of A. The following is a generalization of the Prime Filter Theorem on the existence of prime filters for bounded distributive lattices (Lemma 3.7 of [5]).

Proposition 2.1. *Let A be a WH-algebra, let F be a filter and I an ideal of A and let $X \subseteq A$. If $D_F(X) \cap I = \emptyset$, then there is a prime filter P such that $D_F(X) \subseteq P$, $(F, P) \in T_{\rightarrow}$ and $P \cap I = \emptyset$.*

If T is a binary relation on a set X and $x \in X$ we define $T(x) = \{y \in X : (x, y) \in T\}$. The duality between bounded distributive lattices and Priestley spaces can be specialized to *WH*-algebras. A *WH-space* is a structure $\langle X, \leq, T\rangle$ such that $\langle X, \leq\rangle$ is a Priestley space, $T(x)$ is a closed subset of X for all $x \in X$, and

$$U \Rightarrow V = \{x \in X : T(x) \cap U \subseteq V\} \in D(X),$$

for every $U, V \in D(X)$. It is easy to see that if $\langle X, \leq, T\rangle$ is a *WH*-space, then $(\leq \circ T) \subseteq T$ (i.e., for every $x, y, z \in X$, if $x \leq z$ and $(z, y) \in T$ then $(x, y) \in T$). If $\langle X, \leq, T\rangle$ is a *WH*-space, then the bounded distributive lattice $D(X)$ with the additional operation $\Rightarrow$ is a *WH*-algebra.

Let $\langle X_1, \leq_1, T_1\rangle$ and $\langle X_2, \leq_2, T_2\rangle$ be *WH*-spaces. A function $f\colon X_1 \rightarrow X_2$ is a *WH-morphism* if it is a morphism of Priestley spaces (i.e., it is continuous and monotone), and

1. If $(x, y) \in T_1$ then $(f(x), f(y)) \in T_2$.
2. If $(f(x), z) \in T_2$ then there is $y \in X_1$ such that $(x, y) \in T_1$ and $f(y) = z$.

We denote by $\boldsymbol{WHS}$ the category that has WH-spaces as objects and WH-morphisms as arrows.

The next theorem is proved in [6]. We only give a sketch of the proof. The missing details can be found in [6].

THEOREM 2.2. *The categories* $\boldsymbol{WH}$ *and* $\boldsymbol{WHS}$ *are dually equivalent.*

PROOF. Define a contravariant functor $(-)_*: \boldsymbol{WH} \to \boldsymbol{WHS}$ as follows. If A is a WH-algebra, then $A_* = \langle X(A), \subseteq, T_{\to}, \mathcal{T}_A \rangle$, where $\langle X(A), \subseteq, \mathcal{T}_A \rangle$ is the Priestley space of A, and $T_{\to} \subseteq X(A) \times X(A)$ is the relation defined in (2.1). If $h\colon A_1 \to A_2$ is a homomorphism of WH-algebras, then the mapping $h_*\colon X(A_2) \to X(A_1)$ given by $h_*(P) = h^{-1}(P)$ is a WH-morphism.

Next define the contravariant functor $(-)^*: \boldsymbol{WHS} \to \boldsymbol{WH}$ as follows. For a WH-space $\langle X, \leq, T \rangle$, the structure $\langle X, \leq, T \rangle^* = \langle D(X), \cup, \cap, \Rightarrow, \emptyset, X \rangle$ is a WH-algebra. If $f\colon X_1 \to X_2$ is a WH-morphism, then the map $f^*\colon D(X_2) \to D(X_1)$ given by $f^*(U) = f^{-1}(U)$ is a homomorphism between WH-algebras.

Consequently, $(-)_*$ and $(-)^*$ are well-defined contravariant functors. Moreover, the function $\varphi\colon A \to D(X(A))$ is a natural isomorphism between the WH-algebras A and $(A_*)^* = \langle D(X(A)), \cup, \cap, \Rightarrow, \emptyset, X(A) \rangle$. Moreover, the function $\varepsilon\colon X \to X(D(X))$ given by $\varepsilon(x) = \{U \in D(X) : x \in U\}$ is a natural isomorphism between the WH-spaces $\langle X, \leq, T \rangle$ and $(\langle X, \leq, T \rangle^*)_* = \langle X(D(X)), \subseteq, T_{\Rightarrow} \rangle$. This yields the desired dual equivalence between $\boldsymbol{WH}$ and $\boldsymbol{WHS}$. ∎

The Priestley spaces dual to Heyting algebras were characterized by Esakia in [11] (see also [12]). As Heyting algebras are special WH-algebras, the Priestley spaces of Heyting algebras are WH-spaces with respect to the order. A Priestley space $\langle X, \leq \rangle$ is said to be an *Esakia space* if $(U]$ is clopen, for every clopen U. In particular, a WH-space $\langle X, \leq, T \rangle$ is an Esakia space iff $T = \leq$.

An algebra $\langle A, \tau \rangle$ is a *modal lattice, or a τ-lattice*, if A is a bounded distributive lattice and τ is a unary operator defined on A such that:

1. $\tau(1) = 1$, and
2. $\tau(a \wedge b) = \tau(a) \wedge \tau(b)$ for all $a, b \in A$.

A morphism of bounded lattices which preserve the modal operator will be called a *morphism of modal lattices.*

If X is a set and $R \subseteq X \times X$, for every $U \subseteq X$ we define the set

$$\tau_R(U) = \{x \in X : R(x) \subseteq U\}.$$

Definition 2.3. A *modal Priestley space* ([13], [7], or [8]) is a relational structure $\langle X, \leq, R\rangle$ where $\langle X, \leq\rangle$ is a Priestley space, and R is a binary relation on X such that

1. $R(x)$ is a closed upset, for each $x \in X$.
2. $\tau_R(U) \in D(X)$, for each $U \in D(X)$.

Let $\langle X_1, \leq_1, R_1\rangle$ and $\langle X_2, \leq_2, R_2\rangle$ be two modal Priestley spaces. A *p-morphism* is a monotone and continuous mapping $f\colon X_1 \to X_2$ satisfying the following conditions:

1. If $(x, y) \in R_1$ then $(f(x), f(y)) \in R_2$.
2. If $(f(x), z) \in R_2$ then there is $y \in X_1$ such that $(x, y) \in R_1$ and $f(y) \leq_2 z$.

Let A be a modal lattice. We define a binary relation R_τ on $X(A)$ in the following way:

$$(P, Q) \in R_\tau \text{ iff } \tau^{-1}(P) \subseteq Q, \tag{2.2}$$

with $P, Q \in X(A)$.

We denote by $\boldsymbol{ML}$ the category that has modal lattices as objects and morphisms of modal lattices as arrows. We denote by $\boldsymbol{MS}$ the category that has modal Priestley spaces as objects and p-morphisms as arrows. For the next theorem we only give a sketch of the proof. The missing details can be found in [13], [7], or [8].

Theorem 2.4. *The categories $\boldsymbol{ML}$ and $\boldsymbol{MS}$ are dually equivalent.*

Proof. Define a contravariant functor $\mathcal{F}\colon \boldsymbol{ML} \to \boldsymbol{MS}$ as follows. If $\langle A, \tau\rangle$ is a modal lattice, then $\mathcal{F}(\langle A, \tau\rangle) = \langle X(A), \subseteq, R_\tau\rangle$ is a modal Priestley space. If $h\colon \langle A_1, \tau_1\rangle \to \langle A_2, \tau_2\rangle$ is a morphism of modal lattices, then the mapping $\mathcal{F}(h)\colon \langle X(A_2), \subseteq, R_{\tau_2}\rangle \to \langle X(A_1), \subseteq, R_{\tau_1}\rangle$ given by $\mathcal{F}(h)(P) = h^{-1}(P)$ is a p-morphism.

Next define the contravariant functor $\mathcal{G}\colon \boldsymbol{MS} \to \boldsymbol{ML}$ as follows. For a modal Priestley space $\langle X, \leq, R\rangle$, the structure $\mathcal{G}(\langle X, \leq, R\rangle) = \langle D(X), \tau_R\rangle$ is a modal lattice. If $f\colon \langle X_1, \leq_1, R_1\rangle \to \langle X_2, \leq_2, R_2\rangle$ is a p-morphism, then the map $\mathcal{G}(f) : \langle D(X_2), \tau_{R_2}\rangle \to \langle D(X_1), \tau_{R_1}\rangle$ given by $\mathcal{G}(f)(U) = f^{-1}(U)$ is a morphism of modal lattices.

Consequently, $\mathcal{F}$ and $\mathcal{G}$ are well-defined contravariant functors. If $\langle A, \tau\rangle$ is a modal lattice, then the the mapping $\varphi\colon \langle A, \tau\rangle \to \langle D(X(A)), \tau_{R_\tau}\rangle$ is an isomorphism of modal lattices, i.e., $\varphi(\tau(a)) = \tau_{R_\tau}(\varphi(a))$, for all $a \in A$. Moreover, the function $\varepsilon\colon \langle X, \leq, R\rangle \to \langle X(D(X)), \subseteq, R_{\tau_R}\rangle$ given by $\varepsilon(x) = \{U \in D(X) : x \in U\}$ is an isomorphism in the category of modal Priestley spaces. This yields the desired dual equivalence between $\boldsymbol{ML}$ and $\boldsymbol{MS}$. ■

3. Frontal WH-algebras

In this section we define frontal *WH*-algebras as a generalization of frontal Heyting algebras introduced by L. Esakia in [10]. We give two examples of them: the *WH*-algebras with successor and the *WH*-algebras with gamma.

Definition 3.1. A *frontal WH-algebra* is a pair $\langle A, \tau \rangle$ such that A is a *WH*-algebra and τ is a unary operator satisfying the following equations:

(W1) $\tau(a \wedge b) = \tau(a) \wedge \tau(b)$,

(W2) $a \leq \tau(a)$,

(W3) $\tau(a) \leq b \vee (b \to a)$.

If $\langle A, \tau \rangle$ is a frontal *WH*-algebra we say that τ is a *frontal operator*. Let **FWHA** be the category whose objects are frontal *WH*-algebras and whose morphisms are morphisms of *WH*-algebras which preserve the frontal operator. Note that this category is a subcategory of the category of modal algebras.

We will define some particular classes of frontal *WH*-algebras. First we will define the class of *SWH*-algebras that are a generalization of the *KM-algebras* (also called *fronton*) studied by L. Esakia in [10] (these algebras were introduced in [14] by Kuznetsov. See also [15], [2] and [4]).

Definition 3.2. A *WH-algebra with successor*, or *SWH-algebra*, is a pair $\langle A, S \rangle$ such that $S\colon A \to A$ satisfies the equations (**W2**), (**W3**), and the following equation:

$$S(a) \to a \leq a. \tag{3.1}$$

If $\langle A, S \rangle$ is a *SWH*-algebra, the function S is called the *successor function*. Let $\mathbf{FWHA}_S$ be the category whose objects are *SWH*-algebras and whose morphisms are morphisms of *WH*-algebras which preserve the modal operator.

Lemma 3.3. *If $\langle A, S \rangle$ is a SWH-algebra, then $\langle A, S \rangle$ is a frontal WH-algebra and*

$$S(a) = \min(E_a)_{\to}, \tag{3.2}$$

for each $a \in A$, where $(E_a)_{\to} = \{b \in A : b \to a \leq b\}$.

Proof. First we prove that $S(a \wedge b) = S(a) \wedge S(b)$, for all $a, b \in A$. Observe that S is monotone. In fact, if $c \leq d$ then using **(W3)**, (3.1) and **(W2)** we have that

$$\begin{array}{rcl} S(c) & = & S(c \wedge d) \leq S(d) \vee (S(d) \to (c \wedge d)) \\ & = & S(d) \vee ((S(d) \to c) \wedge (S(d) \to d)) \\ & \leq & S(d) \vee ((S(d) \to c) \wedge S(d)) \qquad = \quad S(d). \end{array}$$

Thus $S(a \wedge b) \leq S(a) \wedge S(b)$. On the other hand we have $S(a) \leq S(a \wedge b) \vee (S(a \wedge b) \to a)$ and $S(b) \leq S(a \wedge b) \vee (S(a \wedge b) \to b)$. Taking meet of these two inequalities we obtain

$$\begin{array}{rcl} S(a) \wedge S(b) & \leq & S(a \wedge b) \vee ((S(a \wedge b) \to a) \wedge (S(a \wedge b) \to b)) \\ & = & S(a \wedge b) \vee (S(a \wedge b) \to (a \wedge b)) \\ & \leq & S(a \wedge b) \vee (a \wedge b) \\ & = & S(a \wedge b). \end{array}$$

Thus $S(a) \wedge S(b) \leq S(a \wedge b)$. Therefore $S(a \wedge b) = S(a) \wedge S(b)$, for every $a, b \in A$.

We now prove that S is given by $S(a) = \min(E_a)_{\to}$, for each $a \in A$. By equations (**W2**) and (3.1) we conclude that $S(a) \in (E_a)_{\to}$. Let $b \in (E_a)_{\to}$. By equation (**W3**) we have that $S(a) \leq b$. Thus, $S(a) = \min(E_b)_{\to}$. ■

Definition 3.4. A *WH-algebra with* γ, or γ *WH-algebra*, is a pair $\langle A, \gamma \rangle$ such that $\gamma \colon A \to A$ satisfies the following equations:

(g1) $\gamma(0) \to 0 = 0$,

(g2) $\gamma(a) \leq b \vee (b \to a)$,

(g3) $\gamma(a) = a \vee \gamma(0)$.

Note that a gamma function on a *WH*-algebra can be characterized by the equations that define a frontal operator, equation (**g1**) and the equation $\gamma(a) \leq a \vee \gamma(0)$. An easy computation proves that if this function exists then it takes the form

$$\gamma(a) = \min\{b \in A : \neg b \vee a \leq b\}, \tag{3.3}$$

where $\neg b = b \to 0$. Let **FWHA**γ be the category whose objects are γ *WH*-algebras and whose morphisms are morphisms of *WH*-algebras which preserve the gamma operator.

In the following theorem we present another axiomatization of the γ *WH*-algebras.

Theorem 3.5. *Let A be a WH-algebra. Then there exists a unary operator $\gamma \colon A \to A$ such that $\langle A, \gamma \rangle$ is a γ WH-algebra iff*

1. $a \leq b \vee (b \to a)$, *for all* $a \in A$,
2. *There exists an element* $c \in A$ *satisfying the following conditions:*
   (a) c *is dense, i.e.,* $c \to 0 = 0$,
   (b) $c \leq a \vee (a \to 0)$, *for all* $a \in A$.

Proof. $\Rightarrow$) If $\langle A, \gamma \rangle$ is a $\gamma\,WH$-algebra, then $a \leq b \vee (b \to a)$, for all $a \in A$, and the element $c = \gamma(0)$ satisfies the conditions (a) and (b).

$\Leftarrow$) Define a unary function $\gamma \colon A \to A$ as

$$\gamma(a) = a \vee c,$$

for each $a \in A$. Then it is clear that $a \leq \gamma(a)$, for all $a \in A$, and $\gamma(0) \to 0 = c \to 0 = 0$. We prove condition (**g2**) of Definition 3.4. As $b \to 0 \leq b \to a$, for all $a, b \in A$, we have that

$$\gamma(a) = a \vee c \leq b \vee (b \to a) \vee c \leq b \vee (b \to a) \vee b \vee (b \to 0) = b \vee (b \to a),$$

for all $a, b \in A$. Therefore, $\langle A, \gamma \rangle$ is a $\gamma\,WH$-algebra. ∎

Remark 3.6. Taking into account the previous theorem and that the identity $a \leq b \vee (b \to a)$ is satisfied in any frontal weak Heyting algebra, it is thus natural to address the question of when the weak Heyting reducts of a subvariety of frontal weak Heyting algebras form a subvariety of weak Heyting algebras.

For example, the Heyting reduct of the variety of frontal Heyting algebras coincides with the variety of Heyting algebras because every Heyting algebra H can be turned into a frontal one: we can just equip H with the trivial operator τ putting $\tau(a) = a$ for all $a \in H$.

If $\mathcal{V}$ is the Heyting reduct of the variety of Heyting algebras with gamma then it is not a variety of Heyting algebras. In order to prove it suppose that $\mathcal{V}$ is a variety. By Theorem 3.1 of [2] we have that there is a unary Heyting term t such that $t = \gamma$. On the other hand we have that γ is not expressible by a Heyting term because in the three-element chain $\mathrm{H}_3 = \{0, a, 1\}$ we have that $\gamma(0) = a$, while $t(0) \in \{0, 1\}$ for any Heyting term t. Therefore $\mathcal{V}$ is not a variety.

We now consider two examples of WH-algebras with successor and WH-algebras with gamma.

Example 3.7. Let A be a bounded distributive lattice and consider the binary operation $\to$ given by $a \to b = 1$, for every $a, b \in A$. Then $\langle A, \to \rangle$

is a *WH*-algebra. There exists the successor function iff A is trivial (it has one element). In particular, if A is not trivial then S and γ are not given by (3.2) and (3.3), respectively.

EXAMPLE 3.8. Consider the chain of three elements $\mathrm{H}_3 = \{0, a, 1\}$ with the following operations:

$\hat{\to}$	0	a	1
0	1	1	1
a	1	1	1
1	0	0	1

$\tilde{\to}$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

It is easy to see that the structures $\langle \mathrm{H}_3, \hat{\to} \rangle$ and $\langle \mathrm{H}_3, \tilde{\to} \rangle$ are *WH*-algebras (in particular, $\langle \mathrm{H}_3, \tilde{\to} \rangle$ is a Heyting algebra). The following operations defined by the tables give examples of *SWH*-algebras and γ*WH*-algebras:

x	$S_{\hat{\to}}(x)$	$\gamma_{\hat{\to}}(x)$
0	1	1
a	1	1
1	1	1

x	$S_{\tilde{\to}}(x)$	$\gamma_{\tilde{\to}}(x)$
0	a	a
a	1	a
1	1	1

4. Representation and duality

Let $\langle X, \leq, T \rangle$ be a *WH*-space. We define an auxiliary relation $\bar{T} \subseteq X \times X$ in the following way:

$$(x, y) \in \bar{T} \text{ iff } (x, y) \in T \text{ and } y \nleq x.$$

DEFINITION 4.1. A *frontal WH-space* is a structure $\langle X, \leq, T, R \rangle$ such that:

1. $\langle X, \leq, T \rangle$ is a *WH*-space and $\langle X, \leq, R \rangle$ is a modal Priestley space.

2. $\bar{T} \subseteq R \subseteq \leq$.

PROPOSITION 4.2. *If $\langle X, \leq, T, R \rangle$ is a frontal WH-space, then*

$$\langle D(X), \cup, \cap, \Rightarrow, \tau_R, \emptyset, X \rangle$$

is a frontal WH-algebra.

PROOF. Let $U, V \in D(X)$. We prove that $\tau_R(U) \subseteq V \cup (V \Rightarrow U)$. Let $x \in \tau_R(U)$, i.e., $R(x) \subseteq U$. Suppose that $x \notin V \Rightarrow U$. Then, $T(x) \cap V \nsubseteq U$. Thus there exists $y \in X$ such that $(x, y) \in T$, $y \in V$ and $y \notin U$. If $y \leq x$ then $x \in V$. If $y \nleq x$, then $(x, y) \in \bar{T}$. By item 2. of Definition 4.1 we conclude that $(x, y) \in R$, so $y \in U$, a contradiction. Therefore $\tau_R(U) \subseteq V \cup (V \Rightarrow U)$.

By the condition $R \subseteq \leq$ we have the equation (**W1**). The equation (**W2**) is easy to verify. ■

Let $\langle A, \tau\rangle$ be a frontal *WH*-algebra. Since τ is a modal operator, we can consider the relation $R_\tau \subseteq X(A) \times X(A)$ defined in (2.2).

Remark 4.3. If A is a *WH*-algebra and $\tau\colon A \to A$ a function satisfying the equations (**W1**) and (**W2**), then $\langle A, \tau\rangle$ is a τ-lattice and the structure $\langle X(A), \subseteq, R_\tau\rangle$ is a modal Priestley space, where we recall that R_τ is defined in (2.2). Moreover, an easy computation shows that in a modal lattice A, we have $a \leq \tau(a)$ for every $a \in A$ iff $R_\tau \subseteq \leq$.

In the next lemma we give a first-order characterization of the equation **(W3)**.

LEMMA 4.4. *Let A be a WH-algebra and $\tau\colon A \to A$ a function satisfying the equations* (**W1**) *and* (**W2**)*. Then $\bar{T}_\to \subseteq R_\tau$ if and only if $\tau(a) \leq b \vee (b \to a)$, for every $a, b \in A$.*

PROOF. $\Rightarrow$) Suppose that there exist $a, b \in A$ such that $\tau(a) \nleq b \vee (b \to a)$. Thus there exists $P \in X(A)$ such that $\tau(a) \in P$, $b \notin P$ and $b \to a \notin P$. If $a \in D_P(\{b\})$ then $b \to a \in P$, a contradiction. Thus we have that $a \notin D_P(\{b\})$. Hence by Theorem 2.1 there is $Q \in X(A)$ such that $(P, Q) \in T_\to$, $D_P(\{b\}) \subseteq Q$ and $a \notin Q$. Using that $b \in D_P(\{b\})$ we have that $b \in Q$. However $b \notin P$, so $Q \nsubseteq P$ and so $(P, Q) \in \bar{T}_\to$. Thus by hypothesis we have that $(P, Q) \in R_\tau$. Then using that $a \in \tau^{-1}(P)$ we conclude that $a \in Q$, a contradiction.

$\Leftarrow$) We will prove that $\bar{T}_\to \subseteq R_\tau$. Let $(P, Q) \in \bar{T}_\to$. Then $(P, Q) \in T_\to$ and $Q \nsubseteq P$. So there exists $b \in A$ such that $b \in Q$ and $b \notin P$. Let $a \in \tau^{-1}(P)$, so $\tau(a) \in P$. Using (**W3**) we conclude that $b \to a \in P$, and so $a \in Q$ (because $b \in Q$ and $(P, Q) \in T_\to$). Therefore $\bar{T}_\to \subseteq R_\tau$. ■

COROLLARY 4.5. *$\langle A, \tau\rangle$ is a frontal WH-algebra if and only if the structure $\langle X(A), \subseteq, T_\to, R_\tau\rangle$ is a frontal WH-space.*

Let **FWHS** be the category whose objects are frontal *WH*-spaces and whose morphisms are functions $f\colon X_1 \to X_2$ such that f is a *WH*-morphism and f is a p-morphism. Then by the results given in [6] for *WH*-algebras and the results given in [13], [7] or [8] for bounded distributive lattices with a modal operator we obtain the following result.

THEOREM 4.6. *The category* **FWHS** *is dually equivalent to the category* **FWHA**.

5. Representation theory for the category of *WH*-algebras with successor

Definition 5.1. A *frontal S-space* is a frontal WH-space $\langle X, \leq, T, R\rangle$ such that

(S) For every $U \in D(X)$ and $x \in X$, if $x \in U^c$ then there exists $y \in U^c$ such that $(x, y) \in T$ and $R(y) \subseteq U$.

The category $\mathbf{FWHS}_S$ consists of all frontal S-spaces and the same morphisms as in **FWHS**.

Lemma 5.2. *Let A be a WH-algebra. Let $S\colon A \to A$ be a function. Then the pair $\langle A, S\rangle$ is a SWH-algebra if and only if $\langle X(A), \subseteq, T_{\rightarrow}, R_S\rangle$ is a frontal S-space.*

Proof. $\Rightarrow$) We will prove that if $P \in X(A)$ and $a \notin P$ then there exists $Q \in X(A)$ such that $a \notin Q$, $(P, Q) \in T_{\rightarrow}$ and $R_S(Q) \subseteq \varphi(a)$. Let $a \notin P$, so $S(a) \to a \notin P$. Thus there is $Q \in X(A)$ such that $a \notin Q$, $S(a) \in Q$ and $(P, Q) \in T_{\rightarrow}$. As $S(a) \in Q$, we have $R_S(Q) \subseteq \varphi(a)$. Thus, $\langle X(A), \subseteq, T_{\rightarrow}, R_S\rangle$ is a frontal S-space.

$\Leftarrow$) We will prove that $S(a) \to a \leq a$, for any $a \in A$. Suppose that there exists $a \in A$ such that $S(a) \to a \nleq a$. Then there exists $P \in X(A)$ such that $S(a) \to a \in P$, and $a \notin P$. By hypothesis there exists $Q \in X(A)$ such that $(P, Q) \in T_{\rightarrow}$, $a \notin Q$, and $S(a) \in Q$. From $(P, Q) \in T_{\rightarrow}$ and $S(a) \to a \in P$, we obtain $a \in Q$, which is a contradiction. Thus, $S(a) \to a \leq a$, for any $a \in A$. ∎

Then by Lemma 5.2 and Theorem 4.6 we have the following

Theorem 5.3. *The category* $\mathbf{FWHS}_S$ *is dually equivalent to the category* $\mathbf{FWHA}_S$.

In the following we will introduce a new type of WH-spaces that are dual to the SWH-algebras.

If X is a set and $T \subseteq X \times X$, for each $U \subseteq X$ we define the set

$$U_T = \{x \in U^c : T(x) \cap U^c \subseteq (x]\}.$$

Definition 5.4. A *WH-space with successor*, or *SWH-space*, is a WH-space $\langle X, \leq, T\rangle$ satisfying the following conditions for every $U \in D(X)$:

(a) $U \cup U_T \in D(X)$.

(b) If $x \in U^c$, then $T(x) \cap U_T \neq \emptyset$.

(c) If $(x, y) \in \bar{T}$, and $x \in U \cup U_T$, then $y \in U$.

We will write **SWHS** for the category whose objects are SWH-spaces and whose morphisms are WH-morphisms $f\colon \langle X_1, \leq_1 T_1\rangle \to \langle X_2, \leq_2, T_2\rangle$ such that

$$f^{-1}(U \cup U_{T_2}) = f^{-1}(U) \cup f^{-1}(U)_{T_1},$$

for each $U \in D(X_2)$. These morphisms will be called *SWH-morphisms.*

PROPOSITION 5.5. *If $\langle X, T\rangle$ is a SWH-space, then $\langle D(X), \cup, \cap, \Rightarrow, S, \emptyset, X\rangle$ is a SWH-algebra where $S(U) = U \cup U_T$, for each $U \in D(X)$.*

PROOF. It is clear that S is monotone, well defined (by condition **(a)** of Definition 5.4) and that $U \subseteq S(U)$, for every $U \in D(X)$. Let $U, V \in D(X)$. We will prove that $S(U \cap V) = S(U) \cap S(V)$. As S is monotone, we have that $S(U \cap V) \subseteq S(U) \cap S(V)$. Let $x \in S(U) \cap S(V)$. Then, $x \in U \cup U_T$ and $x \in V \cup V_T$. Suppose that $x \notin U \cap V$. We will prove that

$$x \in (U \cap V)_T = \{y \in (U \cap V)^c : T(y) \cap (U \cap V)^c \subseteq (y]\}.$$

Consider the case that $x \notin U$. If $x \notin V$ then $x \in (U \cap V)_T$. If $x \in V$ we will prove that $T(x) \cap V^c \subseteq (x]$ which implies that $x \in (U \cap V)_T$. Suppose that there is $y \in X$ such that $(x, y) \in T$, $y \in V^c$ and $y \nleq x$, so $(x, y) \in \overline{T}$. Thus by condition **(c)** of Definition 5.4 we conclude that $y \in V$, which is a contradiction. Therefore $x \in S(U \cap V)$.

We will prove that $S(U) \subseteq V \cup (V \Rightarrow U)$. Suppose that there exists $x \in X$ such that $x \in S(U)$ and $x \notin V \Rightarrow U$. Then there exists $y \in X$ such that $(x, y) \in T$, $y \in V$ and $y \notin U$. If $y \nleq x$, then as $(x, y) \in T$ we have that $(x, y) \in \bar{T}$. As $x \in S(U)$, we have by item **(c)** of Definition 5.4 that $y \in U$, which is impossible. Thus $y \leq x$. As $y \in V$ and S is monotone, $x \in V$.

Let $U \in D(X)$. We will prove that $S(U) \Rightarrow U \subseteq U$. Suppose that there exists $x \in X$ such that $x \in S(U) \Rightarrow U$ but $x \notin U$. Then $T(x) \cap S(U) \subseteq U$ and from condition **(b)** of Definition 5.4, there exists $y \in X$ such that $(x, y) \in T$ and $y \in U_T \subseteq S(U)$. Then $y \in T(x) \cap S(U)$, and consequently $y \in U$, which is impossible because $y \in U_T$. ■

LEMMA 5.6. *Let $\langle A, S\rangle$ be a SWH-algebra. For every $a \in A$ we have that $\varphi(a) \cup \varphi(a)_{T_\to} = \varphi(S(a))$.*

PROOF. Let $a \in A$. We will prove that $\varphi(a) \cup \varphi(a)_{T_\to} = \varphi(S(a))$, for each $a \in A$. Let $a \in A$ and let $P \in X(A)$ be such that $S(a) \in P$ and $a \notin P$. We will prove that $P \in \varphi(a)_{T_\to}$, i.e., $T_\to(P) \cap \varphi(a)^c \subseteq (P]$. If there exists

$Q \in T_{\to}(P) \cap \varphi(a)^c$ such that $Q \nsubseteq P$, then there exists $b \in Q - P$. As $S(a) \leq b \vee (b \to a) \in P$, we have $b \to a \in P$. As $Q \in T_{\to}(P)$ and $b \in Q$, we deduce that $a \in Q$, which is an contradiction. Therefore $\varphi(S(a)) \subseteq \varphi(a) \cup \varphi(a)_{T_{\to}}$. We will prove the other inclusion. Let $P \in \varphi(a) \cup \varphi(a)_{T_{\to}}$. If $a \in P$, then $S(a) \in P$ because $a \leq S(a)$. Assume that $a \notin P$. Then $P \in \varphi(a)_{T_{\to}}$, i.e., $T_{\to}(P) \cap \varphi(a)^c \subseteq (P]$. Suppose that $S(a) \notin P$. As $S(a) \to a \leq a \leq S(a)$, we have that $S(a) \to a \notin P$. Then by Proposition 2.1 there exists $Q \in X(A)$ such that $S(a) \in Q$, $a \notin Q$ and $(P, Q) \in T_{\to}$. So $Q \in T_{\to}(P) \cap \varphi(a)^c$. This implies that $Q \subseteq P$, and consequently $a \in P$, which is a contradiction. ■

Proposition 5.7. *Let $\langle A, S\rangle$ be a SWH-algebra. Then $\langle X(A), \subseteq, T_{\to}\rangle$ is a SWH-space.*

Proof. We will prove conditions **(a)**, **(b)** and **(c)** of Definition 5.4.

(a) It follows from Lemma 5.6.

(b) Let $P \in X(A)$ and let $a \notin P$. Then $S(a) \to a \notin P$. From Proposition 2.1 there exists $Q \in T_{\to}(P) \cap \varphi(a)^c$ such that $S(a) \in Q$. We will prove that $Q \in \varphi(a)_{T_{\to}}$, i.e. $T_{\to}(Q) \cap \varphi(a)^c \subseteq (Q]$. Let $D \in T_{\to}(Q) \cap \varphi(a)^c$. If $D \nsubseteq Q$, then there exists $b \in D - Q$. As $S(a) \leq b \vee (b \to a) \in Q$, we deduce that $b \to a \in Q$, and as $D \in T_{\to}(Q)$ and $b \in D$ we obtain that $a \in D$, which is impossible.

(c) Let $a \in A$ and let $P, Q \in X(A)$ such that $(P, Q) \in T_{\to}$, $Q \nsubseteq P$ and $P \in \varphi(a) \cup \varphi(a)_{T_{\to}}$. From **(a)** we have that $\varphi(a) \cup \varphi(a)_{T_{\to}} = \varphi(S(a))$. Then $S(a) \in P$. As $Q \nsubseteq P$, there exists $b \in Q - P$. So from $S(a) \leq b \vee (b \to a) \in P$, we deduce that $b \to a \in P$, and as $(P, Q) \in T_{\to}$, we have that $a \in Q$. ■

Note that if $f\colon \langle X_1, \leq_1 T_1\rangle \to \langle X_2, \leq_2, T_2\rangle$ is a morphism of SWH-spaces, then $f^*\colon D(X_2) \to D(X_1)$ is a homomorphism of SWH-algebras, because $f^{-1}(S_2(U)) = f^{-1}(U \cup U_{T_2}) = f^{-1}(U) \cup f^{-1}(U)_{T_1} = S_1(f^{-1}(U))$, for each $U \in D(X_2)$.

Proposition 5.8. *Let $\langle A, S_A\rangle$ and $\langle B, S_B\rangle$ be SWH-algebras. Let $h\colon A \to B$ be a homomorphism of SWH-algebras. Then $h_*\colon X(B) \to X(A)$ is a SWH-morphism.*

Proof. Write $\langle X(A), \subseteq, T_A\rangle$ and $\langle X(B), \subseteq, T_B\rangle$ for the SWH-spaces of $\langle A, S_A\rangle$ and $\langle B, S_B\rangle$, respectively. It is clear that h_* is a WH-morphism. We note that in the proof of Proposition 5.7 we have proved that $\varphi(S_A(a)) =$

$\varphi(a) \cup \varphi(a)_{T_A}$, for all $a \in A$. Then

$$\begin{aligned}
h_*^{-1}(\varphi(a) \cup \varphi(a)_{T_A}) &= h_*^{-1}(\varphi(S_A(a)) \\
&= \varphi(h(S_A(a))) \\
&= \varphi(S_B(h(a))) \\
&= \varphi(h(a)) \cup \varphi(h(a))_{T_B} \\
&= h_*^{-1}(\varphi(a)) \cup (h_*^{-1}(\varphi(a))_{T_B}.
\end{aligned}$$

Thus, $h_*\colon X(B) \to X(A)$ is a SWH-morphism. ■

PROPOSITION 5.9. *Let $\langle A, S_A \rangle$ be a SWH-algebra. Then $\varphi : \langle A, S_A \rangle \to \left\langle D(X(A)), S_{D(X(A))} \right\rangle$ is an isomorphism of SWH-algebras.*

PROOF. It follows from Proposition 5.5, Lemma 5.6 and Proposition 5.7. ■

PROPOSITION 5.10. *Let $\langle X, T \rangle$ be a SWH-space. Then $\varepsilon\colon \langle X, \leq, T \rangle \to \langle X(D(X)), \subseteq, T_{\Rightarrow} \rangle$ is a SWH-isomorphism.*

PROOF. Here we use propositions 5.5, 5.7 and the fact that ε is an isomorphism of WH-spaces.

We only need to prove that for every clopen upset of $X(D(X))$ it holds that $\varepsilon^{-1}(U_{T_{\Rightarrow}}) = ((\varepsilon^{-1}(U))_T$. Let $x \in \varepsilon^{-1}(U_{T_{\Rightarrow}})$, so $\varepsilon(x) \in U^c$ and $T_{\Rightarrow}(x) \cap U^c \subseteq (\varepsilon(x)]$. In particular $x \in (\varepsilon^{-1}(U))^c$. Let $y \in T(x) \cap \varepsilon^{-1}(U^c)$. Thus $(x, y) \in T$ and $\varepsilon(y) \in U^c$. Then $(\varepsilon(x), \varepsilon(y)) \in T_{\Rightarrow}$, so $\varepsilon(y) \in T_{\Rightarrow}(\varepsilon(x)) \cap U^c$. Hence $\varepsilon(y) \subseteq \varepsilon(x)$, so $y \leq x$ and consequently $x \in ((\varepsilon^{-1}(U))_T$. Conversely let $x \in ((\varepsilon^{-1}(U))_T$, so $x \in \varepsilon^{-1}(U^c)$ and $T(x) \cap \varepsilon^{-1}(U^c) \subseteq (x]$. In particular $\varepsilon(x) \in U^c$. Let $y \in T_{\Rightarrow}(\varepsilon(x)) \cap U^c$. Using that $y = \varepsilon(z)$ for some $z \in X$ we have that $(\varepsilon(x), \varepsilon(z)) \in T_{\Rightarrow}$, so $(x, z) \in T$. Thus $z \in T(x) \cap \varepsilon^{-1}(U^c) \subseteq (x]$, so $z \leq x$ and hence $y = \varepsilon(z) \subseteq \varepsilon(x)$. Therefore $y \in \varepsilon^{-1}(U_{T_{\Rightarrow}})$. ■

THEOREM 5.11. *The category* **SWHS** *is dually equivalent to the category* $\mathbf{FWHA}_S$.

PROOF. It follows from propositions 5.5, 5.7, 5.8, 5.9, 5.10 and the results given in [6] for WH-algebras. ■

The next aim is to study the connection between frontal S-spaces and SWH-spaces.

LEMMA 5.12. *Let $\langle X, \leq, T \rangle$ be a WH-space. Let R be a binary relation on X that satisfies the following conditions for every $U \in D(X)$ and $x \in X$:*

(i) $\bar{T} \subseteq R \subseteq \leq$.

(ii) *If $x \in U^c$ then there exists $y \in U^c$ such that $(x, y) \in T$ and $R(y) \subseteq U$.*

(iii) $\leq \circ R \subseteq R$.

Then for every $U \in D(X)$ it holds that $\tau_R(U) = U \cup U_T$.

Proof. Note that by item (**iii**) we obtain that the set $\tau_R(U)$ is an upset. Let $x \in U \cup U_T$. If $x \in U$, then $x \in \tau_R(U)$. Let $(x, y) \in R$. By condition (**i**) we have that $x \leq y$, so $y \in U$. Hence $U \subseteq \tau_R(U)$. Suppose that $x \in U_T$. Then $x \in U^c$ and $T(x) \cap U^c \subseteq (x]$. By condition (**ii**) there exists $y \in U^c$ with $(x, y) \in T$ and $R(y) \subseteq U$. In particular $y \leq x$. So, from $y \in \tau_R(U)$ and $y \leq x$, we obtain that $x \in \tau_R(U)$.

Conversely. Suppose that $x \in \tau_R(U)$ and $x \in U^c$. By condition (**ii**) there exists $y \in U^c$ such that $(x, y) \in T$. Note that $(x, y) \notin R$, because $y \notin U$. So by condition (**i**) we have that $(x, y) \notin \bar{T}$. Then by the definition of $\bar{T}$ we have $y \leq x$. Thus, $x \in U_T$. ■

Remark 5.13. If $\langle X, \leq, T\rangle$ is a WH-space such that $\bar{T} \subseteq \leq$, then

$$U \cup U_T = \{x \in X : \bar{T}(x) \subseteq U\},$$

for each $U \in D(X)$. Let $x \in U \cup U_T$. If $x \in U$ and $y \in \bar{T}(x)$, then $x \leq y$, so $y \in U$. Let $x \in U^c$, $T(x) \cap U^c \subseteq (x]$ and $y \in \bar{T}(x)$. If $y \in U^c$ then $y \leq x$, a contradiction. Therefore $U \cup U_T \subseteq \{x \in X : \bar{T}(x) \subseteq U\}$. Conversely let $x \in X$ with $\bar{T}(x) \subseteq U$. Suppose that $x \in U^c$ and take $y \in T(x) \cap U^c$. If $y \nleq x$, then $y \in \bar{T}(x) \subseteq U$, and thus $y \in U$, a contradiction. Therefore $\{x \in X : \bar{T}(x) \subseteq U\} \subseteq U \cup U_T$.

Proposition 5.14. *Let $\langle X, \leq, T\rangle$ be a SWH-space. Then there exists a binary relation R_T on X such that $\langle X, \leq, T, R_T\rangle$ is a frontal S-space.*

Proof. We define a binary relation R_T on X in the following way:

$$(x, y) \in R_T \quad \text{iff} \quad \forall U \in D(X) \text{ (if } x \in U \cup U_T \text{, then } y \in U).$$

First we will prove that $R_T(x)$ is a closed upset of X, for each $x \in X$. Let $x, y, z \in X$ with $y \leq z$ and $y \in R_T(x)$. Let $U \in D(X)$ such that $x \in U \cup U_T$. As $y \in R_T(x)$ we have that $y \in U$, and since $y \leq z$, we obtain $z \in U$ because U is an upset. Hence $z \in R_T(x)$. Let $y \notin R_T(x)$. Then there exists $U \in D(X)$ such that $x \in U \cup U_T$ and $y \notin U$. It is clear that $R_T(x) \subseteq U$. Thus $R_T(x)$ is closed.

In the following we will prove that R_T satisfies conditions (**i**), (**ii**) and (**iii**) of Lemma 5.12.

(**i**) Let $(x, y) \in \bar{T}$, so $(x, y) \in T$ and $y \not\leq x$. Let $U \in D(X)$ such that $x \in U \cup U_T$. By item (**c**) of Definition 5.4 we have that $y \in U$, so $\bar{T} \subseteq R_T$. Let $(x, y) \in R_T$ and suppose that $x \not\leq y$. Then there exists $U \in D(X)$ such that $x \in U$ and $y \notin U$. However $(x, y) \in R_T$ and $x \in U \cup U_T$, so $y \in U$, a contradiction. Therefore $R_T \subseteq \leq$.

(**ii**) Let $U \in D(X)$ and $x \in U^c$. Then by item (**b**) of Definition 5.4 there exists $y \in U_T$ such that $(x, y) \in T$. We will prove that $R_T(y) \subseteq U$. Let $(y, z) \in R_T$. Thus $z \in U$ because $y \in U \cup U_T$. Therefore $R_T(y) \subseteq U$.

(**iii**) First observe that for every $U \in D(X)$ we have that $U \cup U_T$ is an upset. In order to prove it, let $x, y \in X$ such that $x \leq y$ and $x \in U \cup U_T$. If $y \in U$ we are done. Suppose that $y \in U^c$, so $x \in U_T$. We will prove that $T(y) \cap U^c \subseteq (y]$. Let $z \in X$ such that $(y, z) \in T$ and $z \in U^c$. Using that $x \leq y$ and that $(y, z) \in T$, we conclude that $(x, z) \in T$. Then $z \in T(x) \cap U^c \subseteq (x]$, so $z \leq x \leq y$. Thus $z \leq y$, so $y \in U_T$. Then for every $U \in D(X)$ we have that $U \cup U_T$ is an upset. Suppose that there exist $x, y \in X$ such that $(x, y) \in \leq \circ R_T$. Then there exists $z \in X$ such that $x \leq z$ and $(z, y) \in R_T$. Suppose that $z \in U \cup U_T$. We need to prove that $y \in U$. Since $U \cup U_T$ is an upset, $z \in U \cup U_T$ and $(z, y) \in R_T$, we obtain that $y \in U$. Thus, $\leq \circ R_T \subseteq R_T$. From Lemma 5.12 we have that $\tau_{R_T}(U) \in D(X)$, for every $U \in D(X)$. Therefore $\langle X, \leq, T, R_T \rangle$ is a frontal S-space. ■

The frontal S-space $\langle X, \leq, T, R_T \rangle$ built in the previous proof will be called the *associated frontal S-space* of the SWH-space $\langle X, \leq, T \rangle$. Note that $\tau_{R_T}(U) = U \cup U_T$, for each $U \in D(X)$.

PROPOSITION 5.15. *Let $\langle X, \leq, T, R \rangle$ be a frontal S-space. Then $\langle X, \leq, T \rangle$ is a SWH-space such that $R = R_T$.*

PROOF. We will prove the conditions of Definition 5.4.

(**a**) It follows from the fact that for the WH-space $\langle X, \leq, T \rangle$ the relation R satisfies the conditions of Lemma 5.12.

(**b**) Let $x \in U^c$. By Definition 5.1 there exists $y \in U^c$ such that $(x, y) \in T$ and $R(y) \subseteq U$. We will prove that $y \in U_T$. Let $z \in T(y) \cap U^c$. In particular $z \notin R(y)$, and as $\bar{T} \subseteq R$, we have $(y, z) \notin \overline{T}$. Thus $z \leq y$, and consequently $y \in U_T$. Therefore $y \in T(x) \cap U_T$, i.e., $T(x) \cap U_T \neq \emptyset$.

(**c**) Let $(x, y) \in \bar{T}$ and $x \in U \cup U_T$. Then $(x, y) \in T$ and $y \not\leq x$. If $x \in U$ then $y \in U$ because $x \leq y$. Let $x \in U_T$. Thus $x \in U^c$ and $U^c \cap T(x) \subseteq (x]$. If $y \in U^c$ then $y \in U^c \cap T(x)$, so $y \leq x$, a contradiction.

Therefore $\langle X, \leq, T \rangle$ is a SWH-space.

We will prove that $R \subseteq R_T$. Let $(x, y) \in R$ and $x \in U \cup U_T$. By Lemma 5.12, $\tau_R(U) = U \cup U_T$, and as $x \in \tau_R(U)$, we have $R(x) \subseteq U$. Therefore

$y \in U$.

We will prove that $R_T \subseteq R$. From Lemma 5.12 we have that

$$\tau_R(U) = U \cup U_T = \tau_{R_T}(U)$$

for each $U \in D(X)$. Let $(x, y) \in R_T$ and suppose that $(x, y) \notin R$. As $R(x)$ is a closed upset, there exists $U \in D(X)$ such that $R(x) \subseteq U$ and $y \notin U$. So, $x \in \tau_R(U) = \tau_{R_T}(U)$, i.e., $R_T(x) \subseteq U$ which is a contradiction. ∎

In the next proposition we show that a morphism between two *SWH*-spaces can be characterized as a *WH*-morphism that is a p-morphism with respect to the associated frontal S-spaces.

Proposition 5.16. *Let $\langle X_1, \leq_1, T_1\rangle$ and $\langle X_2, \leq_2, T_2\rangle$ be two SWH-spaces. Let $f\colon X_1 \to X_2$ be a WH-morphism. Then f is a SWH-morphism iff f is a p-morphism between the associated frontal S-spaces $\langle X_1, \leq_1, T_1, R_{T_1}\rangle$ and $\langle X_2, \leq_2, T_2, R_{T_2}\rangle$.*

Proof. Let $\langle X_1, \leq_1, T_1\rangle$ and $\langle X_2, \leq_2, T_2\rangle$ be two *SWH*-spaces. The relations R_{T_1} and R_{T_2} will be written as R_1 and R_2, respectively.

$\Rightarrow$) Assume that $f\colon X_1 \to X_2$ is a *SWH*-morphism. Suppose that $(x, y) \in R_1$ but $f(y) \notin R_2(f(x))$. As $R_2(f(x))$ is a closed upset, there exists $U \in D(X_2)$ such that $R_2(f(x)) \subseteq U$ and $y \notin f^{-1}(U)$. So, $x \in f^{-1}(S_{R_2}(U)) = S_{R_1}(f^{-1}(U))$, and thus $R_1(x) \subseteq f^{-1}(U)$. But it implies that $y \in f^{-1}(U)$, which is a contradiction.

Assume that $(f(x), z) \in R_2$. Suppose that $f(y) \nleq_2 z$ for each $y \in R_1(x)$. Then for each $y \in R_1(x)$ there exists $U_y \in D(X_1)$ such that $y \in f^{-1}(U_y)$ and $z \notin U_y$. So, $R_1(x) \subseteq \bigcup \{f^{-1}(U_y) : y \in R_1(x)\}$. As $R_1(x)$ is closed, it is compact. Then there exists a finite sequence $U_{y_1}, ..., U_{y_n} \in D(X_1)$ such that

$$R_1(x) \subseteq f^{-1}(U_{y_1}) \cup \ldots \cup f^{-1}(U_{y_n}) = f^{-1}(U),$$

where $U = U_{y_1} \cup \ldots \cup U_{y_n}$. So, $x \in S_{R_1}(f^{-1}(U)) = f^{-1}(S_{R_2}(U))$, and consequently $R_2(f(x)) \subseteq U$. But this implies that $y \in U$, which is a contradiction. Thus, f is a p-morphism.

$\Leftarrow$) From Theorem 2.4 we conclude that if f is a p-morphism then $S_{R_1}(f^{-1}(U)) = f^{-1}(S_{R_2}(U))$, for each $U \in D(X_1)$. Therefore f is a SWH-morphism. ∎

Proposition 5.17. *Let $\langle X_1, \leq_1, R_1, T_1\rangle$ and $\langle X_2, \leq_2, R_2, T_2\rangle$ be two frontal S-spaces. Let $f\colon X_1 \to X_2$ be a WH-morphism. Then f is a p-morphism iff f is a SWH-morphism.*

PROOF. The proof is similar to the previous proof. ■

Then we have the following

THEOREM 5.18. *The categories* **SWHS** *and* $\mathbf{FWHS}_S$ *are isomorphic.*

6. Representation theory for *WH*-algebras with gamma

DEFINITION 6.1. A frontal *WH*-space $\langle X, \leq, T, R\rangle$ is a *frontal γ-space* if the following conditions are satisfied:

(γ_1) For every $x \in X$ there exists $y \in X$ such that $(x, y) \in T$ and $R(y) = \emptyset$.

(γ_2) For every $x \in X$, $R(x) = \emptyset$ or $x \in R(x)$.

The category $\mathbf{FWHS}_\gamma$ is that whose objects are frontal γ-spaces and whose morphisms are the same as in **FWHS**.

LEMMA 6.2. *Let A be a WH-algebra and let $\gamma\colon A \to A$ be a function. Then the pair $\langle A, \gamma\rangle$ is a γWH-algebra iff $\langle X(A), \subseteq, T_\to, R_\gamma\rangle$ is a frontal γ-space.*

PROOF. $\Rightarrow$) The proof of condition (γ_1) of Definition 6.1 is similar to the proof of Lemma 5.2 (taking $a = 0$). In order to prove condition (γ_2) of Definition 6.1, let $P \in X(A)$ such that $R_\gamma(P) \neq \emptyset$. It implies that $\gamma^{-1}(P) \neq A$. As $\gamma^{-1}(P)$ is a proper filter, $0 \notin \gamma^{-1}(P)$. We will prove that $\gamma^{-1}(P) \subseteq P$. Let $\gamma(a) \in P$. Then $a \vee \gamma(0) \in P$, and as $0 \notin \gamma^{-1}(P)$, we have $a \in P$. Thus $(P, P) \in R_\gamma$.

$\Leftarrow$) By Lemma 5.2 taking $a = 0$ we have that $\gamma(0) \to 0 = 0$. We will prove that $\gamma(a) \leq a \vee \gamma(0)$, for any $a \in A$. Suppose that $\gamma(a) \nleq a \vee \gamma(0)$, then there exists $P \in X(A)$ such that $\gamma(a) \in P$, $a \notin P$ and $\gamma(0) \notin P$. Hence $\gamma^{-1}(P)$ is a proper filter, i.e., $R_\gamma(P) \neq \emptyset$. Then $(P, P) \in R_\gamma$, and as $\gamma(a) \in P$, we obtain $a \in P$, which is a contradiction. Therefore, $\langle A, \gamma\rangle$ is a γ*WH*-algebra. ■

By Lemma 6.2 and Theorem 4.6 we have the following

THEOREM 6.3. *The category* $\mathbf{FWHA}_\gamma$ *is dually equivalent to the category* $\mathbf{FWHS}_\gamma$*.*

Recall that if X is a set and $T \subseteq X \times X$, then $\emptyset_T = \{x \in X : T(x) \subseteq (x]\}$. In what follows we will provide an alternative duality for the category of γ*WH*-algebras.

Definition 6.4. A *WH-space with gamma*, or γWH-space, is a WH-space $\langle X, \leq, T\rangle$ satisfying the following conditions:

(a) $U \cup \emptyset_T \in D(X)$, for each $U \in D(X)$.

(b) $T(x) \cap \emptyset_T \neq \emptyset$, for every $x \in X$.

(c) If $(x, y) \in \bar{T}$ and $x \in U \cup \emptyset_T$, then $y \in U$.

Let γ**WHS** be the category whose objects are γWH-spaces $\langle X, \leq, T\rangle$ and whose morphisms are WH-morphisms $f\colon \langle X_1, \leq_1 T_1\rangle \to \langle X_2, \leq_2, T_2\rangle$ such that $f^{-1}(U \cup \emptyset_{T_2}) = f^{-1}(U) \cup f^{-1}(\emptyset)_{T_1}$, for each $U \in D(X_2)$. These morphisms will be called *γWH-morphisms*.

Proposition 6.5. *If $\langle X, T\rangle$ is a γWH-space, then $\langle D(X), \cup, \cap, \Rightarrow, \gamma, \emptyset, X\rangle$ is a γWH-algebra, where γ is defined as $\gamma(U) = U \cup \emptyset_T$, for each $U \in D(X)$.*

Proof. The proof is similar to the proof of Proposition 5.5. ■

Lemma 6.6. *Let $\langle A, \gamma\rangle$ be a γWH-algebra. For every $a \in A$ we have that $\varphi(a) \cup \varphi(0)_{T_\to} = \varphi(\gamma(a))$.*

Proof. The proof is similar to the proof of Lemma 5.6. ■

Proposition 6.7. *If $\langle A, \gamma\rangle$ is a γWH-algebra, then $\langle X(A), \subseteq, T_\to\rangle$ is a γWH-space.*

Proof. The proof is similar to the proof of Proposition 5.7. Condition **(a)** follows from Lemma 6.6. To prove condition **(b)** of Definition 6.4 we take a prime filter P in A. Since $\gamma(0) \to 0 = 0 \notin P$, there exists $Q \in X(A)$ such that $(P, Q) \in T_\to$ and $\gamma(0) \in Q$. We will prove that $T_\to(Q) \subseteq (Q]$. Suppose that there exists $D \in T_\to(Q)$ but $D \nsubseteq Q$. Then there exists $b \in D - Q$. As $\gamma(0) \leq b \vee (b \to 0) \in Q$, we have that $b \to 0 \in Q$. As $D \in T_\to(Q)$ and $b \in D$, we get $0 \in D$, which is impossible. Thus, $T_\to(P) \cap \emptyset_{T_\to} \neq \emptyset$. It is condition **(b)** of Definition 6.4. Finally we will prove condition **(c)** of Definition 6.4. Let $P, Q \in X(A)$, and $a \in A$ such that $(P, Q) \in T_\to$, $Q \nsubseteq P$, and $P \in \varphi(a) \cup \varphi(0)_{T_\to} = \varphi(\gamma(a))$. Then $\gamma(a) \in P$, and there exists $b \in Q - P$. From $\gamma(a) \leq b \vee (b \to a) \in P$ and $(P, Q) \in T_\to$, we obtain $a \in Q$, i.e., $Q \in \varphi(a)$. ■

Note that if $f\colon \langle X_1, \leq_1 T_1\rangle \to \langle X_2, \leq_2, T_2\rangle$ is a morphism of γWH-spaces, then $f^*\colon D(X_2) \to D(X_1)$ is a homomorphism of γWH-algebras, because $f^{-1}(\gamma(U)) = f^{-1}(U \cup \emptyset_{T_2}) = f^{-1}(U) \cup f^{-1}(\emptyset)_{T_1} = \gamma(f^{-1}(U))$, for each $U \in D(X_2)$.

PROPOSITION 6.8. *Let $\langle A, \gamma_A\rangle$ and $\langle B, \gamma_B\rangle$ be γ WH- algebras. Let $h\colon A \to B$ be a homomorphism of γ WH-algebras. Then $h_*\colon X(B) \to X(A)$ is a γ WH-morphism.*

PROOF. The proof is similar to the proof of Proposition 5.8. ■

PROPOSITION 6.9. *Let $\langle A, \gamma_A\rangle$ be a γ WH-algebra. Then $\varphi : \langle A, \gamma_A\rangle \to \left\langle D(X(A)), \gamma_{D(X(A))}\right\rangle$ is an isomorphism of γ WH-algebras.*

PROOF. It follows from Proposition 6.5, Lemma 6.6 and Proposition 6.7. ■

PROPOSITION 6.10. *Let $\langle X, T\rangle$ be a γ WH-space. Then $\varepsilon\colon \langle X, \leq, T\rangle \to \langle X(D(X)), \subseteq, T_{\Rightarrow}\rangle$ is a γ WH-isomorphism.*

PROOF. The proof is similar to the proof of Proposition 5.10. ■

Then we have the following

THEOREM 6.11. *The category γ**WHS** is dually equivalent to the category **FWHA**$_\gamma$.*

In the following we will study the connection between frontal γ-spaces and γ WH-spaces.

LEMMA 6.12. *Let $\langle X, \leq, T\rangle$ be a WH-space and R a binary relation on X such that the following conditions are satisfied:*

(i) $\bar{T} \subseteq R \subseteq \leq$.

(ii) *For every $x \in X$ there exists $y \in X$ such that $(x, y) \in T$ and $R(y) = \emptyset$.*

(iii) $\leq \circ R \subseteq R$.

(iv) *For every $x \in X$, $R(x) = \emptyset$ or $x \in R(x)$.*

Then for every $U \in D(X)$ it holds that $\tau_R(U) = U \cup \emptyset_T$.

PROOF. Let $U \in D(X)$. The inclusion $U \cup \emptyset_T \subseteq \tau_R(U)$ can be proved using the same idea as in the proof of Lemma 5.12. Conversely let $x \in \tau_R(U)$ and suppose that $x \notin \emptyset_T$, so there is $y \in X$ such that $y \in T(x)$ and $y \nleq x$. As $\bar{T} \subseteq R$, we have $y \in R(x)$. By condition (**iv**) we have $x \in R(x)$, and as $x \in \tau_R(U)$, we obtain that $x \in U$. ■

PROPOSITION 6.13. *Let $\langle X, \leq, T\rangle$ be a γ WH-space. Then there exists a binary relation R_T in X such that $\langle X, \leq, T, R_T\rangle$ is a frontal γ-space.*

Proof. We define a binary relation on X in the following way:

$$(x, y) \in R_T \quad \text{iff} \quad \forall U \in D(X)(\text{if } x \in U \cup \emptyset_T, \text{ then } y \in U).$$

Note that $R_T(x)$ is a closed upset of X, for every $x \in X$.

We will prove that R_T satisfies conditions (**i**)-(**iv**) of Lemma 6.12.

(**i**) The fact that $\bar{T} \subseteq R_T$ is consequence of (**c**) of Definition 6.4, and $R_T \subseteq\, \leq$ is proved like in Lemma 5.12. The proof of the items (**ii**) and (**iii**) is similar to the proof of Lemma 5.12.
(**iv**) Let $x \in X$. Suppose that $x \notin R_T(x)$. Hence there exists $V \in D(X)$ such that $x \in V \cup \emptyset_T$ and $x \notin V$. So $x \in \emptyset_T$, i.e., $T(x) \subseteq (x]$. By condition (**ii**) of Lemma 6.12 we have that there is $y \in T(x)$ with $R(y) = \emptyset$. Thus $y \leq x$, so $R_T(x) \subseteq R_T(y) = \emptyset$. Hence, $R_T(x) = \emptyset$.

It follows from Lemma 6.12 that $\tau_{R_T}(U) \in D(X)$ for every $U \in D(X)$. Therefore $\langle X, \leq, T, R_T \rangle$ is a frontal γ-space. ■

The frontal γ-space $\langle X, \leq, T, R_T \rangle$ built in the previous proof will be called the *associated frontal γ-space* of the γWH-space $\langle X, \leq, T \rangle$. Note that $\tau_{R_T}(U) = U \cup \emptyset_T$, for each $U \in D(X)$.

Proposition 6.14. *If $\langle X, \leq, T, R \rangle$ is a frontal γ-space, then $\langle X, \leq, T \rangle$ is a γWH-space such that $R = R_T$.*

Proof. (**a**) It follows from Lemma 6.12.

(**b**) Let $x \in X$. Then by condition (γ_1) there exists $y \in X$ such that $(x, y) \in T$ and $R(y) = \emptyset$. We will prove that $y \in T(x) \cap \emptyset_T$. Let $z \in T(y)$, so $(y, z) \in T$. In particular $z \notin R(y)$, so $z \leq y$ (because if $z \not\leq y$ then we have that $(y, z) \in R$, a contradiction). Thus $y \in \emptyset_T$.

(**c**) Let $U \in D(X)$, $(x, y) \in \bar{T}$ and $x \in U \cup \emptyset_T$. If $x \in \emptyset_T$ then $y \leq x$, a contradiction. Thus $x \in U$. Besides $x \leq y$, so $y \in U$.

The proof that $R = R_T$ is similar to the proof of Proposition 5.15. ■

The proofs of the following two propositions are similar to the proofs of propositions 5.16 and 5.17, respectively.

Proposition 6.15. *Let $\langle X_1, \leq, T_1 \rangle$ and $\langle X_2, \leq, T_2 \rangle$ be two γWH-spaces. Let $f \colon X_1 \to X_2$ be a WH-morphism. Then f is a γWH-morphism iff f is a p-morphism between the associated frontal γ-spaces $\langle X_1, T_1, R_{T_1} \rangle$ and $\langle X_2, T_2, R_{T_2} \rangle$.*

Proposition 6.16. *Let $\langle X_1, \leq_1, R_1, T_1 \rangle$ and $\langle X_2, \leq_2, R_2, T_2 \rangle$ be two frontal γ-spaces. Let $f \colon X_1 \to X_2$ be a WH-morphism. Then f is a p-morphism iff f is a γWH-morphism.*

Then we have the following

THEOREM 6.17. *The categories $\gamma\mathbf{WHS}$ and $\mathbf{FWHS}_\gamma$ are isomorphic.*

7. Final remarks

A WH-algebra is a Heyting algebra if and only if the relation on its dual space is the inclusion relation (Theorem 4.24 of [6]). In particular, we have that if $\langle X, \leq, T\rangle$ is a WH-space, then $\langle X, \leq\rangle$ is an Esakia space if and only if $T = \leq$. By this fact, and using the results of the previous sections, we can obtain a similar result to that given in Theorem 3.10 of [3]. In the aforementioned article the categories **SHA** (Heyting algebras with successor) and $\mathbf{SH}_S$ (a particular subcategory of the category of Esakia spaces) were defined, and the existence of a categorical dual equivalence between them was proved. Moreover, if $\langle X, \leq\rangle \in \mathbf{SH}_S$, then in $D(X)$ the successor function takes the form

$$S(U) = U \cup (U^c)_M,$$

where $(U^c)_M$ is the set of maximal elements of U^c. This result can be seen as a particular case of Theorem 5.11 since if $\langle X, \leq, T\rangle$ is a WH-space with $T = \leq$, then for every $U \in D(X)$, we have that $(U^c)_M = U_T$.

Note that if $\langle X, \leq, T\rangle$ is a WH-space, then $(U^c)_M \subseteq U_T$. This fact implies that if $\langle A, \rightarrow_1, S_1\rangle$ is a WH-algebra with successor and $\langle A, \rightarrow_2, S_2\rangle$ is a Heyting algebra with successor, then $S_2(x) \leq S_1(x)$ for every $x \in A$. For instance, if we consider Example 3.8 with $\rightarrow_1 = \hat{\rightarrow}$ then $S_2(0) < S_1(0)$.

There are similar results for the case of WH-algebras with gamma.

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